Market Dynamics of Risk-On and Risk-Off Incentives and their Effect on Asset Prices

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Abstract

Mutual fund asset managers' incentives contracts are dynamic: initially, there is a drive to outperform the index. However, as performance improves, the focus shifts from outperforming to maintaining performance. In equilibrium, expressed in closed form, the active share exhibits a U-shape, and the tracking error an inverted U-shape with the active share at its minimum and tracking error at its maximum when performance equals the benchmark. Equilibrium response to frictions introduces two predictability channels due to changes in incentives and performance uncertainty. These two channels lead to momentum in outperformance and reversal in underperformance, both in the time-series and cross-section.

1 Introduction

Institutional investors hold a significant proportion of publicly traded stocks and account for an even larger share of daily trading volume. Consequently, their actions play a crucial role in determining stock prices, and understanding the incentives behind these actions is essential for comprehending price dynamics. In March 2005, the U.S. Securities and Exchange Commission (SEC) implemented a rule requiring mutual funds to disclose the compensation

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structure of their portfolio managers in the Statement of Additional Information. These disclosures indicate that nearly all mutual fund managers' contracts include a performance-based bonus tied to the funds' performance relative to their benchmark index.

This paper develops a dynamic equilibrium model with benchmarking incentives that incorporates a critical element the literature has for the most part overlooked: the incentives to outperform the benchmark (risk-on) are powerful when asset managers underperform in a bid to get the bonus. However, as performance exceeds the benchmark, the incentives to outperform the benchmark become less relevant, and instead, asset managers' incentives shift to maintaining their performance (risk-off) to avoid losing the bonus.

Our first main finding demonstrates that in equilibrium active share exhibits a U-shaped relationship with respect to fund performance, while tracking error shows an inverted U-shaped relationship. As a mutual fund's performance deviates from the benchmark performance, either towards outperformance or underperformance, its active share tends to increase and its tracking error tends to diminish.

Cremers and Petajisto (2009) argue that tracking error serves as a proxy for a factor bet strategy, whereas active share represents a stock selection strategy. The factor bet strategy involves rotating across systemic factors such as sectors and industries. Conversely, the stock selection strategy focuses on choosing individual stocks across all investment classes, rather than systemic factors. Their analysis indicates that the factor bet strategy results in a higher tracking error and lower active share compared to the stock selection strategy. Our theory claims that the variations in active share and tracking error among different funds may be attributed to their performance levels rather than to inherent differences in their investment strategies.

This finding also rationalizes conflicting empirical findings. Hu, Kale, Pagani, and Subramanian (2011) argue a U-shaped relationship between performance in the first 6 months of the year and the fund manager's risk choices in the second half of the year. In contrast, Lee, Trzcinka, and Venkatesan (2019) claim that their findings contrast sharply with the previous findings of Hu et al. (2011) since they document an inverted U-shaped relationship between the performance in the first 6 month and a risk shifting measure in the second half of the year. Our theoretical findings suggest that these two predictions are completely consistent with each other since the risk choice measure of Hu et al. (2011) aligns with the active share, whereas the risk-shift measure of Lee et al. (2019) correlates with the tracking error.

Our second contribution highlights asset management frictions as a potential reason for the observed momentum and reversal in asset markets, both in the time-series and cross-section. We demonstrate that momentum and reversal naturally arise in equilibrium within a complete information environment, where perfectly rational asset managers are incentivized to outperform the benchmark when they are trailing the benchmark, but shift towards maintaining performance when they are outperforming it.

We identify time-series momentum when asset managers outperform the benchmark and time-series reversal when they underperform the benchmark. Time-series momentum is the tendency for expected returns (and future returns) to increase when current returns are rising. Conversely, time-seires reversal is the tendency for expected returns (and future returns) to decrease when current returns are falling.

Furthermore, we show that the predictability patterns obsereved in the time-series of each asset separately also exist in the long-short portfolio. Since the work of Jegadeesh and Titman (1993), the analysis of cross-sectional predictability in asset returns has involved constructing long-short portfolios by purchasing assets with high returns and selling those with low returns. According to this framework, momentum is identified when the long-short portfolio yields a positive return, whereas reversal is identified when the portfolio yields a negative return. Based on our theory, we expect momentum in the long-short portfolio when asset managers outperform their benchmark index and reversal when they underperform it.

Given their time-varying incentives, asset managers' payoffs are convex when they underperform the benchmark but become concave as their returns exceed the benchmark. This results in an S-shaped objective, where the inflexion (or reference) point is the benchmark. Despite this added complexity, we characterize equilibrium prices in closed form.

The S-shaped objective function and its pricing implications are novel in asset management. It was originally designed to explain financial decisions of households that appeared at odds with the predictions of standard utility models. Friedman and Savage (1948) demonstrate that for an agent to both purchase insurance and a lottery ticket, the agent's utility must shift from concave at certain wealth levels to convex at others. Levy (1969) demonstrates that an objective function that only depends on the first three moments of the wealth's distribution leads to an inverted S-shape objective, concave below a reference and convex above it. Kahneman and Tversky (1979) show that an S-shape objective arises when risk-aversion is different towards losses and gains — so that the reference point is current wealth.

The key distinction between these models and ours is that in our model the benchmark, and not the value of the assets under management, is the reference point. This difference significantly enhances the tractability of the equilibrium characterization.

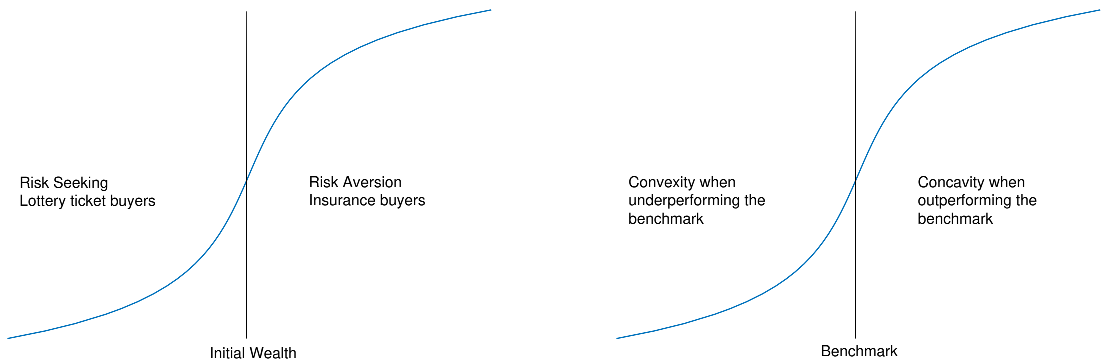

Figure 1. The figure on the left shows that an agent prefers a fair gamble (such as purchasing a lottery ticket) when their utility is convex with respect to absolute wealth. Conversely, when their utility is concave with respect to wealth, the agent prefers to pay a premium to avoid a fair gamble. The figure on the right demonstrates a similar intuition for asset managers. They are incentivized to gamble when their wealth is below the benchmark and they are underperforming. In contrast, they are incentivized to purchase insurance and maintain their wealth when it is above the benchmark.

Generally, pricing predictability is surprising because conventional wisdom suggests that if market participants expect prices to rise, they will quickly buy the asset to capture the return, causing an immediate price increase that neutralizes the expected return. Similarly, if they expect prices to fall, they will sell the asset immediately to avoid losses, causing an immediate price drop that neutralizes the expected decline in return. Consequently, the main explanations for momentum and reversal involve either an asymmetric information environment leading to partial price adjustments or irrational investor behavior. In these scenarios, momentum and reversal occur as participants gradually adjust their positions based on their observations of the market.

The return predictability analyzed in this paper is fully rational and stems directly from the no-arbitrage conditions, where all available information is fully reflected in prices at any given time. Predictability persists in this environment because it is not optimal for asset managers to trade on it. In this setting, asset managers respond immediately to information, but their reactions depend on their current performance relative to the benchmark. These reactions shape return predictability patterns through equilibrium, resulting in patterns that emerge endogenously.

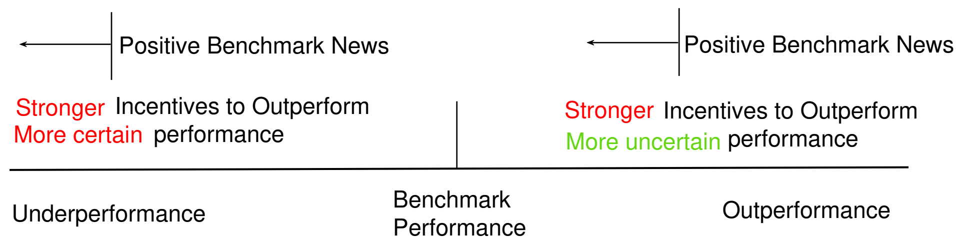

Figure 2. This figure illustrates the asymmetric reaction to news during periods of underperformance and outperformance. The left side of the figure represents scenarios where asset managers underperform, while the right side cases of outperformance. In both cases, positive benchmark news negatively impacts the performance of asset managers. For those outperforming, it reduces their outperformance, while for those underperforming, it exacerbates their underperformance. Lower performance leads to stronger incentives to outperform the benchmark. In contrast, the uncertainty of performance increases for outperforming asset managers but decreases for those underperforming.

There are two equilibrium predictable channels of asset mangers' performance. The first channel involves *incentives*: the current performance predicts the incentives to outperform the benchmark relative to maintaining performance. When performance is below the benchmark, the incentives to outperofrm the benchmark are stronger relative to the incentives to maintain performance, whereas when performance is above the benchmark, the incentives to maintain perfromance are stronger. The second equilibrium channel is the *uncertainty of performance*: When performance diverges from the benchmark, performance is more ceratin. Conversly, when performance converges to the benchmark, it becomes less certain whether the asset manager will perform. These two predictable channels lead to an asymmetric price reaction to news during periods of underperformance and outperformance.

The incentives channel leads to reversal in returns for any level of performance. In contrast, the uncertainty of performance channel leads to momentum when asset managers outperform the benchmark. Consequently, when asset managers underperform, the two predictability channels result in an eventual reversal in returns, whereas when asset managers outperform the benchmark, the overall effect leads to eventual momentum in asset returns.

The drive to outperform the benchmark prompt asset managers to maintain a benchmark hedging position to avoid lagging behind. This hedging generates demand for the benchmark, independent of risk and return considerations. Consequently, this reaction to incentives boosts the benchmark asset price while lowering the prices of assets outside the benchmark. An unpredictable shock that worsens performance relative to the benchmark initially raises the benchmark price and depresses the prices of non-benchmark assets due to the increased demand for the benchmark. However, this shock subsequently reduces the benchmark's expected return and raises the expected returns of other assets, as the market anticipates stronger future price reactions to shocks when the incentives to outperform the benchmark are heightened. The incentives channel thus leads to asset price reversals.

Performance uncertainty results in greater sensitivity of assets to news. From an economic perspective, the asset managers' portfolio is not perfectly aligned with the benchmark. Therefore, when news arrives that aligns the portfolio's performance more closely with the benchmark, it increases performance uncertainty. This, through equilibrium, leads to riskier asset markets.

The equilibrium market prices of risk lead to an increase in expected returns for the benchmark asset and a decrease in expected returns for the non-benchmark assets in both outperformance and underperformance regions. However, when performance further deteriorates, the pricing effects reverse, leading to a decrease in expected returns for the benchmark asset and an increase in expected returns for the non-benchmark assets.

When asset managers outperform the benchmark, an unpredictable shock that raises the benchmark price initially would also increase the expected return of the benchmark because higher benchmark prices imply that the asset manager's relative performance shrinks. The market expects bigger price reactions in the future due to this higher uncertainty of performance, which leads to higher expected returns when asset managers are more likely to outperform. The same shock decreases the non-index prices and their expected returns. As a result, the uncertainty of performance channel generates momentum when asset managers outperform the benchmark.

Conversely, when asset managers underperform the benchmark, an unpredictable shock that raises the benchmark price initially would decrease the expected return of the benchmark because higher benchmark prices imply that performance deteriorates further below the benchmark. The market expects smaller price reactions in the future due to this lower uncertainty of performance. The same shock decreases the non-index prices and increases their expected returns. Therefore, the uncertainty of performance channel generates reversal when asset managers underperform the benchmark. When performance further deteriorates, the market prices of risk reverse, increasing the expected returns of the benchmark and decreasing the expected returns of the non-benchmark assets. As a result, this channel leads to momentum when performance further declines.

The remainder of the paper is organized as follows. Section 2 summarizes the related literature; Section 3 sets up the economy with the time-varying incentives to outperform and maintaining performance ; Section 4 solves for the equilibrium in closed-form; Section 5 discusses the active share and tracking error; Section 6 discusses momentum and reversal in the time-series; Section 7 discusses momentum and reversal in the cross-section, and Section 8 concludes.

2 Related Literature

This paper fits into the asset pricing literature investigating stock returns predictable patterns, and in particular, the drivers of momentum and reversal in stock returns. Jegadeesh and Titman (1993) first documented momentum and reversal in the cross section of the stock market, and Chan, Jegadeesh, and Lakonishok (1996) have thoroughly investigated the phenomenon. They showed that a portfolio that buys past winners and sells past losers will generate a significant positive return over a short-term window of one year or less, and that the strategy's performance partially reverts for longer horizons.

There are several competing theories, falling into a few broad categories, which explain momentum. One of the most prevalent explanations of momentum is behavioural. In this category, Barberis, Shleifer, and Vishny (1998) studies an agent that due to behavioural biases incorrectly forms beliefs about stock earnings. Daniel, Hirshleifer, and Subrahmanyam (1998) study investors that overreact to confirming evidence and underreact to refuting evidence about their prior signal. Hong and Stein (1999) study a two agents model where agents have different restrictions on the way they process information. Barberis and Shleifer (2003) study an economy with two investors. One of these investors groups assets into styles and gradually moves funds from the worst performing style to the best performing style, while the second investor does not act on the predictable pattern. Lastly, Hillert, Jacobs, and Müller (2014) find that news coverage exacerbate momentum because investors' biases are stronger.

The most prevalent rational explanation is based on an information economy. In Holden and Subrahmanyam (2002), the setup induces a sequential information acquisition that feeds into prices. Allen, Morris, and Shin (2006) assumes that agents, instead of forecasting prices, forecast future forecasts. Shin (2006) endogenizes firms' disclosure decision jointly with prices. Banerjee, Kaniel, and Kremer (2009) investigate a heterogenous beliefs economy, where agents agree to disagree about their beliefs. Biais, Bossaerts, and Spatt (2010) investigates an overlapping generation setup with informed and uninformed traders. Cespa and Vives (2012) compares consensus opinion and prices and show that specific patterns of information generates momentum and reversal. Albuquerque and Miao (2014) shows that information about future earnings that is independent from current earning induces momentum and reversal. Ottaviani and Sørensen (2015) investigate a heterogenous beliefs economy where traders exhibit wealth constraints. Andrei and Cujean (2017) show that short-term momentum can be generated by an economy where private information flows at an increasing rate. Cujean and Hasler (2017) investigate an economy with heterogenous agents where some agents continuously update their beliefs while others do so intermittently.

Another strand of literature connects momentum to firms' specific attributes, such as growth options, dividend growth rate, revenues, and costs. Berk, Green, and Naik (1999), Johnson (2002), and Sagi and Seasholes (2007), do so in a partial equilibrium setup. Relatedly, Liu and Zhang (2008) identifies the loading on the growth rate of industrial production as the source for momentum, and Liu and Zhang (2014) study a partial equilibrium production based setup to study momentum.

More recent studies connect the asset management industry to momentum and reversal. This strand of literature is the most relevant to this paper. Dasgupta, Prat, and Verardo (2011) assume that investors observe asset managers' actions and form beliefs about their ability to perform. Asset managers internalize the reputational cost of underperformance, which causes a herding behavior that positively predicts short-term returns. Vayanos and Woolley (2013) suggest that a slow-moving flow between active and passive funds due to active funds' performance is the source of momentum and reversal.

Our paper is the first to show that a predictable pattern in asset managers' risk preferences generates momentum and reversal and that the predictable changes to asset managers' risk preferences arise due to the time-varying nature of their incentive contracts. In particular, the incentives to outperform the benchmark are stronger for asset managers who underperform. However, as performance improves, the incentives to outperform the benchmark become less relevant, and instead, asset managers' incentives shift to maintain their performance in a bid not to lose the bonus.

Furthermore, our article contributes to the expanding body of literature examining the impact of professional asset management on asset prices. In particular, our research aligns with a strand of this literature that investigates the asset pricing implications resulting from benchmarking incentives.

The paper most closely related to ours is Sotes-Paladino and Zapatero (2019). Their setup introduces a kink to the performance base fees of the asset manager, which changes the sensitivity of the objective to the relative (to the benchmark) wealth. Similar to our setup, the kink represents a shift in the performance sensitivity between the end-of-period underperformance and outperformance regions. While the kink introduces a convexity to the asset manager's objective, the slope at the outperformance region could be concave, which aligns with our setup. They find that the asset manager overinvests in assets outside the benchmark and underinvests in the benchmark, which aligns with our predictions when asset managers are likely to outperform (the risk-off region). The main difference from our paper is that they investigate the asset manager's behavior, given prices, in a partial equilibrium setup, while our setup focuses on the pricing implications in an equilibrium setup.

By and large, the asset management literature investigates a setup where asset managers' objective is always convex and does not become concave when outperformance is likely in a bid to maintain performance. The idea to embed the benchmark into the asset manager's objective started by Brennan (1993) and Gómez and Zapatero (2003). Cuoco and Kaniel (2011) were the first to study the equilibrium implications of benchmarking incentives through performance-based fees. Later, Basak and Pavlova (2013) and Basak and Pavlova (2016) introduce a reduced form approach to incorporate the benchmark, which allows for much tractability. Buffa and Hodor (2023) extended their setup to investigate the pricing implications of heterogenous benchmarking incentives and Hodor and Zapatero (2023) to investigate the pricing implications of heterogenous horizons.

Unlike the literature, we investigate the pricing implications when the asset manager's benchmarking incentives are time-varying. The asset manager exhibits incentives to outperform the benchmark when underperformance is likely (similar to the literature), but these incentives weaken, and eventually flips when outperformance is more likely, in a bid to maintain performance (instead of outperforming the benchmark).

Lastly, our paper relates to Friedman and Savage (1948) early work. They investigated a case whereby a wealthy individual is willing to pay an insurance premium to avoid a fair gamble with an equal chance of winning or losing a dollar, while a poor individual is willing to purchase a lottery ticket to participate in such a fair gamble. They introduced the utility characteristics that jointly induce these two behaviors in a rational expected utility maximizing framework. Rationalizing these behaviors implies that when the individual wealth is low, the agent's marginal utility increases with wealth, and the utility is convex. In contrast, when individual wealth is high, the agent's marginal utility decreases with wealth, and utility becomes concave, eventually leading to an S-shape utility function: a convex segment followed by a concave segment.

We incorporate Friedman and Savage (1948) main insights into an asset management framework with relative wealth concerns. In particular, when wealth is expected to fall below the index, asset managers behave similarly to poor individuals and face risk-on incentives (purchasing lottery tickets): convex incentives that increase their marginal utility of wealth the further the wealth falls below the index. In contrast, when wealth is expected to surpass the index, asset managers behave similarly to wealthy individuals and face risk-off incentives (purchasing insurance): concave incentives that reduce their marginal utility of wealth the further the wealth increases above the index.

3 The Economy

This section lays out a simple and tractable model to study the joint equilibrium effect of risk-on and risk-off incentives of asset managers. We consider a simple and tractable standard pure-exchange finite horizon economy in which time t is continuous and goes from zero to T. The economy is populated by two types of investors: passive investors, $\mathcal{P}$, and asset managers, $\mathcal{A}$. While the economy could be generalized and accommodate multiple assets and different specifications of the index against which the asset managers' performance is evaluated, critical aspects of the mechanism can be analyzed with three assets. Accordingly, uncertainty is driven by a (3)-dimensional Brownian motion $\mathbf{Z} = (Z_1, Z_2, Z_3)'$.

3.1 Investment Opportunities

The are 3 risky assets and a single riskless bond in the economy. We set the bond as numeraire and normalize its interest rate to zero. The risky asset is denoted by $S_{kt}, k = 1, 2, 3$, and represents a claim on the dividends D_{kT} arriving at time T. We assume each risky asset is in unit supply and posit that it follows

$$dS_{kt} = S_{kt} \left(\mu^S_{kt} dt + \boldsymbol{\sigma}^{S\prime}_{kt} d\boldsymbol{Z}_t \right). \tag{1}$$

Prices (S_{kt}), the (instantaneous) vector of expected returns $\boldsymbol{\mu}^S_t \equiv (\mu^S_{1t}, \mu^S_{2t}, \mu^S_{3t})'$, and the (instantaneous) volatility matrix $\boldsymbol{\Sigma}^S_t \equiv \left(\boldsymbol{\sigma}^S_{1t}, \boldsymbol{\sigma}^S_{2t}, \boldsymbol{\sigma}^S_{3t} \right)'$ are endogenous and determined in equilibrium. The bold symbols represent vectors and the $\prime$ represents the transpose throughout the analysis.

The dividends (D_{kT}) are determined by the dynamics of the processes

$$dD_{kt} = D_{kt} \left(\mu_k dt + \boldsymbol{\sigma}^{D\prime}_k d\boldsymbol{Z}_t \right), \quad D_{k0} > 0, \tag{2}$$

which we refer to as dividend news. The parameter μ_k and the vector $\boldsymbol{\sigma}^D_k$ are constants for the first two dividends. We follow Menzly, Santos, and Veronesi (2004), Basak and Pavlova (2013), and Buffa and Hodor (2023) and leave the remaining dividend news process unspecified, and, instead, we specify an aggregate process as soon follows. The procedure substantially improves the tractability of the model and provides closed-form precise equilibrium characterization.

We assume that stocks' fundamentals are independent, implying that the vector $\boldsymbol{\sigma}^D_k$ (2) has a positive kth element equaling $\sigma_k > 0$, while the remaining entries are zero. This assumption provides clear interpretation to the type of shocks that affect prices. To simplify the analysis, we assume that the distributional properties of news are the same, $\sigma_1 = \sigma_2 = \sigma_3$ and $\mu_1 = \mu_2 = \mu_3$.

3.2 The Index

For the baseline analysis we assume that the first asset is the index.

$$S_t^I \equiv S_{1t}. \tag{3}$$

The asset manager's performance is evaluated against the performance of this index and the passive investor invests its entire wealth in it. The index does not include the remaining assets in the economy because we are interested in analyzing the asset managers' holdings within and outside its mandate.

For consistency of the analysis, we refer to I_t as the *index news*, which in the baseline economic setup equals the news about asset 1,

$$I_t \equiv D_{1t}, \quad \boldsymbol{\sigma}^I \equiv \boldsymbol{\sigma}_1^D. \tag{4}$$

3.3 Investors

The passive investors ($\mathcal{P}$) are endowed with $(1-\lambda)$ units of the index at time 0

$$W_0^{\mathcal{P}} = (1-\lambda)\, S_0^I \tag{5}$$

and remain passive until time T. The asset managers ($\mathcal{A}$) are endowed with the remaining (λ) units of the index and holds the entire supply of the other assets,

$$W_0^{\mathcal{A}} = \lambda S_0^I + S_{20} + S_{30} \equiv S_0^{\mathcal{A}} \tag{6}$$

The asset managers' initial endowment identifies the active section of the asset market, which is a claim on the total asset market dividends minus the passive index holdings' dividends,

$$V_T \equiv I_{1T} + D_{2T} + D_{3T} - (1-\lambda)\, I_{1T}, \tag{7}$$

where we assume that this terminal value is determined by the process

$$dV_t = V_t \left(\eta dt + \boldsymbol{\nu}' d\boldsymbol{Z}_t\right), \quad V_0 > 0. \tag{8}$$

The parameter η is positive and the vector $\boldsymbol{\nu} \equiv (\nu_1, \nu_2, \nu_3)'$ has positive entries. We assume that $(\nu_1, \nu_2, \nu_3) = (\lambda\sigma_1/3, \sigma_2/3, \sigma_3/3)$ to mimic the aggregate active dividends and following the standard in the literature. We refer to V_t as the *market news*.

3.4 Risk-On and Risk-Off Incentives

Empirical evidence indicates that individuals sometimes choose to pay an insurance premium to avoid a fair gamble, while at other times they prefer to take the gamble. At first glance, these behaviors appear inconsistent with the expected utility maximization theory, which would require a rational agent to pay for insurance in some situations and buy a lottery ticket in others.

Friedman and Savage (1948) addressed this puzzle by introducing utility characteristics that explain these two behaviors within a rational, expected utility-maximizing framework. They demonstrated that rationalizing these behaviors means that, at certain wealth levels, an agent's marginal utility increases with wealth, while at other wealth levels, the marginal utility decreases.

The empirical evidence on asset managers and their benchmarking incentives aligns with the behaviors of individual agents as described by Friedman and Savage (1948). Specifically, when wealth is expected to fall below the index, asset managers act like individuals willing to take a fair gamble, facing risk-on incentives. These convex incentives increase their marginal utility of wealth as it falls further below the index. Conversely, when wealth is expected to exceed the index, asset managers behave like individuals willing to purchase insurance to avoid a fair gamble, facing risk-off incentives. These concave incentives reduce their marginal utility of wealth as it rises above the index.

Crucially, the arguments and insights of Friedman and Savage (1948) apply to asset managers' risk-on and risk-off incentives because the key factor for prices is the marginal utility of wealth. Changes in the index impact prices solely through their effect on the marginal utility of wealth.

Therefore, Friedman and Savage (1948)'s arguments in this context suggest that asset managers' marginal utility of wealth *increases* with the index level when their wealth is below the index, ($U_{WS^I} > 0$), indicating a stronger desire to increase wealth when they are underperforming. Conversely, when wealth is above the index, their marginal utility of wealth *decreases*, ($U_{WS^I} < 0$), indicating a weaker desire to increase wealth when they are outperforming the benchmark. Any changes in the index level that do not affect the marginal utility of wealth have no implications for prices. Figure 3 illustrates this argument.

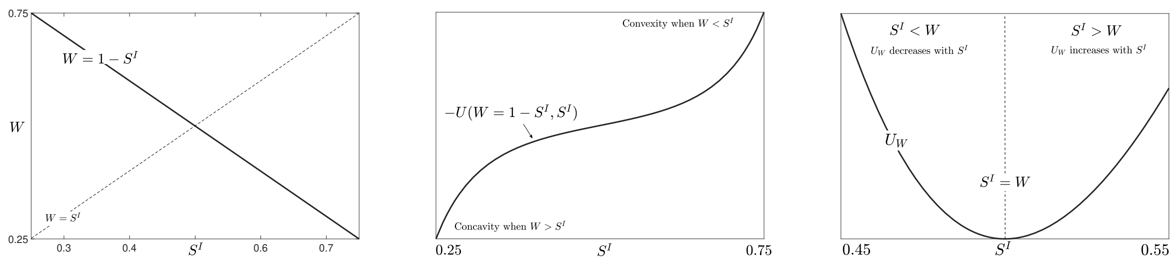

Figure 3. The left figure plots the $S^I - W$ plane. At the top left triangle, the asset managers have concave incentives because $W > S^I$, while at the bottom right triangle, the asset managers have convex incentives because $W < S^I$, where the solid linear line represents $W = -S^I + 1$. As we trace that line from the top left corner to the bottom right corner, incentives shift from concave to convex incentives. Indeed, the middle figure shows that the cross derivative U_{WS^I} is concave initially and becomes convex as the line crosses the dashed line. The cross derivative (U_{WS^I}) in the direction of the solid line (in the left figure) can be captured by the rate of change in $U_S(W, S^I)$ in the direction $(-1, 0)$, leading to a negative cross derivative: $-U_{WS^I}(W, S^I)$. Therefore, we include a minus sign in front of U to capture U_{WS^I}. The right figure shows what the cross derivative captures. When $W < S^I$ (right side of the figure), the marginal utility of wealth increases when S^I increases, indicating the asset manager desire to outperform. In contrast, when $W > S^I$ (left side of the figure), the marginal utility of wealth decreases when S^I increases, indicating the asset manager desire to maintain performance. The function U represents the asset managers' objective (11) with $\alpha_1 = 0.5$ and $\alpha_2 = 4.5$.

While many potential objective functions may satisfy the requirements for risk-on and risk-off incentives. The goal is to introduce an objective that switches between risk-on and risk-off incentives based on the state of asset managers' wealth relative to the index without losing tractability.

To do so, we start with a variation of Basak and Pavlova (2013)s' asset manager objective function, and introduce the *risk-on objective*

$$-E\left[\frac{\left(S_T^I\right)^{1-\gamma_1}\left(W_T^A\right)^{1-\gamma_2}}{(\gamma_2 - 1)(1-\gamma_1)}\right], \quad \gamma_2 > 1 > \gamma_1 > 0. \tag{9}$$

Similar to the original objective function of Basak and Pavlova (2013), the risk-on objective (9) is always convex, $U_{WS^I} = \left(S_T^I\right)^{-\gamma_1}\left(W_T^{\mathcal{A}}\right)^{-\gamma_2} > 0$. The parameter of risk aversion in this case is γ_2.

If we alternate the wealth and the index in the risk-on objective (9) and remove the minus sign, we attain a *risk-off objective*,

$$E\left[\frac{\left(W_T^{\mathcal{A}}\right)^{1-\gamma_1}\left(S_T^I\right)^{1-\gamma_2}}{\left(\gamma_2-1\right)\left(1-\gamma_1\right)}\right], \quad \gamma_2 > 1 > \gamma_1 > 0, \tag{10}$$

which is always concave, $U_{WS^I} = -\left(S_T^I\right)^{-\gamma_2}\left(W_T^{\mathcal{A}}\right)^{-\gamma_1} < 0$. The parameter of risk aversion in this case is γ_1.

Perhaps more importantly, by combining the risk-on (9) and risk-off (10) incentives we attain an objective function that switches between risk-on and risk-off depending on whether the wealth is below or above the index.

$$U\left(W^{\mathcal{A}}, S^I\right) \equiv E\left[\mathbf{1}_{\text{Off}}\frac{\left(S_T^I\right)^{1-\gamma_2}\left(W_T^{\mathcal{A}}\right)^{1-\gamma_1}}{\left(\gamma_2-1\right)\left(1-\gamma_1\right)} - \mathbf{1}_{\text{On}}\frac{\left(S_T^I\right)^{1-\gamma_1}\left(W_T^{\mathcal{A}}\right)^{1-\gamma_2}}{\left(\gamma_2-1\right)\left(1-\gamma_1\right)}\right], \quad \gamma_2 > 1 > \gamma_1 > 0. \tag{11}$$

The indicators $\mathbf{1}_{\text{Off}}, \mathbf{1}_{\text{On}}$ in (11) allow to separate the asset pricing implications of each incentive individually and their joint effect. A fully risk-on asset manager has only the second component in (11), $\mathbf{1}_{\text{Off}} = 0, \mathbf{1}_{\text{On}} = 1$, while a fully risk-off asset manager has only the second component in (11), $\mathbf{1}_{\text{Off}} = 1, \mathbf{1}_{\text{On}} = 0$. When taken together, $\mathbf{1}_{\text{Off}} = 1, \mathbf{1}_{\text{On}} = 1$, the risk-off component dominates when $S^I < W^{\mathcal{A}}$, while the risk-on component dominates when $S^I > W^{\mathcal{A}}$, and the asset manager has both risk-on and risk-off incentives, in line with Friedman and Savage (1948) main insights. The parameter of risk aversion in this case is endogenous and time-varying.

Besides the risk-on and risk-off incentives, the asset manager objective function (11) satisfies other useful characteristics: it increases and concave in wealth ($U_W > 0, U_{WW} < 0$), and satisfies the inada conditions stating that the marginal utility of wealth (U_W) takes the value of ∞ as wealth approaches zero, and the value 0 as wealth approaches infinity. These conditions are at the core of expected utility maximization.

There are two variations to the original Basak and Pavlova (2013) objective function. First, we introduce a risk aversion parameter larger than one (γ_2) instead of the typical log utility of wealth, and second, the objective is a power function of the index (γ_1) instead of being linear. The are two reasons for these variations.

The symmetrical structure of the asset manager's objective function (11) is the primary characteristic, inducing the shift between risk-on and risk-off incentives. If we were to follow Basak and Pavlova (2013) log utility of wealth for the risk-on incentives, the symmetrical structure would require a log of the index for the risk-off incentives. This outcome would, in turn, violates the requirement for the increasing-in-wealth objective function ($U_W > 0$) because the index log becomes negative when the index level falls below one, and U_W would eventually turn negative. Similarly, if we were to follow Basak and Pavlova (2013)s' linearity in the index for risk-on incentives, the symmetrical structure would require linearity in wealth for the risk-off incentives. This outcome would, in turn, violates the requirement for a decreasing marginal utility in wealth, $U_{WW} < 0$. Due to these two reasons, we introduce the two variations to the original asset manager objective function, as laid out by Basak and Pavlova (2013).

Starting with their initial endowments, the asset managers dynamically choose a portfolio $\boldsymbol{\pi}_t^{\mathcal{A}}$, which represents the fraction of wealth invested in each of the risky assets. The wealth process of the asset managers, therefore, follows the dynamics

$$\frac{dW_t^{\mathcal{A}}}{W_t^{\mathcal{A}}} = \boldsymbol{\pi}_t^{\mathcal{A}\prime} \left(\boldsymbol{\mu}_{St} dt + \boldsymbol{\Sigma}_t^S d\boldsymbol{Z}_t \right), \quad t \leq T. \tag{12}$$

4 The Equilibrium

We define the equilibrium in a standard way: equilibrium prices and portfolio holdings are such that (i) the asset managers choose their optimal portfolio for given prices, and (ii) stocks, the bond, and consumption-good markets clear.

Asset managers' risk-on and risk-off objectives are of a constant relative risk aversion type individually. However, when taken together, the relative risk aversion of the asset managers' objective (11) is a weighted average of the two risk aversion coefficients (γ_1, γ_2), with time-varying weights that depend on the likelihood of the asset managers to outperform or underperform. We denote the relative risk aversion by $RA(W, S^I)$. By applying the definition of relative risk aversion to the asset managers' objective function, we find that:

$$RA(W, S^I) = \gamma_1 \underbrace{\left(1 - \frac{1}{1 + \frac{(1-\gamma_1)}{(\gamma_2-1)} (W/S^I)^{(\gamma_2-\gamma_1)}}\right)}_{\equiv 1-w} + \gamma_2 \underbrace{\left(\frac{1}{1 + \frac{(1-\gamma_1)}{(\gamma_2-1)} (W/S^I)^{(\gamma_2-\gamma_1)}}\right)}_{\equiv w}, \quad (13)$$

where $w \in (0, 1)$ since $0 < \gamma_1 < 1 < \gamma_2$. The formulation shows that risk aversion, $RA(W, S^I)$, is endogenous, time-varying, and dependent on the ratio of the asset managers' wealth to the index. When asset managers' wealth is low relative to the index, they are in the risk-on region since $W < S^I$. In this case, the risk aversion aligns more closely to the risk-aversion coefficient in the risk-on region (γ_2) since the weight on the risk-off risk-aversion (γ_1) declines.

As the performance of asset managers improve, their total wealth increases relative to the index. Consequently, the weight on the risk-off risk-aversion parameter rises while the weight on the risk-on risk-aversion parameter falls, resulting in an overall objective function that is decreasing in relative risk aversion: the risk aversion decreases as asset managers becomes wealthier.

Since market clearing requires asset managers to hold the entire active supply, the index price must equal its dividends at maturity, and in equilibrium, $RA(W, S^I) = RA(V, I)$. Interestingly, the state variable (V/I) that determines the variations in risk aversion also dictates how asset managers hedge against shifts between the risk-off and the risk-on regions, ultimately determining the equilibrium quantities.

Before we start the equilibrium analysis we first define the asset managers' *risk-on hedging demand*, $\mathcal{H}_{it}$, which is given by

$$\mathcal{H}_{it} \equiv \frac{1}{1 + \frac{1-\gamma_1}{\gamma_2-1} (V_t/I_t)^{\gamma_2-\gamma_1} \mathcal{E}_{it}}, \quad i = 1, 2, 3, 4, \quad (14)$$

where i refer to four different determinstic functions of time, $\mathcal{E}_{1t}$, $\mathcal{E}_{2t}$, $\mathcal{E}_{3t}$, $\mathcal{E}_{4t}$, given in (A.3), (A.6), (A.9), (A.12), respectively. We refer to $\mathcal{H}_{it}$ as the risk-on hedging demand because the closer it is to one, the more correlated the equilibrium quantities of the joint incentives case are with those of the purely risk-on incentives case. The deterministic functions $\mathcal{E}_{it}$ determines the speeds at which the hedging demand changes with the market-index ratio,

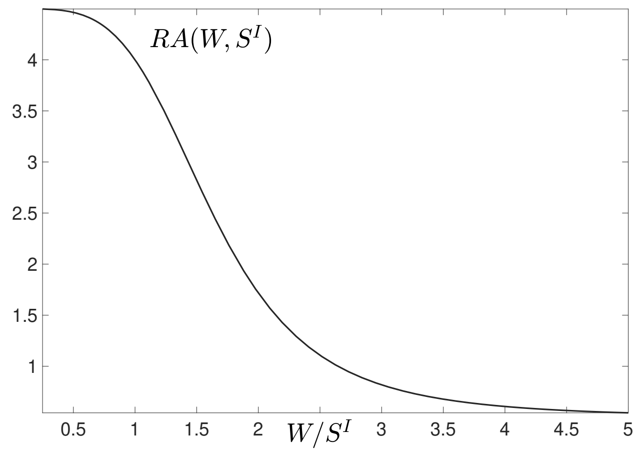

Figure 4. This figure plots the asset managers' risk aversion as a function of W/S^I. It converges to γ_2, the risk aversion coefficient of the risk-on region when $W/S^I \downarrow 0$, and to γ_1, the risk aversion coefficient of the risk-off region when $W/S^I \uparrow \infty$, resulting in a decreasing relative risk aversion utility.

$\partial\mathcal{H}_{it}/\partial(V/I)$. Thus, we refer to $\mathcal{H}_{it}$ for $i = 1, 2, 3, 4$ as the risk-on hedging demand. The market-index ratio governs both the risk-on hedging demand and the risk aversion parameter, and it serves as the key state variable that determines all equilibrium quantities.

We begin by analyzing the partial equilibrium effect of how shocks impact the wealth of asset managers, given prices. This approach allows us to identify the different shock propagation channels for risk-on and risk-off incentives, both individually and jointly. To facilitate this, let us denote the (equilibrium) discount factor by ξ_t and its dynamics by

$$d\xi_t = -\xi_t \left(\boldsymbol{\theta}'_t d\boldsymbol{Z}_t \right). \tag{15}$$

By definition of the discount factor, the vector of market prices of risk process, $\boldsymbol{\theta}_t$, can be represented as

$$\boldsymbol{\theta}_t = \left(\boldsymbol{\Sigma}_t^S\right)^{-1} \boldsymbol{\mu}_t^S, \tag{16}$$

which in a single asset economy simplifies to the ratio of an asset's (excess) expected return and its standard deviation.

Lemma 1 (**Risk Exposure**). *The asset managers' risk exposure due to risk-on incentives is given by*

$$\mathbf{\Sigma}_t^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}} = \boldsymbol{\theta}_t - (\gamma_2 - 1)\,\boldsymbol{\nu} + (1-\gamma_1)\,\boldsymbol{\sigma}^I, \tag{17}$$

while the risk exposure due to risk-off incentives is given by

$$\mathbf{\Sigma}_t^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}} = \boldsymbol{\theta}_t + (1-\gamma_1)\,\boldsymbol{\nu} - (\gamma_2 - 1)\,\boldsymbol{\sigma}^I. \tag{18}$$

In the presence of both risk-on and risk-off incentives the risk exposure is given by

$$\mathbf{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}} = \boldsymbol{\theta}_t + \left[-(\gamma_2 - 1)\,\boldsymbol{\nu} + (1-\gamma_1)\,\boldsymbol{\sigma}^I\right]\mathcal{H}_{2t} + \left[(1-\gamma_1)\,\boldsymbol{\nu} - (\gamma_2 - 1)\,\boldsymbol{\sigma}^I\right](1-\mathcal{H}_{2t}) \tag{19}$$

The function $\mathcal{H}_{2t}$ is defined in (14).

There are three shock propagation channels. The first is the myopic mean-variance channel, where asset managers seek a risk exposure proportional to the assets' risk-return tradeoff, correlating their wealth with the market price of risk, $\boldsymbol{\theta}_t$. Lemma 1 demonstrates that this channel remains unaffected regardless of the type of incentives. However, the exact exposure will be determined in equilibrium once the market prices of risk are identified.

The second channel is a risk-aversion adjustment channel, and Lemma 1 indicates that it has different implications for risk-on and risk-off incentives. In the pure risk-on incentives case (17), asset managers are risk averse with respect to the market and desire to hedge against adverse market moves. They do so by requiring higher wealth following adverse market movements since the risk aversion coefficient is greater than one, $\gamma_2 > 1$. The logic reverses for risk-off incentives. In pure risk-off incentives (18), asset managers are risk takers with respect to the market and correlate their wealth with the market. They do so because the risk aversion coefficient is less than one, $\gamma_1 < 1$.

The third channel is the benchmark consideration. Similar to the risk aversion adjustment channel, it has opposing implications for risk-on and risk-off incentives. When asset managers trail the index and desire to outperform the index (17), they are risk-takers with respect to the index and correlate their wealth with the index since $\gamma_1 < 1$. Conversely, when asset managers outperform the index and desire to maintain performance (18), they are risk averse with respect to the benchmark and desire to hedge against adverse benchmark moves.

The joint effect is a time-varying weighted average of the incentives to outperform the benchmark and maintain performance. The weights are determined by the likelihood of outperformance relative to underperformance and controlled by the risk-on hedging demand, $\mathcal{H}_{2t}$. As asset managers' wealth declines relative to the index, the demand for risk-on hedging rises. Consequently, asset managers sell part of their risk-off portfolios and purchase more of their risk-on portfolios.

We proceed by uncovering the closed-form expressions for the equilibrium quantities, beginning with the market prices of risk.

Proposition 1 (**Market Prices of Risk**). *The market prices of risk in the presence of risk-on incentives is given by*

$$\boldsymbol{\theta}_t = -\left(1-\gamma_1\right)\boldsymbol{\sigma}^I + \gamma_2\boldsymbol{\nu}, \tag{20}$$

while in the presence of risk-off incentives, it is given by

$$\boldsymbol{\theta}_t = \left(\gamma_2-1\right)\boldsymbol{\sigma}^I + \gamma_1\boldsymbol{\nu}. \tag{21}$$

In the presence of both risk-on and risk-off incentives the market prices of risk are given by

$$\boldsymbol{\theta}_t = \left[-\left(1-\gamma_1\right)\boldsymbol{\sigma}^I + \gamma_2\boldsymbol{\nu}\right]\left(\mathcal{H}_{1t}\right) + \left[\left(\gamma_2-1\right)\boldsymbol{\sigma}^I + \gamma_1\boldsymbol{\nu}\right]\left(1-\mathcal{H}_{1t}\right). \tag{22}$$

The function $\mathcal{H}_{1t}$ is defined in (14).

The market prices of risk in the pure risk-on incentives case, (20), align with findings documented in the literature over the past decade: the incentives to outperform the benchmark decreases the market prices of risk that correlate with benchmark. Asset managers are risk takers with respect to the index and seek to buy it with a proportion of their wealth, creating additional demand for the index. Equilibrium is restored by making these assets less attractive from a risk-return tradeoff perspective, thereby reducing their market prices of risk.

The logic reverses for pure risk-off incentives (21). In this case, asset managers aim to maintain performance and, therefore, are risk-averse with respect to the benchmark. They do so by hedging against adverse benchmark moves, which introduces an extra supply for the index. Equilibrium is restored by enhancing the risk-return tradeoff of the index, leading to a higher market price of risk.

The joint incentives case (22) is a time-varying weighted average of the two individual effects, with $\mathcal{H}_{1t}$ determining the weights. The greater the outperformance, the closer the market price of risk aligns with the maintaining performance incentive case (21). Conversely, the more significant the underperformance, the closer it aligns with the incentives to outperform the benchmark (20).

Consequently, the market prices of risk that correlate with the benchmark are *cyclical* with asset managers' performance: as their relative performance improves, market prices of risk increase. Previous literature has demonstrated that benchmarking reduces the market price of risk for benchmarked assets. This paper refines that statement by asserting that it is true when asset managers underperform the benchmark. However, the statement reverses when they outperform the benchmark.

We proceed by characterizing the risk exposures.

Proposition 2 (Volatility). *The return volatilities in the presence of either risk-on incentives or risk-off incentives, separately, are given by*

$$\boldsymbol{\sigma}^S_{1t} = \boldsymbol{\sigma}^D_{1t}, \qquad \boldsymbol{\sigma}^S_{2t} = \boldsymbol{\sigma}^D_{2t}. \tag{23}$$

The return risk exposures in the presence of joint risk-on and risk-off incentives are given by

$$\boldsymbol{\sigma}^S_{1t} = \boldsymbol{\sigma}^D_{1} + (\gamma_2 - \gamma_1)\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right)\left(\mathcal{H}_{3t} - \mathcal{H}_{1t}\right), \tag{24}$$

$$\boldsymbol{\sigma}^S_{2t} = \boldsymbol{\sigma}^D_{2} - (\gamma_2 - \gamma_1)\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right)\left(\mathcal{H}_{1t} - \mathcal{H}_{4t}\right). \tag{25}$$

where $\mathcal{H}_{1t}$, $\mathcal{H}_{3t}$, $\mathcal{H}_{4t}$ are given in (14), and $\mathcal{H}_{3t} - \mathcal{H}_{1t} > 0$, $\mathcal{H}_{1t} - \mathcal{H}_{4t} > 0$.

The index asset is the most sensitive to news when $\mathcal{H}_{1t} + \mathcal{H}_{3t} = 1$, and the non-index asset is the most sensitive to news when $\mathcal{H}_{1t} + \mathcal{H}_{4t} = 1$.

Positive index news increases the index price, while positive non-index news decreases the index price:

$$\boldsymbol{\sigma}^S_{1t}(1) > 0, \ \boldsymbol{\sigma}^S_{1t}(2) < 0, \ \boldsymbol{\sigma}^S_{1t}(3) < 0. \tag{26}$$

In contrast, positive index news decreases the non-index price, while positive non-index news *increases the non-index price:*

$$\boldsymbol{\sigma}_{2t}^{S}(1) < 0,\ \boldsymbol{\sigma}_{2t}^{S}(2) > 0,\ \boldsymbol{\sigma}_{2t}^{S}(3) > 0. \tag{27}$$

In a traditional endowment economy without benchmark incentives, prices only reflect news about the assets' fundamentals. In such environments, optimal portfolios remain unchanged by the arrival of news, and, therefore, equilibrium does not alter the exposure of prices to future news, leading to the immediate reflection of news in prices.

Proposition 2 reveals that pure incentives cause asset managers to respond to news differently than market participants in a traditional setup. However, similar to a traditional endowment economy, optimal portfolios (and risk exposures) remain unchanged by the arrival of news. Therefore, equilibrium does not alter the exposure of prices to future news.

In contrast, in previous pure risk-on models, such as those by Basak and Pavlova (2013), and Buffa and Hodor (2023), asset managers' portfolios change with the arrival of news, since the incentives to outperform the benchmark increases as performance deteriorates. As a result, equilibrium changes how the index price will react to future news. In the pure risk-on (and risk-off) setups of this paper, this behavior does not occur.

In the joint incentives case, equilibrium reacts to changes in the incentives to outperform the benchmark relative to the incentives to maintain performance. As the performance of asset managers converges to the benchmark, $\mathcal{H}_{1t} + \mathcal{H}_{3t}$ and $\mathcal{H}_{1t} + \mathcal{H}_{4t}$ get closer to one, and assets become riskier due to the higher uncertainty of performance.

Furthermore, when index news arrives, the index price increases, making asset managers more likely to fall behind the benchmark and underperform. Anticipating this, asset managers aim to buy more of the benchmark as positive news arrives, increasing the demand for the index. Consequently, equilibrium causes the index price to overreact to index news. To purchase more of the index, asset managers must sell other assets they hold, leading non-index assets to underreact to index news.

Lastly, when non-index news arrives, the non-index price increases, boosting the overall market relative to the index. In this scenario, asset managers are more likely to outperform. Anticipating this, asset managers buy less of (or sell) the benchmark as positive non-index news arrives, decreasing the demand for the index. Consequently, equilibrium causes the index price to underreact to non-index news.

5 Active Share and Tracking Error

In this section, we analyze the equilibrium predictions on two primary empirical measures designed to assess the asset manager's activity level in relation to their benchmark.

The literature distinguishes between two main types of measures used to assess asset management activity: tracking error and active share. Tracking error, the earliest method of the two, quantifies the volatility of the difference between the portfolio's return and that of its benchmark index. Subsequently, the active share measure was developed to offer a direct comparison between the portfolio's holdings and those of the benchmark index.

By empirically comparing these two measures, one might hypothesize that they capture different aspects of management activity. Cremers and Petajisto (2009) argue that tracking error serves as a proxy for a factor bet strategy, whereas active share represents a stock selection strategy. The factor bet strategy involves rotating across systemic factors such as sectors and industries. Conversely, the stock selection strategy focuses on choosing individual stocks across all investment classes, rather than systemic factors. Their analysis indicates that the factor bet strategy results in a higher tracking error and lower active share compared to the stock selection strategy.

Our theory indicates that the cross-sectional empirical analysis they conducted may be influenced by inherent differences in mutual funds' performance. Specifically, our theory demonstrates that active share exhibits a U-shaped relationship with fund performance, while tracking error shows an inverted U-shaped relationship. As a mutual fund's performance deviates from the benchmark performance, either towards outperformance or underperformance, its active share tends to increase and its tracking error to diminish. Consequently, the variations in active share and tracking error among different funds may be attributed to differences in their current performance levels rather than to inherent differences in their investment strategies.

Our theory also rationalizes conflicting empirical findings. Hu et al. (2011) argue a U-shaped relationship between performance in the first 6 months of the year and the fund manager's risk choices in the second half of the year. In contrast, Lee et al. (2019) claim that their findings contrast sharply with the previous findings of Hu et al. (2011) since they document an inverted U-shaped relationship between the performance in the first 6 month and a risk shifting measure in the second half of the year. Our theoretical findings suggest that these two predictions are completely consistent with each other since the risk choice measure of Hu et al. (2011) aligns with the active share, whereas the risk-shift measure of Lee et al. (2019) correlates with the tracking error.

From a partial equilibrium perspective, the inverse relationship between active share and tracking error appears contradictory, as one measure suggests increased management activity while the other measure indicates reduced management activity. The equilibrium perspective resolves the conflict because asset managers' trading strategies propagate to prices. The logic is as follows.

When the benchmark performance aligns closely with the market performance, it is equally likely that the asset manager will outperform or underperform the benchmark. In such scenarios, the asset manager's portfolio is equally weighted between risk-on and risk-off strategies. This balanced approach is reflected in balanced prices, indicating that assets in the economy have similar risk and return trade-offs. Consequently, asset managers tend to have a lower active share compared to situations where they either significantly underperform or outperform the benchmark.

The tracking error depicts a different picture. When it is equally likely that the asset manager will outperform or underperform the benchmark, equilibrium increases assets' sensitivities to news due to the uncertainty of performance. This result implies that the asset market becomes more volatile overall. However, different assets in the economy have different sensitivities to news. Since the asset manager's portfolio is equally weighting risk-on and risk-off strategies, this result implies that the volatility of the difference between the portfolio return and the benchmark return becomes large when the active share becomes low.

The difference between the portfolio exposure and the benchmark exposure is given by

$$\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}} - \boldsymbol{\sigma}_{1t}^{S}. \tag{28}$$

By plugging the market prices of risk, (22), into the asset manager's portfolio exposure, (19), and taking the variance, we find that the equilibrium tracking error per unit of time is given by

$$\text{Tracking Error} = \left[(\gamma_2 - \gamma_1)\left(\mathcal{H}_{3t} - \mathcal{H}_{2t}\right) + 1\right]^2 \|\boldsymbol{\sigma}^I - \boldsymbol{\nu}\|^2. \tag{29}$$

The formulation of the tracking error reveals that it attains its maximum when the difference between $\mathcal{H}_{2t}$ and $\mathcal{H}_{3t}$ is maximized. The derivative of the difference increases initially, becomes zero when $1 = \mathcal{H}_{3t} + \mathcal{H}_{2t}$, and negative afterwards.

The active share compares the portfolio weight of all assets to their respective weight in the benchmark index. However, market participants may attain the exposure of the benchmark from the exposures of the replicating portfolio. So, in principle, the asset manager may hold a portfolio that has no weight on the benchmark, but generates precisely the same exposure to news, and by no arbitrage has the same value. Since our equilibrium is closed-form, we account for the replicating portfolio by directly measuring the correlation between the asset manager's portfolio and the benchmark. Therefore, the active share is defined as the correlation between the portfolio and the benchmark exposures per unit of time,

$$\frac{\boldsymbol{\sigma}_{1t}^{S\prime}\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)}{\|\boldsymbol{\sigma}_{1t}^{S}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)\|}. \tag{30}$$

For example, if the asset manager holds either the benchmark or its replicating portfolio, the correlation is 100%, despite the fact that the asset manager does not necessarily hold the benchmark. Conversely, if the asset manager's exposure is orthogonal to the benchmark exposure, the correlation is 0%. When applied to the asset manager portfolio relative to the benchmark, the correlation measures the active share.[1] By plugging the market prices of risk, (22), into the asset manager's portfolio exposure, (19), and applying the definition, we find that the active share per unit of time is given by

$$\begin{aligned}\text{Active Share} =& \frac{\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu} - (\gamma_2-\gamma_1)\left(\mathcal{H}_{1t}-\mathcal{H}_{2t}\right)\left[1+(\gamma_2-\gamma_1)\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)\right]\left(\|\boldsymbol{\sigma}^{I}\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}_{1t}^{S}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)\|} \\ &- \frac{(\gamma_2-\gamma_1)\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)\left[1+(\gamma_2-\gamma_1)\left(\mathcal{H}_{1t}-\mathcal{H}_{2t}\right)\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}_{1t}^{S}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)\|}.\end{aligned} \tag{31}$$

The active share attains its minimum in the neighborhood of the location that the tracking error attains its maximum. First, notice that the active share decreases when the differences $\mathcal{H}_{1t} - \mathcal{H}_{2t}$ and $\mathcal{H}_{3t} - \mathcal{H}_{1t}$ increase. Second, the two differences governing the tracking error and the difference governing the active share all attain their maximum roughly at the

[1] This measure is similar to co-sign similarity between the benchmark and the portfolio holdings of asset managers. See Buffa and Hodor (2023) for details.

same location, as we show in the following proposition.

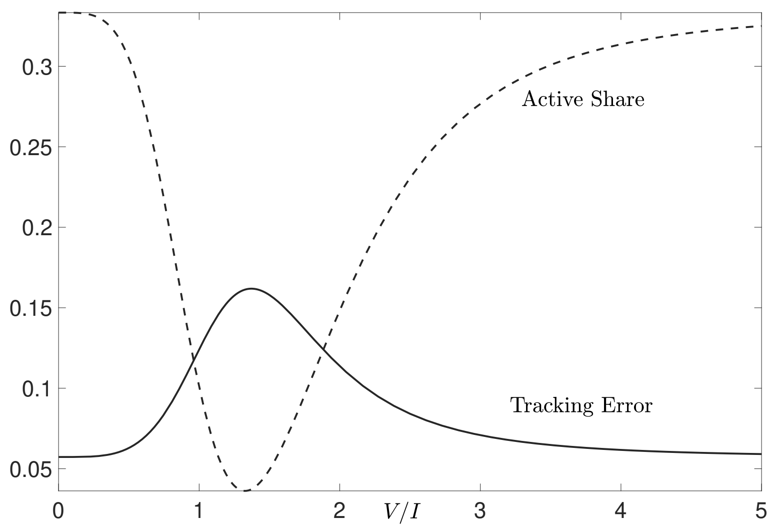

Figure 5. The figure plots the tracking error (29) and the active share (31). It shows that in equilibrium, when the active share is minimized the tracking error is maximized. The parameters are $\gamma_2 = 4.5$, $\gamma_1 = 0.5$, $\mu_1 = \mu_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.25$, $\lambda = 0.5$, $V_t = 5$, $V_0 = 2.5$, $I_0 = 1$, $T = 5$, $t = 2$.

Proposition 3 (**Tracking Error and Active Share**). *The tracking error attains its maximum in the neighborhood of the minimum active share. The tracking error is given in (29) and the active share in (31).*

6 Time-Series Predictability

This section examines how an asset's past returns can predict its current return. Positive autocorrelation is known as time-series *momentum*, while negative autocorrelation is referred to as time-series *reversal*. Predictability arises from predictable changes to both quantities of risk and market prices of risk. We separate these two predictability channels throughout the analysis.

To deepen our understanding of the equilibrium mechanism, we start by analyzing how the market prices of risk and the quantities of risk react to news.

6.1 Equilibrium Response to Frictions

Prices not only react to information about the fundamentals but also react to news about management frictions.

To illustrate the intuitions clearly, we rewrite the vector of market prices of risk, (22), as a sum of the *index news exposure* ($\boldsymbol{\sigma}^I$) and the *market news exposure* ($\boldsymbol{\nu}$),

$$\boldsymbol{\theta}_t = [(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\sigma}^I + [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\nu}. \tag{32}$$

When asset managers are purely risk-on or purely risk-off (or do not have incentives altogether), the market prices of risk do not reflect news about the management frictions because the risk-on hedging demand ($\mathcal{H}_{1t}$) is a constant that equals one or zero, respectively, and never changes. As a result, prices fully incorporate fundamental information. However, when asset managers balance both strategies, market prices of risk reflect not only fundamentals but also news about asset management frictions. Note that from a purely fundamental perspective, the price reaction to management friction might appear as incorrect pricing, or an overreaction.

As asset managers' performance deteriorates, risk-on incentives become more pronounced than risk-off incentives, leading asset managers to invest more in the benchmark and less in assets outside the benchmark. To restore equilibrium, market prices of risk associated with the benchmark are depressed to make the index assets less attractive and more expensive. Conversely, the market prices of risk associated with assets outside the benchmark increase to make these assets more attractive and less expensive.

Similar to the market price of risk, we rewrite the vector of return exposures (quantities of risk) as a sum of index news exposure ($\boldsymbol{\sigma}^I$) and market news exposure ($\boldsymbol{\nu}$),

$$\boldsymbol{\sigma}^S_{1t} = \boldsymbol{\sigma}^D_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{3t} - \mathcal{H}_{1t})\,\boldsymbol{\sigma}^I - (\gamma_2 - \gamma_1)(\mathcal{H}_{3t} - \mathcal{H}_{1t})\,\boldsymbol{\nu}, \tag{33}$$

$$\boldsymbol{\sigma}^S_{2t} = \boldsymbol{\sigma}^D_2 - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t} - \mathcal{H}_{4t})\,\boldsymbol{\sigma}^I + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t} - \mathcal{H}_{4t})\,\boldsymbol{\nu}. \tag{34}$$

The representations reveal that when asset managers are purely risk-on or purely risk-off, asset prices react only to news about fundamentals since the risk-on hedging demands are constant, either equaling one or zero, respectively. In contrast, when asset managers weigh both strategies, assets incorporate news about management frictions in addition to the news about fundamentals.

The difference between the risk-on hedging demands attain its maximum when the sum of the risk-on hedging damands equals one, $\mathcal{H}_{it} + \mathcal{H}_{jt} = 1$, $i, j = 1, 2, 3, 4$. The maximum values for different i and j are reached close to each other and identify the state at which asset managers have an equal likelihood of outperforming or underperforming the benchmark.

This likelihood influences prices. The arrival of news can cause the asset managers' performance to either align more closely with the benchmark or deviate away from the benchmark. The uncertainty about performance is strongest when asset managers' performance closely matches the benchmark, compared to when their performances diverge. Consequently, the price sensitivity to news is the largest in this scenario.

When asset managers' performance diverges towards underperformance, their portfolios weigh more towards the risk-on strategy relative to the risk-off strategy. In this case, it becomes more likely that asset managers will underperform. Equilibrium reflects this higher certainty of underperformance by reducing the risks in the asset markets. Conversely, when asset managers' performance diverges towards outperformance, their portfolios weigh more towards the risk-off strategy relative to the risk-on strategy. Equilibrium reflects this higher certainty of outperformance by reducing the risks in the asset markets.

The equilibrium response to news about management frictions, through both the market prices of risk and quantities of risk, reveals two predictability channels.

6.2 Time-Series Predictability

So far, we have shown that market prices of risk and quantities of risk react to news about asset management frictions.

Before we delve into why these responses are predictable, let us first establish how we identify predictability. Let's begin by defining the expected return dynamics for both the index and non-index assets as follows

$$d\mu_{1t}^S = \alpha_{1t}^\mu dt + \boldsymbol{b}_{1t}^{\mu'} d\boldsymbol{Z}_t, \qquad d\mu_{2t}^S = \alpha_{2t}^\mu dt + \boldsymbol{b}_{2t}^{\mu'} d\boldsymbol{Z}_t. \tag{35}$$

The covariance between the expected returns in time t and $t+\Delta$ (for $\Delta > 0$) is given by

$$\begin{aligned}\mathbb{C}\text{ov}_t\left(\frac{dS_{kt}}{S_{kt}}, \frac{dS_{kt+\Delta}}{S_{kt+\Delta}}\right) =& \mathbb{C}\text{ov}_t\left(\left(\mu^S_{kt}dt + \boldsymbol{\sigma}^{S\prime}_{kt}d\boldsymbol{Z}_t\right), \left(\mu^S_{kt+\Delta}d\left(t+\Delta\right) + \boldsymbol{\sigma}^{S\prime}_{kt+\Delta}d\boldsymbol{Z}_{t+\Delta}\right)\right) \\ =& \mathbb{C}\text{ov}_t\left(\boldsymbol{\sigma}^{S\prime}_{kt}d\boldsymbol{Z}_t, \mu^S_{kt+\Delta}d\left(t+\Delta\right)\right) = \boldsymbol{\sigma}^{S\prime}_{kt}\mathbb{E}_t\left[\left(d\boldsymbol{Z}_t\right)\mu^S_{kt+\Delta}\right]d\left(t+\Delta\right).\end{aligned} \tag{36}$$

This covariance indicates that the predictability of asset k depends on how today's news ($d\boldsymbol{Z}_t$) influences the asset's expected return in $t+\Delta$. It reflects the response of the expected return $\mu^S_{kt+\Delta}$ to the news $d\boldsymbol{Z}_t$.

To obtain a closed-form characterization, we examine predictability as Δ approaches 0. Plugging the dynamics of μ^S_{kt} and taking the covariance leads to

$$\mathbb{C}\text{ov}_t\left(\frac{dS_{kt}}{S_{kt}}, \frac{dS_{kt+\Delta\downarrow 0}}{S_{kt+\Delta\downarrow 0}}\right) = \boldsymbol{\sigma}^{S\prime}_{kt}\boldsymbol{b}^{\mu}_{kt}dt, \tag{37}$$

where $\boldsymbol{b}^{\mu}_{kt}$ is the exposure of asset k expected return to news, (35). The following definition characterizes momentum and reversal.

Definition 1 (**Momentum - Reversal**). *Asset k exhibits* momentum *if both the asset return and its expected return move in the same direction,*

$$\boldsymbol{\sigma}^{S\prime}_{kt}\boldsymbol{b}^{\mu}_{kt} > 0, \tag{38}$$

and exhibits reversal *if the asset return and the expected return move in opposite directions,*

$$\boldsymbol{\sigma}^{S\prime}_{kt}\boldsymbol{b}^{\mu}_{kt} < 0, \tag{39}$$

where $k = 1, 2$ and identifies the index and non-index assets, respectively.

The definition is straightforward. We identify momentum when returns rise (fall) and the equilibrium expectation is that they will continue to rise (fall). Conversely, we identify reversal when returns rise (fall) but the equilibrium expectation is that they will fall (rise) in the future. However, identifying the direction of predictability in equilibrium is challenging because the definition captures the average effect across all shocks.

There are two potential predictable channels that emerge from the no arbitrage condition, $\boldsymbol{\theta}'_t\boldsymbol{\sigma}^S_{it} = \mu^S_{it}$. The first channel is a *pricing effect*, which is driven by the predictable change in the market prices of risk, $\boldsymbol{\theta}_t$. The second channel is a *quantity-of-risk effect*, which is driven by the predictable change in the asset's risk exposures, $\boldsymbol{\sigma}_{it}^S$.

To disentangle these two channels and examine how shocks in each channel propagate to expected returns, we adopt the approach of Buffa and Hodor (2023). This approach introduces the concept of elasticity between two quantities within a dynamic system, akin to an impulse-response function.[2] Accordingly, we define the shock elasticity at time t of an equilibrium quantity X at time t as the time t expectation of the Malliavin derivative $\mathcal{D}_t$ of X_t, normalized by the absolute value of the time t expectation of X_t,

$$\varepsilon_t(X,\ell) \equiv \frac{\mathbb{E}_t[\mathcal{D}_t X_t]}{|\mathbb{E}_t[X_t]|} = \frac{\mathcal{D}_t^\ell X_t}{|X_t|} = \frac{\sigma_t^X(\ell)}{|X_t|}, \tag{40}$$

where $\sigma_t^X(\ell)$ is entry ℓ of the diffusion term of dX_t. The shock elasticity $\varepsilon_t(X,\ell)$ is well-defined for $X_t \neq 0$. News in the economy can be separated into two mutually exclusive types: index news and non-index news, which implies that there are two shock elasticities, one for each type of news. By applying the chain rule, we find that

$$\varepsilon_t(\mu_{it}^S,\ell) = \varepsilon_t(\boldsymbol{\sigma}_{it}^{S\prime}\boldsymbol{\theta}_t,\ell) = \frac{\boldsymbol{\sigma}_{it}^{S\prime}}{|\mu_{it}^S|}\left(\underbrace{(\mathcal{D}_t^\ell \boldsymbol{\theta}_t)}_{\substack{\text{benchmarking}\\ \text{incentives}}}\right) + \frac{\boldsymbol{\theta}_t'}{|\mu_{it}^S|}\left(\underbrace{(\mathcal{D}_t^\ell \boldsymbol{\sigma}_{it}^S)}_{\substack{\text{uncertainty of}\\ \text{performance}}}\right), \tag{41}$$

where the vector of Malliavin derivatives is the vector of derivatives for every entry. For example, $\mathcal{D}_t^\ell \boldsymbol{\theta}_t$ is the Malliavin derivative of each entry of the market price of risk vector with respect to news ℓ, which could be either index news or non-index news.

Let $\boldsymbol{\varepsilon}_t(\mu_{it}^S)$ be the vector of shock-elasticities. The definition of momentum implies a positive cross product between the shock elasticities of the expected return and the asset price. Conversely, the definition of reversal is indicated by a negative cross product,

$$\text{Momentum} \Rightarrow \boldsymbol{\varepsilon}_t(\mu_{it}^S)'\boldsymbol{\varepsilon}_t(S_{it}) > 0, \quad \text{Reversal} \Rightarrow \boldsymbol{\varepsilon}_t(\mu_{it}^S)'\boldsymbol{\varepsilon}_t(S_{it}) < 0. \tag{42}$$

[2]This methodology was originally presented by Borovička, Hansen, and Scheinkman (2014), which, in continuous time, builds on Malliavin calculus. Please refer to Detemple, Garcia, and Rindisbacher (2003) for further discussion on Malliavin calculus with Markovian processes in finance.

6.3 Incentives Channel

This channel causes a reversal in asset returns. Whenever the index asset return reacts positively to news, it also strengthens the incentives to outperform the benchmark versus the incentives to maintain performance. The shift in incentives means that asset managers require a stronger benchmark hedging position. As a result, equilibrium market clearing reduces the expected return of the benchmark asset. In contrast, whenever the index asset return reacts negatively to news, it also weakens the incentives to outperform the benchmark versus the incentives to maintain performance, and equilibrium increases expected returns. Similar logic applies to the non-index assets.

Let's examine this channel more closely, which is derived by the Malliavin derivative of the market prices of risk, which is given by

$$\mathcal{D}_t^\ell \boldsymbol{\theta}_t = \left\{ -\left(\gamma_2 - \gamma_1\right)^2 \mathcal{H}_{1t} \left(1 - \mathcal{H}_{1t}\right) \left(\boldsymbol{\sigma}^I(\ell) - \boldsymbol{\nu}(\ell)\right) \right\} \left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right). \tag{43}$$

The factor in the curly brackets determines how every entry of the market price of risk vector changes with the arrival of news. It shows that index news ($\ell = 1$) decreases the market prices of risk that correlate with the benchmark ($\boldsymbol{\sigma}^I > 0$) and non-index news ($\ell = 2, 3$) increases these market prices of risk. The opposite effect applies for market prices of risk that are not correlated with the benchmark ($\boldsymbol{\sigma}^I = 0$).

The logic is as follows. When positive index news arrives, the likelihood of underperformance increases because the portfolio of the asset manager is not fully correlated with the benchmark. Asset managers respond by hedging against further declines, increasing their allocation to the benchmark. To sustain equilibrium, the benchmark asset must become less attractive to offset the asset manager's demand, thereby reducing the market price of risk and the expected return of the benchmark. In contrast, when positive news unrelated to the benchmark arrives, the likelihood of underperformance decreases, and the logic follows in the opposite direction.

When asset managers desire more from the benchmark, their desire for other assets is reduced, which implies that the effects described here are reversed for assets excluded from the benchmark.

When cross multiplying $\mathcal{D}_t^\ell \boldsymbol{\theta}_t$ by $\boldsymbol{\sigma}_{it}^S$, we obtain the *incentives* shock elasiticity, $\varepsilon_t(\mu_{it}^S, \ell)$. This cross multiplication reveals that for any enrty whereby $\mathcal{D}_t^\ell \boldsymbol{\theta}_t$ is negative, $\boldsymbol{\sigma}_{1t}^S$ is positive,

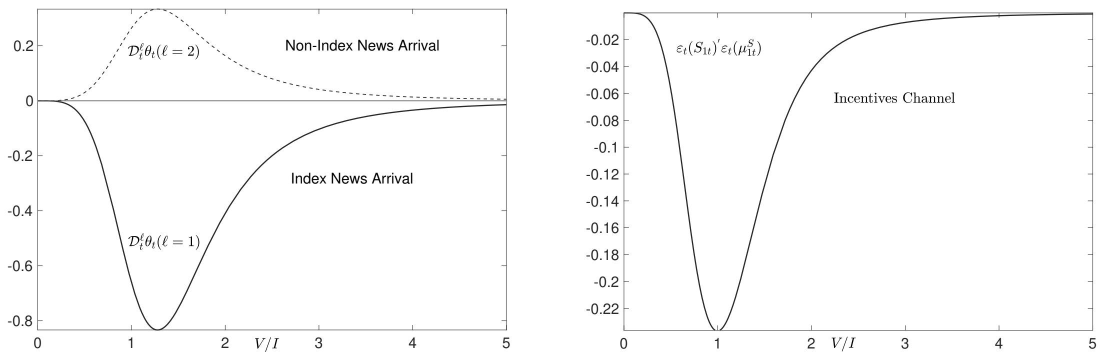

Figure 6. The left figure illustrates the curly brackets in (43). Positive index news decreases the market prices of risk that covary with the benchmark, and positive non-index news increases these market prices of risk. The reverse applies for the market prices of risk that do not covary with the benchmark. The right figure shows that the overall effect coming from the *incentives* channel produces a reversal in asset returns. The parameters are as in Figure 5.

and vice versa,

$$\begin{aligned}
\left(\mathcal{D}_t^\ell \boldsymbol{\theta}_t\right)' \boldsymbol{\sigma}_{1t}^S = &-(\gamma_2-\gamma_1)^3\,\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})\,(\mathcal{H}_{3t}-\mathcal{H}_{1t})\left(\boldsymbol{\sigma}^I(\ell)-\boldsymbol{\nu}(\ell)\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
&-(\gamma_2-\gamma_1)^2\,\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})\left(\boldsymbol{\sigma}^I(\ell)-\boldsymbol{\nu}(\ell)\right)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right), \qquad (44)\\
\left(\mathcal{D}_t^\ell \boldsymbol{\theta}_t\right)' \boldsymbol{\sigma}_{2t}^S = &-(\gamma_2-\gamma_1)^3\,\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})\,(\mathcal{H}_{1t}-\mathcal{H}_{4t})\left(\boldsymbol{\sigma}^I(\ell)-\boldsymbol{\nu}(\ell)\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
&+(\gamma_2-\gamma_1)^2\,\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})\left(\boldsymbol{\sigma}^I(\ell)-\boldsymbol{\nu}(\ell)\right)\left(\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D\right). \qquad (45)
\end{aligned}$$

To determine whether the *incentives* channel generates price momentum or reversal, we first need to identify the index asset shock elasticity to index and non-index news. This elasticity is given by

$$\varepsilon_t(S_{1t},\ell)=\boldsymbol{\sigma}_{1t}^S(\ell), \qquad \varepsilon_t(S_{2t},\ell)=\boldsymbol{\sigma}_{2t}^S(\ell), \tag{46}$$

which reveals that positive index news boosts the index return, while positive non-index news reduces it. Conversely, positive index news reduces the non-index asset return, whereas positive non-index news increases it.

In sum, the analysis reveals that the *incentives* channel induce an opposite reaction on the expected return and asset return shock elasticities: whenever one shock elasticity is positive the other is negative, and vice versa. Therefore, this channel causes asset price *reversals* during periods of both underperformance and outperformance.

$$
\begin{array}{l} \text{Underperformance and} \\ \text{Outperformance} \end{array} : \frac{\begin{array}{ccc} \varepsilon_t(S_{it},\ell) > 0 & \Rightarrow & S_{1t} \uparrow \\ \varepsilon_t(\mu^S_{it},\ell) < 0 & \Rightarrow & \mu^S_{1t} \downarrow \end{array}}{\begin{array}{ccc} \varepsilon_t(S_{it},\ell) < 0 & \Rightarrow & S_{1t} \downarrow \\ \varepsilon_t(\mu^S_{it},\ell) > 0 & \Rightarrow & \mu^S_{1t} \uparrow \end{array}} \quad \Rightarrow \quad \text{Reversal.} \tag{47}
$$

6.4 Uncertainty of Performance Channel

This mechanism leads to momentum in asset returns when asset managers outperform the benchmark, and can result in either reversal or momentum when they underperform the benchmark. When performance converges to the benchmark, it becomes less certain whether asset managers will outperform or underperform the benchmark. This uncertainty increases the assets response to news because asset managers balance risk-on and risk-off strategies more closely. Equilibrium market clearing compensates asset managers for the heightened uncertainty by weighing the return correlation with the market prices of risk.

When asset managers outperform the benchmark, news that brings their performance more closely in line with the benchmark boosts the benchmark asset return and its expected return. This is because the pricing impact of the correlation with the index news contributes to a positive expected return, which outweighs the negative pricing impact of the correlation with the market news. In contrast, similar news depresses the non-index asset return and also its expected return since the pricing impact of the correlation with the index news contributes to negative expected returns for the non-index asset. Therefore, asset returns exhibit momentum when asset managers outperform the benchmark.

When asset managers underperform relative to the benchmark, news that aligns their performance more closely with the benchmark decreases the benchmark asset return and increases the non-index asset return, causing reversals in both. However, as performance further declines, the pricing impact of the correlation with both the market news and the index news contribute to a negative price impact. Consequently, in such cases, news that brings performance closer to the benchmark reduces the benchmark's expected return and increases the non-index asset's expected return, leading to momentum when performance is significantly below the benchmark.

This channel is driven by the predictable change in the benchmark asset risk exposures, (41), which is given by the derivatives

$$\mathcal{D}_t^{\ell}\boldsymbol{\sigma}_{1t}^S = (\gamma_2 - \gamma_1)(\mathcal{H}_{3t} - \mathcal{H}_{1t})\left\{(\gamma_2 - \gamma_1)(1 - \mathcal{H}_{1t} - \mathcal{H}_{3t})\left(\boldsymbol{\sigma}^I(\ell) - \boldsymbol{\nu}(\ell)\right)\right\}\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right), \tag{48}$$

$$\mathcal{D}_t^{\ell}\boldsymbol{\sigma}_{2t}^S = -(\gamma_2 - \gamma_1)(\mathcal{H}_{1t} - \mathcal{H}_{4t})\left\{(\gamma_2 - \gamma_1)(1 - \mathcal{H}_{1t} - \mathcal{H}_{4t})\left(\boldsymbol{\sigma}^I(\ell) - \boldsymbol{\nu}(\ell)\right)\right\}\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right). \tag{49}$$

The factors in the curly brackets determines how every entry of the risk exposure vector changes with the arrival of news. It shows that the direction of the effect depends on whether the asset manager outperforms or underperforms the benchmark because, in outperformance, the risk-on hedging demands ($\mathcal{H}_{1t}$, $\mathcal{H}_{3t}$, and $\mathcal{H}_{4t}$) are close to zero, whereas in underperformance, they are close to one.

The curly brackets further indicates that news pushing the state variable V/I towards the center increases assets sensitivity (further away from zero sensitivity), while news pushing the state variable V/I away from the center reduces assets sensitivity (closer to zero sensitivity). In the outperformance region ($\mathcal{H}_{1t} + \mathcal{H}_{3t} < 1$ and $\mathcal{H}_{1t} + \mathcal{H}_{4t} < 1$), positive index news pushes the state variable towards the center, whereas in underperformance, positive index news pushes it away from the center. Conversely, positive non-index news pushes the state variable away from the center in outperformance, whereas in underperformance, positive non-index news pushes it towards the center. See Figure (7) for an illustration.

To determine whether this predictability channel induces momentum or reversal, we incorporate the pricing effect and analyze the shock elasticity of the expected return attributable to the *uncertainty of performance* channel, which is driven by the cross multiplication $\boldsymbol{\theta}_t'\left(\mathcal{D}_t^{\ell}\boldsymbol{\sigma}_{it}^S\right)$. These inner products are given by

$$\begin{aligned}\boldsymbol{\theta}_t'\left(\mathcal{D}_t^{\ell}\boldsymbol{\sigma}_{1t}^S\right) = &\overbrace{\left\{(\gamma_2 - \gamma_1)(1 - \mathcal{H}_{1t} - \mathcal{H}_{3t})\left(\boldsymbol{\sigma}^I(\ell) - \boldsymbol{\nu}(\ell)\right)\right\}}^{\textit{uncertainty of performance}} \times \\ &\underbrace{(\gamma_2 - \gamma_1)(\mathcal{H}_{3t} - \mathcal{H}_{1t})\{\,[(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\sigma}^{I\prime}\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right) - [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\nu}'\left(\boldsymbol{\nu} - \boldsymbol{\sigma}^I\right)\,\}}_{\text{expected return of the price-dividend ratio}},\end{aligned} \tag{50}$$

$$\begin{aligned}\boldsymbol{\theta}_t'\left(\mathcal{D}_t^{\ell}\boldsymbol{\sigma}_{2t}^S\right) = &\overbrace{\left\{(\gamma_2 - \gamma_1)(1 - \mathcal{H}_{1t} - \mathcal{H}_{4t})\left(\boldsymbol{\sigma}^I(\ell) - \boldsymbol{\nu}(\ell)\right)\right\}}^{\textit{uncertainty of performance}} \times \\ &\underbrace{-(\gamma_2 - \gamma_1)(\mathcal{H}_{1t} - \mathcal{H}_{4t})\{\,[(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\sigma}^{I\prime}\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right) - [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})]\,\boldsymbol{\nu}'\left(\boldsymbol{\nu} - \boldsymbol{\sigma}^I\right)\,\}}_{\text{expected return of the price-dividend ratio}}.\end{aligned} \tag{51}$$

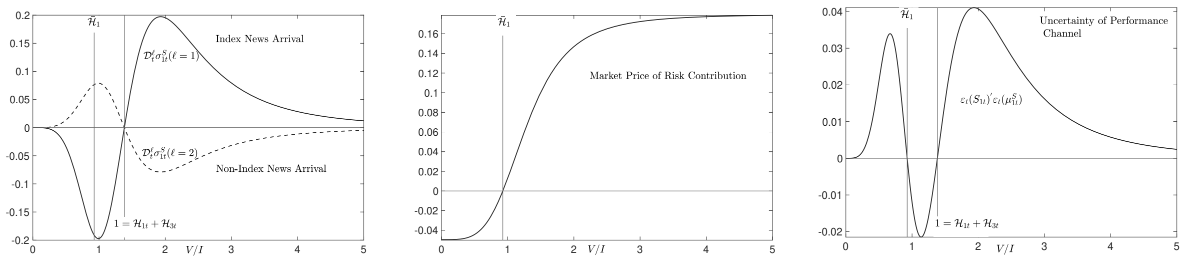

Figure 7. The left figure illustrates the predictable change in the quantities of risk of the benchmark asset, capture by the curly brackets in (48). When V/I is to the right of the vertical line $1 = \mathcal{H}_{1t} + \mathcal{H}_{3t}$, positive index news increases the sensitivities of the benchmark asset: the positive sensitivities becomes more positive and the negative ones becomes more negative. When V/I is to the left of the vertical line, the roles are reversed: positive index news decreases the sensitivities of the benchmark asset, and positive non-index news increases the sensitivities of the benchmark asset. The middle figure considers the contribution of the market prices of risk to the shock elasticity of the expected return, (50). When V/I is to the right of the vertical line indicated by $\bar{\mathcal{H}}_{1t}$ (52), the predictable change in the quantities of risk passes through to the expected return shock elasticity because the contribution of the market prices of risk is positive. However, when V/I falls below the vertical line $\bar{\mathcal{H}}_{1t}$, the predictable change in the quantities of risk does not pass through: the market price of risk contribution flips the effects coming from the predictable change in the quantities of risk. The right figure shows the overall effect coming from the *unceratinty of performance* channel. When V/I is above the vertical line $\bar{\mathcal{H}}_{1t}$, the predictability patterns in the quantity of risk pass to the expected return shock elasticity because the contribution of the market prices of risk is positive. In this case, the *unceratinty of performance* generates momentum when V/I is above the vertical line $1 = \mathcal{H}_{1t} + \mathcal{H}_{3t}$ (indicating the outperformance region), and reversal when it is below (indicating the underperformance region). However, when V/I is below the vertical line $\bar{\mathcal{H}}_{1t}$, the predictability patterns in the quantity of risk reverse since the contribution of the market prices of risk are negative. In this case, the *unceratinty of performance* generates momentum in asset returns. The parameters are as in Figure 5.

The representations reveal that the predictability patterns can be represented by the factor representing the *uncertainty of performance* multiplied by the expected return of the price-dividend ratio, which factors in the correlation of the asset return with index news and market news. When performance is above a threshold, the correlation between asset returns and the index contributes to a positive expected return, outweighing the negative contribution of the correlation with the market. However, as performance further declines and falls below a threshold, the correlation between asset returns and both the index and the market becomes negative, leading to a negative price impact. Despite the increase in uncertainty, expected returns may fall due to an adverse price reaction. The threshold is given by,

$$\bar{\mathcal{H}}_1 \equiv \frac{(\gamma_2 - 1)\, \boldsymbol{\sigma}^{I\prime} \left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right) - \gamma_1 \boldsymbol{\nu}' \left(\boldsymbol{\nu} - \boldsymbol{\sigma}^I\right)}{(\gamma_2 - \gamma_1)\, \|\boldsymbol{\sigma}^I - \boldsymbol{\nu}\|^2}, \tag{52}$$

and it exists when the numerator is strictly positve, which is true when the risk-on risk aversion parameter is sufficiently large,

$$\gamma_2 > 1 + \gamma_1 \frac{\boldsymbol{\nu}' \left(\boldsymbol{\nu} - \boldsymbol{\sigma}^I\right)}{\boldsymbol{\sigma}^{I\prime} \left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right)}. \tag{53}$$

Overall, when the asset manager outperforms the benchmark, we find that the *uncertainty of performance* induces the same reaction on the expected return and asset return shock elasticities: whenever one shock elasticity is positive the other is also positive, and vice versa. Therefore, this channel creates asset price *momentum* during periods of outperformance.

$$\text{Outperformance:} \quad \Rightarrow \quad \frac{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) > 0 & \Rightarrow & S_{1t} \uparrow \\ \varepsilon_t(\mu^S_{it}, \ell) > 0 & \Rightarrow & \mu^S_{1t} \uparrow \end{array}}{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) < 0 & \Rightarrow & S_{1t} \downarrow \\ \varepsilon_t(\mu^S_{it}, \ell) < 0 & \Rightarrow & \mu^S_{1t} \downarrow \end{array}} \quad \Rightarrow \text{Momentum.} \tag{54}$$

We proceed with the discussion of underperformance. In this case, the *uncertainty of performance* contributes to price reversals as long the contribution of the market prices of risk is positive, $\mathcal{H}_{1t} \leq \bar{\mathcal{H}}_1$. The reason is that news that pushes the asset manager performance towards the benchmark performance have a negative price impact today. Conversely, any news that makes underperfomance more ceratin have a positive price impact today.

$$\text{Underperformance, } \mathcal{H}_{1t} \leq \bar{\mathcal{H}}_{1t} : \Rightarrow \quad \frac{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) > 0 & \Rightarrow & S_{1t} \uparrow \\ \varepsilon_t(\mu^S_{it}, \ell) < 0 & \Rightarrow & \mu^S_{1t} \downarrow \end{array}}{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) < 0 & \Rightarrow & S_{1t} \downarrow \\ \varepsilon_t(\mu^S_{it}, \ell) > 0 & \Rightarrow & \mu^S_{1t} \uparrow \end{array}} \quad \Rightarrow \text{Reversal.} \tag{55}$$

When underperformance deteriorates further and $\mathcal{H}_{1t} > \bar{\mathcal{H}}_1$, the pricing effects flips the reversal predictability patterns, resulting in momentum when performance deteriorate further.

$$\text{Underperformance, } \mathcal{H}_{1t} > \bar{\mathcal{H}}_{1t} : \Rightarrow \frac{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) > 0 & \Rightarrow & S_{1t} \uparrow \\ \varepsilon_t(\mu^S_{it}, \ell) > 0 & \Rightarrow & \mu^S_{1t} \uparrow \end{array}}{\begin{array}{ccc} \varepsilon_t(S_{it}, \ell) < 0 & \Rightarrow & S_{1t} \downarrow \\ \varepsilon_t(\mu^S_{it}, \ell) < 0 & \Rightarrow & \mu^S_{1t} \downarrow \end{array}} \Rightarrow \text{Momentum.} \quad (56)$$

6.5 The Joint Effect of the Two Predictable Channels

The two predictable channels, *incentives* and *uncertainty of performance*, work in opposite directions when asset managers outperform the benchmark, $\mathcal{H}_{1t} + \mathcal{H}_{3t} < 1$, and when the contribution of the market prices of risk becomes negative, which occurs when asset managers underperform the benchmark, $\mathcal{H}_{1t} > \bar{\mathcal{H}}_1$.

The sum of the two predictable channels lead to an asymmetric price reaction: when asset managers underperform the benchmark, the benchmarking incentives channel dominates and lead to price reversals. However, when asset managers outperform the benchmark, the uncertainty of performance channel dominates and leads to price momentum.

In the following proposition, we identify a parametric restriction on the risk-on risk aversion parameter, γ_2. This ensures that the *benchmarking incentives* is at least as significant as the *uncertainty of performance.* Specifically, when the risk-on risk-aversion parameter falls within the specified range, $\underline{\gamma}_2 \leq \gamma_2 \leq \bar{\gamma}_2$, prices show overall momentum during outperformance periods and reversal in underperformance periods.

The same two mechanisms apply similarly to the non-index asset, though with a slightly different parametric restriction on the risk-on risk-aversion parameter. When asset managers underperform, the reversal channel is more dominant. Conversely, when outperformance is more likely, the momentum channel is stronger under the parametric restriction $\underline{\underline{\gamma}}_2 \leq \gamma_2 \leq \bar{\bar{\gamma}}_2$. The following proposition summarizes our main findings.

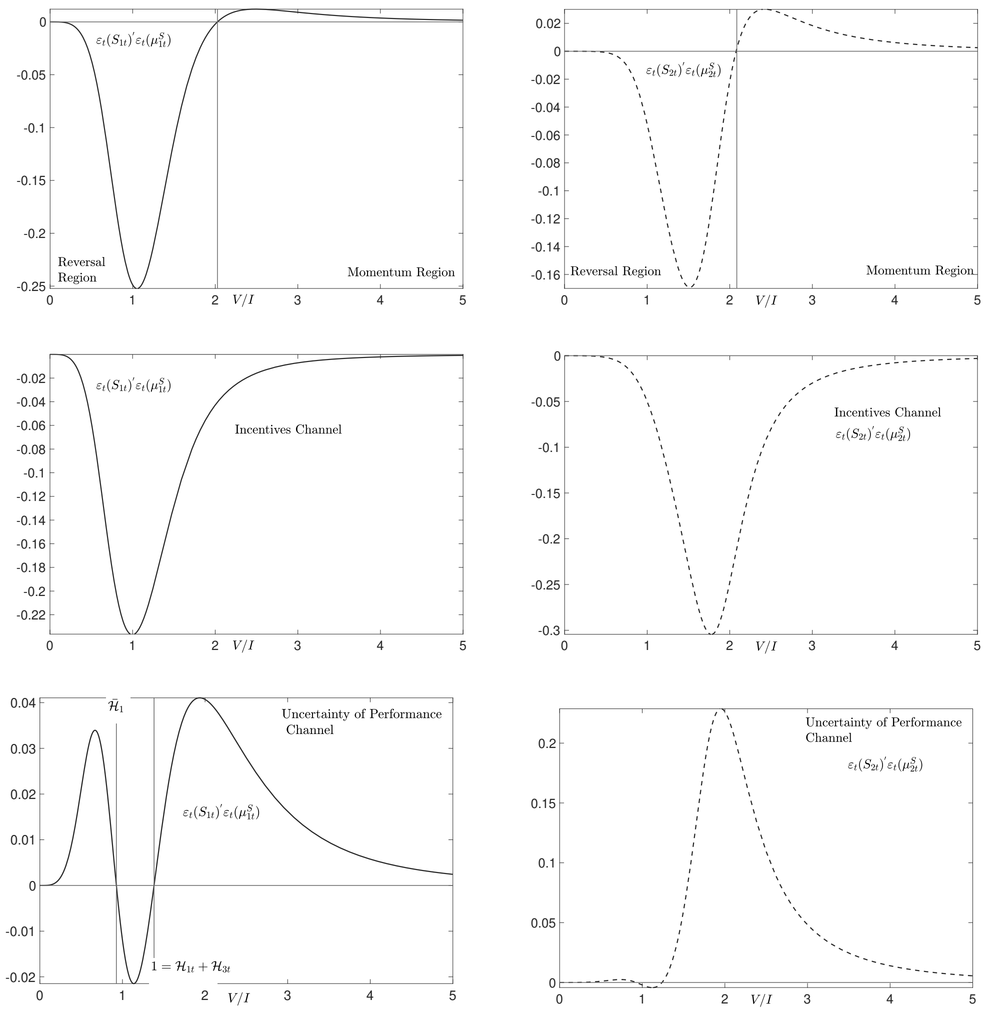

Figure 8. The top figures sums the two separate predictability channels: the *benchmarking incentives* channel in the middle figures, and the *uncertainty of performance* channel in the bottom figures. The left column figures are for the index asset and the right column figure is for the non-index asset. These figures illustrate the cross product between the shock elasticities of an asset return vector and the shock elasticities of its expected return. They demonstrate that the index and non-index assets exhibit reversal during underperformance and momentum during outperformance. Overall, momentum comes from the uncertainty of performance channel, and it dominates when asset managers outperform. Conversely, reversal comes from the benchmarking incentives channel, and it dominates when asset managers underperform. The parameters are as in Figure 5.

Proposition 4 (Time-Series Momentum and Reversal). *The index asset exhibits momentum when asset managers outperform and reversal if they underperform, if and only if*

$$\underline{\gamma}_2 < \gamma_2 < \bar{\gamma}_2. \tag{57}$$

Similarly, the non-index asset exhibits momentum when asset managers outperform and reversal if they underperform, if and only if

$$\underline{\underline{\gamma}}_2 < \gamma_2 < \bar{\bar{\gamma}}_2. \tag{58}$$

The risk-on risk-aversion parameter thresholds $\underline{\gamma}_2$, $\bar{\gamma}_2$, $\underline{\underline{\gamma}}_2$, and $\bar{\bar{\gamma}}_2$ depend on the risk-off risk aversion parameter, γ_1, and the active management share λ. For a given γ_1, there always exist an active management share λ such that (57) holds. The same is true for the non-index asset under the parametric restriction in (A.97). The thresholds are given in (A.77), (A.79), (A.82), and (A.83), respectively.

7 Cross-Sectional Predictability

Since the work of Jegadeesh and Titman (1993), the analysis of predictability in asset returns has involved constructing long-short portfolios by purchasing assets with high returns and selling those with low returns. According to this framework, momentum is identified when the long-short portfolio yields a positive return, whereas reversal is identified when the portfolio yields a negative return.

In this section, we show that the predictability patterns obsereved in the time-series of each asset separately also exist in the long-short portfolio: our theory implies that we expect momentum in the long-short portfolio when asset managers outperform their benchmark index and reversal when they underperform it.

It is neither obvious nor immediate that time-series predictability would manifest in the cross-section. This phenomenon primarily occurs because time-series momentum and reversal work in opposite directions for index and non-index assets. The long-short portfolio buys the asset that has risen and sells the asset that has fallen, as the former has a positive return and the latter a negative return. When asset managers outperform the benchmark, momentum causes the rising asset to continue rising and the falling asset to continue falling. Consequently, this strategy results in a positive return, aligning with traditional cross-sectional momentum. Conversely, when asset managers underperform the benchmark, reversal causes the rising asset to fall in the future and the falling asset to rise. As a result, this strategy leads to a negative return, aligning with traditional cross-sectional reversal.

Proposition 5 (**Momentum and Reversal in the Cross-Section**). *The shock sensitivity of the index asset is positive if and only if the shock sensitivity of the non-index asset is negative,*

$$\varepsilon_t(S_{1t}, \ell) > 0 \iff \varepsilon_t(S_{2t}, \ell) < 0, \quad \ell = 1, 2, 3. \tag{59}$$

Suppose that the risk-on risk aversion parameter satisfies the conditions (57) and (58), which guarentee time-series momentum in outperformance and time-series reversal in underperformance.

A long-short portfolio that takes a long position on the asset with positive shock sensitivity and a short position on the asset with negative shock sensitivity generates momentum when asset managers outperform the benchmark and reversal when they underperform the benchmark.

7.1 Prices and Expected Returns

To clearify the mechanism, this section characterizes the index and non-index prices and their expected returns. This analysis provides further intuitions for why the long-short portfolio generates momentum in the outperformance region and reversal in the underperformance region. We start by characterizing the index and non-index prices.

Proposition 6 (**Stock Prices**). *The index and non-index price-dividend ratios in the presence of risk-on incentives are given by*

$$S_t^I / I_t = \frac{\mathcal{E}_{3t}^{On}}{\mathcal{E}_{1t}^{On}}, \qquad S_{2t}/D_{2t} = \frac{\mathcal{E}_{4t}^{On}}{\mathcal{E}_{1t}^{On}}, \tag{60}$$

while in the presence of risk-off incentives, they are given by

$$S_t^I/I_t = \frac{\mathcal{E}_{3t}^{Off}}{\mathcal{E}_{1t}^{Off}}, \qquad S_{2t}/D_{2t} = \frac{\mathcal{E}_{4t}^{Off}}{\mathcal{E}_{1t}^{Off}}. \tag{61}$$

The index and non-index prices in the presence of both risk-on and risk-off incentives are given by

$$S_t^I/I_t = \frac{\mathcal{E}_{3t}^{On}}{\mathcal{E}_{1t}^{On}} \left[\frac{\mathcal{H}_{1t}}{\mathcal{H}_{3t}}\right], \qquad S_{2t}/D_{2t} = \frac{\mathcal{E}_{4t}^{On}}{\mathcal{E}_{1t}^{On}} \left[\frac{\mathcal{H}_{1t}}{\mathcal{H}_{4t}}\right]. \tag{62}$$

The comparison between the index and non-index price-dividend ratios reveals that the index price-dividend ratio

- *is higher than the non-index one with pure risk-on incentives.*
- *is lower than the non-index one with pure risk-off incentives, if the difference between risk-on and risk-off risk aversion parameters is greater than one:* $\gamma_2 - \gamma_1 \geq 1$.

In the joint incentives case, if $\gamma_2 - \gamma_1 \geq 1$*, the index price-dividend ratio is higher than the non-index one if and only if the ratio of the market news to the index news is below a deterministic threshold,*

$$\frac{V_t}{I_t} \leq \left[\frac{\left(\mathcal{E}_{3t}^{On} - \mathcal{E}_{4t}^{On}\right)}{\left(\mathcal{E}_{4t}^{Off} - \mathcal{E}_{3t}^{Off}\right)} \frac{(\gamma_2 - 1)}{(1 - \gamma_1)}\right]^{\frac{1}{\gamma_2 - \gamma_1}}. \tag{63}$$

Further, in the joint incentives case, the index price-dividend ratio S_t^I/I_t

- *decreases following news about the market:* $\partial\left(S_t^I/I_t\right)/\partial V_t < 0$,
- *increases following news about the index:* $\partial\left(S_t^I/I_t\right)/\partial I_t > 0$.

The results flip for the non-index price-dividend ratio, S_{2t}/D_{2t}*; it*

- *increases following news about the market:* $\partial\left(S_{2t}/D_{2t}\right)/\partial V_t > 0$,
- *decreases following news about the index:* $\partial\left(S_{2t}/D_{2t}\right)/\partial I_t < 0$,

where $\mathcal{H}_{1t}$, $\mathcal{H}_{3t}$, *and* $\mathcal{H}_{4t}$ *are given in (14), and* $\mathcal{E}_{1t}^{Off}$, $\mathcal{E}_{1t}^{On}$, $\mathcal{E}_{3t}^{Off}$, $\mathcal{E}_{3t}^{On}$, $\mathcal{E}_{4t}^{Off}$, $\mathcal{E}_{4t}^{On}$ *are given in (A.1), (A.2), (A.7), (A.8), (A.10), and (A.11), respectively.*

In pure risk-on incentives, asset managers desire to post higher returns when the benchmark is high rather than when it is low. These risk-on incentives induce asset managers to buy the benchmark to hedge against unexpected increases in the benchmark, which, in turn, creates extra demand for the benchmark. Equilibrium is restored by raising the prices of assets within the benchmark and lowering the prices of assets outside the benchmark, thereby diminishing the benchmark's attractiveness relative to the market.

In pure risk-off incentives, asset managers desire to maintain their performance. In this case, asset managers are risk-averse and protect their portfolio against scenarios where the benchmark declines. To achieve that goal, asset managers substantially reduce their benchmark position, which increases the available supply for the benchmark. Equilibrium is restored by decreasing the prices of benchmark assets and increasing the prices of non-benchmark assets, which in turn, enhances the attractiveness of the benchmark relative to the market.

In the joint incentives case, asset managers weigh the risk-on and risk-off strategies, depending on the likelihood that they will outperform the benchmark. Since their portfolio is not fully correlated with the benchmark, the likelihood of outperformance is influenced by the ratio of market news to index news, V/I. When V is large relative to I, the asset managers are likely to outperform, whereas when V is small relative to I, the asset managers are likely to underperform.

Thus far, we have demonstrated a negative correlation between index and non-index prices, an increase in the index price leads to a decrease in the non-index price, and vice versa. Next, we discuss the implications for expected returns.

By applying the covariaion between between the discount factor exposure (32) and the assets exposure, (24) and (25), we find that the expected return (per unit of time) of the index is given by

$$
\begin{aligned}
\mu_{1t}^{S} = & (\gamma_2 - \gamma_1)(\mathcal{H}_{3t} - \mathcal{H}_{1t}) \times \\
& \left\{ [(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \boldsymbol{\sigma}^{I\prime} \left(\boldsymbol{\sigma}^{I} - \boldsymbol{\nu} \right) - [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \boldsymbol{\nu}' \left(\boldsymbol{\nu} - \boldsymbol{\sigma}^{I} \right) \right\} \\
& + \left\{ [(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \boldsymbol{\sigma}^{I\prime} \left(\boldsymbol{\sigma}_1^{D} \right) + [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \boldsymbol{\nu}' \left(\boldsymbol{\sigma}_1^{D} \right) \right\},
\end{aligned}
\tag{64}
$$

and the expected return (per unit of time) of the non-index asset is given by

$$\mu_{2t}^{S} = (\gamma_2 - \gamma_1)(\mathcal{H}_{1t} - \mathcal{H}_{4t}) \times \\ \left\{ -\left[(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})\right] \boldsymbol{\sigma}^{I\prime}\left(\boldsymbol{\sigma}^{I} - \boldsymbol{\nu}\right) + \left[\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})\right] \boldsymbol{\nu}'\left(\boldsymbol{\nu} - \boldsymbol{\sigma}^{I}\right) \right\} \\ + \left\{ \left[(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})\right] \boldsymbol{\sigma}^{I\prime}\left(\boldsymbol{\sigma}_2^{D}\right) + \left[\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})\right] \boldsymbol{\nu}'\left(\boldsymbol{\sigma}_2^{D}\right) \right\}. \tag{65}$$

There are two curly brackets in each representation. These brackets identify the contribution of the market prices of risk, which reflects the *incentives* channel. As asset managers' performance declines, equilibrium lowers the market prices of risk that correlate with the benchmark and raises the market prices of risk orthogonal to the benchmark. Consequently, this channel reduces the expected return of benchmark assets and increases the expected return of assets outside the benchmark.

The leading factor multiplying the first curly brackets in each representation highlights assets' response due to the *uncertainty of performance*. As asset managers' performance aligns more closely with the benchmark, assets become increasingly sensitive to news in equilibrium, which increases the expected returns of both assets.

The expected returns are influenced by two channels, causing the benchmark asset's expected return to increase as asset managers' performance aligns more closely with the benchmark. Conversely, this alignment leads to a decrease in the non-index asset's expected return. Figure 9 illustrates that when asset managers underperform, the two channels work together for the index asset but operate in opposite directions for the non-index asset. When asset managers outperform the benchmark, the effects are reversed: the two channels work together for the non-index asset but operate in opposite directions for the index asset.

The fact that the expected returns of the index and non-index assets work in opposite direction is the key driver of the cross-sectional predictability: if the arrival of news increases the expected return of the index, it also decreases the expected return of assets outside the index. Figure 10 illustrates the predictability in the cross-section.

The following proposition summarizes our key findings.

Proposition 7 (**Expected Return**). *The expected returns for the index and non-index assets are given by (64) and (65), respectively. The expected return of the index in the case of extreme outperformance is strictly higher than in the case of extreme underperformance* ($\mu_{1t}^{S}\,|_{\mathcal{H}_{it}\downarrow 0} - \mu_{1t}^{S}\,|_{\mathcal{H}_{it}\uparrow 1} > 0$)*. In contrast, the expected return of the non-index asset in extreme*

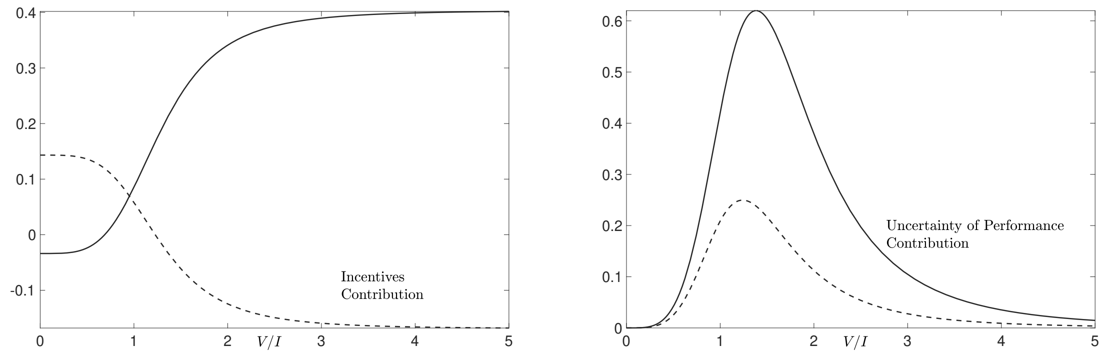

Figure 9. The solid line in the left figure plots the curly brackets for the index asset (64), and the dashed line represents the non-index asset (65). The figure shows that as performance deteriorates, the benchmarking incentives (changes to the market prices of risk) lead the benchmark asset's expected return to decline and the non-index asset's expected return to rise. The solid line in the right figure plots the leading factor multiplying the first curly brackets for the index asset (64), and the dashed line represents the non-index asset (65). The figure shows that expected returns of both assets rise due to the uncertainty of performance. The figures illustrate that when asset managers underperform, the two channels work together for the index asset but operate in opposite directions for the non-index asset. When asset managers outperform the benchmark, the effects are reversed: the two channels work together for the non-index asset but operate in opposite directions for the index asset. The parameters are as in Figure 5.

outperformance is strictly lower than in the case of extreme underperformance ($\mu^S_{2t}\ |_{\mathcal{H}_{it}\downarrow 0} - \mu^S_{2t}\ |_{\mathcal{H}_{it}\uparrow 1} < 0$)*. Furthermore, the index expected return*

- *decreases in outperformance if and only if* $\underline{\gamma}_2 \leq \gamma_2$*,*
- *increases in underperformance if and only if* $\bar{\gamma}_2 \geq \gamma_2$*.*

Conversly, the non-index expected return

- *increases in outperformance if and only if* $\underline{\underline{\gamma}}_2 \leq \gamma_2$*,*
- *decreases in underperformance if and only if* $\bar{\bar{\gamma}}_2 \geq \gamma_2$*,*

where $\underline{\gamma}_2$*,* $\bar{\gamma}_2$*,* $\underline{\underline{\gamma}}_2$*, and* $\bar{\bar{\gamma}}_2$ *are given in (A.77), (A.79), (A.82), and (A.83), respectively.*

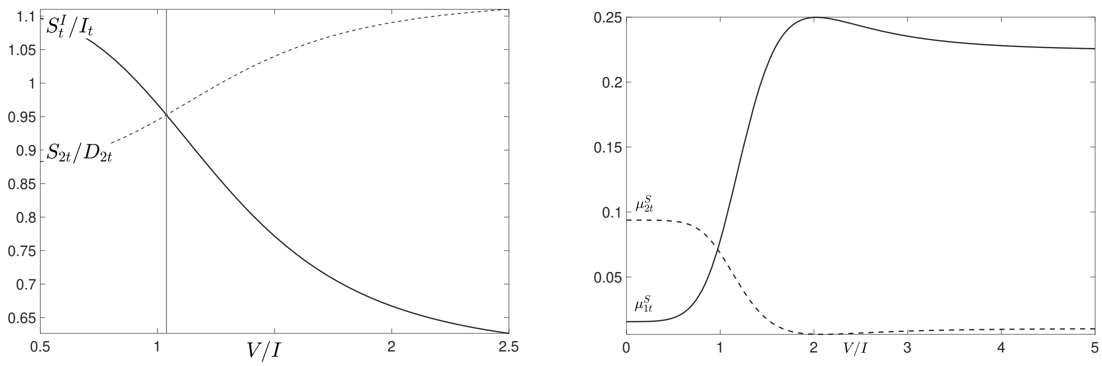

Figure 10. The left figure plots the index and non-index price-dividend ratios in the joint incentives case as a function of (V/I), the state variable governing the likelihood of performance. The vertical line is given in (63) and captures the location at which the price-dividend ratios flip. The figure shows that when the arrival of news causes one asset to rise, it also causes the other asset to fall. The right figure plots the expected returns of index and non-index assets as a function of (V/I). The long-short portfolio buys the asset that has risen (positive slope in the right figure) and sells the asset that has fallen (negative slope in the left figure). When asset managers outperform the benchmark, momentum causes the rising asset to continue rising (positive slope in the left figure) and the falling asset to continue falling (negative slope in the left figure). Consequently, this strategy results in a positive return, aligning with traditional cross-sectional momentum. Conversely, when asset managers underperform the benchmark, reversal causes the rising asset to fall (negative slope in the right figure) in the future and the falling asset to rise (positive slope in the right figure). As a result, this strategy leads to a negative return, aligning with traditional cross-sectional reversal. The parameters are as in Figure 5.

8 Conclusion

Mutual fund managers aim to balance incentives to outperform the index and incentives to maintain performance levels. This balance leads asset managers to optimally adjust their risk-on and risk-off portfolio positions based on their current performance. When managers are likely to underperform the index, they increase the weight of the risk-on portfolio to boost their performance above the benchmark. As their performance improves, they shift towards a risk-off portfolio, reducing the weight of the risk-on portfolio.

This paper incorporates these time-varying incentives into a dynamic continuous-time asset pricing model and derives equilibrium quantities in closed form.

The paper demonstrates that the active share follows a U-shaped pattern, while the tracking error follows an inverted U-shaped pattern in relation to fund performance. This prediction reconciles conflicting empirical findings: some researchers claim a U-shaped relationship between early-year performance and later-year risk choices, while others claim an inverted U-shape. We contend that U-shaped risk measures are associated with the active share, whereas inverted U-shaped risk measures are linked to the tracking error.

In addition, previous research suggests that the differences between active-share and tracking-error measures stem from varying investment strategies. Our theory posits that these differences can be partially explained by the performance of the funds, rather than by inherent differences in their investment strategies. Furthermore, early research attributes the difference between active-share and tracking-error measures to different investment strategies. Our theory claims this difference can be partially reconciled by the performance of the funds rather than inherent differences in their investment strategies.

Perhaps more importantly, asset managers' aim to balance between risk-on and risk-off investment strategies serves as a potential reason for the observed momentum and reversal in asset markets, both in the time-series and cross-section. We observe time-series momentum when asset managers outperform the benchmark and time-series reversal when they fall behind the benchmark. Furthermore, we show that the predictability patterns observed in the time-series also exist in the long-short portfolio. A portfolio that buys the asset with a higher return and sells the asset with a lower return generates a positive return (momentum) when asset managers outperform the benchmark and results in a negative return (reversal) when they underperform the benchmark.

Return predictability emerges from two primary channels. The first channel is driven by predictable changes in incentive contracts: as new information becomes available, asset managers adjust the optimal balance between risk-on and risk-off portfolios. This adjustment impacts returns through market clearing. The second channel is related to performance uncertainty. When the chances of outperformance and underperformance are equal, there is heightened uncertainty about asset managers' performance. This uncertainty amplifies asset sensitivity to news through equilibrium.

A Proofs

Before we begin with the formal proofs, we solve the following expected values.

$$\begin{aligned}\mathcal{E}_{1t}^{\text{Off}} &= (\mathbf{1}_{\text{Off}})\, \mathcal{E}_{1t}\left(1-\gamma_2, -\gamma_1\right) \equiv \frac{E_t\left[\left(I_T\right)^{1-\gamma_2}\left(V_T\right)^{-\gamma_1}\right]}{\left(I_t\right)^{1-\gamma_2}\left(V_t\right)^{-\gamma_1}} \\ &= e^{\left((-\gamma_1)(\eta)-(\gamma_2-1)(\mu_1)+(\sigma_1\nu_1)(\gamma_1)(\gamma_2-1)+\frac{1}{2}(\sigma_1)^2(\gamma_2)(\gamma_2-1)+\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_1)(1+\gamma_1)\right)(T-t)},\end{aligned} \tag{A.1}$$

$$\begin{aligned}\mathcal{E}_{1t}^{\text{On}} &= (\mathbf{1}_{\text{On}})\, \mathcal{E}_{1t}\left(1-\gamma_1, -\gamma_2\right) \equiv \frac{E_t\left[\left(I_T\right)^{1-\gamma_1}\left(V_T\right)^{-\gamma_2}\right]}{\left(I_t\right)^{1-\gamma_1}\left(V_t\right)^{-\gamma_2}} \\ &= e^{\left((-\gamma_2)(\eta)+(1-\gamma_1)(\mu_1)-(\sigma_1\nu_1)(\gamma_2)(1-\gamma_1)-\frac{1}{2}(\sigma_1)^2(\gamma_1)(1-\gamma_1)+\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_2)(1+\gamma_2)\right)(T-t)},\end{aligned} \tag{A.2}$$

$$\frac{\mathcal{E}_{1t}^{\text{Off}}}{\mathcal{E}_{1t}^{\text{On}}} \equiv \mathcal{E}_{1t} = e^{\left((\gamma_2-\gamma_1)\left(\eta-\mu_1+\sigma_1\nu_1+\frac{1}{2}(\sigma_1)^2(\gamma_1+\gamma_2-1)-\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_1+\gamma_2+1)\right)\right)(T-t)}, \tag{A.3}$$

$$\begin{aligned}\mathcal{E}_{2t}^{\text{Off}} &= (\mathbf{1}_{\text{Off}})\, \mathcal{E}_{2t}\left(1-\gamma_2, 1-\gamma_1\right) \equiv \frac{E_t\left[\left(I_T\right)^{1-\gamma_2}\left(V_T\right)^{1-\gamma_1}\right]}{\left(I_t\right)^{1-\gamma_2}\left(V_t\right)^{1-\gamma_1}} \\ &= e^{\left((1-\gamma_1)(\eta)-(\gamma_2-1)(\mu_1)-(\sigma_1\nu_1)(1-\gamma_1)(\gamma_2-1)+\frac{1}{2}(\sigma_1)^2(\gamma_2)(\gamma_2-1)+\frac{1}{2}\|\boldsymbol{\nu}\|^2(-\gamma_1)(1-\gamma_1)\right)(T-t)},\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathcal{E}_{2t}^{\text{On}} &= (\mathbf{1}_{\text{On}})\, \mathcal{E}_{2t}\left(1-\gamma_1, 1-\gamma_2\right) \equiv \frac{E_t\left[\left(I_T\right)^{1-\gamma_1}\left(V_T\right)^{1-\gamma_2}\right]}{\left(I_t\right)^{1-\gamma_1}\left(V_t\right)^{1-\gamma_2}} \\ &= e^{\left((1-\gamma_2)(\eta)+(1-\gamma_1)(\mu_1)-(\sigma_1\nu_1)(\gamma_2-1)(1-\gamma_1)-\frac{1}{2}(\sigma_1)^2(\gamma_1)(1-\gamma_1)+\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_2)(\gamma_2-1)\right)(T-t)},\end{aligned} \tag{A.5}$$

$$\frac{\mathcal{E}_{2t}^{\text{Off}}}{\mathcal{E}_{2t}^{\text{On}}} \equiv \mathcal{E}_{2t} = e^{\left((\gamma_2-\gamma_1)\left(\eta-\mu_1+\frac{1}{2}\left[(\sigma_1)^2-\|\boldsymbol{\nu}\|^2\right](\gamma_1+\gamma_2-1)\right)\right)(T-t)}, \tag{A.6}$$

$$\begin{aligned}\mathcal{E}_{3t}^{\text{Off}} &= (\mathbf{1}_{\text{Off}})\, \mathcal{E}_{3t}\left(2-\gamma_2, -\gamma_1\right) \equiv \frac{E_t\left[\left(I_T\right)^{2-\gamma_2}\left(V_T\right)^{-\gamma_1}\right]}{\left(I_t\right)^{2-\gamma_2}\left(V_t\right)^{-\gamma_1}} \\ &= e^{\left(-\gamma_1\eta+(2-\gamma_2)(\mu_1)-(\sigma_1\nu_1)\gamma_1(2-\gamma_2)+\frac{1}{2}(\sigma_1)^2(\gamma_2-1)(\gamma_2-2)+\frac{1}{2}\|\boldsymbol{\nu}\|^2\gamma_1(1+\gamma_1)\right)(T-t)},\end{aligned} \tag{A.7}$$

$$\mathcal{E}_{3t}^{\text{On}} = (\mathbf{1}_{\text{On}})\,\mathcal{E}_{3t}\,(2-\gamma_1,-\gamma_2) \equiv \frac{E_t\left[(I_T)^{2-\gamma_1}\,(V_T)^{-\gamma_2}\right]}{(I_t)^{2-\gamma_1}\,(V_t)^{-\gamma_2}}$$
$$= e^{\left(-\gamma_2\eta+(2-\gamma_1)(\mu_1)-(\sigma_1\nu_1)\gamma_2(2-\gamma_1)+\frac{1}{2}(\sigma_1)^2(\gamma_1-1)(\gamma_1-2)+\frac{1}{2}\|\boldsymbol{\nu}\|^2\gamma_2(1+\gamma_2)\right)(T-t)}, \tag{A.8}$$

$$\frac{\mathcal{E}_{3t}^{\text{Off}}}{\mathcal{E}_{3t}^{\text{On}}} \equiv \mathcal{E}_{3t} = e^{(\gamma_2-\gamma_1)\left(\eta-\mu_1+2(\sigma_1\nu_1)+\frac{1}{2}(\sigma_1)^2(\gamma_2+\gamma_1-3)-\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_2+\gamma_1+1)\right)(T-t)}, \tag{A.9}$$

$$\mathcal{E}_{4t}^{\text{Off}} = (\mathbf{1}_{\text{Off}})\,\mathcal{E}_{4t}\,(1-\gamma_2,-\gamma_1) \equiv \frac{E_t\left[(I_T)^{1-\gamma_2}\,(V_T)^{-\gamma_1}\,D_{2T}\right]}{(I_t)^{1-\gamma_2}\,(V_t)^{-\gamma_1}\,D_{2t}}$$
$$= e^{\left[-(\gamma_2-1)\mu_1-\gamma_1\eta+\mu_2+(\sigma_1\nu_1)(\gamma_1)(\gamma_2-1)-(\sigma_2\nu_2)(\gamma_1)+\frac{1}{2}(\sigma_1)^2(\gamma_2-1)(\gamma_2)+\frac{1}{2}\|\boldsymbol{\nu}\|^2\gamma_1(1+\gamma_1)\right](T-t)}, \tag{A.10}$$

$$\mathcal{E}_{4t}^{\text{On}} = (\mathbf{1}_{\text{On}})\,\mathcal{E}_{4t}\,(1-\gamma_1,-\gamma_2) \equiv \frac{E_t\left[(I_T)^{1-\gamma_1}\,(V_T)^{-\gamma_2}\,D_{2T}\right]}{(I_t)^{1-\gamma_1}\,(V_t)^{-\gamma_2}\,D_{2t}}$$
$$= e^{\left[-(\gamma_1-1)\mu_1-\gamma_2\eta+\mu_2+(\sigma_1\nu_1)(\gamma_2)(\gamma_1-1)-(\sigma_2\nu_2)(\gamma_2)+\frac{1}{2}(\sigma_1)^2(\gamma_1-1)(\gamma_1)+\frac{1}{2}\|\boldsymbol{\nu}\|^2\gamma_2(1+\gamma_2)\right](T-t)}, \tag{A.11}$$

$$\frac{\mathcal{E}_{4t}^{\text{Off}}}{\mathcal{E}_{4t}^{\text{On}}} \equiv \mathcal{E}_{4t} = e^{(\gamma_2-\gamma_1)\left[-\mu_1+\eta+(\sigma_1\nu_1)+(\sigma_2\nu_2)+\frac{1}{2}(\sigma_1)^2(\gamma_2+\gamma_1-1)-\frac{1}{2}\|\boldsymbol{\nu}\|^2(\gamma_2+\gamma_1+1)\right](T-t)}. \tag{A.12}$$

We reference to these expected values throughout the proofs. Note that $\mathcal{E}_{it}$, $i = 1,2,3,4$ is not well defined in the risk-off incentive case, since $\mathcal{E}_{it}^{\text{On}} = 0$. In pure risk-on incentives $\mathcal{E}_{it} = 0$, $i = 1,2,3,4$ since $\mathcal{E}_{it}^{\text{Off}} = 0$. By plugging the definition of $\mathcal{E}_{it}$ into $\mathcal{H}_{it}$ we find that

$$\mathcal{H}_{it} \equiv \frac{1}{1+\frac{1-\gamma_1}{\gamma_2-1}\,(V_t/I_t)^{\gamma_2-\gamma_1}\,\mathcal{E}_{it}} = \frac{\mathcal{E}_{it}^{\text{On}}}{\mathcal{E}_{it}^{\text{On}}+\frac{1-\gamma_1}{\gamma_2-1}\,(V_t/I_t)^{\gamma_2-\gamma_1}\,\mathcal{E}_{it}^{\text{Off}}}, \tag{A.13}$$

implying that $\mathcal{H}_{it} = 0$ in the pure risk-off incentives and $\mathcal{H}_{it} = 1$ in the pure risk-on incentives. Notice that

$$\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu} = \frac{\sigma_1^2}{N}\,(N-\lambda) > 0, \tag{A.14}$$
$$\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I = \frac{\sigma_1^2}{N^2}\,(1-\lambda)\,((N-1)-\lambda)\,, \tag{A.15}$$

implying that $\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I$ approached zero as $\lambda \to 1$, and that

$$\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu} > \|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I. \tag{A.16}$$

By using the inequalities (A.14) and (A.15), and the that positive news have a positive impact on fundamentals, $\sigma_2 > 0$ and $\nu_2 > 0$, we find that

$$\mathcal{H}_{3t} > \mathcal{H}_{1t} \iff \mathcal{E}_{1t} > \mathcal{E}_{3t} \iff e^{(\gamma_2-\gamma_1)\left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)(T-t)} > 1, \tag{A.17}$$

$$\mathcal{H}_{1t} > \mathcal{H}_{4t} \iff \mathcal{E}_{4t} > \mathcal{E}_{1t} \iff e^{(\gamma_2-\gamma_1)(\sigma_2\nu_2)(T-t)} > 1, \tag{A.18}$$

$$\mathcal{H}_{1t} > \mathcal{H}_{2t} \iff \mathcal{E}_{2t} > \mathcal{E}_{1t} \iff e^{(\gamma_2-\gamma_1)\left(\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)(T-t)} > 1, \tag{A.19}$$

$$\mathcal{H}_{3t} > \mathcal{H}_{2t} \iff \mathcal{E}_{2t} > \mathcal{E}_{3t} \iff e^{(\gamma_2-\gamma_1)\left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)(T-t)} > 1. \tag{A.20}$$

The definition of the risk-on hedging demand (14) and the ensuing analysis implies that in extreme underperformance the ratio of market news to index news approaches zero, $V/I \downarrow 0$, while in extreme outperformance the ratio approaches infinity, $V/I \downarrow \infty$, implying that

$$\lim_{V/I \to 0} \mathcal{H}_{it} - \mathcal{H}_{jt} \to 0, \qquad \lim_{V/I \to \infty} \mathcal{H}_{it} - \mathcal{H}_{jt} \to 0. \tag{A.21}$$

The derivative of the risk-on hedging demand with respect to V/I equals

$$\frac{d\mathcal{H}_{it}}{d\frac{V}{I}} = -\left(\gamma_2 - \gamma_1\right)\mathcal{H}_{it}\left(1 - \mathcal{H}_{it}\right)\left(V_t/I_t\right)^{-1} < 0, \tag{A.22}$$

and the derivative of $\mathcal{H}_{it} - \mathcal{H}_{jt}$ with respect to V/I is given by

$$\frac{d\mathcal{H}_{it}}{d\frac{V}{I}} - \frac{d\mathcal{H}_{jt}}{d\frac{V}{I}} = -\left(\gamma_2 - \gamma_1\right)\left(V_t/I_t\right)^{-1}\left(\mathcal{H}_{it} - \mathcal{H}_{jt}\right)\left(1 - \mathcal{H}_{it} - \mathcal{H}_{jt}\right), \tag{A.23}$$

which approaches zero at extreme outperformance and underperformance due to (A.21). Next, we find that $\mathcal{H}_{3t} - \mathcal{H}_{1t}$ converges to zero as fast as $\mathcal{H}_{1t}\left(1 - \mathcal{H}_{1t}\right)$ converges to zero, both when $\mathcal{H}_{it} \to 0$ and $\mathcal{H}_{it} \to 1$, and the limits are given by

$$\lim_{\mathcal{H}_{it}\to 0} = \lim_{V/I\to\infty} \frac{\mathcal{H}_{jt} - \mathcal{H}_{1t}}{\mathcal{H}_{1t}\left(1 - \mathcal{H}_{1t}\right)} = \frac{\mathcal{E}_{1t} - \mathcal{E}_{jt}}{\mathcal{E}_{jt}} > 0, \quad j = 3, 4, \tag{A.24}$$

$$\lim_{\mathcal{H}_{it}\to 1} = \lim_{V/I\to 0} \frac{\mathcal{H}_{jt} - \mathcal{H}_{1t}}{\mathcal{H}_{1t}\left(1 - \mathcal{H}_{1t}\right)} = \frac{\mathcal{E}_{1t} - \mathcal{E}_{jt}}{\mathcal{E}_{1t}} > 0. \tag{A.25}$$

Lastly,

$$\lim_{V/I\to 0} \frac{\mathcal{H}_{1t}\left(1 - \mathcal{H}_{1t}\right)}{V/I} = \lim_{V/I\to\infty} \frac{\mathcal{H}_{1t}\left(1 - \mathcal{H}_{1t}\right)}{V/I} = 0, \tag{A.26}$$

$$\lim_{V/I\to 0} \frac{\mathcal{H}_{jt} - \mathcal{H}_{1t}}{V/I} = \lim_{V/I\to\infty} \frac{\mathcal{H}_{jt} - \mathcal{H}_{1t}}{V/I} = 0, \quad j = 3, 4. \tag{A.27}$$

Proof of Proposition 8 (Discount Factor). The security market is dynamically complete. As such, there exists a unique state price density process, ξ, and the no arbitrage relations

$$\xi_t S_{kt} = \mathbb{E}_t\left[\xi_T D_{kT}\right], \quad t \in [0, T], k = 1, 2, 3, \tag{A.28}$$

is always satisfied. In our setup, we set $r = 0$ for $t \leq T$, and thus, the state price density evolves according to

$$d\xi_t = -\xi_t\left(\boldsymbol{\theta}_t' d\boldsymbol{Z}_t\right), \quad t \leq T. \tag{A.29}$$

The vector process $\boldsymbol{\theta}_t$ is the cash-flow news market risk prices. Restating the dynamic budget constraint as

$$\xi_t W_t^{\mathcal{A}} = \mathbb{E}_t\left[\xi_T W_T^{\mathcal{A}}\right], \quad t \in [0, T], \tag{A.30}$$

and maximizing the asset manager's objective function (11), subject to (A.30) at time $t = 0$, we obtain the first order condition

$$y^{\mathcal{A}}\xi_T = \mathbf{1}_{\text{Off}} \frac{\left(I_T\right)^{1-\gamma_2}\left(W_T^{\mathcal{A}}\right)^{-\gamma_1}}{\left(\gamma_2 - 1\right)} + \mathbf{1}_{\text{On}} \frac{\left(I_T\right)^{1-\gamma_1}\left(W_T^{\mathcal{A}}\right)^{-\gamma_2}}{\left(1 - \gamma_1\right)}, \tag{A.31}$$

where $y^{\mathcal{A}}$ is the Lagrange multiplier, and due to no-arbitrage condition at time T, $S_T^I = I_T$.

To clear markets, the asset managers must hold the entire active supply S_t^A, which at time T equals V_T, leading to

$$y^{\mathcal{A}}\xi_T = \mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}. \tag{A.32}$$

Since ξ_t is a martingale, we have

$$\xi_{0,t} = \frac{\mathbb{E}_t\left[\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}\right]}{\mathbb{E}\left[\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}\right]}. \tag{A.33}$$

By plugging (A.1) and (A.2) into (A.33) we find that

$$\xi_{0,t} = \frac{\frac{\mathbf{1}_{\text{Off}}}{\gamma_2-1}(I_t)^{1-\gamma_2}(V_t)^{-\gamma_1}\mathcal{E}_{1t}^{\text{Off}} + \frac{\mathbf{1}_{\text{On}}}{1-\gamma_1}(I_t)^{1-\gamma_1}(V_t)^{-\gamma_2}\mathcal{E}_{1t}^{\text{On}}}{\frac{\mathbf{1}_{\text{Off}}}{\gamma_2-1}(I_0)^{1-\gamma_2}(V_0)^{-\gamma_1}\mathcal{E}_{10}^{\text{Off}} + \frac{\mathbf{1}_{\text{On}}}{1-\gamma_1}(I_0)^{1-\gamma_1}(V_0)^{-\gamma_2}\mathcal{E}_{10}^{\text{On}}}. \tag{A.34}$$

To find the Lagrange multiplier, we plug the equilibrium discount factor (A.32) into the dynamic budget constraint (A.30) evaluated at time $t = 0$, apply the market clearing condition so that the asset managers must hold the entire active supply and find that

$$\xi_0 S_0^A = \mathbb{E}\left[\frac{1}{y^{\mathcal{A}}}\left(\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{1-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{1-\gamma_2}}{(1-\gamma_1)}\right)\right], \tag{A.35}$$

which eventually leads to

$$\begin{aligned} y^{\mathcal{A}} &= \frac{1}{\xi_0 S_0^A}\mathbb{E}\left[\left(\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{1-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{1-\gamma_2}}{(1-\gamma_1)}\right)\right] \\ &= \frac{1}{\xi_0 S_0^A}\left[\frac{\mathbf{1}_{\text{Off}}}{\gamma_2-1}(I_0)^{1-\gamma_2}(V_0)^{1-\gamma_1}\mathcal{E}_{20}(1-\gamma_2, 1-\gamma_1) + \frac{\mathbf{1}_{\text{On}}}{1-\gamma_1}(I_0)^{1-\gamma_1}(V_0)^{1-\gamma_2}\mathcal{E}_{20}(1-\gamma_1, 1-\gamma_2)\right], \end{aligned} \tag{A.36}$$

where the second equality is due to (A.4) and (A.5) evaluated at time $t = 0$.

We conclude the proof with the discount factor derivatives. It is straightforward to show that $\xi_{0,t}$ is decreasing in V_t and increasing in I_t in the risk-on incentives case, (B.1), and $\xi_{0,t}$ is decreasing in V_t and I_t in the risk-off incentives case, (B.2). The derivative of the discount factor with respect to I_t in the joint incentives case equals

$$\frac{\partial \xi_{0,t}}{\partial I_t} = (1-\gamma_1)\frac{1}{I_t}\xi_{0,t} - \xi_{0,t}\frac{1}{\mathcal{H}_{1t}}\frac{\partial \mathcal{H}_{1t}}{\partial I_t}, \tag{A.37}$$

where the derivative of the risk-on hedging demand with respect to I_t equals

$$\frac{\partial \mathcal{H}_{1t}}{\partial I_t} = (\mathcal{H}_{1t})^2 \left(\frac{1-\gamma_1}{\gamma_2 - 1}(\gamma_2 - \gamma_1)(I_t)^{\gamma_1-\gamma_2-1}(V_t)^{\gamma_2-\gamma_1}\mathcal{E}_{1t}\right) > 0. \tag{A.38}$$

By plugging the hedging demand derivative (A.38) into the discount factor derivative (A.37), and by removing strictly positive terms and rearranging, we find that $\partial \xi_{0,t}/\partial I_t < 0$ if and only if

$$(1-\gamma_1) - (\gamma_2 - \gamma_1)(-\mathcal{H}_{1t} + 1) < 0, \tag{A.39}$$

which leads to our desired result in (B.4). Similarly, the derivative of the discount factor with respect to V_t in the joint incentives case equals

$$\frac{\partial \xi_{0,t}}{\partial V_t} = (-\gamma_2)\frac{1}{V_t}\xi_t - \xi_t\frac{1}{\mathcal{H}_{1t}}\frac{\partial \mathcal{H}_{1t}}{\partial V_t}, \tag{A.40}$$

where the derivative of the risk-on hedging demand with respect to V_t equals

$$\frac{\mathcal{H}_{it}}{\partial V_t} = -(\mathcal{H}_{it})^2 \left(\frac{1-\gamma_1}{\gamma_2 - 1}(\gamma_2 - \gamma_1)(I_t)^{\gamma_1-\gamma_2}(V_t)^{\gamma_2-\gamma_1-1}\mathcal{E}_{1t}\right) < 0, \quad i = 1,2,3,4. \tag{A.41}$$

By plugging the hedging demand derivative (A.41) into the discount factor derivative (A.40) and rearranging, we find that $\partial \xi_{0,t}/\partial V_t < 0$ if and only if

$$(\gamma_2 - \gamma_1)\mathcal{H}_{1t} + \gamma_1 > 0, \tag{A.42}$$

which is always true because $\gamma_2 > \gamma_1$ and $\mathcal{H}_{1t} > 0$.

□

Proof of Lemma 1 (Risk Exposures). We utilize the no-arbitrage condition (A.30) at time t, the market clearing condition, and write

$$\xi_t W_t^{\mathcal{A}} = \mathbb{E}_t\left[\xi_T V_T\right] = \frac{1}{y^{\mathcal{A}}}\mathbb{E}_t\left[\left(\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{1-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{1-\gamma_2}}{(1-\gamma_1)}\right)\right]. \tag{A.43}$$

By plugging (A.4) and (A.5) we obtain

$$\xi_t W_t^{\mathcal{A}} = \frac{1}{y^{\mathcal{A}}}\left[\frac{\mathbf{1}_{\text{Off}}}{\gamma_2-1}(I_t)^{1-\gamma_2}(V_t)^{1-\gamma_1}\mathcal{E}_{2t}^{\text{Off}} + \frac{\mathbf{1}_{\text{On}}}{1-\gamma_1}(I_t)^{1-\gamma_1}(V_t)^{1-\gamma_2}\mathcal{E}_{2t}^{\text{On}}\right] \tag{A.44}$$

Applying Itô's Lemma on both sides of the above equation leads to the desired result.

$$\boldsymbol{\pi}_t^{\mathcal{A}\prime}\boldsymbol{\Sigma}^S = \boldsymbol{\theta}_t + (1-\gamma_1)\boldsymbol{\nu} - (\gamma_2-1)\boldsymbol{\sigma}_1^D + (\gamma_2-\gamma_1)\mathcal{H}_{2t}^{\text{On}}\left(\boldsymbol{\sigma}_1^D - \boldsymbol{\nu}\right) \tag{A.45}$$

□

Proof of Proposition 1 (Market Prices of Risk). Applying Itô's Lemma on both sides of (A.34) leads to the desired result.

□

Proof of Proposition 6 (Stock Prices). We start from the no arbitrage condition (A.28), plug $\xi_{0,T}$ (A.34), and find that the deflated index price satisfies

$$S_t^I \xi_{0,t} = \frac{\mathbb{E}_t\left[\left(\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}\right)I_T\right]}{\mathbb{E}\left[\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}\right]}. \tag{A.46}$$

By plugging the expected values from (A.7) and (A.8), $\xi_{0,t}$ (A.34), and rearranging we obtain the desired result. Next, we characterize the non-index price. Similar to the index price, we find that the deflated non-index price satisfies

$$S_{2t}\xi_{0,t} = \frac{\mathbb{E}_t\left[\left(\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}D_{2T}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}D_{2T}}{(1-\gamma_1)}\right)\right]}{\mathbb{E}\left[\mathbf{1}_{\text{Off}}\frac{(I_T)^{1-\gamma_2}(V_T)^{-\gamma_1}}{(\gamma_2-1)} + \mathbf{1}_{\text{On}}\frac{(I_T)^{1-\gamma_1}(V_T)^{-\gamma_2}}{(1-\gamma_1)}\right]}. \tag{A.47}$$

By plugging the expected values from (A.10) and (A.11) and $\xi_{0,t}$ (A.34), and rearranging we obtain the desired result. We conclude the proof by taking the derivative of the index and non-index price dividend ratios, and find that

$$\partial\left(S_t^I/I_t\right)/\partial V_t = \frac{\mathcal{E}_{3t}^{\mathrm{On}}}{\mathcal{E}_{1t}^{\mathrm{On}}\left(\mathcal{H}_{3t}\right)^2}\left[\left(\partial\mathcal{H}_{1t}/\partial V_t\right)\mathcal{H}_{3t} - \left(\partial\mathcal{H}_{3t}/\partial V_t\right)\mathcal{H}_{1t}\right] \tag{A.48}$$

$$\partial\left(S_{2t}/D_{2t}\right)/\partial V_t = \frac{\mathcal{E}_{4t}^{\mathrm{On}}}{\mathcal{E}_{1t}^{\mathrm{On}}\left(\mathcal{H}_{4t}\right)^2}\left[\left(\partial\mathcal{H}_{1t}/\partial V_t\right)\mathcal{H}_{4t} - \left(\partial\mathcal{H}_{4t}/\partial V_t\right)\mathcal{H}_{1t}\right]. \tag{A.49}$$

By plugging the derivative of $\mathcal{H}_{it}$, $i = 1, 3, 4$, (14) with respect to V_t, into the price-dividend ratios (A.48) and (A.49), and by removing strictly positive terms and rearranging, we find that

$$\partial\left(S_t^I/I_t\right)/\partial V_t < 0 \iff \mathcal{E}_{3t} - \mathcal{E}_{1t} < 0, \tag{A.50}$$

$$\partial\left(S_{2t}/D_{2t}\right)/\partial V_t > 0 \iff \mathcal{E}_{4t} - \mathcal{E}_{1t} > 0. \tag{A.51}$$

The right hand side inequalities are true due to (A.17) and (A.18), respectively, implying that the left hand side inequality are also true, which is our desire result. In similar fashion, we obtain

$$\partial\left(S_t^I/I_t\right)/\partial I_t > 0 \iff \mathcal{E}_{3t} - \mathcal{E}_{1t} < 0, \tag{A.52}$$

$$\partial\left(S_{2t}/D_{2t}\right)/\partial I_t < 0 \iff \mathcal{E}_{4t} - \mathcal{E}_{1t} > 0. \tag{A.53}$$

The right hand side is always satisfied due to (A.17) and (A.18), respectively, implying that the left hand side inequality are also true, which is our desire result.

We continue with the comparison of the index price-dividend ratio and the non-index price-dividend ratio. Dividing the index price-dividend ratio by the non-index price-dividend ratio (60) in the pure risk-on incentives, we obtain

$$\begin{aligned}\frac{S_t^I/I_t}{S_{2t}/D_{2t}} = \frac{\mathcal{E}_{3t}^{\mathrm{On}}}{\mathcal{E}_{4t}^{\mathrm{On}}} &= e^{\left(\mu_1-\mu_2-(\sigma_1\nu_1)\gamma_2+(\sigma_2\nu_2)\gamma_2+(\sigma_1)^2(1-\gamma_1)\right)(T-t)} \\ &= e^{\left(-\left(\lambda\sigma^2/3\right)\gamma_2+\left(\sigma^2/3\right)\gamma_2+(\sigma)^2(1-\gamma_1)\right)(T-t)} > 1.\end{aligned} \tag{A.54}$$

We obtain the second equality by plugging the definitions of $\mathcal{E}_{3t}^{\mathrm{On}}$ (A.8) and $\mathcal{E}_{4t}^{\mathrm{On}}$ (A.11) and cancelling similar terms, and the last equality by using the assumptions stating that $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\nu_1 = \lambda\sigma/3$, $\nu_2 = \sigma/3$. In similar fashion, dividing the index price-dividend ratio by the non-index price-dividend ratio (61) in the pure risk-off incentives, we obtain

$$\frac{S_t^I/I_t}{S_{2t}/D_{2t}} = \frac{\mathcal{E}_{3t}^{\text{Off}}}{\mathcal{E}_{4t}^{\text{Off}}} = e^{\left(\mu_1-\mu_2-(\sigma_1\nu_1)\gamma_1+(\sigma_2\nu_2)\gamma_1-(\sigma_1)^2(\gamma_2-1)\right)(T-t)}$$
$$= e^{\left(-\left(\lambda\sigma^2/3\right)\gamma_1+\left(\sigma^2/3\right)\gamma_1-(\sigma)^2(\gamma_2-1)\right)(T-t)} < 1. \tag{A.55}$$

We obtain the second equality by plugging the definitions of $\mathcal{E}_{3t}^{\text{Off}}$ (A.7) and $\mathcal{E}_{4t}^{\text{Off}}$ (A.10) and cancelling similar terms, and the last equality by using the assumptions stating that $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\nu_1 = \lambda\sigma/3$, $\nu_2 = \sigma/3$. The inequality is true because the exponent simplifies to $\sigma^2\left(\frac{\gamma_1}{3}(1-\lambda)+1-\gamma_2\right) < 0$, which is negative because we assume that $\gamma_2 - \gamma_1 > 1$.

Lastly, dividing the index price-dividend ratio by the non-index price-dividend ratio (62) in the joint incentives case, we obtain

$$\frac{S_t^I/I_t}{S_{2t}/D_{2t}} = \frac{\frac{\mathcal{E}_{3t}^{\text{On}}}{\mathcal{H}_{3t}}}{\frac{\mathcal{E}_{4t}^{\text{On}}}{\mathcal{H}_{4t}}} = \frac{\left(\mathcal{E}_{3t}^{\text{On}} + \frac{1-\gamma_1}{\gamma_2-1}\left(V_t/I_t\right)^{\gamma_2-\gamma_1}\mathcal{E}_{3t}^{\text{Off}}\right)}{\left(\mathcal{E}_{4t}^{\text{On}} + \frac{1-\gamma_1}{\gamma_2-1}\left(V_t/I_t\right)^{\gamma_2-\gamma_1}\mathcal{E}_{4t}^{\text{Off}}\right)}, \tag{A.56}$$

where the last equality is obtained by plugging the definitions of $\mathcal{E}_{3t}^{\text{On}}$ (A.8) and $\mathcal{E}_{4t}^{\text{On}}$ (A.11) and the definitions of $\mathcal{H}_{3t}$ and $\mathcal{H}_{4t}$, (14), and cancelling similar terms. Thus, we find that $\frac{S_t^I/I_t}{S_{2t}/D_{2t}} > 1$ if and only if V/I is below the deterministic threshold in (63). The threshold is well defined because $\mathcal{E}_{3t}^{\text{On}} > \mathcal{E}_{4t}^{\text{On}}$ and $\mathcal{E}_{4t}^{\text{Off}} > \mathcal{E}_{3t}^{\text{Off}}$ since $\gamma_2 - \gamma_1 > 1$. □

Proof of Proposition 2 (Risk Exposures). We start with the joint incentives case. By applying Itô's Lemma on both sides of the deflated index price (A.46), given (A.7) and (A.8), we obtain

$$\boldsymbol{\sigma}_{1t}^S = \boldsymbol{\theta}_t + (2-\gamma_2)\,\boldsymbol{\sigma}^I - \gamma_1\boldsymbol{\nu} + (\gamma_2-\gamma_1)\left(\mathcal{H}_{3t}\right)\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right). \tag{A.57}$$

And, plugging θ_t (22) and rearranging leads to the desired result.

$$\boldsymbol{\sigma}_{1t}^S = \boldsymbol{\sigma}_1^D + (\gamma_2-\gamma_1)\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right)\left(\mathcal{H}_{3t} - \mathcal{H}_{1t}\right). \tag{A.58}$$

Note that $\mathcal{H}_{3t} > \mathcal{H}_{1t}$ since $\mathcal{E}_{1t} > \mathcal{E}_{3t}$, as (A.17) shows. Therefore, the first entry is always positive, $\boldsymbol{\sigma}_{1t}^S(1) > 0$ since $\nu_1 = \lambda\sigma/3 < \sigma$, while the second and third entries are always negative because $\boldsymbol{\sigma}^I \equiv \boldsymbol{\sigma}_1^D = 0$ for these two entries. In similar fashion, by applying Itô's

Lemma on both sides of the deflated non-index price (A.47), given (A.7) and (A.8), we obtain

$$\boldsymbol{\sigma}_{2t}^{S} = \boldsymbol{\theta}_t + \boldsymbol{\sigma}_2^{D} - (\gamma_2 - 1)\,\boldsymbol{\sigma}^{I} - \gamma_1 \boldsymbol{\nu} + (\gamma_2 - \gamma_1)\left(\boldsymbol{\sigma}^{I} - \boldsymbol{\nu}\right)(\mathcal{H}_{4t}). \tag{A.59}$$

And, plugging θ_t (22) and rearranging leads to the desired result. Note that $\mathcal{H}_{4t} < \mathcal{H}_{1t}$ since $\mathcal{E}_{1t} < \mathcal{E}_{4t}$, as (A.18) shows. Therefore, the first entry is always negative, $\boldsymbol{\sigma}_{2t}^{S}(1) < 0$, because $\nu_1 = \lambda\sigma/3 < \sigma$ and $\boldsymbol{\sigma}_2^{D}(1) = 0$, while the second and third entries are always positive because $\boldsymbol{\sigma}^{I} = 0$ and $\boldsymbol{\nu} > 0$ for the second and third entries, and $\boldsymbol{\sigma}_2^{D}(2) = \sigma > 0$.

$$\boldsymbol{\sigma}_{2t}^{S} = \boldsymbol{\sigma}_2^{D} + (\gamma_2 - \gamma_1)\left(\boldsymbol{\nu} - \boldsymbol{\sigma}^{I}\right)(\mathcal{H}_{1t} - \mathcal{H}_{4t}). \tag{A.60}$$

The pure incentives cases are two knife-edge cases of the joint incentives case. In the pure risk-off incentives ($\mathbf{1}_{\text{Off}} = 1, \mathbf{1}_{\text{On}} = 0$), the risk-on hedging demand equals zero, $\mathcal{H}_{it} = 0$, which leads to the simplifications in (23). Similarly, in the pure risk-on incentives ($\mathbf{1}_{\text{Off}} = 0, \mathbf{1}_{\text{On}} = 1$), the risk-on hedging demand equals one, $\mathcal{H}_{it} = 1$, which leads to the same simplifications. □

Proof of Proposition 3 (Tracking Error and Active Share). The difference between the portfolio exposure and the benchmark exposure is given by

$$\frac{dW_t^{\mathcal{A}}}{W_t^{\mathcal{A}}} - \frac{dS_{1t}}{S_{1t}} = \boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}} - \boldsymbol{\sigma}_{1t}^{S}. \tag{A.61}$$

By plugging the market prices of risk, (22), into the asset manager's portfolio exposure, (19), and taking the variance, we find that the tracking error per unit of time is given by (29).

Similarly, the active share is defined as the correlation between the portfolio exposure and the benchmark per unit of time,

$$\frac{\boldsymbol{\sigma}_{1t}^{S\prime}\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)}{\|\boldsymbol{\sigma}_{1t}^{S}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}_t^{\mathcal{A}}\right)\|}. \tag{A.62}$$

By plugging the market prices of risk, (22), into the asset manager's portfolio exposure, (19), and applying the definition, we find that the active share per unit of time is given by (31).

The tracking error is a monotonic trasnformation of $(\mathcal{H}_{3t} - \mathcal{H}_{2t})$, which implies that it attains its maximum when this difference is maximized,

$$\frac{d\text{Tracking Error}}{dV/I} = 2\left[(\gamma_2-\gamma_1)(\mathcal{H}_{3t}-\mathcal{H}_{2t})+1\right]\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2(\gamma_2-\gamma_1)\frac{d(\mathcal{H}_{3t}-\mathcal{H}_{2t})}{dV/I} = 0. \tag{A.63}$$

The derivative of the difference is given in (A.23). It is positive initially, becomes zero when $1=\mathcal{H}_{3t}+\mathcal{H}_{2t}$, and negative afterwards.

The derivative of the active share is given by

$$\begin{aligned}\frac{d\text{Active Share}}{V/I} =& \frac{-(\gamma_2-\gamma_1)\frac{d\left(\mathcal{H}_{1t}-\mathcal{H}_{2t}\right)}{dV/I}\left[1+(\gamma_2-\gamma_1)(\mathcal{H}_{3t}-\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}^S_{1t}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}^{\mathcal{A}}_t\right)\|}\\ &\frac{-(\gamma_2-\gamma_1)(\mathcal{H}_{1t}-\mathcal{H}_{2t})(\gamma_2-\gamma_1)\frac{d\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)}{dV/I}\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}^S_{1t}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}^{\mathcal{A}}_t\right)\|}\\ &-\frac{(\gamma_2-\gamma_1)\frac{d\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)}{dV/I}\left[1+(\gamma_2-\gamma_1)(\mathcal{H}_{1t}-\mathcal{H}_{2t})\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}^S_{1t}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}^{\mathcal{A}}_t\right)\|}\\ &-\frac{(\gamma_2-\gamma_1)(\mathcal{H}_{3t}-\mathcal{H}_{1t})(\gamma_2-\gamma_1)\frac{d\left(\mathcal{H}_{1t}-\mathcal{H}_{2t}\right)}{dV/I}\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\|\boldsymbol{\sigma}^S_{1t}\|\|\left(\boldsymbol{\Sigma}^{S\prime}\boldsymbol{\pi}^{\mathcal{A}}_t\right)\|}.\end{aligned} \tag{A.64}$$

Similar to the previous difference, the differences $(\mathcal{H}_{1t}-\mathcal{H}_{2t})$ and $(\mathcal{H}_{3t}-\mathcal{H}_{1t})$ are positive initially, attain their maximum when $\mathcal{H}_{1t}+\mathcal{H}_{2t}=1$ and $\mathcal{H}_{3t}+\mathcal{H}_{1t}=1$, respecitvely, and decrease hereafter.

Suppose that the maximum of $(\mathcal{H}_{1t}-\mathcal{H}_{2t})$ and $(\mathcal{H}_{3t}-\mathcal{H}_{1t})$ coincides with the maximum of $(\mathcal{H}_{3t}-\mathcal{H}_{2t})$, and they all achieve their maximum exactly at the same location. Since there is a leading minus sign in all the active share derivatives, it attains its minimum when these differences attain their maximum. Therefore, the tracking error achieves its maximum when the tracking error its minimum.

While the maximums of $(\mathcal{H}_{1t}-\mathcal{H}_{2t})$, $(\mathcal{H}_{3t}-\mathcal{H}_{1t})$ and $(\mathcal{H}_{3t}-\mathcal{H}_{2t})$ are not precisely at the same location, they are close to each other. We find that $\mathcal{H}_{1t}\geq\mathcal{H}_{2t}$, $\mathcal{H}_{3t}\geq\mathcal{H}_{1t}$, and $\mathcal{H}_{3t}\geq\mathcal{H}_{2t}$, as (A.19), (A.17), (A.20) respectively show. By plugging the definition of the risk-on hedging demand (14), we find that

$$1 = \mathcal{H}_{2t} + \mathcal{H}_{1t} \iff (V_t/I_t) = \left(\frac{\gamma_2 - 1}{1 - \gamma_1} \frac{1}{\sqrt{\mathcal{E}_{2t}\mathcal{E}_{1t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}}, \tag{A.65}$$

$$1 = \mathcal{H}_{1t} + \mathcal{H}_{3t} \iff (V_t/I_t) = \left(\frac{\gamma_2 - 1}{1 - \gamma_1} \frac{1}{\sqrt{\mathcal{E}_{1t}\mathcal{E}_{3t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}}, \tag{A.66}$$

$$1 = \mathcal{H}_{2t} + \mathcal{H}_{3t} \iff (V_t/I_t) = \left(\frac{\gamma_2 - 1}{1 - \gamma_1} \frac{1}{\sqrt{\mathcal{E}_{2t}\mathcal{E}_{3t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}}. \tag{A.67}$$

To assess how far these locations are from each other, we derive their ratios. The ratio of the first maximum location to the second maximum location is given by

$$\left(\sqrt{\frac{\mathcal{E}_{3t}}{\mathcal{E}_{2t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} = e^{-\frac{1}{2}\left(\|\boldsymbol{\sigma}^I - \boldsymbol{\nu}\|^2\right)(T-t)} \approx 0.91. \tag{A.68}$$

Similarly, the ratio of the first maximum location to the third maximum location is given by

$$\left(\sqrt{\frac{\mathcal{E}_{3t}}{\mathcal{E}_{1t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} = e^{-\frac{1}{2}\left(\boldsymbol{\sigma}^{I\prime}\left(\boldsymbol{\sigma}^I - \boldsymbol{\nu}\right)\right)(T-t)} \approx 0.92. \tag{A.69}$$

Lastly, the ratio of the third maximum location to the second is given by

$$\left(\sqrt{\frac{\mathcal{E}_{1t}}{\mathcal{E}_{2t}}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} = e^{-\frac{1}{2}\boldsymbol{\nu}'\left(\boldsymbol{\nu} - \boldsymbol{\sigma}^I\right)(T-t)} \approx 0.99. \tag{A.70}$$

Notice that the ratios do not depend on the difference between the risk-on and the risk-off risk aversion parameters. They only depend on the volatility coeficients of the exogenous news and the time horizon. The ratios reveal that any reasonable parametric choice leads to maximum locations that are close to each other. □

Proof of Proposition 7 (Expected Return). The index and non-index expected returns are given by (64) and (65), respectively, and obtained by taking the inner product between the market price of risk (32) and the return exposures (33) and (34). The six channels are identified by the six combinations of inner products between the two components of the market price of risk (σ^I, ν) and the three components of the asset exposures (33) and (34) (σ_i^D, σ^I, ν). By multiplying the characterizations for θ_t and $\boldsymbol{\sigma}_{1t}^S$, we obtain our desired result in (64) and (65)

Next, we show that the expected return in extreme undeperformance is strictly lower than the expected return in extreme outperformance. First, notice that the term,

$$(\gamma_2 - \gamma_1)\left(\mathcal{H}_{3t} - \mathcal{H}_{1t}\right), \tag{A.71}$$

in the first four channels is strictly positive (A.17) and approaches zero in extreme outperformance and underperformance (A.21). Therefore, in extreme outperformance the index asset expected return becomes

$$\mu_{1t}^S \mid_{\mathcal{H}_{it}\downarrow 0} = (\gamma_2 - 1)\, \boldsymbol{\sigma}^{I'} \boldsymbol{\sigma}_1^D + \gamma_1 \boldsymbol{\nu}' \boldsymbol{\sigma}_1^D, \tag{A.72}$$

while in extreme underperformance it becomes

$$\mu_{1t}^S \mid_{\mathcal{H}_{it}\uparrow 1} = -\,(1 - \gamma_1)\, \boldsymbol{\sigma}^{I'} \boldsymbol{\sigma}_1^D + \gamma_2 \boldsymbol{\nu}' \boldsymbol{\sigma}_1^D.$$

The difference between extreme outperformance and underperformance is strictly positive,

$$\mu_{1t}^S \mid_{\mathcal{H}_{it}\downarrow 0} - \mu_{1t}^S \mid_{\mathcal{H}_{it}\uparrow 1} = (\gamma_2 - \gamma_1)\left(\boldsymbol{\sigma}^{I'} \boldsymbol{\sigma}_1^D - \boldsymbol{\nu}' \boldsymbol{\sigma}_1^D\right) > 0, \tag{A.73}$$

because $\gamma_2 - \gamma_1 > 0$, and (A.14). In similar fashion, the difference between extreme outperformance and underperformance for the non-index asset is

$$\mu_{2t}^S \mid_{\mathcal{H}_{it}\downarrow 0} - \mu_{2t}^S \mid_{\mathcal{H}_{it}\uparrow 1} = -\,(\gamma_2 - \gamma_1)\, \boldsymbol{\nu}' \boldsymbol{\sigma}_2^D < 0. \tag{A.74}$$

Next, we establish that the index expected return decreases in outperformance and increases in underperformance. We achieve it by showing the derivative of the expected return with respect to V/I is negative as $V/I \to \infty$ ($\mathcal{H}_{it} \to 0$) and positive as $V/I \to 0$ ($\mathcal{H}_{it} \to \infty$). We derive the derivative, $d\mu_{1t}^S/d(V/I)$, by plugging the derivatives of $\mathcal{H}_{it}$ and $\mathcal{H}_{it} - \mathcal{H}_{jt}$, given in (A.22) and (A.23), respectively, and grouping similar terms, which leads to

$$
\begin{aligned}
d\mu_{1t}^S/d(V/I) = & \left(V_t/I_t\right)^{-1}\left(\gamma_2-\gamma_1\right)^2 \mathcal{H}_{1t}\left(1-\mathcal{H}_{1t}\right) \times \Bigg\{ \\
& -\frac{\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)}{\mathcal{H}_{1t}\left(1-\mathcal{H}_{1t}\right)}\left(1-\mathcal{H}_{1t}-\mathcal{H}_{3t}\right)\left[\left(\gamma_2-1\right)-\left(\gamma_2-\gamma_1\right)\left(\mathcal{H}_{1t}\right)\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right) \\
& +\frac{\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)}{\mathcal{H}_{1t}\left(1-\mathcal{H}_{1t}\right)}\left(1-\mathcal{H}_{1t}-\mathcal{H}_{3t}\right)\left[\gamma_1+\left(\gamma_2-\gamma_1\right)\left(\mathcal{H}_{1t}\right)\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right) \\
& +\left(\gamma_2-\gamma_1\right)\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2+\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)\Bigg\}. \qquad \text{(A.75)}
\end{aligned}
$$

The function multiplying the curly brackets is always positive and approaches zero as $V/I \to 0$ and $V/I \to \infty$ due to (A.26) and (A.27). Therefore, to show that the derivative is negative as $V/I \to \infty$, we establish that the function inside the curly brackets is negative as $V/I \to \infty$. Accordingly, the limit $V/I \to \infty$ of the function inside the curly brackets equals

$$
-\frac{\mathcal{E}_{1t}-\mathcal{E}_{3t}}{\mathcal{E}_{3t}}\left(\gamma_2-1\right)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)+\frac{\mathcal{E}_{1t}-\mathcal{E}_{3t}}{\mathcal{E}_{3t}}\gamma_1\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)+\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right) \qquad \text{(A.76)}
$$

because the limit of $\frac{\left(\mathcal{H}_{3t}-\mathcal{H}_{1t}\right)}{\mathcal{H}_{1t}\left(1-\mathcal{H}_{1t}\right)}$ is a strictly positive constant (A.25). Simplifying the term, we find that the derivative is negative if and only if

$$
\underline{\gamma}_2 \equiv \frac{\mathcal{E}_{1t}}{\mathcal{E}_{1t}-\mathcal{E}_{3t}}+\gamma_1\frac{\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)}{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}<\gamma_2. \qquad \text{(A.77)}
$$

Similarly, the limit $V/I \to 0$ of the function inside the curly brackets equals

$$
-\frac{\mathcal{E}_{1t}-\mathcal{E}_{3t}}{\mathcal{E}_{1t}}\left(1-\gamma_1\right)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)-\frac{\mathcal{E}_{1t}-\mathcal{E}_{3t}}{\mathcal{E}_{1t}}\gamma_2\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)+\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right). \qquad \text{(A.78)}
$$

Simplifying the term, we find that the derivative is positive if and only if

$$
\gamma_2<\frac{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}{\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)}\left(\frac{\mathcal{E}_{3t}}{\mathcal{E}_{1t}-\mathcal{E}_{3t}}+\gamma_1\right)\equiv\bar{\gamma}_2. \qquad \text{(A.79)}
$$

Similar to $d\mu_{1t}^S/d(V/I)$, we plug the derivatives of $\mathcal{H}_{it}$ and $\mathcal{H}_{it}-\mathcal{H}_{jt}$, given in (A.22) and

(A.23), respectively, and group similar terms, leading to

$$
\begin{aligned}
d\mu^S_{2t}/d(V/I) = &(V_t/I_t)^{-1}(\gamma_2-\gamma_1)^2\mathcal{H}_{1t}(1-\mathcal{H}_{1t})\times\Bigg\{ \\
&\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\left[(\gamma_2-1)-(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right) \\
&-\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\left[\gamma_1+(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right) \\
&-(\gamma_2-\gamma_1)(\mathcal{H}_{1t}-\mathcal{H}_{4t})\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^D_2\Bigg\}.
\end{aligned}
\tag{A.80}
$$

The function multiplying the curly brackets is always positive and approaches zero as $V/I\to 0$ and $V/I\to\infty$ due to (A.26) and (A.27). Therefore, to show that the derivative is positive as $V/I\to\infty$, we establish that the function inside the curly brackets is positive as $V/I\to\infty$. Accordingly, the limit $V/I\to\infty$ of the function inside the curly brackets equals

$$
\frac{\mathcal{E}_{4t}-\mathcal{E}_{1t}}{\mathcal{E}_{4t}}(\gamma_2-1)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)-\frac{\mathcal{E}_{4t}-\mathcal{E}_{1t}}{\mathcal{E}_{4t}}\gamma_1\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)-\boldsymbol{\nu}'\boldsymbol{\sigma}^D_2, \tag{A.81}
$$

which is positive if and only if

$$
\gamma_2 > 1+\frac{1}{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}\left[\gamma_1\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)+\frac{\mathcal{E}_{4t}}{\mathcal{E}_{4t}-\mathcal{E}_{1t}}\boldsymbol{\nu}'\boldsymbol{\sigma}^D_2\right]\equiv\underline{\underline{\gamma}}_2. \tag{A.82}
$$

Similarly, the limit $V/I\to 0$ of the function inside the curly brackets is negative if and only if

$$
\gamma_2 < \frac{1}{\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)}\left[\frac{\mathcal{E}_{1t}}{\mathcal{E}_{4t}-\mathcal{E}_{1t}}\boldsymbol{\nu}'\boldsymbol{\sigma}^D_2-(1-\gamma_1)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)\right]\equiv\bar{\bar{\gamma}}_2 \tag{A.83}
$$

□

Proof of Proposition 4 (Momentum and Reversal). By taking Itó's Lemma of the index asset expected return (μ^S_{1t}), given in (64), and matching the $d\boldsymbol{Z}_t$ terms with (35) we find that

$$
\begin{aligned}
\boldsymbol{b}_{1t}^{\mu} = & (\gamma_2-\gamma_1)^2\left[(\mathcal{H}_{3t}-\mathcal{H}_{1t})(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\right]\left[(\gamma_2-1)-(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)(\boldsymbol{\sigma}^I-\boldsymbol{\nu}) \\
& -(\gamma_2-\gamma_1)^2\left[(\mathcal{H}_{3t}-\mathcal{H}_{1t})(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\right]\left[\gamma_1+(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)(\boldsymbol{\sigma}^I-\boldsymbol{\nu}) \\
& -(\gamma_2-\gamma_1)^3(\mathcal{H}_{3t}-\mathcal{H}_{1t})\left[\mathcal{H}_{1t}(1-\mathcal{H}_{1t})\right]\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2(\boldsymbol{\sigma}^I-\boldsymbol{\nu}) \\
& -\left[(\gamma_2-\gamma_1)^2\mathcal{H}_{1t}(1-\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)(\boldsymbol{\sigma}^I-\boldsymbol{\nu}). \qquad \text{(A.84)}
\end{aligned}
$$

By taking the inner product of $\boldsymbol{b}_{1t}^{\mu}$ and $\boldsymbol{\sigma}_{1t}^{S}$, we find that

$$
\begin{aligned}
\boldsymbol{\sigma}_{1t}^{S'}\boldsymbol{b}_{1t}^{\mu} = & (\gamma_2-\gamma_1)^2\mathcal{H}_{1t}(1-\mathcal{H}_{1t})\times\Bigg\{ \\
& (\gamma_2-\gamma_1)\frac{(\mathcal{H}_{3t}-\mathcal{H}_{1t})^2}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\left[(\gamma_2-1)-(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)\frac{(\mathcal{H}_{3t}-\mathcal{H}_{1t})^2}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\left[\gamma_1+(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)^2(\mathcal{H}_{3t}-\mathcal{H}_{1t})^2\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)(\mathcal{H}_{3t}-\mathcal{H}_{1t})\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& \frac{(\mathcal{H}_{3t}-\mathcal{H}_{1t})}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\left[(\gamma_2-1)-(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)^2 \\
& -\frac{(\mathcal{H}_{3t}-\mathcal{H}_{1t})}{\mathcal{H}_{1t}(1-\mathcal{H}_{1t})}(1-\mathcal{H}_{1t}-\mathcal{H}_{3t})\left[\gamma_1+(\gamma_2-\gamma_1)(\mathcal{H}_{1t})\right]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right) \\
& -(\gamma_2-\gamma_1)(\mathcal{H}_{3t}-\mathcal{H}_{1t})\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right) \\
& -\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)^2\Bigg\}. \qquad \text{(A.85)}
\end{aligned}
$$

The function multiplying the curly brackets is always positive and approaches zero as $V/I \to 0$ and $V/I \to \infty$ due to (A.26) and (A.27). Therefore, to show that the momentum condition is satisfied as $V/I \to \infty$, we establish that the function inside the curly brackets is positive as $V/I \to \infty$ ($\mathcal{H}_{it} \to 0$). Accordingly, the limit $V/I \to \infty$ ($\mathcal{H}_{it} \to 0$) of the function inside the curly brackets is positive if and only if

$$\gamma_2 > \underline{\gamma}_2. \qquad \text{(A.86)}$$

Similarly, the reversal condition is satisfied as $V/I \to 0$ ($\mathcal{H}_{it} \to 1$), if and only if

$$\gamma_2 < \bar{\gamma}_2. \qquad \text{(A.87)}$$

We conclude by showing that there always exist an active share λ such that

$$\underline{\gamma}_2 < \bar{\gamma}_2. \tag{A.88}$$

First, comparing the factors multiplying γ_1 in (A.77) and (A.79), it is clear that

$$\gamma_1 \frac{\left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}{\left(\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)} > \gamma_1 \frac{\left(\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)}{\left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)},$$

due to (A.16). Second, comparing the remaining factors in (A.77) and (A.79), it is clear that

$$\frac{\left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}{\left(\|\boldsymbol{\nu}\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)} \frac{\mathcal{E}_{3t}}{\mathcal{E}_{1t} - \mathcal{E}_{3t}} > \frac{\mathcal{E}_{1t}}{\mathcal{E}_{1t} - \mathcal{E}_{3t}} \tag{A.89}$$

because the denominator shrinks to zero as the active increases, (A.15). Therefore, there is always exist a γ_2 such that index asset exhibits momentum and reversal.

We continue with the non-index asset. By taking Itó's Lemma of the non-index asset expected return (μ^S_{2t}), given in (65), and matching the $d\boldsymbol{Z}_t$ terms with (35) we find that

$$\begin{aligned}
\boldsymbol{b}^{\mu}_{2t} = &- (\gamma_2 - \gamma_1)^2 (\mathcal{H}_{1t} - \mathcal{H}_{4t}) (1 - \mathcal{H}_{1t} - \mathcal{H}_{4t}) [(\gamma_2 - 1) - (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \left(\|\boldsymbol{\sigma}^I\|^2 - \boldsymbol{\nu}'\boldsymbol{\sigma}^I\right) (\boldsymbol{\sigma}^I - \boldsymbol{\nu}) \\
&+ (\gamma_2 - \gamma_1)^2 (\mathcal{H}_{1t} - \mathcal{H}_{4t}) (1 - \mathcal{H}_{1t} - \mathcal{H}_{4t}) [\gamma_1 + (\gamma_2 - \gamma_1)(\mathcal{H}_{1t})] \left(\|\boldsymbol{\nu}\|^2 - \boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right) (\boldsymbol{\sigma}^I - \boldsymbol{\nu}) \\
&+ (\gamma_2 - \gamma_1)^3 (\mathcal{H}_{1t} - \mathcal{H}_{4t}) \mathcal{H}_{1t} (1 - \mathcal{H}_{1t}) \|\boldsymbol{\sigma}^I - \boldsymbol{\nu}\|^2 (\boldsymbol{\sigma}^I - \boldsymbol{\nu}) \\
&+ (\gamma_2 - \gamma_1)^2 (\mathcal{H}_{1t}(1 - \mathcal{H}_{1t})) \boldsymbol{\nu}' \boldsymbol{\sigma}^D_2 (\boldsymbol{\sigma}^I - \boldsymbol{\nu}).
\end{aligned} \tag{A.90}$$

By taking the inner product of $\boldsymbol{b}_{2t}^{\mu}$ and $\boldsymbol{\sigma}_{2t}^{S}$, we find that

$$\begin{aligned}
\boldsymbol{\sigma}_{2t}^{S'}\boldsymbol{b}_{2t}^{\mu} = & (\gamma_2-\gamma_1)^2\,\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})\times\Bigg\{ \\
& (\gamma_2-\gamma_1)\,\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})^2}{\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})}\,(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\,[(\gamma_2-1)-(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t})]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)\,\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})^2}{\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})}\,(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\,[\gamma_1+(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t})]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t}-\mathcal{H}_{4t})\,\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -(\gamma_2-\gamma_1)^2\,(\mathcal{H}_{1t}-\mathcal{H}_{4t})^2\,\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2 \\
& -\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})}{\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})}\,(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\,[(\gamma_2-1)-(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t})]\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)\boldsymbol{\sigma}_2^{D\prime}\,(\boldsymbol{\sigma}^I-\boldsymbol{\nu}) \\
& +\frac{(\mathcal{H}_{1t}-\mathcal{H}_{4t})}{\mathcal{H}_{1t}\,(1-\mathcal{H}_{1t})}\,(1-\mathcal{H}_{1t}-\mathcal{H}_{4t})\,[\gamma_1+(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t})]\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)\boldsymbol{\sigma}_2^{D\prime}\,(\boldsymbol{\sigma}^I-\boldsymbol{\nu}) \\
& +\left(\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D\right)\boldsymbol{\sigma}_2^{D\prime}\,(\boldsymbol{\sigma}^I-\boldsymbol{\nu})+(\gamma_2-\gamma_1)\,(\mathcal{H}_{1t}-\mathcal{H}_{4t})\,\|\boldsymbol{\sigma}^I-\boldsymbol{\nu}\|^2\boldsymbol{\sigma}_2^{D\prime}\,(\boldsymbol{\sigma}^I-\boldsymbol{\nu})\Bigg\}.
\end{aligned} \tag{A.91}$$

The function multiplying the curly brackets is always positive and approaches zero as $V/I \to 0$ and $V/I \to \infty$ due to (A.26) and (A.27). Therefore, to show that the reversal condition is satisfied as $V/I \to \infty$, we establish that the function inside the curly brackets is negative as $V/I \to \infty$ ($\mathcal{H}_{it} \to 0$). Accordingly, the limit $V/I \to \infty$ ($\mathcal{H}_{it} \to 0$) of the function inside the curly brackets is positive if and only if

$$\gamma_2 > \underline{\underline{\gamma}}_2. \tag{A.92}$$

Similarly, the reversal condition is satisfied as $V/I \to 0$ ($\mathcal{H}_{it} \to 1$), if and only if

$$\gamma_2 < \bar{\bar{\gamma}}_2 \tag{A.93}$$

We conclude the proof by showing that there always exist an active share λ such that

$$\underline{\underline{\gamma}}_2 < \bar{\bar{\gamma}}_2. \tag{A.94}$$

First, comparing the factors multiplying γ_1 in (A.82) and (A.83), it is clear that

$$\gamma_1\frac{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}{\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)} > \gamma_1\frac{\left(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I\right)}{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I'}\boldsymbol{\nu}\right)}, \tag{A.95}$$

due to (A.16). Second, comparing the remaining factors in (A.82) and (A.83), it is clear that

$$1+\frac{1}{(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu})}\left[\frac{\mathcal{E}_{4t}}{\mathcal{E}_{4t}-\mathcal{E}_{1t}}\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D\right]<\frac{1}{(\|\boldsymbol{\nu}\|^2-\boldsymbol{\nu}'\boldsymbol{\sigma}^I)}\left[\frac{\mathcal{E}_{1t}}{\mathcal{E}_{4t}-\mathcal{E}_{1t}}\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D-\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)\right] \tag{A.96}$$

is true if

$$\frac{\mathcal{E}_{1t}}{\mathcal{E}_{4t}-\mathcal{E}_{1t}}\geq\frac{\left(\|\boldsymbol{\sigma}^I\|^2-\boldsymbol{\sigma}^{I\prime}\boldsymbol{\nu}\right)}{\boldsymbol{\nu}'\boldsymbol{\sigma}_2^D}. \tag{A.97}$$

The denominator in the right hand side of (A.96) shrinks to zero as the active increases due to (A.15). Therefore, there is always exist a γ_2 such that the non-index asset exhibits momentum in outperformance and reversal in underperformance. □

B Further Discussion

Proposition 8 (**Discount Factor**). *The discount factor in the presence of risk-on incentives is given by*

$$\xi_{0,t}=(I_t/I_0)^{1-\gamma_1}(V_t/V_0)^{-\gamma_2}\,\mathcal{E}_{1t}^{On}/\mathcal{E}_{10}^{On}, \tag{B.1}$$

while it is given by

$$\xi_{0,t}=(I_t/I_0)^{-(\gamma_2-1)}(V_t/V_0)^{-\gamma_1}\,\mathcal{E}_{1t}^{Off}/\mathcal{E}_{10}^{Off} \tag{B.2}$$

in presence of risk-off incentives. In the presence of both risk-on and risk-off incentives, the discount factor is given by

$$\xi_{0,t}=(I_t/I_0)^{1-\gamma_1}(V_t/V_0)^{-\gamma_2}\,\mathcal{E}_{1t}^{On}/\mathcal{E}_{10}^{On}\frac{1}{\mathcal{H}_{1t}/\mathcal{H}_{10}}, \tag{B.3}$$

where $\mathcal{E}_{1t}^{Off}$ and $\mathcal{E}_{1t}^{On}$ are determinstic functions of time given in (A.1) and (A.2), respectively, and $\mathcal{H}_{1t}$ is given in (14). The discount factor exhibits the following characteristics:

- *It decreases following market news: $\partial\xi_t/\partial V_t<0$.*
- *It decreases following index news in the risk-off incentives case: $\partial\xi_t/\partial I_t<0$.*

- *It increases following index news in the risk-on incentives case:* $\partial \xi_t / \partial I_t > 0$.
- *It has a U-shape with index news in the joint incentives case. It decreases with the index news* ($\partial \xi_t / \partial I_t < 0$) *if and only if the risk-on hedging demand is below a threshold,*

$$\mathcal{H}_{1t} < \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} < 1, \tag{B.4}$$

and increases with the index news ($\partial \xi_t / \partial I_t > 0$) *when* $\mathcal{H}_{1t}$ *passes the threshold.*

Regardless of the asset managers' incentives, Proposition 8 reveals that the state price density is inversely related to the market news, V_t — a feature that is similar to a traditional asset pricing model: an asset that pays off in bad states get a high value.

The risk-on incentives imply that assets that pay off when the index is high get a high value because asset managers with risk-on incentives are concerned about underperforming the index and are willing to pay high prices for assets that pay in these states. This feature is typical to an asset pricing model with benchmarking concerns, such as Basak and Pavlova (2013), Buffa and Hodor (2023), Hodor and Zapatero (2023), among others.

The logic flips with risk-off incentives: assets that pay off when the index is high get a low value. With risk-off incentives, asset managers are outperforming the index and do not desire to hold assets that pay off when the benchmark is high. Instead of hedging unexpected increases in the benchmark, outperforming asset managers hedge unexpected falls by taking a short position on the benchmark and investing the proceeds in other assets — a feature unique to risk-off incentives.

When asset managers face both risk-on and risk-off incentives, the joint incentives discount factor fluctuates between risk-on and risk-off pricing environments. When the benchmark news (I) is low relative to the market news (V), the risk-on hedging demand ($\mathcal{H}_{1t}$) is small, and the asset prices correlate with the purely risk-off prices. In these states, asset managers are more likely to outperform, and prices adjust to accommodate their risk-off portfolios. As the benchmark news (I) increases, the risk-on hedging demand ($\mathcal{H}_{1t}$) becomes more pronounced, and prices adjust to accommodate the heightened likelihood that asset managers will underperform. Eventually, the risk-on hedging demand ($\mathcal{H}_{1t}$) cross the threshold in (B.4) and becomes so pronounced that prices correlate with the purely risk-on prices. Figure 11 illustrates this logic.

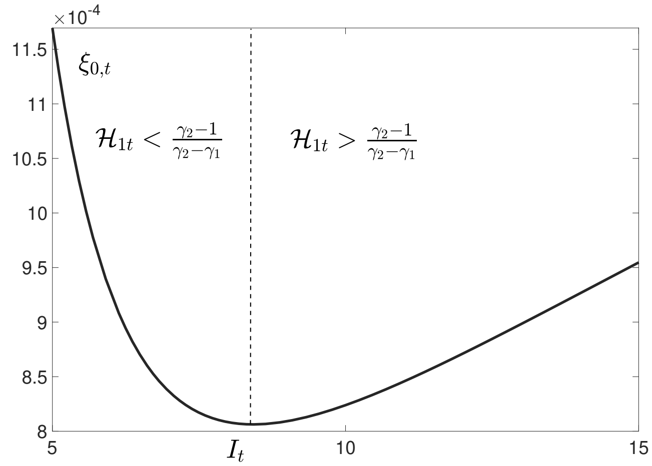

Figure 11. This figure plots the discount factor as a function of index news in the joint incentives case. The discount factor sets prices similar to the risk-off environment when the risk-on hedging demand is below a threshold. In contrast, the discount factor sets prices similar to the risk-on environment when the risk-on hedging demand crosses that threshold. The parameters are as in Figure 5.

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