Asset prices and exchange rates with short-selling risk*

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Abstract

We study the role of short selling risk (volatility of stock lending fees) in determining real exchange rates and international asset prices. We construct a model of the world economy, where heterogeneous investors can borrow and lend stocks across countries. The equilibrium shorting fee endogenously clears the associated markets. Aggregate shocks transmit internationally through short selling fee movements. The strength of this transmission is governed by the distribution of wealth between short sellers and stock lenders. We show that short selling risk amplifies the effects of shocks and that adjustments to shorting fee leads to high stock correlations.

1 Introduction

Short selling market is a market where investors borrow and lend a wide variety of stocks. A key feature of short selling market is that shorting stock is more costly compared to standard trade of securities. When the shorting cost is high, investors choose not to engage in stocks borrowing or lending which limits arbitrage. As a result, costly short selling could introduce imperfections in the financial markets and influence efficient risk sharing. This friction is even more compelling in an international finance context where

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the effects of differences in informational advantages, which could justify the needs to short assets in the first place, are more pronounced. But the current body of literature studying short selling is vastly limited to close economy environment.

Despite the wide interest in the effects of imperfection in international financial markets on capital flow, asset prices, and exchange rate determination (Maggiori, 2022; Simsek, 2021), almost no paper has explored the effects of costly short selling on global investors' portfolio allocation, international prices, and quantities. All existing papers either: (i) assume that investors could costlessly short assets across country given that market are usually assumed to be complete; (ii) or impose an exogenous short selling constraint that limit the size of a negative position in a given asset. We depart from these two polar cases and provide, in this paper, an international asset pricing model in which short selling fee emerge endogenously as a result of demand and supply forces in the short selling market. We use our model to assess the interacting implications of real exchange rate and costly short selling on countries' stock prices.

Our model is a straightforward generalization of the Atmaz et al. (2022) costly short-selling model to a setting with two countries. Specially, in our model, the world economy contains two Lucas trees and three group of investors within each country having heterogeneous beliefs about the expected foreign output growth and may be a borrower or lender in the foreign equity short selling market. A constant fraction of optimist investors are stock lenders who pay a fee to participate in the stock lending market where they meet short-sellers with a constant probability. These short-sellers have a pessimistic outlook on expected foreign output growth. To facilitate aggregation and tractability, we assume that all investors have finite lives captured in a continuous time overlapping generations setting.

The starting point of our analysis is a complete characterization of equilibrium allocation and prices in closed-economy driven by the dynamics of consumption share of stock lenders and borrowers. We show that stock lenders' consumption share is procyclical contrasting with that of stock borrowers which is countercyclical. Stocks lenders thus experience large consumption losses in bad times while enjoying high consumption in good times. Furthermore, when stock lenders command a larger share of wealth in the economy, they allocate a substantial fraction to risky assets hoping to earn extra revenues in the securities lending market. The short-selling fee provides the appropriate signals to stocks lenders. A high short-selling fee signals that potential revenues generated by lending the stocks are high. In equilibrium, the resulting market price of risk is lower than what would prevail in a log-economy driven by the heterogeneity in beliefs and the possibility of short-selling stocks.

Our second set of results concerns the characterization of real exchange rate in the open economy model which generalized the analysis in Pavlova and Rigobon (2007) for the possibility of shorting assets across countries. We make two additional assumptions common in international macrofinance: (i) consumption home bias where investors prefer their domestic good relatively more compared to foreign good and (ii) domestic informational advantage whereby investors have more precise information on their domestic stock compare to foreign stock. The second assumption is equivalent to saying that investors only have heterogeneous beliefs with respect to foreign stocks. As such they can only short foreign stocks. Under these two assumptions the key closed-economy insights carryover to the open economy model. In particular, the within-country distribution of consumption between stock lenders and borrowers remain the key state variables together with the relative supply of goods.

We find that movements in short-selling cost transmits shocks across countries. For positive domestics supply shocks, we show that the real exchange rate response combines three effects: a term of trade channel, a belief heterogeneity channel, and a short-selling channel. The term of trade channel captures the fact that a positive supply shocks lowers domestic good prices leading to a real exchange rate depreciation. The belief heterogeneity channel and short-selling channel are new channels contributing to further real exchange rate depreciation. Intuitively, when foreign stock lenders are wealthier relative to borrowers, they hold relatively more home stocks compared to their holdings in an environment without short-selling. A booming domestic stock market following a positive supply shocks leads to transfer of wealth to the foreign stock lender which will be mostly spent on foreign good. As a results, foreign goods prices rise contributing to a further deterioration of home terms of trade and booming foreign stock market.

Our paper is related to three streams of the literature. First, it is related to the empirical work on the effects of short-selling on equity markets using cross-country data. Important contributions in this literature exploit quasi-natural experiments to establish causal effects of short-selling restrictions policies or short-selling disclosure policies on market or stock level performance. The literature has found that short selling policies has mixed effects on the distribution of stock returns depending on the nature of the short-selling proxies employed. Using difference in short-selling regulations covering 46 equity markets, Bris et al. (2007) show that market returns display less negative skewnees with short-selling restrictions. By contrast, Saffi and Sigurdsson (2011) use equity lending supply proxies the severity of the short-selling restrictions on a given stock and found opposite results. That is, a higher level of lending supply is associated with a greater degree of negative skewness.

Some recent work have investigated how investors circumvent short-selling bans on their home markets by trading sophisticated financial instruments like Deposits Receipts. Deposits Receipts provide a way for firm to access foreign capital market without directly listing shares overseas. American depositary receipts (ADRs) are an example of such instruments traded in the United States. Blau et al. (2012) show that short turnover is about 40% higher for ADRs with binding home-market constraints than for ADRs without constraints. Among other factors, greater dispersion of opinion related to the ADRs performance contribute to intense shorting activities. They also report that short selling activities predict returns more so for ADRs with home-market constraints. More recently, Boehmer et al. (2022) conducted a systematic analysis of short selling predictability using data from 38 countries from 2006 to 2014. They document that various shorting measure predict stock returns with the predictive power of these measures varying across countries. Moreover, they find that the predictive power of short selling is higher for countries and firms with higher cost of shorting and tighter regulations. Gorbenko (2021) provide evidence that short interest significantly but negatively predicts aggregate stocks returns in most country worldwide out-of-sample. However, none of these papers investigates the role of equity short selling activities on exchange rates. A void this paper fills.

Second, we contribute to literature on the role of financial intermediaries for external adjustment operating through the limits to international arbitrage. One strand of this literature emphasizes the role of financial frictions facing financial intermediaries in shaping the capital flow across countries and exchange rate determination. In Gabaix and Maggiori (2015), equilibrium exchange is driven by supply and demand of assets denominated in different currencies and the tightness of intermediaries balance sheet constraints. A different strand of the literature studies models in which financial intermediaries are risk averse and needed to be compensated for taking risk in assets denominated in different currencies (Itskhoki and Mukhin, 2021). In this alternative paradigm, financial intermediaries risk-bearing capacity, shaped by their preferences, play a key role in equilibrium exchange determination. We complement this literature by focusing on the costs of short-selling which is among the most important limits to arbitrage (Gromb and Vayanos, 2010; Muravyev et al., 2022).

Finally, we also contribute to an emerging theoretical literature that stresses the importance of jointly modeling asset prices and shorting fees in general equilibrium. In fact, Gârleanu et al. (2021) show how high short interests (fraction of outstanding shares held by short-sellers) are effectively a subsidy for long positions that can generate "runs-type" behavior. Atmaz et al. (2022) build a tractable model to match the abundant cross-sectional and predictability evidence on the effects of short-selling. The economic mechanism that produces these results are based on time-varying wealth shares of different types of agents (optimists and pessimists) together with their access status to lending market. Evgeniou et al. (2022) show how costly short-selling increases the contribution of the shortable assets valuation to the overall market portfolio given that the price of shortable assets include not only the present value of its future dividends but also its resale-option value (present value of future lending revenues). So far, this literature has mostly focused on closed-economy. I extent to the open economy.

Our approach to shorting fee is close to Atmaz et al. (2022) who use a CARA-Gaussian model to obtain closed-form solutions event with multiple type of agents. The main innovation in our paper compared to their approach is that we use an overlapping generation framework with logarithm utility as in Gârleanu et al. (2021), Panageas (2020) or Pavlova and Rigobon (2007) which allows us to easily characterize the equilibrium market of prices and shorting fee. Although, Atmaz et al. (2022) and Evgeniou et al. (2022) consider a set-up with two risky assets in a closed-economy, the open-economy term of trade adjustments present in our model provide an additional economic channel that is absent in a closed-economy.

The rest of the notes proceed as follows. Section 3 recasts the model of Atmaz et al. (2022) in an overlapping generations setting. Section 4 presents and characterizes properties of the open economy model. Section 6 concludes. All proofs are relegated to the Appendix.

2 Empirical Motivation

This section examines the exchange rate response to aggregate short selling activities conditional on the financial intermediaries networth shocks. We start this section by discussing our data sources and how we measure short selling activities. We then discuss our main empirical specification, before presenting our empirical results and robustness checks.

2.1 Data

Measure of short selling activities. We obtain data on stock borrowing and lending from S&P Global (formerly IHS Markit), covering 27 OECD countries. This dataset includes daily observations from July 2006 to September 2020. S&P Global provides various shorting variables, including the number of shares out on loan and the number of shares available for lending, sourced from major equity loan market participants. We consolidate the datasets into a monthly frequency by calculating the average of all relevant variables for each stock and year-month, aggregating across days. Following common practices in the literature, our stock-level measure of short-selling activities is the utilization rate defined as the ratio of the number of shares on loan over the shares available for lending. Let $UR_{i,c,t}$ be the utilization rate of security i from country c at time t. We formally have:

$$UR_{i,c,t} = \frac{\text{Number of shares on loan}_{i,c,t}}{\text{Number of shares available for lending}_{i,c,t}} \tag{2.1}$$

Similar to labour market tightness, the utilization rate could provide insights into the state of the stock lending market. An increase in utilization rate indicates a tight stock lending market, where short-sellers would struggle to locate a share to borrow. We construct the country-level aggregate utilization rate as the equal-weighted average of all stock-level utilization rate variables given by:

$$UR_{c,t} = \frac{1}{N_c} \sum_{i=1}^{N_c} UR_{i,c,t} \tag{2.2}$$

where N_c is the number of stocks in country c. For each country c in month t, we normalize the country-month utilization rate by adjusting it to its sample's average and standard deviation. This approach allows us to assess whether a specific short selling market in a country is relatively tight or loose in comparison to its sample average.

Financial intermediary shocks. We use the high-frequency financial intermediary shocks of Ottonello and Song (2022). These shocks correspond to changes in stock prices of large U.S. financial intermediaries within a 60-minute window of their earnings announcements. This approach mirrors the method used for monetary policy shocks whereby only the disclosed information could plausibly caused the movement in stock prices within the narrow time window. The change in stock prices directly impact financial intermediaries networth which in turn will affect the broader economy.

Country-level variables. We collected monthly series consisting of exchange rates, stock prices, long-term government bonds and consumer price indices (CPI) for each country over the time frame July 2006 to September 2020. Spot, $S_{c,t}$ and one month forward, $F_{c,t}$ exchange rates of country c at time t are quoted in units of US dollar so that an increase corresponds to a depreciation country'c currency. Stock prices are MSCI country level equity return indices. We use the 10-Year Government Bond Yield as our measure of long-term bonds. All these series comes from Datastream, but long term government bonds obtained from Haver.

World-level variables. Our global variables consists of the price of oil taken to be the spot price of Cushing West Texas Intermediate (WTI) crude oil and the gold price corresponding to the London Bullion Market spot prices.

2.2 Estimation by local projections

We assess the exchange rate responses to heightened aggregate short selling activities in the equity market using the local projections method of Jordà (2005). Our main empirical specification is:

$$\Delta_h \log S_{c,t+h} = \alpha_{c,h} + \beta_h UR_{c,t-1} \times \varepsilon_t^{FI} + \gamma_h UR_{c,t-1} + \delta_h X_{c,t-1} + \eta_h Z_{t-1} + e_{c,t+h} \tag{2.3}$$

where h is the horizon at which the impact is being estimated, $\Delta_h \log S_{c,t+h} = \log S_{c,t+h} - \log S_{c,t-1}$ denotes the cumulative exchange rate change, $\alpha_{c,h}$ is a country fixed effect, $UR_{c,t}$ is the country c utilization rate in month t, ε_t^{FI} is the monthly financial intermediary shock, $X_{c,t-1}$ denotes additional controls and the term $e_{c,t+h}$ is the error term assumed to be heteroskedastic, independent across country c, and serially correlated. Following Engel (2016), we include 12 lags of monthly control variable that captures various aspects of the state of the country, $X_{c,t}$, and world, Z_t, economy including news about US monetary policy (*gold prices*), expectation of global economic growth (*oil prices*), county-level local monetary policy and growth prospects (*stock prices*, and *long-term government bonds*).

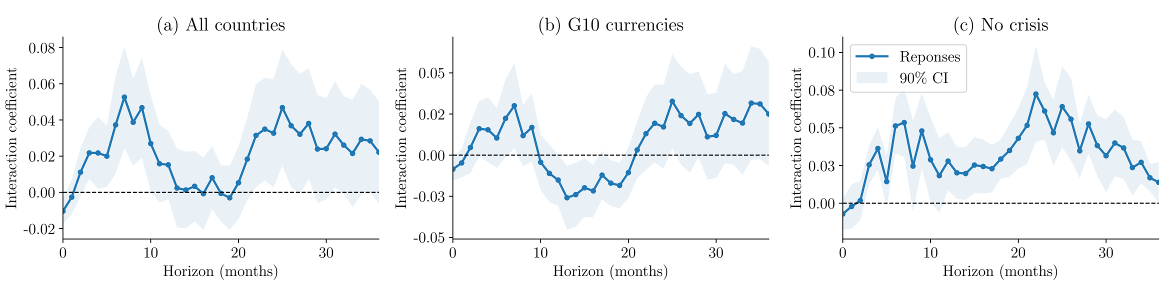
We estimate (2.3) by fixed-effects panel regression method with Driscoll and Kraay (1998) standard errors that corrects for the serial correlation at the country level and cross-country correlation. This approach of computing standard errors in panel fixed effects model is appropriate for large time periods T (Vogelsang, 2012). In our case, we have large T with an average of $T = 162$ time series data per country. The coefficient of interest

β_h captures the exchange rates responses the changes in aggregate short selling activities measured by the utilization rate.

2.3 Exchange rate responses to aggregate short-selling activities

Baseline results. Figure 1 presents the results of our local projection specification in equation (2.3) estimated for different countries set and time periods. Panel (a) shows the point estimates of interest for all countries in our sample. We find that the coefficient of the interaction term, β_h, is positive and statistically significant for most horizons. The positive estimates imply that exchange rate depreciates significantly more in countries with high aggregate short selling activities following a rise in financial intermediaries networth, with peak impact occurring in two sup-horizons including between 6 to 12 months and 22 to 30 months after the shock. On impact, a country has approximately a 0.01 units lower semielasticity of exchange rate to financial intermediary networth shock when it has one standard deviation more utilization rate than its typical average. These impacts are economically sizable as they will correspond to an average monthly exchange rate appreciation of 8 basis points on impact and a cumulative 41 basis points depreciation 9 months later.[1]

Figure 1: Exchange rate response to changes in utilization rate



Notes: The figure plots the exchange rate responses, β_h in equation (2.3), to a 1% increase in financial intermediary networth shock by Ottonello and Song (2022) at month t. The responses are driven by cross-country variation in utilization rate. Utilization rate are demeaned and standardized so that units are standard deviations. The x-axes show the horizon h (months). The vector of controls includes 12 lags of forward discounts, gold prices returns, oil prices returns, MSCI returns, 10-year government bonds yields and inflation differentials. All controls are standardized. Standard errors are Driscoll and Kraay (1998) standard errors with a bandwidth lag 16. The sample period is from July 2006 to September 2020. Shaded area are the 90% confidence interval.

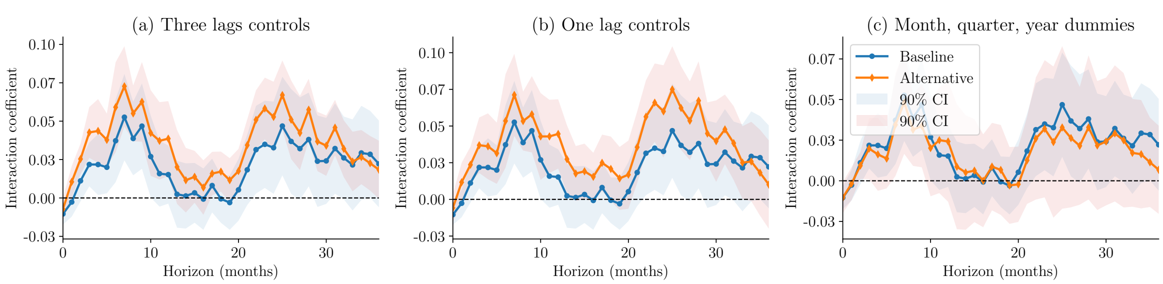
In panel (b), we restrict the set of countries to G10 currencies with the most liquid foreign exchange rate markets. The point estimates decrease for each horizon while becoming insignificant from 10 month horizon forward. Panel (c) replicates the previous

[1] We compute these average effects as $\beta_h \times sd(UR_t) \times sd(\varepsilon_t^{FI}) = -0.01 \times 0.18 \times 0.45 = -0.0008$ and $0.05 \times 0.18 \times 0.45 = 0.00405$

results for a sample that excludes the three major crisis including the 2008-2009 financial crisis (September 2008 to June 2009), the European debt crisis (July 2009 to December 2013) and the Covid pandemic (February 2020 to June 2020). The coefficient of interest increase in magnitude across horizon with tighter confidence interval. This shows that our results are not driven by short selling regulations usually introduced during a crisis (Edwards et al., 2023).

Additional Empirical Results. We present a number of checks to make sure that our findings are robust. We start by investigating the robustness of our baseline results to the alternative controls specification. Figure 2 presents results for a case where we include three, one lagged controls and include no controls other than the fixed effects. The results for this specification closely resemble those obtained using our standard set of controls. The exchange rate response is marginally higher and more precisely estimated with smaller standard errors in the results with shorter lagged controls. Additionally, the response derived without controls are estimated with somewhat less precision for horizons above 10 months.

Figure 2: Responses with different controls



Notes: The figure plots the exchange rate responses, β_h in equation (2.3), to a 1% increase in financial intermediary networth shock by Ottonello and Song (2022) at month t. The responses are driven by cross-country variation in utilization rate. Utilization rate are demeaned and standardized so that units are standard deviations. The x-axes show the horizon h (months). The vector of controls includes forward discounts, gold prices returns, oil prices returns, MSCI returns, 10-year government bonds yields and inflation differentials. All controls are standardized. Standard errors are Driscoll and Kraay (1998) standard errors with a bandwidth lag 16. The sample period is from July 2006 to September 2020. Shaded area are the 90% confidence interval.

Recently, Mei et al. (2023) show that fixed-effects panel local projection estimator suffers from the Nickell bias, regardless of the inclusion or not of lagged dependent variables, a bias inherent in the panel predictive specification. To address this concern, they propose to use the split-panel jackknife estimator for inference.[2] Figure 3 in the appendix shows that the responses are more precisely estimated and larger in magnitude using the split-panel jackknife estimator. The initial exchange rate appreciation following the financial

[2]The split-panel jackknife estimator equally solves the Nickel biais arising when the panel data has larger units N than time periods T (e.g. Kahn et al. (2021)).

intermediary shocks is not however statistically significant. These results suggest that not correcting for the Nickel bias provides a more estimated of our coefficient of interest.

We next analyze the robustness of our results to alternative choice of the level of clustering of the standard errors. We compute standard errors clustered by country, time and two-way clustered on time and country. The results are reported in Figure 4 in the appendix. Relative to Driscoll and Kraay (1998) standard errors, clustering by country produce the smallest standard errors and the coefficient of interaction term is always significant at 10% level. However, these results might lead to incorrect inference as the number of cluster is small. Furthermore, clustering either by time or two-way clustering by country produce standard errors very similar to our baseline case.

In summary, the results of this section provide evidence that aggregate equity short-selling activities in equity markets impact exchange rate movements. We demonstrated that exchange rates tend to depreciate in countries experiencing a higher-than-usual growth rate in short-selling activities. There are at least two channels through which short-selling can affect exchange rates. First, the capital flow channel: increased short selling may initially attract foreign investors, leading to capital inflows and potential immediate currency appreciation. However, the future repatriation of short-selling revenues could lead to subsequent depreciation.

Second, the market sentiment channel: intense short-selling activities might be interpreted as a lack of confidence in a country's economic growth prospects or corporate sector, deterring foreign investment. This reduction in foreign capital inflows can lead to a decrease in demand for the domestic currency, resulting in depreciation. In the next section, we build an open economy asset pricing model with endogenous short-selling to interpret our motivating facts and clarify the mechanisms through which aggregate short-selling activities affect exchange rates.

3 Closed-economy model

In this section, we introduce a closed-economy model with a lending and borrowing market of stocks in which the cost of shorting risky assets arise endogenously. Our framework boils down to a simple extension of the closed-economy version of the model in Atmaz et al. (2022), expanded to an overlapping generations setup with logarithmic preferences. We start with this simple framework because we can derive analytical solutions which provide intuition for how consumption share of different investors type shape the behavior of asset prices and shorting fee. These insights will be helpful to highlight the contribution of open economies features in later sections.

3.1 Environment

Preferences. We consider a continuous time overlapping generations economy of Yaari (1965) and Blanchard (1985). This economy is populated by a continuum of households of total mass equal to one. Households are finitely lived with a random time of death which follows an exponential distribution leading to a constant hazard rate. During each small time period dt, a mass πdt of the population dies and a new cohort of mass πdt is born, so the total population size stays constant at $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} ds = 1$. A household born at time s derives utility from consumption plan $C_s = \{c_{s,t} : t \geq s\}$ as

$$U_s = \mathbb{E}_s^i \int_s^{\infty} e^{-(\rho+\pi)(t-s)} \log(c_{s,t}^i) dt, \tag{3.1}$$

where ρ denotes the subjective discount factor. The superscript in the in the expectation operator accounts for the individual-specific subjective subjective beliefs about the aggregate output.

Technology. The output process Y_t evolves according to

$$dY_t / Y_t = \mu^Y dt + \sigma^Y dB_t, \tag{3.2}$$

where the expected endowment growth is given by μ^Y with σ^Y, μ^Y are positive parameters and B_t is a standard Brownian motion.

Heterogeneous beliefs. There are three types of investors indexed i who differ by their beliefs about the expected output μ^Y. A fraction ν is optimist and perceives the expected return to be $\mu^Y + \Delta$, while the complementary fraction is pessimist with perceived mean equal to $\mu^Y - \Delta$. Thus, type-i investor believes that the output follows:

$$dY_t / Y_t = \mu^{i,Y} dt + \sigma^Y dB_t^i \tag{3.3}$$

where $\mu^{i,Y}$ the perceived long-run mean and B_t^i is the corresponding perceive Brownian motion equal to $dB_t - \frac{\Delta}{\sigma^Y} dt$ for optimists and $dB_t + \frac{\Delta}{\sigma^Y} dt$ for pessimists.

Short-selling market. There exist a short selling and lending market for the stock where investors can sell the stock they do not own. In particular a fraction λ of optimists investors are stock lenders indexed by ℓ. They pay a fee $\sigma_t^S \phi_t$ to participate in stock lending market where they successfully find a short seller to lend the asset to with probability α. The remaining fraction $\nu - \lambda$ of optimists investors, denoted by c simply hold the stock and do not participate in the stock lending market. Short sellers are pessimists investors denoted by b who can borrow the stock by paying a fee ϕ_t. We denote by ν^i the measure of investors of type i given by:

$$\nu^i = \begin{cases} \lambda, & \text{if } i = \ell \\ \nu - \lambda, & \text{if } i = c \\ 1 - \nu, & \text{if } i = b. \end{cases} \tag{3.4}$$

We assume without loss of generality that $\nu = \frac{1}{2}$ so that average perceived expected output is unbiased. That is $\sum_i \nu^i \mu^{i,Y} = \mu^Y + \Delta(2\nu - 1) = \mu^Y$.

Financial market structure. Investors can trade two types of financial assets. First, they have access to a riskless asset which pays an interest rate r_t and is in zero net supply. Second, there is an infinitely lived risky asset whose supply is normalized to one. Its prices S_t evolves as

$$dR_t \equiv \frac{dS_t + Y_t dt}{S_t} = \mu_t^S dt + \sigma_t^S dB_t = (\mu_t^S + \sigma_t^S \eta^i) dt + \sigma_t^S dB_t^i \tag{3.5}$$

where the volatility σ_t^S the expected stock return μ_t^S, are determined in equilibrium. The last equality gives the investor-specific perceived returns process where η^i conveniently corresponds to:

$$\eta^i = \begin{cases} \frac{\Delta}{\sigma^Y}, & \text{if } i = \ell \\ \frac{\Delta}{\sigma^Y}, & \text{if } i = c \\ -\frac{\Delta}{\sigma^Y}, & \text{if } i = b \end{cases} \quad \text{and} \quad dB_t^i = dB_t - \eta^i dt. \tag{3.6}$$

Finally, investors can also access an actuarially fair annuities through competitive insurance companies as in Blanchard (1985). An investor would receive an income stream proportional to its financial wealth when it is alive, $\pi w_{s,t}^i$. In exchange, its insurance company collects the household's financial wealth at its death.

Budget constraints. Each type of investor is endowed with the same initial wealth at birth $w^i_{t,t} = \frac{\delta}{\rho+\pi} Y_t$. Let $\psi^i_{s,t}$ be the dollar amount invested in the stock S_t. The budget constraint of the type-i investors is given by:

$$dw^i_{s,t} = \left(r_t w^i_{s,t} + \psi^i_{s,t}(\mu^S_t + \sigma^S_t \eta^i + \alpha^i \sigma^S_t \phi_t - r_t) + \pi w^i_{s,t} - c^i_{s,t}\right)dt + \psi^i_{s,t}\sigma^S_t dB^i_t. \tag{3.7}$$

In this expression, $c^i_{s,t}$ is the investors' consumption, $w^i_{s,t}$ the financial wealth, $\alpha^i\phi_t$ is the additional income of per stock share held earned in the stock lending market, and $\eta^i\sigma^S_t$ accounts for the degree of disagreement. In equilibrium, short sellers have a negative position on the risky assets i.e. $\psi^b_{s,t} < 0$. For notation convenience, we let the parameter α^i and η^i be type specific given by :

$$\alpha^i = \begin{cases} \alpha, & \text{if } i = \ell \\ 0, & \text{if } i = c \\ 1, & \text{if } i = b. \end{cases} , \tag{3.8}$$

corresponding respectively to the probability that a stock lenders meets a short sellers in the lending market and the degree of disagreement.

3.2 Equilibrium

A competitive equilibrium is defined by a set of adapted processes for prices $\{r_t, S_t, \phi_t\}$ and consumption and portfolio allocations $\{c^i_{s,t}, \psi^i_{s,t}\}$ such that

1. Given prices $\{r_t, S_t, \phi_t\}$, policies $\{c^i_{s,t}, \psi^i_{s,t}\}$ solve each household's utility maximization problem

2. The goods market clears:

$$\sum_{i\in\{\ell,c,b\}} \nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c^i_{s,t} ds = Y_t; \tag{3.9}$$

3. The bond market clears:

$$\sum_{i\in\{\ell,c,b\}} \nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} (w^i_{s,t} - \psi^i_{s,t}) ds = 0; \tag{3.10}$$

4. The stock market clears:

$$\sum_{i\in\{\ell,c,b\}} \nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \psi^i_{s,t} ds = S_t. \tag{3.11}$$

5. The short-selling and lending market clears:

$$\sum_{i\in\{\ell,c,b\}} \nu^i \alpha^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \psi^i_{s,t} \phi_t ds = 0. \tag{3.12}$$

3.3 Characterization of equilibrium

In this section, we first solve a household's consumption-portfolio choice problem and then characterize equilibrium asset prices and the dynamics of wealth distribution. In our framework, market are dynamically complete leading us to specify the stochastic discount factor (SDF) under the objective probability measure as:

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \theta_t dB_t. \tag{3.13}$$

where θ_t it the market price of risk. Following Ehling et al. (2018), we further specify the individual specific pricing kernel which takes into considerations the heterogeneity in beliefs about the endowment as:

$$\frac{d\xi^i_t}{\xi^i_t} = -r_t dt - \theta^i_t dB^i_t = -r_t dt - \Big(\theta_t + \eta^i + \alpha^i \phi_t\Big) dB^i_t. \tag{3.14}$$

Investor optimization problem. We use the martingale method to solve for the optional consumption allocation. In this method, we exploit the fact that markets are dynamically complete to transform the sequence of budget constraints into a single lifetime budget constraint. As such, the equivalent household's problem can be reformulated as:

$$\max_{C_s,\psi_s} \mathbb{E}^i_s \int_s^{\infty} e^{-(\rho+\pi)(t-s)} \log(c^i_{s,t}) dt \tag{3.15}$$

$$\text{s.t. } \mathbb{E}^i_s \int_s^{\infty} e^{-\pi(t-s)} \xi^i_t c^i_{s,t} dt = \xi^i_s w^i_{s,s}. \tag{3.16}$$

The first order conditions of this problem equates the marginal utilities and the stochastic discount factor and is given by:

$$\frac{e^{-(\rho+\pi)(t-s)}}{c^i_{s,t}} = \kappa_s e^{-\pi(t-s)} \xi^i_t, \tag{3.17}$$

where κ_s is the Lagrange multiplier of the lifetime budget constraint. This condition holds for all time period $t \geq s$. And so we can eliminate the Lagrange multiplier and to get the following recursive consumption plan as function of the disagreement process and the stochastic discount factor under the true probability measure:

$$c^i_{s,t} = c^i_{s,s} \left(e^{-\rho(t-s)} \frac{\xi^i_s}{\xi^i_t} \right). \tag{3.18}$$

In order to solve for the equilibrium consumption as function of the exogenous process, we define the consumption share accruing to type-i investor as follow:

$$x^i_t = \frac{\nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c^i_{s,t} ds}{Y_t} \tag{3.19}$$

We can further simplify the expression of consumption share by substituting (3.18) into (3.19) so as to express it in term of crossectional average of new born consumption which allow to define a new Markovian equilibrium characterized by the state vector (x^ℓ_t, x^b_t) where the good market equilibrium is now given by:

$$x^\ell_t + x^b_t + x^c_t = 1.$$

Lemma 1. *In equilibrium, the consumption share of type-i agents follows the diffusion process given by:*

$$\frac{dx^i_t}{x^i_t} = \mu^{i,x}_t dt + \sigma^{i,x}_t dB_t \tag{3.20}$$

where

$$\mu^{i,x}_t = r_t - \rho - \mu^Y + (\sigma^Y)^2 + \theta^i_t(\theta^i_t - \sigma^Y - \eta^i) - \pi\left(1 - \nu^i \frac{c^i_{t,t}}{Y_t x^i_t}\right),$$

$$\sigma^{i,x}_t = \theta^i_t - \sigma^Y.$$

The law of motion of consumption shares (3.19) are summarized in the lemma 1. The drift of these law of motions have an overlapping generation component given by $\pi(1 - \nu^i \frac{c^i_{t,t}}{Y_t x^i_t})$ which enables non-degenerate stationary distributions. The remaining terms in these drifts move endogenous driven by differences in consumption-portfolio choices and access to the short selling and lending market across investors type.

Lemma 2. *In equilibrium, the optimal consumption and investment in the risky assets are linear in wealth:*

$$\begin{aligned} c^i_{s,t} =&(\rho + \pi) w^i_{s,t}, \\ \psi^i_{s,t} =&\frac{\theta^i_t}{\sigma^S_t} w^i_{s,t}. \end{aligned} \tag{3.21}$$

Before finding the equilibrium prices in this economy, let us notice that the assumption of logarithm preferences simplifies tremendously their optimal policy functions reported in lemma 2. This lemma shows that consumption-wealth ratio and portfolio shares are all independent of the cohort s. At the same time, each investor's portfolio share has two components: a myopic demand, $\frac{\theta_t}{\sigma^S_t}$, common for all investors and a hedging demand, $\frac{\eta^i \sigma^S_t + \alpha^i \phi_t}{(\sigma^S_t)^2}$, which differs across investor type.

Market price of risk and stock price. The interaction of different type of investors in each market determine the equilibrium prices. In particular, the consumption good market $\sum_{i\in\{\ell,c,b\}} x^i_t = 1$ implies that $\sum_{i\in\{\ell,c,b\}} dx^i_t = 0$. This last equality imposes a joint restrictions on drifts and diffusion terms which allow to find the equilibrium market price of risk and interest rate. Furthermore, combining the constant consumption-wealth ratio in lemma 2 with the bond and stock market clearing conditions gives the price-dividend ratio. All these results are reported in proposition 1.

Proposition 1. *In equilibrium, the market price of risk, the risk free rate and the stock volatility are given by*

$$\begin{aligned} \theta_t =&\sigma^Y - \sum_{i\in\{\ell,c,b\}} x^i_t \eta^i - \phi_t \sum_{i\in\{\ell,c,b\}} x^i_t \alpha^i \\ r_t =&\rho + \mu^Y - \sigma^Y \theta_t + \pi \left(1 - \sum_{i\in\{\ell,c,b\}} \nu^i \frac{c^i_{t,t}}{Y_t} \right) \\ \sigma^S_t =&\sigma^Y \end{aligned} \tag{3.22}$$

It should be emphasized that the results of proposition 1 are very general in a sense that they apply to economies where the short selling and lending market is either present or not. To get further insights, let us rewrite the market price of risk in equation (3.22) as:

$$\theta_t = \underbrace{\sigma^Y}_{\text{Log utility economy}} - \underbrace{\frac{\Delta}{\sigma^Y}(1-2x_t^b)}_{\text{Heterogenous beliefs}} - \underbrace{\phi_t(x_t^b + \alpha x_t^\ell)}_{\text{Short selling}}.$$

The expression of the market price of risk thus deviates from the one in an economy with infinitely lived investor endowed with log preferences by two additional terms. The term $\frac{\Delta}{\sigma^Y}(1-2x_t^b)$ accounts for beliefs distortions. If $x_t^b < 1/2$, the market price of risk is lower than σ^Y; reflecting the optimism of lenders. The term $\phi_t(x_t^b + \alpha x_t^\ell)$ is due to short selling activities which always contribute to lower the market price of risk.

Substituting, the market of price (3.22) into the consumption share diffusion term in (3.20), rearranging reveals that:

$$\sigma_t^{i,x} = \eta^i - \sum_{j\in\{\ell,c,b\}} x_t^j \eta^j + \phi_t\left(\alpha^i - \sum_{j\in\{\ell,c,b\}} x_t^j \alpha^j\right), \tag{3.23}$$

so that for any $\phi_t \leq \frac{2\Delta}{(1-\alpha)\sigma^Y}$, $\sigma_t^{b,x} \leq 0 \leq \sigma_t^{\ell,x}$. In this case, x_t^b is countercyclical and x_t^ℓ is procyclical. Put differently, in bad times, stock lenders, ℓ, experiences large consumption losses, while in good times they receive greater consumption gains.

Short selling fee and risk. The short selling fee is such that the supply and demand of shares in the short selling market are equals. The next proposition states the equilibrium shorting fee together with short interest which is the fraction of its outstanding shares held by short-sellers.

Proposition 2. *In the costly stock short-selling economy with a constant disagreement the shorting fee is given by*

$$\phi_t = -\frac{\sum_i \frac{x_t^i \alpha^i}{\sum_j x_t^j \alpha^j}\eta^i + \theta_t}{\sum_i \frac{x_t^i \alpha^i}{\sum_j x_t^j \alpha^j}\alpha^i}, \tag{3.24}$$

*the **short interest** defined as the fraction of outstanding stocks shares held by short-sellers is*

$$\frac{SI_t}{Y_t} = -\frac{1}{\rho+\pi}\frac{\theta_t^b}{\sigma_t^S}x_t^b \tag{3.25}$$

Equation (3.24) shows that as in Atmaz et al. (2022), market price of risk is negatively related to shorting fee. Intuitively, an increase in the shorting fee incentivizes the optimistic stock lenders to increase their demand of the stocks of which a fraction will be lent out to earn extra income. The increase in stock demand leads to higher current stock prices and lower subsequent returns. Equation (3.25) reveals that short interest rate decrease with shorting fee. In fact, exploiting the market price of risk (3.22) together with definition of θ_t^b, we show that $\frac{1}{Y_t}\frac{\partial SI_t}{\partial \phi_t} \propto x_t^b + \alpha x_t^\ell - 1 \leq 0$.

One measure of the short selling risk is the diffusion term of the change in shorting fee process. Since the equilibrium in our economy is Markovian characterized by the state vector (x_t^b, x_t^ℓ) whose dynamics equations are given by lemma 1, we can easily determine short selling risk by directly treating ϕ_t as function of (x_t^b, x_t^ℓ) and using the Ito's lemma. We find that:

$$\text{Short selling risk} \equiv \sigma_t^\phi = \sum_{i\in\{\ell,b\}} \frac{\partial \phi_t}{\partial x_t^i} x_t^i \sigma_t^{i,x}. \tag{3.26}$$

The byproduct of solving for the short selling risk involves finding also the diffusion term of the change in sharp ratio that we labeled sharp ratio risk and given by:

$$\text{Sharp ratio risk} \equiv \sigma_t^\theta = \sum_{i\in\{\ell,b\}} \frac{\partial \theta_t}{\partial x_t^i} x_t^i \sigma_t^{i,x}. \tag{3.27}$$

Proposition 3. *In the costly stock short-selling economy with a constant disagreement:*

1. *The sharp ratio risk is given by:*

$$\sigma_t^\theta = -\sigma_t^\phi \sum_i x_t^i \alpha^i - \sum_i x_t^i (\sigma_t^{i,x})^2 \tag{3.28}$$

2. *If the short selling risk is procyclical (i.e. $\sigma_t^\phi \geq 0$), then the sharp ratio risk is countercyclical (i.e. $\sigma_t^\theta \leq 0$).*

Proposition 3 highlights how the presence of the short selling frictions affects the sharp ratio risk. In point 1, the sharp ratio risk is decomposed in two intuitive terms. The first term represents the contribution of short selling activities featured by the sharp selling risk. In an economy without short selling activities, the sharp ratio risk will be given only by the second term which is always negative, suggesting that the sharp ratio will be generically countercyclical (Gârleanu and Panageas, 2015; Panageas, 2020). The intuition for Point 2 is as follows. When $\sigma_t^\phi \geq 0$, a negative aggregate output shock leads to a decline of the short selling fee making the net revenue from borrowing securities higher and thus increasing stock demand by short sellers. To see this, let us examine diffusive term of the change in portfolio weights in equation (3.21) which is proportional to $\sigma_t^{\theta,i}$. Using the definition of θ_t^i, we have:

$$\sigma_t^{\theta,i} = \sigma_t^{\theta} + \alpha^i \sigma_t^{\phi} \tag{3.29}$$

so that $\sigma_t^{\theta,c} \leq \sigma_t^{\theta,\ell} \leq \sigma_t^{\theta,b}$. As such, stock short sellers rebalance relatively more their portfolio compared to stock lenders and increase their stock holding. Consequently, in a bad times, short sellers are more exposed to aggregate shocks and need to be compensated for bearing this risk. As a results, the sharp ratio risk need to be countercyclical, so that $\sigma_t^{\theta} \leq 0$.

4 Open Economy model

To understand the role of short selling frictions for exchange rate dynamics, we now introduce a two countries model, home and foreign indexed by $n, m \in \{H, F\}$. The structure of each economy is in many respect similar to the closed-economy setup and we only add the minimal ingredients necessary to isolate the role of short selling for international asset prices and exchange rate.

4.1 Environment

Endowments. Each country possess a tree that produces a differentiated good and its output follow a geometric Brownian motion given by:

$$\frac{dY_{nt}}{Y_{nt}} = \mu_n^Y dt + (\sigma_n^Y)' \mathbf{dB_t}, \quad \forall n \in \{H, F\}. \tag{4.1}$$

The shock $\mathbf{B_t} = (B_{Ht}, B_{Ft})'$ is a two dimensional vector of standard Brownian motion that are independent of each other, while the expected endowment growth is given by μ_n^Y. The diffusion coefficient are two dimensional vectors $\sigma_H^Y = (\sigma_{HH}^Y, 0)'$ and $\sigma_F^Y = (0, \sigma_{FF}^Y)'$.

Preferences. In both countries, investors now have logarithm preferences over a basket of two goods given by:

$$c^i_{n,st} = \prod_{m \in \{H,F\}} (c^i_{nm,st})^{\varepsilon_{nm}} \quad \forall n \in \{H, F\} \tag{4.2}$$

where $c^i_{nm,st}$ denotes the time t consumption of good m by type-i investors born at s residing in country n, $\varepsilon_{nn} = \varepsilon \quad \forall n$; $\varepsilon_{nm} = 1 - \varepsilon \quad \forall n \neq m$ with $\varepsilon \in [\frac{1}{2}, 1)$ to account for home bias whereby each investor prefers their domestic good relatively more. The solution to the investor's intratemporal problem yields the standard conditions for consumption demand

$$c^i_{nm,st} = \varepsilon_{nm} \left(\frac{p_{m,t}}{P_{n,t}} \right)^{-1} c^i_{n,st}, \quad \forall n, m \in \{H, F\} \tag{4.3}$$

where $p_{n,t}$ denotes the price of country n good and the aggregate price index consistent with households preferences is given by:

$$P_{n,t} = \prod_{m \in \{H,F\}} \left(\frac{p_{m,t}}{\varepsilon_{nm}} \right)^{\varepsilon_{nm}}. \tag{4.4}$$

International prices. The terms of trade, q_t, is the ratio between the foreign price and the home such that an increase in q_t represents a deterioration in the home terms of trade:

$$q_t = \frac{p_{F,t}}{p_{H,t}}. \tag{4.5}$$

Intuitively, this relative price denotes the quantity of foreign goods a domestic country can afford by exporting one unit of the domestic good. We postulate that the law of motion of the term of trade q_t evolves according to the stochastic differential equation:

$$\frac{dq_t}{q_t} = \mu^q_t dt + (\sigma^q_t)' \mathbf{dB_t} \tag{4.6}$$

where μ^q_t and σ^q_t are endogenously determined. The real exchange rate, $\mathcal{E}_t$, is expressed as the home price of foreign currency and is given by the ratio of Foreign to Home price indices:

$$\mathcal{E}_t = \frac{P_{F,t}}{P_{H,t}} = q_t^{2\varepsilon-1}. \tag{4.7}$$

Without loss of generality, the global numeraire good is a basket of goods composed of φ units of home goods and $1-\varphi$ units of foreign good. This assumption implies that $(p_{Ht})^{\varphi}(p_{Ft})^{1-\varphi} = 1$ and $p_{Ht} = q_t^{\varphi-1}$, $p_{Ft} = q_t^{\varphi}$.

Beliefs distortion. We assume that investors have distorted beliefs about the domestic and foreign expected output. However, each investor have relative informational advantage over their home stock. Let η_n^i be the two-dimensional column vector of beliefs distortion of type-i investor residing in country n. The subjective Brownian motion is given by:

$$\mathbf{dB^i_{n,t}} = \mathbf{dB_t} - \eta_n^i dt \quad \text{with } \eta_{nm}^i = \begin{cases} \frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = \ell \\ \frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = c \\ -\frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = b. \end{cases} \tag{4.8}$$

where the term Δ^{nm} denotes a bilateral beliefs distortion. We do not impose any particular restriction on these distortions as we will ultimately estimate them in the next section. Asymmetric beliefs distortion is sometimes used in international finance to explain international equity prices and quantities. Informational advantage rationalizes these asymmetries(Brennan and Cao, 1997; Dumas et al., 2017).[3] In this class of models, domestic investor have more precise signals of their own country home securities compare to foreign securities.

Stock prices and returns. In the world economy, investor can now trade three assets: two stocks, one for each country, and a global risk-free asset in the unit of the global numeraire. The supply of stock n is normalized to one while the risk-free asset is in zero net supply. Each country's stock market is a claim to the aggregate output produced in that country. The country-specific returns are given by the processes:

$$dR_{n,t} \equiv \frac{dS_{n,t} + p_{n,t}Y_{n,t}dt}{S_{n,t}} = \mu_{n,t}^S dt + (\sigma_{n,t}^S)'\mathbf{dB_t} \tag{4.9}$$

[3] Coval and Moskowitz (2001) show that U.S. mutual funds they tend to have better performance on local stocks. Similarly, Allen et al. (2022) show that from 2000 to 2018, Chinese companies listed in China, A shares firms, had smaller stock returns than Chinese companies listed in other countries and companies in both developed and emerging markets. They suggest that investor sentiment could account for these facts.

for $n \in \{H, F\}$, where $\mu^S_{n,t}$ is scalar and $\sigma^S_{n,t}$ is a 2×1 vector of diffusion, all determined in equilibrium.

Let dR_t denote the 2-dimensional column vector of returns for investing in home and foreign stock given by:

$$dR_t = \mu_t dt + \sigma^S_t \mathbf{dB_t} \tag{4.10}$$

where $\mu_t = (\mu^S_{H,t}, \mu^S_{F,t})'$ is the vector of expected returns and $\sigma^S_t = (\sigma^S_{H,t}, \sigma^S_{F,t})'$ is an 2×2 matrix. Let also $\psi^i_{n,st}$ denote the 2-dimensional column vector of dollar amount investor i born at s hold in each component of R_t.

Budget constraint. Each type of investor is endowed with the same initial wealth at birth $w^i_{n,tt} = \frac{\delta}{\rho+\pi} \sum_m p_{m,t} Y_{m,t}$ and maximizes their lifetime utility subject to the dynamic budget constraint:

$$dw^i_{n,st} = \left(r_t w^i_{n,st} + (\psi^i_{n,st})'(\mu^S_t + \alpha^i \sigma^S_t \phi_t + \sigma^S_t \eta^i_n - r_t \iota) + \pi w^i_{n,st} - P_{n,t} c^i_{n,st} \right) dt + (\psi^i_{n,st})' \sigma^S_t \mathbf{dB^i_{n,t}} \tag{4.11}$$

for $n \in \{H, F\}$, where ι is 2×1 vector of ones, $\phi_t = (\phi_{H,t}, \phi_{F,t})'$ is the vector of fee associated with shorting domestic and foreign stocks. To ease some derivations, we have normalized the fees by the stock volatility matrix σ^S_t.

4.2 Equilibrium

A competitive equilibrium of this economy consists of a set of adapted processes for prices $\{r_t, S_{Ht}, S_{Ft}\}$ and consumption and portfolio allocations $\{c^i_{n,st}, \psi^i_{n,st}\}$ such that:

1. Given prices $\{r_t, S_{Ht}, S_{Ft}\}$, policies $\{c^i_{n,st}, \psi^i_{n,st}\}$ solve each household's utility maximization problem

2. The goods market clears:

$$\sum_{m \in \{H,F\}} \sum_{i \in \{\ell,c,b\}} \nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} c^i_{mn,st} ds = Y_{n,t}, \quad \forall n \in \{H, F\}; \tag{4.12}$$

3. The bond market clears:

$$\sum_{n \in \{H,F\}} \sum_{i \in \{\ell,c,b\}} \nu^i \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left(w^i_{n,st} - \sum_{m \in \{H,F\}} \psi^i_{nm,st} \right) ds = 0; \tag{4.13}$$

4. All stock markets clear:

$$\sum_{m\in\{H,F\}}\sum_{i\in\{\ell,c,b\}}\nu^i\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\psi^i_{mn,st}ds = S_{n,t}, \quad \forall n\in\{H,F\} \tag{4.14}$$

5. The short-selling and lending markets clear:

$$\sum_{m\in\{H,F\}}\sum_{i\in\{\ell,c,b\}}\nu^i\alpha^i\int_{-\infty}^{t}\pi e^{-\pi(t-s)}\psi^i_{mn,st}\phi_{n,t}ds = 0, \quad \forall n\in\{H,F\} \tag{4.15}$$

4.3 Characterization of the open economy equilibrium

Relative supply. In this open economy version of our model, the relative supply of the home good constitutes an additional state variable and capture the relative importance of the domestic economy in the world economy. Following Sauzet (2022), we explicitly have:

$$y_t = \frac{Y_{H,t}}{Y_{H,t}+Y_{F,t}} \tag{4.16}$$

Its law of motion can be expressed as

$$dy_t = y_t(1-y_t)\left(\mu^y_t dt + (\sigma^y_t)'\mathbf{dB_t}\right) \tag{4.17}$$

where $\mu^y_t = \mu^Y_H - \mu^Y_F - (\sigma^Y_H - \sigma^Y_F)'(y_t\sigma^Y_H + (1-y_t)\sigma^Y_F)$ and $\sigma^y_t = (\sigma^Y_H - \sigma^Y_F)$.

Stochastic discount factor. In the world economy version of our model, markets are still dynamically complete. As such, we can now define the stochastic discount factor in terms of the global numeraire as follows:

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \theta_t'\mathbf{dB_t}. \tag{4.18}$$

where θ_t is the vector of market price of risks. The

$$\frac{d\xi^i_{n,t}}{\xi^i_{n,t}} = -r_t dt - (\theta^i_{n,t})'\mathbf{dB^i_{n,t}} = -r_t dt - \left(\theta_t + \eta^i_n + \alpha^i\phi_t\right)'\mathbf{dB^i_{n,t}}. \tag{4.19}$$

Investor optimization. The investors choose consumption and investment plans so as to maximize his lifetime utility subject to the static budget constraint. This choice problem is very similar to closed-economy version except for the introduction of price level and consumption good basket. We formally have:

$$\max_{C_s} \mathbb{E}_s \int_s^\infty e^{-(\rho+\pi)(t-s)} \log(c^i_{n,st})dt \tag{4.20}$$

$$\text{s.t. } \mathbb{E}_s \int_s^\infty e^{-\pi(t-s)} \xi^i_{n,t} P_{n,t} c^i_{n,st} dt = \xi^i_{n,s} w^i_{n,ss}. \tag{4.21}$$

Taking the first order conditions of this problem and rearranging in term consumption of the newborn gives:

$$c^i_{n,st} = c^i_{n,ss} \left(e^{-\rho(t-s)} \frac{\xi^i_{n,s}}{\xi^i_{n,t}} \frac{P_{n,s}}{P_{n,t}} \right). \tag{4.22}$$

Consumption shares. The new definition of consumption shares now takes into account both the origin of production and the residence of expenditure. Let $x^i_{mn,t}$ be the share of good n consumed in country m at time t by all the generations alive given by:

$$x^i_{mn,t} = \frac{\nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c^i_{mn,st} ds}{Y_{n,t}} \tag{4.23}$$

As such the new market clearing conditions are $\sum_{m \in \{H,F\}} \sum_{i \in \{\ell,c,b\}} x^i_{mn,t} = 1$ and the following lemma characterizes the process $dx^i_{mn,t}$.

Lemma 3. *In equilibrium, the consumption share of type-i agents follows the diffusion process given by:*

$$\frac{dx^i_{mn,t}}{x^i_{mn,t}} = \mu^{i,x}_{mn,t} dt + (\sigma^{i,x}_{mn,t})' \mathbf{dB_t}, \tag{4.24}$$

where

$$\begin{aligned}
\mu^{i,x}_{mn,t} =& r_t - \rho - \mu^Y_n - \mu^p_{n,t} + (\sigma^Y_n)'\sigma^Y_n + (\theta^i_{m,t})'(\sigma^p_{n,t} - \eta^i_m) + (\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n)'(\theta^i_{m,t} - \sigma^p_{n,t}) \\
& - \pi\left(1 - \nu^i \frac{\varepsilon_{mn} P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t} x^i_{mn,t}}\right), \\
\sigma^{i,x}_{mn,t} =& \theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n.
\end{aligned}$$

A few comments are in order. First, the evolution of consumption share in open economy includes all the different channels present in the closed-economy. Second, the open economy features operate through changes in the terms of trade. In fact, the drift of $x^i_{mn,t}$

depends on the final good prices through the terms $\mu^p_{n,t}$ and $\sigma^p_{n,t}$, which depends in turn on the term of trade.

In line with Section 3, each investor's policy functions are still linear in their wealth, stemming primarily from the household preferences specification; see Lemma 4. The hedging demand of the portfolio investment is however now country-type specific and impacted by term of trades movements. We clarify this channel in the next paragraphs.

Lemma 4. *In equilibrium, the optimal consumption and investment in the risky assets are linear in wealth*

$$\begin{aligned} c^i_{n,st} &= (\rho + \pi) w^i_{n,st} \\ \psi^i_{n,st} &= ((\sigma^S_t)')^{-1} \theta^i_{n,t} w^i_{n,st} \end{aligned} \tag{4.25}$$

Market price of risk and terms of trade. Given the evolution of consumption share summarized in Lemma 3, we can apply Ito's lemma to each good market to obtain the market price of risks and risk free rate. The following proposition reports the results of this procedure in the open economy context.

Proposition 4. *For all $n \in \{H, F\}$, the equilibrium market price of risk and the risk free rate are given by:*

$$\begin{aligned} \theta_t &= \sigma^Y_n + \sigma^p_{n,t} - \sum_m \sum_i x^i_{mn,t} \eta^i_m - \sum_m \sum_i x^i_{mn,t} \alpha^i \phi_t, \\ r_t &= \rho + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n - (\sigma^p_{n,t} + \sigma^Y_n)'(\theta_t - \sigma^Y_n) \\ &\quad - \sum_m \sum_i x^i_{mn,t} \alpha^i (\theta^i_{m,t})' \phi_t + \pi \left(1 - \delta \frac{\sum_k p_{k,t} Y_{k,t}}{p_{n,t} Y_{n,t}} \right), \quad \forall n \end{aligned} \tag{4.26}$$

Proposition 4 establishes that the short term interest rate and the market price of risk both take the same form as in the closed-economy. They state that these equations have to hold for home and foreign country because the same stochastic discount factor written in terms of numeraire good is used by both domestic and foreign investors to price assets. For this to happen, the good prices, and thus term of trade, will also adjust in response to shocks. The next proposition describes the term of trade processes.

Proposition 5. *The terms of trades, q_t, follows the diffusion process given by:*

$$\frac{dq_t}{q_t} = \mu^q_t dt + (\sigma^q_t)' \mathbf{dB_t}, \tag{4.27}$$

where

$$\sigma_t^q = \sigma_H^Y - \sigma_F^Y - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\eta^i_m - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\alpha^i \phi_t \tag{4.28}$$

and the drift, μ_t^q, is given by equation (C.13) in the appendix.

Intuitively, the diffusion coefficients of the term of trades, σ_t^q, is time-varying driven by the average of beliefs distortion drifts, η^i_m, weighted by the difference in type-specific consumption share between the two countries, see equation (4.28), which all depends in turn on the shorting fees and asset prices. Hence, shorting fees influence the evolution of the relative price, which feeds back into asset prices. Beliefs distortion heterogeneity thus interacts with participation in securities lending and borrowing activities to produce terms of trade movements. Expanding the diffusion coefficient of the term of trade sheds more lights on each of these forces. We have :

$$\sigma_t^q = \begin{pmatrix} \sigma^Y_{HH} - \underbrace{\left(\sum_i (x^i_{HH,t} - x^i_{HF,t})\eta^i_{HH} + \sum_i (x^i_{FH,t} - x^i_{FF,t})\eta^i_{FH}\right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{H,t} \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \\ -\sigma^Y_{FF} - \underbrace{\left(\sum_i (x^i_{HH,t} - x^i_{HF,t})\eta^i_{HF} + \sum_i (x^i_{FH,t} - x^i_{FF,t})\eta^i_{FF}\right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{F,t} \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \end{pmatrix}. \tag{4.29}$$

Three key insights emerge from this expansion. First, the equilibrium term of trade depends on the within-country heterogeneity in wealth and consequently in consumption shares. Second, an active short selling market amplifies the effect of shocks on terms of trade. To see this, since $\varepsilon > 0.5$, each investor prefers relatively more their domestic goods compared to foreign ones so that the term $x^i_{nn} - x^i_{nm} > 0$ with n different from m. It then follows that $\partial\sigma^q_{Ht}/\partial\phi_{H,t} > 0$ and $\partial\sigma^q_{Ft}/\partial\phi_{F,t} < 0$. Third, when short sellers are not too wealthy, the sign of the diffusion coefficient are as follows: $\sigma^q_{Ht} > 0, \quad \sigma^q_{Ft} < 0$. In this case, a positive domestic productivity shock leads to deterioration of home terms of trades. Symmetrically, a positive foreign shock improves home's term of trade. Intuitively, this standard result follows from the fact that a positive shock increases the supply of domestic goods and lowers domestic prices relative to foreign prices.

Open economy shorting fee and short interest. We complete the characterization of the equilibrium by solving for the shorting fee of each country stock that clears the short selling market. To this end, we proceed exactly as in the case of the closed-economy model to replace open economy portfolio holding summarized in lemma 4 into the the short selling market clearing conditions. The next proposition summarizes this procedure and expresses the short selling fee and the short interest as function of wealth to ease notation and to highlight the similarities with the closed-economy version.[4]

Proposition 6. *In the costly stock short-selling economy with a constant disagreement,*

1. *the shorting fee is given by:*

$$\phi_{n,t} = -\frac{\theta_{n,t} + \sum_m \sum_i \frac{w^i_{m,t}\alpha^i}{\sum_k \sum_j w^j_{k,t}\alpha^j}\eta^i_{mn}}{\sum_m \sum_i \frac{w^i_{m,t}\alpha^i}{\sum_k \sum_j w^j_{k,t}\alpha^j}\alpha^i} \tag{4.30}$$

2. *the weights are*

$$\frac{w^i_{m,t}\alpha^i}{\sum_k \sum_j w^j_{k,t}\alpha^j} = \frac{y_t x^i_{mH,t}\alpha^i + (1-y_t)q_t x^i_{mF,t}\alpha^i}{y_t \sum_k \sum_j x^j_{kH,t}\alpha^j + (1-y_t)q_t \sum_j x^j_{kF,t}\alpha^j} \tag{4.31}$$

3. *the vector of **short interest** is given by:*

$$SI_t = -\sum_m \frac{y_t x^i_{mH,t} + (1-y_t)q_t x^i_{mF,t}}{q_t^{1-\varphi}}\theta^b_{m,t} \tag{4.32}$$

Proposition 6 reveals that the same transmission channels underlying the closed-economy are present in the open economy. All else equal, an increase in the disagreement parameter Δ or the decrease of the market price of risk loading on the domestic shocks $\theta_{H,t}$ all contribute to increase the domestic shorting cost, $\phi_{H,t}$.

[4]Given that consumption-wealth ratio are constant for each investor, an alternative expression of the group-specific wealth expressed as function of relative supply and consumption share is $(\rho + \pi)w^i_{n,t}/q_t^{\varphi-1}\sum_m Y_{m,t} = y_t x^i_{nH,t} + (1-y_t)q_t x^i_{nF,t}$.

5 Estimation

5.1 Data and observation equation.

We estimate the model parameters with maximum likelihood methods using industrial production as proxy for the output, MSCI indices to measure stock market returns, utilization rate or shorting fee as the measure for short-selling activities and bilateral spot exchange rate. We take the United States to be the home country and the Eurozone represented by Germany to be the foreign country. We focus on the Eurozone because the Euro is the second most liquid foreign exchange markets coupled with an active security lending market. The sample spans July 2006 to September 2020 to be consistent with the time periods used in for our motivating evidences. We formulate our structural model in state-space representation with observation equations given by:

$$\begin{aligned}
\log(RER_t/RER_{t-1}) =& \bar{\mu}^q + \mu_t^q + \sigma^{o,q} o_t^q \\
\log(Relative_output_t/Relative_output_{t-1}) =& \bar{y} + (1-y_t)\mu_t^y + \sigma^{o,y} o_t^y \\
Excess_returns_t =& \bar{\mu}^S + \sigma_t^S \theta_t + (\sigma^{o,s})' o_t^s \\
\log(Utilization_rates_t/Utilization_rates_{t-1}) =& \bar{UR} + \beta(SI_t - SI_{t-1}) + (\sigma^{o,ur})' o_t^{ur}
\end{aligned} \tag{5.1}$$

where $o_t = (o_t^y, (o_t^s)', (o_t^{ur})')'$ is a vector of measurement errors associated with relative output growth, excess returns, utilization rates and expected exchange rates changes data. The relative output series construction mirrors its structural model counterpart computed as $IP_{US,t}/(IP_{US,t} + IP_{DEU,t})$. The equity excess returns are the difference between the MSCI equity returns and the US one month T-bills. The utilization rate and real exchange are the same series used in section 2.

5.2 Maximum likelihood estimation by iterated filtering.

Our model's nonlinearities prevent us from directly applying the Kalman filter to determine the likelihood of our state-space model. As an alternative, Ionides et al. (2006, 2011, 2015) introduced the iterated filtering method for nonlinear state-space model inference. Iterated filtering is characterized by two key elements: (i) converting all fixed model parameters into time-varying parameters, represented as random walks, and (ii) estimating likelihood using a particle filter. This method involves a series of recursive filtering operations that ultimately converge to the maximum likelihood estimates of the pa-

Table 1: Maximum likelihood estimates

Parameter	Baseline			$\alpha = 0.001$		
	MLE	Lower CI	Upper CI	MLE	Lower CI	Upper CI
α	0.882	0.844	0.976	0.001	0.001	0.001
λ	0.495	0.355	0.497	0.074	0.074	0.457
δ	0.104	0.090	0.533	0.003	0.000	0.003
φ	0.988	0.961	0.994	0.539	0.523	0.690
Δ_{HH}	0.015	0.000	0.115	0.005	0.005	0.023
Δ_{FH}	0.129	0.035	0.419	0.020	0.004	0.020
Δ_{HF}	3.026	2.572	3.791	0.155	0.001	0.155
Δ_{FF}	0.173	0.046	0.647	0.197	0.000	0.197
σ^y_{HH}	0.189	0.032	0.343	2.930	0.107	2.930
σ^y_{FF}	2.040	1.636	2.162	0.806	0.028	0.806
$\bar{\mu}^q$	-4.540	-5.119	-2.950	-0.607	-0.607	0.207
$\bar{\mu}^y$	0.009	-0.013	0.009	-0.642	-0.642	0.003
$\bar{\mu}^S_H$	1.140	0.509	1.833	-3.641	-3.641	0.018
$\bar{\mu}^S_F$	1.004	0.538	1.576	-3.703	-3.703	-0.029
$u\bar{r}_H$	-0.074	-0.349	-0.074	0.129	-0.130	0.129
$u\bar{r}_F$	-0.055	-0.285	0.032	-0.053	-0.151	-0.053
β_H	4.275	2.368	6.119	-2.954	-2.954	12.282
β_F	7.560	5.261	8.273	-1.295	-1.295	15.390
$\sigma^{o,y}$	0.019	0.016	0.020	0.018	0.017	0.019
$\sigma^{o,q}$	0.072	0.070	0.078	0.090	0.090	0.093
$\sigma^{o,s}_H$	0.074	0.070	0.109	0.086	0.086	0.178
$\sigma^{o,s}_F$	0.080	0.059	0.081	0.075	0.075	0.155
$\sigma^{o,ur}_H$	0.194	0.170	0.248	0.183	0.143	0.183
$\sigma^{o,ur}_F$	0.202	0.200	0.265	0.280	0.204	0.280
$logLik$	2517.875	2514.987	2517.875	2456.920	2453.821	2456.920

rameters. The details iterated filtering method are incorporated into the pomp R-package (King et al., 2016).

5.3 Estimated parameters.

We calibrate the home bias parameter, ε and the random death probability, π, given that cannot be separately identified given the structure of the model. We set $\varepsilon = 0.85$ corresponding to standard value in the literature and follow Gârleanu and Panageas (2015) to set $\pi = 0.02$. We estimate the remaining eleven parameters.

6 Conclusion

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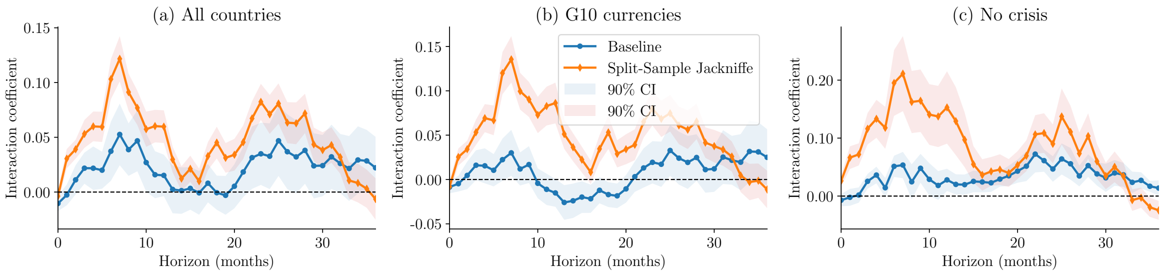
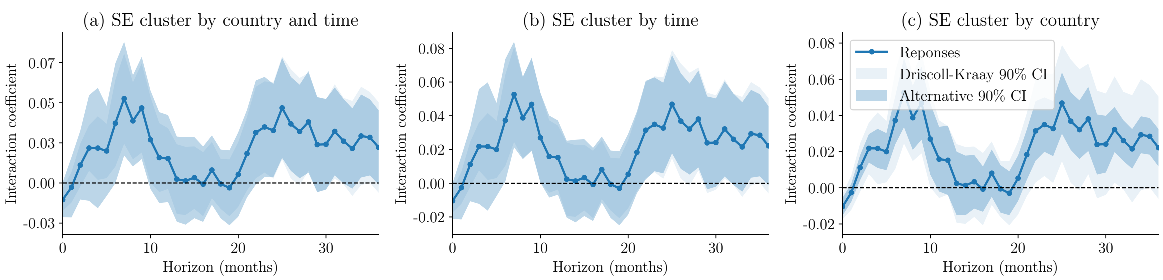
A Additional empirical results

Figure 3: Responses with different estimator



Notes: The figure plots the exchange rate responses, β_h in equation (2.3), to a 1% increase in financial intermediary networth shock by Ottonello and Song (2022) at month t. The responses are driven by cross-country variation in utilization rate. Utilization rate are demeaned and standardized so that units are standard deviations. The x-axes show the horizon h (months). The vector of controls includes 12 lags of forward discounts, gold prices returns, oil prices returns, MSCI returns, 10-year government bonds yields and inflation differentials. All controls are standardized. The sample period is from July 2006 to September 2020. Shaded area are the 90% confidence interval.

Figure 4: Responses with different standard errors



Notes: The figure plots the exchange rate responses, β_h in equation (2.3), to a 1% increase in financial intermediary networth shock by Ottonello and Song (2022) at month t. The responses are driven by cross-country variation in utilization rate. Utilization rate are demeaned and standardized so that units are standard deviations. The x-axes show the horizon h (months). The vector of controls includes 12 lags of forward discounts, gold prices returns, oil prices returns, MSCI returns, 10-year government bonds yields and inflation differentials. All controls are standardized. The sample period is from July 2006 to September 2020. Shaded area are the 90% confidence interval.

B closed-economy model proofs

B.1 Proof of Lemma 1

Proof. The proof of this lemma consists of applying Ito's lemma on the consumption share definition,

$$x_t^i = \frac{\nu^i \int_{-\infty}^{t} \pi e^{-(\rho+\pi)(t-s)} c_{s,s}^i \xi_s^i ds}{Y_t \xi_t^i},$$

using the definition of the stochastic discount factor processes, (3.14) and the endowment processes (3.2). To do so, let $U_t^i = (Y_t \xi_t^i)^{-1}$ and K_t^i is such that $x_t^i = K_t^i U_t^i$. Then by Ito's lemma we have:

$$
\begin{aligned}
\frac{dU_t^i}{U_t^i} &= \left(r_t - \mu^Y + \sigma_Y^2 + \theta_t^i(\theta_t^i - \sigma^Y - \eta^i)\right) dt + \left(\theta_t^i - \sigma^Y\right) dB_t \\
\frac{dK_t^i}{K_t^i} &= \left(\pi \nu^i \frac{c_{t,t}^i}{Y_t x_t^i} - (\rho + \pi)\right) dt \\
\frac{dx_t^i}{x_t^i} &= \frac{dK_t^i}{K_t^i} + \frac{dU_t^i}{U_t^i} \\
&= \left[\left(\pi \nu^i \frac{c_{t,t}^i}{Y_t x_t^i} - (\rho + \pi) + \mu_{U,t}^i\right) dt + \sigma_{U,t}^i dB_t\right] \\
&= \mu_t^{i,x} dt + \sigma_t^{i,x} dB_t
\end{aligned}
\tag{B.1}
$$

□

B.2 Proof of Proposition 1

Proof. **Market price of risk.** The use of consumption share of each type of households as a state variable delivers a clean and transparent proof for asset prices moments. In particular, given the market clearing conditions $\sum_{i \in \{\ell,c,b\}} x_t^i = 1$ we deduce that in equilibrium, by applying the Ito's lemma to this condition yields $\sum_{i \in \{\ell,c,b\}} dx_t^i = 0$ so that the drift and volatility must be zero. Using the restrictions of the diffusion gives:

$$
\sum_{i \in \{\ell,c,b\}} x_t^i \sigma_t^{i,x} = 0 \Rightarrow \theta_t = \sigma^Y - \sum_{i \in \{\ell,c,b\}} x_t^i \eta^i - \phi_t \sum_{i \in \{\ell,c,b\}} x_t^i \alpha^i \tag{B.2}
$$

Interest rate. We next plug this back into the consumption share process to find the diffusion term given by:

$$
\sigma_t^{i,x} = \eta^i - \sum_{i \in \{\ell,c,b\}} x_t^i \eta^i + \phi_t \left(\alpha^i - \sum_{i \in \{\ell,c,b\}} x_t^i \alpha^i\right) \tag{B.3}
$$

$$\begin{cases} \phi_t < \frac{2\Delta}{\sigma^Y} & \Rightarrow \sigma_t^{b,x} < \sigma_t^{c,x} < \sigma_t^{\ell,x} \\ \frac{2\Delta}{\sigma^Y} \leq \phi_t < \frac{2\Delta}{(1-\alpha)\sigma^Y} & \Rightarrow \sigma_t^{c,x} < \sigma_t^{b,x} < \sigma_t^{\ell,x} \\ \phi_t \geq \frac{2\Delta}{(1-\alpha)\sigma^Y} & \Rightarrow \sigma_t^{c,x} < \sigma_t^{\ell,x} < \sigma_t^{b,x} \end{cases} \tag{B.4}$$

We next solve for the risk free interest rate by using the restriction on the drift term:

$$\begin{aligned} &\sum_{i \in \{\ell,c,b\}} x_t^i \mu_t^{i,x} = 0 \\ \Rightarrow & r_t = \rho + \mu^Y - \sigma^Y \theta_t - \phi_t \sum_{i \in \{\ell,c,b\}} x_t^i \alpha^i \theta_t^i + \pi \left(1 - \sum_{i \in \{\ell,c,b\}} \nu^i \frac{c_{t,t}^i}{Y_t}\right) \end{aligned} \tag{B.5}$$

Consumption-wealth ratio. The total wealth of an agent at time t born at s is:

$$\begin{aligned} w_{s,t}^i &= \mathbb{E}_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u^i}{\xi_t^i} c_{s,u}^i du \\ &= c_{s,t}^i \mathbb{E}_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u^i}{\xi_t^i} \frac{c_{s,u}^i}{c_{s,t}^i} du \\ &= \frac{c_{s,t}^i}{\rho + \pi} \quad \text{Using the FOCs (3.18) and integrating} \end{aligned} \tag{B.6}$$

Price-Dividend ratio. The group-specific total wealth will then be:

$$\begin{aligned} w_t^i &= \nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} w_{s,t}^i ds \\ &= \frac{\nu^i}{\rho + \pi} \int_{-\infty}^t \nu e^{-\nu(t-s)} c_{s,t}^i ds \\ &= \frac{Y_t x_t^i}{\rho + \pi} \quad \text{Using the definition } x_t^i, \text{(B.1)} \end{aligned} \tag{B.7}$$

Combing the bond and stock market clearing conditions gives:

$$\sum_{i \in \{\ell,c,b\}} w_t^i = S_t \iff \sum_{i \in \{\ell,c,b\}} \frac{Y_t x_t^i}{\rho + \pi} = S_t \tag{B.8}$$

We first apply Ito's lemma on both sides of this last equality to obtain

$$\sigma_t^S = \sigma^Y \tag{B.9}$$

□

B.3 Proof of Lemma 2

Proof. The fact consumption-wealth ratio is constant (B.6) implies that $\sigma^i_{w,s,t} = \sigma^i_{c,s,t}$. We applying Ito's lemma on the consumption first order condition (3.18):

$$\frac{dc^i_{s,t}}{c^i_{s,t}} = (\cdots)dt - \frac{d\xi^i_t}{\xi^i_t}. \tag{B.10}$$

Matching the diffusion terms of $dc^i_{s,t}$ and the budget constraint gives:

$$\psi^i_{s,t} = \frac{\theta^i_t}{\sigma^S_t} w^i_{s,t} \tag{B.11}$$

□

B.4 Proof of Proposition 2

Proof. **Short-selling fee.** We solve for the short selling fee using the short selling market clearing condition. We rewrite this condition in term of group-specific wealth, (B.7), and the the optimal portfolio of each type of investor in proposition 2 as follow:

$$\sum_i x^i_t \alpha^i \theta^i_t = 0 \tag{B.12}$$

We finally expand the expression above and solve for shorting fee to have

$$\phi_t = \frac{1}{\alpha^2 x^\ell_t + x^b_t} \left(\frac{\Delta}{\sigma^Y} (x^b_t - \alpha x^\ell_t) - \theta_t (x^b_t + \alpha x^\ell_t) \right) \tag{B.13}$$

Short interest. By definition the fraction of outstanding stocks shares held by all short-sellers is given by:

$$
\begin{aligned}
SI_t &= -(1-\nu)\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \psi_{s,t}^{b} ds \\
&= -(1-\nu)\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \frac{\theta_t^b}{\sigma_t^S} w_{s,t}^{b} ds \\
&= -\frac{\theta_t^b}{\sigma_t^S} w_t^b \\
&= -\frac{\theta_t^b}{\sigma_t^S}\frac{Y_t x_t^b}{\rho+\pi}
\end{aligned}
\tag{B.14}
$$

□

B.5 Proof of Proposition 3

Proof. **Simplified consumption share process.** We will start by solving explicitly for the dynamics of consumption shares by substituting out the equilibrium interest rate and market price of risk in equation (3.22) back into equation (3.20) to have:

$$
\frac{dx_t^i}{x_t^i} = \mu_t^{i,x} dt + \sigma_t^{i,x} dB_t \tag{B.15}
$$

where

$$
\mu_t^{i,x} = (\theta_t^i - \sigma^Y)(\theta_t^i - \eta^i - \sigma^Y) + \sigma^Y \alpha^i \phi_t + \pi\delta\left(\frac{\nu^i}{x_t^i} - 1\right) \tag{B.16}
$$

$$
\sigma_t^{i,x} = \theta_t^i - \sigma^Y.
$$

$$
\frac{c_{t,t}^i}{Y_t} = \frac{(\rho+\pi) w_{t,t}}{Y_t} = \delta \tag{B.17}
$$

Closed form expression of shorting fee. In proposition B.4, the short selling fee is written as a function of the market price of risk θ_t. We can further simplify this expression and write it only as function of the consumption share process. To do so, we recognize that the market price of risk and the shorting fee are solution to the system of equations given by the short selling market clearing condition and the expression of θ_t in equation

(3.22). This system is given by:

$$\begin{aligned}\phi_t \sum_i x_t^i (\alpha^i)^2 + \theta_t \sum_i x_t^i \alpha^i =& - \sum_i x_t^i \alpha^i \eta^i \\ \phi_t \sum_i x_t^i \alpha^i + \theta_t =& \sigma^Y - \sum_i x_t^i \eta^i.\end{aligned} \tag{B.18}$$

Solving this system of equation for (ϕ_t, θ_t) yields:

$$\begin{aligned}\phi_t =& \frac{-\sum_i x_t^i \alpha^i \eta^i - \sum_i x_t^i \alpha^i \left(\sigma^Y - \sum_i x_t^i \eta^i\right)}{\sum_i x_t^i (\alpha^i)^2 - (\sum_i x_t^i \alpha^i)^2} = \frac{-\sum_i \left(\frac{x_t^i \alpha^i}{\sum_i x_t^j \alpha^j} - x_t^i\right)\eta^i - \sigma^Y}{\sum_i \left(\frac{x_t^i \alpha^i}{\sum_j x_t^j \alpha^j} - x_t^i\right)\alpha^i} \\ \theta_t =& \frac{\sum_i x_t^i (\alpha^i)^2 \left(\sigma^Y - \sum_i x_t^i \eta^i\right) + \sum_i x_t^i \alpha^i \sum_i x_t^i \alpha^i \eta^i}{\sum_i x_t^i (\alpha^i)^2 - (\sum_i x_t^i \alpha^i)^2}\end{aligned} \tag{B.19}$$

Note that: $\lim_{(x_t^b, x_t^\ell) \to (0,1)} \phi_t = \lim_{(x_t^b, x_t^\ell) \to (1,0)} \phi_t = -\infty < 0$.

Shorting fee process. In this section and without loss of generality we will solve for the normalized $\sigma_t^S \phi_t$ given that the volatility is constant. We can apply the Ito's lemma to the closed form expressions of both the shorting fee and the market price of risk $\phi_t(x^b, x^\ell)$, $\theta_t(x^b, x^\ell)$ using the dynamics processes of consumption shares, x_t^i to find the diffusing term of $d\phi_t$. We have:

$$\begin{aligned}d\phi_t =& \sum_{i \in \{\ell, b\}} \frac{\partial \phi_t}{\partial x_t^i} dx_t^i \\ =& \sum_{i \in \{\ell, b\}} \frac{\partial \phi_t}{\partial x_t^i} x_t^i \mu_t^{i,x} dt + \sum_{i \in \{\ell, b\}} \frac{\partial \phi_t}{\partial x_t^i} x_t^i \sigma_t^{i,x} dB_t \\ =& \mu_t^\phi dt + \sigma_t^\phi dB_t\end{aligned} \tag{B.20}$$

Proceeding in a similar way we can find the drift and diffusion terms of $d\theta_t = \mu_t^\theta dt + \sigma_t^\theta dB_t$. Exploiting the implicit function theorem, we can differentiate the systems of equations (B.18) with respect to x_t^i and obtain the following system of four equations and four unknowns $(\frac{\partial \phi_t}{\partial x_t^i}, \frac{\partial \theta_t}{\partial x_t^i})$:

$$\begin{aligned}
\frac{\partial \phi_t}{\partial x_t^b}(x_t^b + \alpha^2 x_t^\ell) + \frac{\partial \theta_t}{\partial x_t^b}(x_t^b + \alpha x_t^\ell) &= \frac{\Delta}{\sigma^Y} - \phi_t - \theta_t \\
\frac{\partial \phi_t}{\partial x_t^b}(x_t^b + \alpha x_t^\ell) + \frac{\partial \theta_t}{\partial x_t^b} &= \frac{2\Delta}{\sigma^Y} - \phi_t \\
\frac{\partial \phi_t}{\partial x_t^\ell}(x_t^b + \alpha^2 x_t^\ell) + \frac{\partial \theta_t}{\partial x_t^\ell}(x_t^b + \alpha x_t^\ell) &= -\alpha(\frac{\Delta}{\sigma^Y} + \alpha\phi_t + \theta_t) \\
\frac{\partial \phi_t}{\partial x_t^\ell}(x_t^b + \alpha x_t^\ell) + \frac{\partial \theta_t}{\partial x_t^\ell} &= -\alpha\phi_t
\end{aligned} \tag{B.21}$$

The next steps consist of substituting these partial derivatives back into the diffusion term. We have:

$$\begin{aligned}
\sigma_t^\theta =& \sum_{i\in\{\ell,b\}} x_t^i \sigma_t^{i,x} \frac{\partial \theta_t}{\partial x_t^i} \\
=& x_t^b \sigma_t^{b,x}\left(\frac{2\Delta}{\sigma^Y} - \phi_t - \frac{\partial \phi_t}{\partial x_t^b}(x_t^b + \alpha x_t^\ell)\right) + x_t^\ell \sigma_t^{\ell,x}\left(-\alpha\phi_t - \frac{\partial \phi_t}{\partial x_t^\ell}(x_t^b + \alpha x_t^\ell)\right) \\
=& x_t^b \sigma_t^{b,x}\left(\frac{2\Delta}{\sigma^Y} - \phi_t\right) + x_t^\ell \sigma_t^{\ell,x}\left(-\alpha\phi_t\right) - (x_t^b + \alpha x_t^\ell)\left(x_t^b \sigma_t^{b,x}\frac{\partial \phi_t}{\partial x_t^b} + x_t^\ell \sigma_t^{\ell,x}\frac{\partial \phi_t}{\partial x_t^\ell}\right) \\
=& x_t^b \sigma_t^{b,x}\left(\frac{2\Delta}{\sigma^Y} - \phi_t\right) + x_t^\ell \sigma_t^{\ell,x}\left(-\alpha\phi_t\right) - (x_t^b + \alpha x_t^\ell)\sigma_t^\phi \\
=& x_t^b \sigma_t^{b,x}\left(\sigma_t^{c,x} - \sigma_t^{b,x}\right) + x_t^\ell \sigma_t^{\ell,x}\left(\sigma_t^{c,x} - \sigma_t^{\ell,x}\right) - (x_t^b + \alpha x_t^\ell)\sigma_t^\phi \\
=& \sigma_t^{c,x}\left(x_t^b \sigma_t^{b,x} + x_t^\ell \sigma_t^{\ell,x}\right) - \left(x_t^b(\sigma_t^{b,x})^2 + x_t^\ell(\sigma_t^{\ell,x})^2\right) - (x_t^b + \alpha x_t^\ell)\sigma_t^\phi
\end{aligned} \tag{B.22}$$

where we have substituted the second and fourth equation of (B.21) to get the second line, used the expression of $\sigma_t^{i,x}$ to get the fifth line. We finally exploit the identity $\sum_{i\in\{c,\ell,b\}} x_t^i \sigma_t^{i,x} = 0$ to have:

$$\sigma_t^\theta = -\left(x_t^c(\sigma_t^{c,x})^2 + x_t^b(\sigma_t^{b,x})^2 + x_t^\ell(\sigma_t^{\ell,x})^2\right) - (x_t^b + \alpha x_t^\ell)\sigma_t^\phi \tag{B.23}$$

This last equation expresses σ_t^θ as a function of σ_t^ϕ. We now use the same techniques by exploiting the remaining two equations of (B.21).

$$\begin{aligned}
\sigma_t^\theta =& \sum_{i\in\{\ell,b\}} x_t^i \sigma_t^{i,x} \frac{\partial \theta_t}{\partial x_t^i} \\
=& \frac{x_t^b \sigma_t^{b,x}}{x_t^b+\alpha x_t^\ell}\left(\frac{\Delta}{\sigma^Y}-\phi_t-\theta_t-\frac{\partial \phi_t}{\partial x_t^b}(x_t^b+\alpha x_t^\ell)\right)+\frac{x_t^\ell \sigma_t^{\ell,x}}{x_t^b+\alpha x_t^\ell}\left(-\frac{\alpha\Delta}{\sigma^Y}-\alpha^2\phi_t-\alpha\theta_t-\frac{\partial \phi_t}{\partial x_t^\ell}(x_t^b+\alpha x_t^\ell)\right) \\
=& -\frac{x_t^b \sigma_t^{b,x}}{x_t^b+\alpha x_t^\ell}\left(\sigma^Y+\sigma_t^{b,x}+\frac{\partial \phi_t}{\partial x_t^b}(x_t^b+\alpha x_t^\ell)\right)-\frac{x_t^\ell \sigma_t^{\ell,x}}{x_t^b+\alpha x_t^\ell}\left(\alpha\sigma^Y+\alpha\sigma_t^{\ell,x}+\frac{\partial \phi_t}{\partial x_t^\ell}(x_t^b+\alpha x_t^\ell)\right) \\
=& -\frac{\sigma^Y}{x_t^b+\alpha x_t^\ell}\left(x_t^b\sigma_t^{b,x}+\alpha x_t^\ell \sigma_t^{\ell,x}\right)-\frac{1}{x_t^b+\alpha x_t^\ell}\left(x_t^b(\sigma_t^{b,x})^2+\alpha x_t^\ell(\sigma_t^{\ell,x})^2\right)-\frac{x_t^b+\alpha^2 x_t^\ell}{x_t^b+\alpha x_t^\ell}\sigma_t^\phi
\end{aligned} \tag{B.24}$$

where we have substituted the first and third equation of (B.21) to get the second line, used the expression of $\sigma_t^{i,x}$ together with the expression of θ_t in (3.22) to get the third line. This last equation gives another expression of σ_t^θ as a function of σ_t^ϕ.

$$\sigma_t^\theta(x_t^b+\alpha x_t^\ell) = -\sigma^Y\left(x_t^b\sigma_t^{b,x}+\alpha x_t^\ell\sigma_t^{\ell,x}\right)-\left(x_t^b(\sigma_t^{b,x})^2+\alpha x_t^\ell(\sigma_t^{\ell,x})^2\right)-(x_t^b+\alpha^2 x_t^\ell)\sigma_t^\phi \tag{B.25}$$

Solving the system of equation composed of (B.23) and (B.25) for σ_t^θ and σ_t^ϕ gives:

$$
\begin{aligned}
\sigma_t^\theta &= \frac{(x_t^b + \alpha x_t^\ell)\left(\left(x_t^b(\sigma_t^{b,x})^2 + \alpha x_t^\ell(\sigma_t^{\ell,x})^2\right) + \sigma^Y\left(x_t^b\sigma_t^{b,x} + \alpha x_t^\ell\sigma_t^{\ell,x}\right)\right) - (x_t^b + \alpha^2 x_t^\ell)\sum_i x_t^i(\sigma_t^{i,x})^2}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2} \\
&= \frac{(\sum_i x_t^i\alpha^i)\left(\sum_i x_t^i\alpha^i(\sigma_t^{i,x})^2 + \sigma^Y\sum_i x_t^i\alpha^i\sigma_t^{i,x}\right) - (\sum_i x_t^i(\alpha^i)^2)\sum_i x_t^i(\sigma_t^{i,x})^2}{\sum_i x_t^i(\alpha^i)^2 - (\sum_i x_t^i\alpha^i)^2} \\
&= \frac{\sum_i x_t^i\alpha^i(\sigma_t^{i,x})^2 + \sigma^Y\sum_i x_t^i\alpha^i\sigma_t^{i,x} - \left(\sum_i \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\alpha^i\right)\sum_i x_t^i(\sigma_t^{i,x})^2}{\sum_i\left(\frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j} - x_t^i\right)\alpha^i} \\
\sigma_t^\phi &= \frac{(x_t^b + \alpha x_t^\ell)\sum_i x_t^i(\sigma_t^{i,x})^2 - \left(x_t^b(\sigma_t^{b,x})^2 + \alpha x_t^\ell(\sigma_t^{\ell,x})^2\right) - \sigma^Y\left(x_t^b\sigma_t^{b,x} + \alpha x_t^\ell\sigma_t^{\ell,x}\right)}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2} \\
&= \frac{(\sum_i x_t^i\alpha^i)\sum_i x_t^i(\sigma_t^{i,x})^2 - \sum_i x_t^i\alpha^i(\sigma_t^{i,x})^2 - \sigma^Y\sum_i x_t^i\alpha^i\sigma_t^{i,x}}{\sum_i x_t^i(\alpha^i)^2 - (\sum_i x_t^i\alpha^i)^2} \\
&= \frac{\sum_i\left(x_t^i - \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\right)(\sigma_t^{i,x})^2 - \sigma^Y\sum_i \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\sigma_t^{i,x}}{\sum_i\left(\frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j} - x_t^i\right)\alpha^i} \\
&= \frac{\sum_i\left(x_t^i - \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\right)(\sigma_t^{i,x})^2 + \sigma^Y\sum_i\left(x_t^i - \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\right)\sigma_t^{i,x}}{\sum_i\left(\frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j} - x_t^i\right)\alpha^i} \quad \text{Since } \sum_i x_t^i\sigma_t^{i,x} = 0 \\
&= \frac{\sum_i\left(x_t^i - \frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j}\right)\sigma_t^{i,x}(\sigma^Y + \sigma_t^{i,x})}{\sum_i\left(\frac{x_t^i\alpha^i}{\sum_j x_t^j\alpha^j} - x_t^i\right)\alpha^i}
\end{aligned}
\tag{B.26}
$$

After some tedious but straightforward algebra we can also show that:

$$\begin{aligned}
\frac{\partial \phi_t}{\partial x_t^b} =& \frac{(\frac{\Delta}{\sigma_Y} - \phi_t - \theta_t) - (x_t^b + \alpha x_t^\ell)(2\frac{\Delta}{\sigma_Y} - \phi_t)}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2} \\
\frac{\partial \theta_t}{\partial x_t^b} =& \frac{(x_t^b + \alpha^2 x_t^\ell)(2\frac{\Delta}{\sigma_Y} - \phi_t) - (\frac{\Delta}{\sigma_Y} - \phi_t - \theta_t)(x_t^b + \alpha x_t^\ell)}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2} \\
\frac{\partial \phi_t}{\partial x_t^\ell} =& \alpha \frac{\phi_t(x_t^b + \alpha x_t^\ell) - (\frac{\Delta}{\sigma_Y} + \alpha\phi_t + \theta_t)}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2} \\
\frac{\partial \theta_t}{\partial x_t^\ell} =& \alpha \frac{(\frac{\Delta}{\sigma_Y} + \alpha\phi_t + \theta_t)(x_t^b + \alpha x_t^\ell) - \phi_t(x_t^b + \alpha^2 x_t^\ell)}{(x_t^b + \alpha^2 x_t^\ell) - (x_t^b + \alpha x_t^\ell)^2}
\end{aligned} \tag{B.27}$$

□

C Open economy model proofs

C.1 Proof of Lemma 3

Proof. We start by substituting the expression static consumption allocation (4.3) together with the first order conditions (4.22) back into the definition of $x^i_{mn,t}$ and simplify to obtain:

$$x^i_{mn,t} = \frac{\nu^i \int_{-\infty}^t \pi e^{-(\rho+\pi)(t-s)} \varepsilon_{mn} c^i_{m,ss} P_{m,s} \xi^i_{m,s} ds}{p_{n,t} Y_{n,t} \xi^i_{m,t}} \tag{C.1}$$

We finally apply Ito's lemma to this last expression using exactly the same procedure as in the closed-economy model. Let $U^i_{mn,t} = p^{-1}_{n,t} Y^{-1}_{n,t} (\xi^i_{m,t})^{-1}$ and $K^i_{mn,t}$ is such that $x^i_{mn,t} =$

$K^i_{mn,t}U^i_{mn,t}$. By Ito's lemma we have:

$$\begin{aligned}
\frac{dU^i_{mn,t}}{U^i_{mn,t}} &= \Big(r_t - \mu^Y_n - \mu^p_{n,t} + (\sigma^Y_n)'\sigma^Y_n + (\theta^i_{m,t})'(\sigma^p_{n,t} - \eta^i_m) + (\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n)'(\theta^i_{m,t} - \sigma^p_{n,t})\Big)\, dt \\
&\quad + \Big(\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n\Big)' \mathbf{dB_t} \\
\frac{dK^i_{mn,t}}{K^i_{mn,t}} &= \left(\pi\nu^i\varepsilon_{mn}\frac{P_{m,t}c^i_{m,tt}}{p_{n,t}Y_{n,t}x^i_{mn,t}} - (\rho+\pi)\right) dt \\
\frac{dx^i_{mn,t}}{x^i_{mn,t}} &= \frac{dK^i_{mn,t}}{K^i_{mn,t}} + \frac{dU^i_{mn,t}}{U^i_{mn,t}} \\
&= \mu^{i,x}_{mn,t}dt + (\sigma^{i,x}_{mn,t})'\mathbf{dB_t} \\
\mu^{i,x}_{mn,t} &= r_t - \rho - \pi\left(1 - \nu^i\frac{\varepsilon_{mn}P_{m,t}c^i_{m,tt}}{p_{n,t}Y_{n,t}x^i_{mn,t}}\right) - \mu^Y_n - \mu^p_{n,t} \\
&\quad + (\sigma^Y_n)'\sigma^Y_n + (\theta^i_{m,t})'(\sigma^p_{n,t} - \eta^i_m) + (\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n)'(\theta^i_{m,t} - \sigma^p_{n,t}), \\
\sigma^{i,x}_{mn,t} &= \theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n.
\end{aligned}$$

□

C.2 Proof of Proposition 4

Proof. **Market price of risk.** Applying to the market clearing condition in each country, $\sum_m \sum_i x^i_{mn,t} = 1$ gives $\sum_m \sum_i dx^i_{mn,t} = 0$. We next match both the drift and the diffusion of the this last equality and obtain system of two equations given by:

$$\begin{aligned}
\sum_m \sum_i x^i_{mn,t}\sigma^{i,x}_{mn,t} &= 0 \\
\sum_m \sum_i x^i_{mn,t}\mu^{i,x}_{mn,t} &= 0
\end{aligned} \tag{C.2}$$

Using the results of lemma 3 we solve the first equation of this system for θ_t to have:

$$\theta_t = \sigma^Y_n + \sigma^p_{n,t} - \sum_m \sum_i x^i_{mn,t}\eta^i_m - \sum_m \sum_i x^i_{mn,t}\alpha^i\phi_t, \quad \forall n \tag{C.3}$$

Interest rate. We next move to solve the second equation of the system for r_t using a combination of definition of the individual-specific SDF and the short-selling market clearing conditions. The different steps are:

$$
\begin{aligned}
&\sum_m \sum_i x^i_{mn,t} \mu^{i,x}_{mn,t} = 0 \\
\iff & r_t = \rho + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right) + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n \\
& - \sum_m \sum_i x^i_{mn,t} \left[(\theta^i_{m,t})'(\sigma^p_{n,t} - \eta^i_m) + (\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n)'(\theta^i_{m,t} - \sigma^p_{n,t})\right] \quad \text{Using definition of } \mu^{i,x}_{mn,t} \text{ in 4.24} \\
\iff & r_t = \rho + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right) + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n \\
& - \sum_m \sum_i x^i_{mn,t} \left[(\theta^i_{m,t})'(\sigma^p_{n,t} - \eta^i_m) + (\theta^i_{m,t} - \sigma^p_{n,t} - \sigma^Y_n)' \theta^i_{m,t}\right] \quad \text{Using eq. C.2} \\
\iff & r_t = \rho + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right) + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n \\
& - \sum_m \sum_i x^i_{mn,t} \left[(\theta^i_{m,t})'(\theta^i_{m,t} - \eta^i_m - \sigma^Y_n)\right] \quad \text{By rearranging.} \\
\iff & r_t = \rho + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right) + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n \\
& - \sum_m \sum_i x^i_{mn,t} (\theta^i_{m,t})'(\theta_t - \sigma^Y_n) - \sum_m \sum_i x^i_{mn,t} \alpha^i (\theta^i_{m,t})' \phi_t \quad \text{Using definition of } \theta^i_{m,t} \text{ in 4.19} \\
\iff & r_t = \rho + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right) + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n \\
& - (\sigma^p_{n,t} + \sigma^Y_n)'(\theta_t - \sigma^Y_n) - \sum_m \sum_i x^i_{mn,t} \alpha^i (\theta^i_{m,t})' \phi_t \quad \text{By solution to } \sum_m \sum_i x^i_{mn,t} \sigma^{i,x}_{mn,t} = 0
\end{aligned}
\tag{C.4}
$$

In conclusion:

$$
\begin{aligned}
r_t =& \rho + \mu^Y_n + \mu^p_{n,t} - \theta_t'(\sigma^p_{n,t} + \sigma^Y_n) + (\sigma^p_{n,t})' \sigma^Y_n \\
& - \sum_m \sum_i x^i_{mn,t} \alpha^i (\theta^i_{m,t})' \phi_t + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right), \quad \forall n \\
r_t =& \rho + \mu^Y_n + \mu^p_{n,t} - (\sigma^Y_n)' \sigma^Y_n - (\sigma^p_{n,t} + \sigma^Y_n)'(\theta_t - \sigma^Y_n) \\
& - \sum_m \sum_i x^i_{mn,t} \alpha^i (\theta^i_{m,t})' \phi_t + \pi\left(1 - \sum_m \sum_i \nu^i \varepsilon_{mn} \frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}}\right), \quad \forall n
\end{aligned}
\tag{C.5}
$$

Simplified consumption shares. We substitute these expressions of r_t and θ_t back into

$\mu^{i,x}_{mn,t}$ and $\mu^{i,x}_{mn,t}$:

$$\begin{aligned}
\sigma^{i,x}_{mn,t} =& \eta^i_m - \sum_k \sum_j x^j_{kn,t} \eta^j_k + \left(\alpha^i - \sum_k \sum_j x^j_{kn,t} \alpha^j \right) \phi_t \\
\mu^{i,x}_{mn,t} =& \pi \delta \frac{\sum_k p_{k,t} Y_{k,t}}{p_{n,t} Y_{n,t}} \left(\frac{\nu^i \varepsilon_{mn}}{x^i_{mn,t}} - 1 \right) + (\sigma^{i,x}_{mn,t})' (\theta_t - \sigma^p_{n,t} - \sigma^y_n) \\
& + \left(\alpha^i (\sigma^{i,x}_{mn,t} + \sigma^p_{n,t} + \sigma^Y_n) - \sum_k \sum_j x^j_{kn,t} \alpha^j (\sigma^{j,x}_{kn,t} + \sigma^p_{n,t} + \sigma^y_n) \right)' \phi_t \\
\frac{P_{m,t} c^i_{m,tt}}{p_{n,t} Y_{n,t}} =& \delta \frac{\sum_k p_{k,t} Y_{k,t}}{p_{n,t} Y_{n,t}}
\end{aligned} \tag{C.6}$$

Term of trade. We leverage the results of proposition related to the market price of risk, θ_t and the interest rate, r_t to find the equilibrium real exchange rate. Given that these expressions holds for all n, we have that:

$$\begin{aligned}
\theta_t =& \sigma^Y_F + \sigma^p_{F,t} - \sum_m \sum_i x^i_{mF,t} \eta^i_m - \sum_m \sum_i x^i_{mF,t} \alpha^i \phi_t \\
\theta_t =& \sigma^Y_H + \sigma^p_{H,t} - \sum_m \sum_i x^i_{mH,t} \eta^i_m - \sum_m \sum_i x^i_{mH,t} \alpha^i \phi_t
\end{aligned} \tag{C.7}$$

Taking the difference of these two expressions gives:

$$\sigma^p_{F,t} - \sigma^p_{H,t} = \sigma^Y_H - \sigma^Y_F - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t}) \eta^i_m - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t}) \alpha^i \phi_t \tag{C.8}$$

In order to identify the drift and diffusion term of the real exchange rate, we apply Ito's lemma onto the definition of $q_t = \frac{p_{F,t}}{p_{H,t}}$:

$$\frac{dq_t}{q_t} = \left(\mu^p_{F,t} - \mu^p_{H,t} - (\sigma^p_{H,t})' (\sigma^p_{F,t} - \sigma^p_{H,t}) \right) + (\sigma^p_{F,t} - \sigma^p_{H,t})' \mathbf{dB_t} \tag{C.9}$$

Matching the diffusion terms of this last expression with that of (C.8) yields:

$$\sigma^q_t = \sigma^Y_H - \sigma^Y_F - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t}) \eta^i_m - \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t}) \alpha^i \phi_t \tag{C.10}$$

To find the drift term of the real exchange rate, we evaluate r_t, in equation (C.5), respectively at $n = H$ and $n = F$ and take the difference of the resulting equations, rearrange them to have:

$$
\begin{aligned}
0 =&\mu_F^Y - \mu_H^Y + \mu_{F,t}^p - \mu_{H,t}^p - (\theta_t)'(\sigma_{F,t}^p - \sigma_{H,t}^p + \sigma_F^Y - \sigma_H^Y) + (\sigma_{F,t}^p)'\sigma_F^Y - (\sigma_{H,t}^p)'\sigma_H^Y \\
&- \pi \sum_m \sum_i \nu^i P_{m,t} c_{m,tt}^i \left(\frac{\varepsilon_{mF}}{p_{F,t} Y_{F,t}} - \frac{\varepsilon_{mH}}{p_{H,t} Y_{H,t}} \right) \\
&- \sum_m \sum_i (x_{mF,t}^i - x_{mH,t}^i) \alpha^i (\theta_{m,t}^i)' \phi_t
\end{aligned}
\tag{C.11}
$$

An examination of the above equation shows that it depends on $\sigma_{n,t}^p$ which can be eliminated by exploiting the price normalization of the global numeraire $p_{Ht} = q_t^{\varphi-1}$, $p_{Ft} = q_t^{\varphi}$. Applying Ito's lemma yields:

$$
\begin{aligned}
\frac{dp_{Ht}}{p_{Ht}} =&\left((\varphi - 1)\mu_t^q + \frac{1}{2}(\varphi - 1)(\varphi - 2)(\sigma_t^q)'\sigma_t^q \right) dt + (\varphi - 1)(\sigma_t^q)' \mathbf{dB_t} \\
\mu_{H,t}^p =&(\varphi - 1)\mu_t^q + \frac{1}{2}(\varphi - 1)(\varphi - 2)(\sigma_t^q)'\sigma_t^q \\
\sigma_{H,t}^p =&(\varphi - 1)\sigma_t^q \\
\frac{dp_{Ft}}{p_{Ft}} =&\left(\varphi\mu_t^q + \frac{1}{2}\varphi(\varphi - 1)(\sigma_t^q)'\sigma_t^q \right) dt + \varphi(\sigma_t^q)' \mathbf{dB_t} \\
\mu_{F,t}^p =&\varphi\mu_t^q + \frac{1}{2}\varphi(\varphi - 1)(\sigma_t^q)'\sigma_t^q \\
\sigma_{F,t}^p =&\varphi\sigma_t^q
\end{aligned}
\tag{C.12}
$$

Using the above expressions of the drifts and diffusion terms of $dp_{n,t}$ together with the initial wealth of the newborn we finally have:

$$
\begin{aligned}
\mu_t^q = \mu_H^Y - \mu_F^Y + (1 - \varphi)(\sigma_t^q)'\sigma_t^q + (\theta_t)'(\sigma_t^q + \sigma_F^Y - \sigma_H^Y) - (\sigma_t^q)'(\varphi\sigma_F^Y + (1 - \varphi)\sigma_H^Y) \\
+ \pi\delta \frac{y_t^2 - (q_t(1 - y_t))^2}{q_t y_t (1 - y_t)} \\
- \sum_m \sum_i (x_{mH,t}^i - x_{mF,t}^i) \alpha^i (\theta_{m,t}^i)' \phi_t
\end{aligned}
\tag{C.13}
$$

Simplified market price of risk. To simplify the market price of risk vector, we start from the equation characterizing the numeraire price index, $(p_{H,t})^{\varphi}(p_{F,t})^{1-\varphi} = 1$, apply the Ito's lemma to both sides to find a restriction on the diffusion terms given by $\varphi\sigma_{H,t}^p + (1 - \varphi)\sigma_{F,t}^p = 0$. Hence, we multiply both sides of the first equation in (C.7) by $1 - \varphi$ and the second by φ and add them side by side to have:

$$\begin{aligned}\theta_t = \varphi\sigma_H^Y + (1-\varphi)\sigma_F^Y - \sum_m \sum_i \left(\varphi x^i_{mH,t} + (1-\varphi)x^i_{mF,t}\right)\eta^i_m \\ - \sum_m \sum_i \left(\varphi x^i_{mH,t} + (1-\varphi)x^i_{mF,t}\right)\alpha^i \phi_t\end{aligned} \tag{C.14}$$

Consumption-wealth ratio. Under the objective measure, the total wealth of an agent at time t born at s and residing in country n is:

$$\begin{aligned}w^i_{n,st} &= \mathbb{E}_t \int_t^\infty e^{-\pi(u-t)} \frac{P_{n,u}\xi_u}{\xi_t} c^i_{n,su} du \\ &= P_{n,t}c^i_{n,st}\mathbb{E}_t \int_t^\infty e^{-\pi(u-t)} \frac{P_{n,u}}{P_{n,t}} \frac{\xi_u}{\xi_t} \frac{c^i_{n,su}}{c^i_{n,st}} du \\ &= \frac{P_{n,t}c^i_{n,st}}{\rho+\pi} \quad \text{Using the FOCs (4.22), and integrating}\end{aligned} \tag{C.15}$$

The next step consists of finding the group specific wealth share. We start by swapping indexes and substituting out the static consumption allocation budget constraint to have:

$$\begin{aligned}w^i_{m,st} &= \frac{1}{\rho+\pi}\sum_n p_{n,t}c^i_{mn,st} \\ \nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} w^i_{m,st} ds &= \frac{\nu^i}{\rho+\pi}\sum_n \int_{-\infty}^t \pi e^{-\pi(t-s)} p_{n,t}c^i_{mn,st} ds, \quad \text{By aggregating} \\ w^i_{m,t} &= \frac{1}{\rho+\pi}\sum_n x^i_{mn,t} p_{n,t} Y_{n,t}, \quad \text{By definition of } x^i_{mn,t} \\ \frac{\rho+\pi}{q_t^{\varphi-1}\sum_n Y_{n,t}} w^i_{m,t} &= y_t x^i_{mH,t} + (1-y_t) q_t x^i_{mF,t}\end{aligned} \tag{C.16}$$

This last equation shows that the wealth of each investor in our log economy is entirely determined by consumption shares, $x^i_{mn,t}$, relative output supply y_t and the term of trade q_t.

□

C.3 Proof of Lemma 4

By virtue of the constant consumption-wealth ratio we have:

$$w^i_{n,st} = \frac{P_{n,t}c^i_{n,st}}{\rho+\pi} \Rightarrow \frac{dw^i_{n,st}}{w^i_{n,st}} = \frac{d(P_{n,t}c^i_{n,st})}{P_{n,t}c^i_{n,st}} \Rightarrow \frac{(\psi^i_{n,st})'\sigma^S_t}{w^i_{n,st}} = (\sigma^P_{n,t} + \sigma^{i,c}_{n,st})' \tag{C.17}$$

where the last implication arises from matching the diffusion term of the dynamic budget constraint in the LHS and applying the g Ito's lemma to the RHS. Applying Ito's lemma to the consumption first order conditions (4.22) and matching diffusion terms gives:

$$\sigma^P_{n,t} + \sigma^{i,c}_{n,st} = \theta^i_{n,t}. \tag{C.18}$$

We finally replace this last expression into (C.17) and rearrange to have:

$$\psi^i_{n,st} = ((\sigma^S_t)')^{-1}\theta^i_{n,t}w^i_{n,st}. \tag{C.19}$$

C.4 Proof of Proposition 6

Proof. **Short-selling fee.** Exploiting the results of lemma 4 allows to rewrite the two short-selling market clearing conditions compactly in matrix form as:

$$((\sigma^S_t)')^{-1}\sum_m\sum_i w^i_{m,t}\alpha^i\theta^i_{m,t} = 0 \Rightarrow \sum_m\sum_i w^i_{m,t}\alpha^i\theta^i_{m,t} = 0 \tag{C.20}$$

where the first row of this equation corresponds to the domestic short-selling market clearing condition and the second its foreign counterpart.[5] The second equation above holds when the volatility matrix, σ^S_t, is invertible.

We rewrite this system using the definition of $\theta^i_{m,t}$ in equation (4.19) as:

$$\theta_{n,t}\sum_m\sum_i w^i_{m,t}\alpha^i + \sum_m\sum_i w^i_{m,t}\alpha^i\eta^i_{mn} + \phi_{n,t}\sum_m\sum_i w^i_{m,t}(\alpha^i)^2 = 0, \tag{C.21}$$

[5] If the two goods and stocks markets clear, then the bond market automatically clears by Walras law. This bond market clearing condition is explicitly given by $\sum_n p_{n,t}Y_{n,t} = (\rho+\pi)\sum_n S_{n,t}$.

for any country n. Solving for $\phi_{n,t}$ yields:

$$\begin{aligned}\phi_{n,t} &= -\frac{\theta_{n,t}\sum_m\sum_i w^i_{m,t}\alpha^i + \sum_m\sum_i w^i_{m,t}\alpha^i\eta^i_{mn}}{\sum_m\sum_i w^i_{m,t}(\alpha^i)^2} \\ &= -\frac{\theta_{n,t} + \sum_m\sum_i \frac{w^i_{m,t}\alpha^i}{\sum_k\sum_j w^j_{k,t}\alpha^j}\eta^i_{mn}}{\sum_m\sum_i \frac{w^i_{m,t}\alpha^i}{\sum_k\sum_j w^j_{k,t}\alpha^j}\alpha^i}\end{aligned} \tag{C.22}$$

We next exploit the group-specific wealth in (C.16) to write the weights as function of consumption shares given by:

$$\frac{w^i_{m,t}\alpha^i}{\sum_k\sum_j w^j_{k,t}\alpha^j} = \frac{y_t x^i_{mH,t}\alpha^i + (1-y_t)q_t x^i_{mF,t}\alpha^i}{y_t\sum_k\sum_j x^j_{kH,t}\alpha^j + (1-y_t)q_t\sum_j x^j_{kF,t}\alpha^j} \tag{C.23}$$

Short interest. By definition, the vector fraction of outstanding stocks shares held by all short-sellers is given by:

$$\begin{aligned} SI_t &= -\sum_m \psi^b_{m,t} = -\sum_m((\sigma^S_t)')^{-1}\theta^b_{m,t}w^b_{m,t} \\ \Rightarrow \frac{\rho+\pi}{\sum_n Y_{n,t}}(\sigma^S_t)'SI_t &= -\sum_m \frac{y_t x^b_{mH,t} + (1-y_t)q_t x^b_{mF,t}}{q_t^{1-\varphi}}\theta^b_{m,t}\end{aligned} \tag{C.24}$$

□

C.5 Planning problem

In this section, we solve for the stock price volatility using a planner's problem.

Planner's objective. We assume that the world planner maximizes a utilitarian welfare function with constant weights ω^i_n on lifetime utility of type-i investor in country n and a discount rate ρ_p for each generation within a country. As such, the planner welfare trade-off weighted lifetime utility across countries, types and generation. The social welfare at date $t=0$, following Calvo and Obstfeld (1988), is thus given by:

$$V_0 = \mathbb{E}_0\sum_n\sum_i \nu^i\omega^i_n\int_0^\infty\left[\int_{-\infty}^t \pi e^{-\rho_p s}e^{-(\rho+\pi)(t-s)}\log(c^i_{n,st})ds\right]dt. \tag{C.25}$$

The term in square brackets represent aggregate welfare across all generation of type i in country n at time t. Replacing the term $t-s$ by age denoted by a, V_0 becomes:

$$V_0 = \mathbb{E}_0 \sum_n \sum_i \nu^i \omega_n^i \int_0^\infty e^{-\rho_p t} \left[\int_0^\infty \pi e^{-\pi a} e^{-(\rho-\rho_p)a} \log(c_{n,at}^i) da \right] dt. \tag{C.26}$$

Planner's optimization. Optimizing directly V_0 poses some challenges as it involves double integration. To overcome this issue, Calvo and Obstfeld (1988) proposes a two-steps procedure. The first step solves a static allocation of a given level of consumption across generation. In our case, this problem amounts to solving:

$$\max_{c_{n,at}^i} \nu^i \mathbb{E}_0 \int_0^\infty \pi e^{-\pi a} e^{-(\rho-\rho_p)a} \log(c_{n,at}^i) da \tag{C.27}$$

$$\text{s.t. } \nu^i \int_0^\infty \pi e^{-\pi a} c_{n,at}^i da = c_{n,t}^i \tag{C.28}$$

We consider the case where $\rho = \rho_p$ so that the social planner puts equal weights on all generations and do not have an incentive to change the consumption planned for the unborn after their birth. The optimal allocation therefore corresponds to an equal distribution of consumption across all generations irrespective of age at time t. Equipped with these results, we can now solve the dynamic allocation problem while abstracting from intergenerational considerations. The second step problem is formally given by:

$$\max_{c_{nm,t}^i} \mathbb{E}_0 \sum_n \sum_i \omega_n^i \int_0^\infty e^{-\rho t} \log(c_{n,t}^i) dt \tag{C.29}$$

$$\text{s.t. } \sum_m \sum_i c_{mn,t}^i = Y_{n,t}, \quad \forall n \in \{H, F\}; \tag{C.30}$$

Letting $\gamma_{n,t}$ the Lagrange multiplier on the resource allocation constraint of good n and taking the first order conditions leads to:

$$\begin{aligned} [c_{nm,t}^i] &: \frac{\varepsilon_{nm} \omega_n^i e^{-\rho t}}{c_{nm,t}^i} = \gamma_{m,t} \\ [\gamma_{n,t}] &: \sum_m \sum_i c_{mn,t}^i = Y_{n,t} \end{aligned} \tag{C.31}$$

Solving for the consumption allocation gives:

$$c_{nm,t}^{i} = \frac{\omega_n^i \varepsilon_{nm}}{\sum_k \sum_i \omega_k^i \varepsilon_{km}} Y_{m,t} \tag{C.32}$$

Stock price. We replace the expression of optimal consumption allocation (C.32) into the first order conditions (C.31) and use the fact $\gamma_{H,t} = p_{H,t}\xi_t$ and $\gamma_{F,t} = p_{F,t}\xi_t$ to have:

$$\xi_t = e^{-\rho t} \frac{\sum_k \sum_i \omega_k^i \varepsilon_{kH}}{p_{H,t} Y_{H,t}} = e^{-\rho t} \frac{\sum_k \sum_i \omega_k^i \varepsilon_{kF}}{p_{F,t} Y_{F,t}}. \tag{C.33}$$

We normalize the arithmetic average of the price of the global numeraire to one (Pavlova and Rigobon, 2007). That is, we assume that $\varphi p_{H,t} + (1-\varphi) p_{F,t} = 1$. Using this normalization and the definition of q_t we have:

$$\begin{aligned} p_{H,t} &= \frac{1}{\varphi + (1-\varphi) q_t} \\ p_{F,t} &= \frac{q_t}{\varphi + (1-\varphi) q_t} \end{aligned} \tag{C.34}$$

The stock price can then be expressed as:

$$\begin{aligned} S_{H,t} =& p_{H,t} \mathbb{E}_t \int_t^\infty \frac{\xi_u p_{H,u} Y_{H,u}}{\xi_t} du \\ =& p_{H,t} \mathbb{E}_t \int_t^\infty e^{-\rho u} \frac{\sum_k \sum_i \omega_k^i \varepsilon_{kH}}{\xi_t} du \quad \text{By equation (C.33)} \\ =& p_{H,t} e^{-\rho t} \frac{\sum_k \sum_i \omega_k^i \varepsilon_{kH}}{\rho \xi_t} \\ =& \frac{Y_{H,t}}{\rho(\varphi + (1-\varphi) q_t)} \quad \text{By equation (C.34)} \quad \text{and (C.33)} \\ =& \frac{1}{\rho(\varphi + (1-\varphi) q_t)} Y_{H,t} \end{aligned} \tag{C.35}$$

We apply the same procedure to have:

$$S_{F,t} = \frac{q_t}{\rho(\varphi + (1-\varphi) q_t)} Y_{F,t} \tag{C.36}$$

Stock price volatility. Taking the logarithm on both sides of equations (C.35) and (C.36) gives:

$$\begin{aligned} \log S_{H,t} =& -\log(\rho(\varphi + (1-\varphi)q_t)) + \log Y_{H,t} \\ \log S_{F,t} =& \log q_t - \log(\rho(\varphi + (1-\varphi)q_t)) + \log Y_{F,t}. \end{aligned} \tag{C.37}$$

We next apply Ito's lemma to these equations to have:

$$\begin{aligned} d\log S_{H,t} =& (\cdots)dt - \frac{1-\varphi}{\varphi + (1-\varphi)q_t} dq_t + \frac{dY_{H,t}}{Y_{H,t}} \\ d\log S_{F,t} =& (\cdots)dt + \frac{dq_t}{q_t} - \frac{1-\varphi}{\varphi + (1-\varphi)q_t} dq_t + \frac{dY_{F,t}}{Y_{F,t}}. \end{aligned} \tag{C.38}$$

Matching diffusion terms yields:

$$\begin{aligned} \sigma^S_{H,t} =& -\frac{(1-\varphi)q_t}{\varphi + (1-\varphi)q_t} \sigma^q_t + \sigma^Y_H \\ \sigma^S_{F,t} =& \left(1 - \frac{(1-\varphi)q_t}{\varphi + (1-\varphi)q_t}\right) \sigma^q_t + \sigma^Y_F. \end{aligned} \tag{C.39}$$

If we normalize the global numeraire using the geometric average as before we have:

$$\begin{aligned} \sigma^S_{H,t} =& (\varphi - 1)\sigma^q_t + \sigma^Y_H \\ \sigma^S_{F,t} =& \varphi \sigma^q_t + \sigma^Y_F. \end{aligned} \tag{C.40}$$

D Model summary

Table 2: Open economy model summary

Object	Equation
	Technology and preferences
Output	$\frac{dY_{nt}}{Y_{nt}} = \mu_n^Y dt + (\sigma_n^Y)' \mathbf{dB_t}, \quad \sigma_H^Y = (\sigma_{HH}^Y, 0)', \quad \sigma_F^Y = (0, \sigma_{FF}^Y)'$
Expenditures	$c_{nm,st}^i = \varepsilon_{nm} \left(\frac{p_{m,t}}{P_{n,t}} \right)^{-1} c_{n,st}^i, \quad P_{n,t} = \prod_m \left(\frac{p_{m,t}}{\varepsilon_{nm}} \right)^{\varepsilon_{nm}}, \quad \varepsilon_{nn} = \varepsilon, \quad \varepsilon_{n,-n} = 1 - \varepsilon$
Terms of trade	$q_t = \frac{p_{F,t}}{p_{H,t}}$
	$\frac{dq_t}{q_t} = \mu_t^q dt + (\sigma_t^q)' \mathbf{dB_t}$
Normalization	$(p_{Ht})^{\varphi} (p_{Ft})^{1-\varphi} = 1, \quad p_{Ht} = q_t^{\varphi - 1}, p_{Ft} = q_t^{\varphi}$
	Assets
Stock returns	$dR_{n,t} = \mu_{n,t}^S dt + (\sigma_{n,t}^S)' \mathbf{dB_t}, \quad dR_t = \mu_t dt + \sigma_t^S \mathbf{dB_t}, \quad \mu_t = (\mu_{H,t}^S, \mu_{F,t}^S)', \quad \sigma_t^S = (\sigma_{H,t}^S, \sigma_{F,t}^S)'$
	Pricing system
Pricing Kernel	$\frac{d\xi_t}{\xi_t} = -r_t dt - \theta_t' \mathbf{dB_t}$
Type-*i* pricing kernel	$\frac{d\xi_{n,t}^i}{\xi_{n,t}^i} = -r_t dt - (\theta_{n,t}^i)' \mathbf{dB_{n,t}^i} = -r_t dt - \left(\theta_t + \eta_n^i + \alpha^i \phi_t \right)' \mathbf{dB_{n,t}^i}$
Subjective BM	$\mathbf{dB_{n,t}^i} = \mathbf{dB_t} - \eta_n^i dt, \quad \eta_{nm}^i = \begin{cases} \frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = \ell \\ \frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = c \\ -\frac{\Delta^{nm}}{\sigma_{mm}^Y}, & \text{if } i = b. \end{cases}$
	State variables and market clearing conditions
Relative supply	$y_t = \frac{Y_{H,t}}{Y_{H,t} + Y_{F,t}}$
	$dy_t = y_t (1 - y_t) \left(\mu_t^y dt + (\sigma_t^y)' \mathbf{dB_t} \right)$
	$\mu_t^y = \mu_H^Y - \mu_F^Y - (\sigma_H^Y - \sigma_F^Y)' (y_t \sigma_H^Y + (1 - y_t) \sigma_F^Y), \quad \sigma_t^y = (\sigma_H^Y - \sigma_F^Y)$
Consumption shares	$x_{mn,t}^i = \frac{\nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{mn,st}^i ds}{Y_{n,t}}$
	$\frac{dx_{mn,t}^i}{x_{mn,t}^i} = \mu_{mn,t}^{i,x} dt + (\sigma_{mn,t}^{i,x})' \mathbf{dB_t}$
	$\sigma_{mn,t}^{i,x} = \eta_m^i - \sum_k \sum_j x_{kn,t}^j \eta_k^j + \left(\alpha^i - \sum_k \sum_j x_{kn,t}^j \alpha^j \right) \phi_t$
	$\mu_{mn,t}^{i,x} = \pi \delta \frac{\sum_k p_{k,t} Y_{k,t}}{p_{n,t} Y_{n,t}} \left(\frac{\nu^i \varepsilon_{mn}}{x_{mn,t}^i} - 1 \right) + (\sigma_{mn,t}^{i,x} - \eta_m^i)' \sigma_{mn,t}^{i,x} + \alpha^i (\sigma_{n,t}^p + \sigma_n^Y)' \phi_t - \sum_k \sum_j x_{kn,t}^j \alpha^j (\theta_{kt}^j)' \phi_t$
	$\frac{P_{m,t} c_{m,tt}^i}{p_{n,t} Y_{n,t}} = \delta \frac{\sum_k p_{k,t} Y_{k,t}}{p_{n,t} Y_{n,t}}, \quad \frac{\sum_k p_{k,t} Y_{k,t}}{p_{H,t} Y_{H,t}} = \frac{y_t + q_t (1 - y_t)}{y_t}, \quad \frac{\sum_k p_{k,t} Y_{k,t}}{p_{F,t} Y_{F,t}} = \frac{y_t + q_t (1 - y_t)}{q_t (1 - y_t)}$
Good markets	$\sum_m \sum_i x_{mn,t}^i = 1$
Stock markets	$\sum_m \sum_i \psi_{mn,t}^i = S_{n,t}$
Stock lending market	$\sum_m \sum_i \alpha^i \psi_{mn,t}^i = 0$
	Equilibrium
Market price of risk	$\theta_t = \varphi \sigma_H^Y + (1 - \varphi) \sigma_F^Y - \sum_m \sum_i \left(\varphi x_{mH,t}^i + (1 - \varphi) x_{mF,t}^i \right) \eta_m^i - \sum_m \sum_i \left(\varphi x_{mH,t}^i + (1 - \varphi) x_{mF,t}^i \right) \alpha^i \phi_t$
ToT vol	$\sigma_t^q = \sigma_H^Y - \sigma_F^Y - \sum_m \sum_i (x_{mH,t}^i - x_{mF,t}^i) \eta_m^i - \sum_m \sum_i (x_{mH,t}^i - x_{mF,t}^i) \alpha^i \phi_t$
Stock vol	$\sigma_{H,t}^S = -\frac{(1 - \varphi) q_t}{\varphi + (1 - \varphi) q_t} \sigma_t^q + \sigma_H^Y$
	$\sigma_{F,t}^S = \left(1 - \frac{(1 - \varphi) q_t}{\varphi + (1 - \varphi) q_t} \right) \sigma_t^q + \sigma_F^Y$
Shorting fee	$\phi_{n,t} = -\frac{\theta_{n,t} + \sum_m \sum_i \frac{w_{m,t}^i \alpha^i}{\sum_k \sum_j w_{k,t}^j \alpha^j} \eta_{mn}^i}{\sum_m \sum_i \frac{w_{m,t}^i \alpha^i}{\sum_k \sum_j w_{k,t}^j \alpha^j} \alpha^i}, \quad \frac{w_{m,t}^i \alpha^i}{\sum_k \sum_j w_{k,t}^j \alpha^j} = \frac{y_t x_{mH,t}^i \alpha^i + (1 - y_t) q_t x_{mF,t}^i \alpha^i}{y_t \sum_k \sum_j x_{kH,t}^j \alpha^j + (1 - y_t) q_t \sum_j x_{kF,t}^j \alpha^j}$
Wealth	$\frac{(\rho + \pi) w_{n,t}^i}{q_t^{\varphi - 1} \sum_m Y_{m,t}} = y_t x_{nH,t}^i + (1 - y_t) q_t x_{nF,t}^i$

E Estimation

Let us denote the state vector by:

$$z_t = (q_t, y_t, x^{\ell}_{HH,t}, x^{c}_{HH,t}, x^{b}_{HH,t}, x^{\ell}_{FH,t}, x^{b}_{FH,t}, x^{\ell}_{HF,t},, x^{b}_{HF,t}, x^{\ell}_{FF,t}, x^{c}_{FF,t}, x^{b}_{FF,t}) \tag{E.1}$$

where we ignored $x^c_{FH,t}$ and $x^c_{HF,t}$ as they could be rewritten respectively in term of other consumption shares arising from the equilibrium conditions $\sum_m \sum_i x^i_{mH,t} = 1$ and $\sum_m \sum_i x^i_{mF,t} = 1$.

$$\Phi = (\alpha, \lambda, \delta, \varphi, \Delta^{HH}, \Delta^{FH}, \Delta^{FF}, \Delta^{HF}, \sigma^Y_{HH}, \sigma^Y_{FF}) \tag{E.2}$$

Observational equations. The observational equations are:

$$\begin{aligned}
\log(RER_t/RER_{t-1}) =& \bar{\mu}^q + \mu^q_t + \sigma^{o,q} o^q_t \\
\log(Relative_output_t/Relative_output_{t-1}) =& \bar{y} + (1-y_t)\mu^y_t + \sigma^{o,y} o^y_t \\
Excess_returns_t =& \bar{\mu}^S + \sigma^S_t \theta_t + (\sigma^{o,s})' o^s_t \\
\log(Utilization_rates_t/Utilization_rates_{t-1}) =& \bar{UR} + \beta(SI_t - SI_{t-1}) + (\sigma^{o,ur})' o^{ur}_t
\end{aligned} \tag{E.3}$$

State equations. We discretized state equations as:

$$\begin{aligned}
\log(q_{t+\tau}/q_t) =& \left[\mu^q_t - \frac{1}{2}\left((\sigma^q_{H,t})^2 + (\sigma^q_{F,t})^2\right)\right]\tau + \sigma^q_{H,t}\sqrt{\tau} v_{H,t+\tau} + \sigma^q_{F,t}\sqrt{\tau} v_{F,t+\tau} \\
\log(y_{t+\tau}/y_t) =& -\left[(1-y_t)\left((\sigma^Y_{HH})^2 y_t - (\sigma^Y_{FF})^2(1-y_t)\right) + \frac{1}{2}(1-y_t)^2\left((\sigma^Y_{HH})^2 + (\sigma^Y_{FF})^2\right)\right]\tau \\
& + \sigma^Y_{HH}\sqrt{\tau} v_{H,t+\tau} - \sigma^Y_{FF}\sqrt{\tau} v_{F,t+\tau} \\
\log(x^i_{mn,t+\tau}/x^i_{mn,t}) =& \left[\mu^{i,x}_{mn,t} - \frac{1}{2}\left((\sigma^{i,x}_{H,mn,t})^2 + (\sigma^{i,x}_{F,mn,t})^2\right)\right]\tau \\
& + \sigma^{i,x}_{H,mn,t}\sqrt{\tau} v_{H,t+\tau} + \sigma^{i,x}_{F,mn,t}\sqrt{\tau} v_{F,t+\tau}
\end{aligned} \tag{E.4}$$

where the relevant variables are explicitly given by

$$\sigma_t^q = \begin{pmatrix} \sigma^Y_{HH} - \underbrace{\left(\sum_i (x^i_{HH,t} - x^i_{HF,t})\eta^i_{HH} + \sum_i (x^i_{FH,t} - x^i_{FF,t})\eta^i_{FH} \right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{H,t} \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \\ -\sigma^Y_{FF} - \underbrace{\left(\sum_i (x^i_{HH,t} - x^i_{HF,t})\eta^i_{HF} + \sum_i (x^i_{FH,t} - x^i_{FF,t})\eta^i_{FF} \right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{F,t} \sum_m \sum_i (x^i_{mH,t} - x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \end{pmatrix} \tag{E.5}$$

$$\theta_t = \begin{pmatrix} \varphi\sigma^Y_{HH} - \underbrace{\left(\sum_i (\varphi x^i_{HH,t} + (1-\varphi) x^i_{HF,t})\eta^i_{HH} + \sum_i (\varphi x^i_{FH,t} + (1-\varphi) x^i_{FF,t})\eta^i_{FH} \right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{H,t} \sum_m \sum_i (\varphi x^i_{mH,t} + (1-\varphi) x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \\ (1-\varphi)\sigma^Y_{FF} - \underbrace{\left(\sum_i (\varphi x^i_{HH,t} + (1-\varphi) x^i_{HF,t})\eta^i_{HF} + \sum_i (\varphi x^i_{FH,t} + (1-\varphi) x^i_{FF,t})\eta^i_{FF} \right)}_{\text{Beliefs heterogeneity effect}} - \underbrace{\phi_{F,t} \sum_m \sum_i (\varphi x^i_{mH,t} + (1-\varphi) x^i_{mF,t})\alpha^i}_{\text{Short selling effect}} \end{pmatrix} \tag{E.6}$$

$$\sigma^{\ell,x}_{HH,t} = \begin{pmatrix} \frac{\Delta^{HH}-\Delta^{FH}}{\sigma^Y_{HH}} - \frac{\Delta^{HH}-\Delta^{FH}}{\sigma^Y_{HH}}(x^\ell_{HH,t} + x^c_{HH,t}) + \frac{\Delta^{HH}+\Delta^{HF}}{\sigma^Y_{HH}} x^b_{HH,t} + 2\frac{\Delta^{FH}}{\sigma^Y_{HH}} x^b_{FH,t} + \phi_{H,t}\big(\alpha - \alpha(x^\ell_{HH,t} + x^\ell_{FH,t}) - (x^b_{HH,t} + x^b_{FH,t})\big) \\ -\frac{\Delta^{FF}-\Delta^{HF}}{\sigma^Y_{FF}} + \frac{\Delta^{FF}-\Delta^{HF}}{\sigma^Y_{FF}}(x^\ell_{HH,t} + x^c_{HH,t}) + \frac{\Delta^{FF}+\Delta^{HF}}{\sigma^Y_{FF}} x^b_{HH,t} + 2\frac{\Delta^{FF}}{\sigma^Y_{FF}} x^b_{FH,t} + \phi_{F,t}\big(\alpha - \alpha(x^\ell_{HH,t} + x^\ell_{FH,t}) - (x^b_{HH,t} + x^b_{FH,t})\big) \end{pmatrix} \tag{E.7}$$

$$\begin{aligned} \sigma^{c,x}_{HH,t} =& \sigma^{\ell,x}_{HH,t} - \alpha\phi_t \\ \sigma^{b,x}_{HH,t} =& \sigma^{\ell,x}_{HH,t} - 2\eta^\ell_H + (1-\alpha)\phi_t \\ =& \sigma^{\ell,x}_{HH,t} + \begin{pmatrix} -2\frac{\Delta^{HH}}{\sigma^Y_{HH}} + (1-\alpha)\phi_{H,t}) \\ -2\frac{\Delta^{HF}}{\sigma^Y_{FF}} + (1-\alpha)\phi_{F,t}) \end{pmatrix} \end{aligned} \tag{E.8}$$

$$\sigma^{\ell,x}_{FH,t} = \begin{pmatrix} -\frac{\Delta^{HH}-\Delta^{FH}}{\sigma^Y_{HH}}(x^\ell_{HH,t} + x^c_{HH,t}) + \frac{\Delta^{HH}+\Delta^{FH}}{\sigma^Y_{HH}} x^b_{HH,t} + 2\frac{\Delta^{HH}}{\sigma^Y_{HH}} x^b_{FH,t} + \phi_{H,t}\big(\alpha - \alpha(x^\ell_{HH,t} + x^\ell_{FH,t}) - (x^b_{HH,t} + x^b_{FH,t})\big) \\ \frac{\Delta^{FF}-\Delta^{HF}}{\sigma^Y_{FF}}(x^\ell_{HH,t} + x^c_{HH,t}) + \frac{\Delta^{FF}+\Delta^{HF}}{\sigma^Y_{FF}} x^b_{HH,t} + 2\frac{\Delta^{FF}}{\sigma^Y_{FF}} x^b_{FH,t} + \phi_{F,t}\big(\alpha - \alpha(x^\ell_{HH,t} + x^\ell_{FH,t}) - (x^b_{HH,t} + x^b_{FH,t})\big) \end{pmatrix} \tag{E.9}$$

$$\begin{aligned} \sigma^{c,x}_{FH,t} =& \sigma^{\ell,x}_{FH,t} - \alpha\phi_t \\ \sigma^{b,x}_{FH,t} =& \sigma^{\ell,x}_{FH,t} - 2\eta^\ell_F + (1-\alpha)\phi_t \\ =& \sigma^{\ell,x}_{FH,t} + \begin{pmatrix} -2\frac{\Delta^{FH}}{\sigma^Y_{HH}} + (1-\alpha)\phi_{H,t}) \\ -2\frac{\Delta^{FF}}{\sigma^Y_{FF}} + (1-\alpha)\phi_{F,t}) \end{pmatrix} \end{aligned} \tag{E.10}$$

$$\sigma^{\ell,x}_{FF,t} = \begin{pmatrix} -\frac{\Delta^{HH}-\Delta^{FH}}{\sigma^Y_{HH}} + \frac{\Delta^{HH}-\Delta^{FH}}{\sigma^Y_{HH}}(x^\ell_{FF,t} + x^c_{FF,t}) + \frac{\Delta^{HH}+\Delta^{FH}}{\sigma^Y_{HH}} x^b_{FF,t} + 2\frac{\Delta^{HH}}{\sigma^Y_{HH}} x^b_{HF,t} + \phi_{H,t}\big(\alpha - \alpha(x^\ell_{FF,t} + x^\ell_{HF,t}) - (x^b_{FF,t} + x^b_{HF,t})\big) \\ \frac{\Delta^{FF}-\Delta^{HF}}{\sigma^Y_{FF}} - \frac{\Delta^{FF}-\Delta^{HF}}{\sigma^Y_{FF}}(x^\ell_{FF,t} + x^c_{FF,t}) + \frac{\Delta^{FF}+\Delta^{HF}}{\sigma^Y_{FF}} x^b_{FF,t} + 2\frac{\Delta^{HF}}{\sigma^Y_{FF}} x^b_{HF,t} + \phi_{F,t}\big(\alpha - \alpha(x^\ell_{FF,t} + x^\ell_{HF,t}) - (x^b_{FF,t} + x^b_{HF,t})\big) \end{pmatrix} \tag{E.11}$$

$$
\begin{aligned}
\sigma^{c,x}_{FF,t} =& \sigma^{\ell,x}_{FF,t} - \alpha\phi_t \\
\sigma^{b,x}_{FF,t} =& \sigma^{\ell,x}_{FF,t} - 2\eta^{\ell}_{F} + (1-\alpha)\phi_t \\
=& \sigma^{\ell,x}_{FF,t} + \begin{pmatrix} -2\frac{\Delta^{FH}}{\sigma^{Y}_{HH}} + (1-\alpha)\phi_{H,t} \\ \\ -2\frac{\Delta^{FF}}{\sigma^{Y}_{FF}} + (1-\alpha)\phi_{F,t} \end{pmatrix}
\end{aligned} \tag{E.12}
$$

$$
\sigma^{\ell,x}_{HF,t} = \begin{pmatrix} \frac{\Delta^{HH}-\Delta^{FH}}{\sigma^{Y}_{HH}}(x^{\ell}_{FF,t} + x^{c}_{FF,t}) + \frac{\Delta^{HH}+\Delta^{FH}}{\sigma^{Y}_{HH}} x^{b}_{FF,t} + 2\frac{\Delta^{HH}}{\sigma^{Y}_{HH}} x^{b}_{HF,t} + \phi_{H,t}\big(\alpha - \alpha(x^{\ell}_{FF,t} + x^{\ell}_{HF,t}) - (x^{b}_{FF,t} + x^{b}_{HF,t})\big) \\ \\ -\frac{\Delta^{FF}-\Delta^{HF}}{\sigma^{Y}_{FF}}(x^{\ell}_{FF,t} + x^{c}_{FF,t}) + \frac{\Delta^{FF}+\Delta^{HF}}{\sigma^{Y}_{FF}} x^{b}_{FF,t} + 2\frac{\Delta^{HF}}{\sigma^{Y}_{FF}} x^{b}_{HF,t} + \phi_{F,t}\big(\alpha - \alpha(x^{\ell}_{FF,t} + x^{\ell}_{HF,t}) - (x^{b}_{FF,t} + x^{b}_{HF,t})\big) \end{pmatrix} \tag{E.13}
$$

$$
\begin{aligned}
\sigma^{c,x}_{HF,t} =& \sigma^{\ell,x}_{HF,t} - \alpha\phi_t \\
\sigma^{b,x}_{HF,t} =& \sigma^{\ell,x}_{HF,t} - 2\eta^{\ell}_{H} + (1-\alpha)\phi_t \\
=& \sigma^{\ell,x}_{HF,t} + \begin{pmatrix} -2\frac{\Delta^{HH}}{\sigma^{Y}_{HH}} + (1-\alpha)\phi_{H,t} \\ \\ -2\frac{\Delta^{HF}}{\sigma^{Y}_{FF}} + (1-\alpha)\phi_{F,t} \end{pmatrix}
\end{aligned} \tag{E.14}
$$

$$
\begin{aligned}
A_{n,t} =& \sum_k \sum_j x^{j}_{kn,t} \alpha^{j} (\theta^{j}_{k,t})' \phi_t \\
A_{H,t} =& \sum_k \sum_j x^{j}_{kH,t} \alpha^{j} (\theta^{j}_{k,t})' \phi_t \\
=& \Big(\theta_t\big(\alpha(x^{\ell}_{HH,t} + x^{\ell}_{FH,t}) + (x^{b}_{HH,t} + x^{b}_{FH,t})\big) + \phi_t\big(\alpha^2(x^{\ell}_{HH,t} + x^{\ell}_{FH,t}) + (x^{b}_{HH,t} + x^{b}_{FH,t})\big) + \eta^{\ell}_{H}(\alpha x^{\ell}_{HH,t} - x^{b}_{HH,t}) + \eta^{\ell}_{F}(\alpha x^{\ell}_{FH,t} - x^{b}_{FH,t})\Big)' \phi_t
\end{aligned} \tag{E.15}
$$