Risk for Price: Using Generalized Demand System for Asset Pricing *

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Abstract

I decompose Stochastic Discount Factor (SDF) using detailed consumption prices and expenditure. Prices and expenditure work as sufficient statistics for marginal utility, as indirect utility function describes consumer's preference. This decomposition is general and flexible, applies for a wide set of consumer utility functions. When marginal utility has different sensitivities to prices, fluctuations in detailed prices help to measure variation in SDF. I estimate the pricing kernel in a two-sector economy of goods and services. Pricing kernel is summarized by two time-series factors, expenditure relative to price of services, price of goods relative to price of services. This new model explains the cross-section variation of expected returns in equity portfolios, better than simple consumption-based asset pricing models and traded-factor models. Variation in relative price of goods contributes to volatile SDF. Point estimate of relative risk-aversion coefficient is small. Equity assets in U.S. market have negative risk exposure to relative price of goods. Additional negative risk exposure yields higher return.

Keywords: Revealed Preference, Consumer Price, Systematic Risk, Consumption-based Asset Pricing

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1 Introduction

Traditional consumption-based asset pricing models typically identify the stochastic discount factor (SDF) based on aggregate consumption. However, it has become evident that these models struggle to explain both the time-series variation in the equity premium (Mehra and Prescott, 1985; Hansen and Singleton, 1983) and the cross-section of stock returns (Mankiw and Shapiro, 1986). Literature (Savov, 2011; Kroencke, 2017) try to accurately measure the quantity of consumption. They assume that there is the single sector in the economy. On the other hand, barring a few exceptions (Lochstoer, 2009; Roussanov et al., 2021), the asset pricing implications of detailed prices among different consumption commodities have received less attention.

Intuitively, high prices reduce consumer's utility, marginal utility is higher. How to quantify the impact from price of each commodity? In this paper, I develop a general approach to study the role of detailed consumption prices in asset pricing. I use indirect utility function to derive consumer's marginal utility in expression of prices and expenditure. Approximated pricing kernel, variation in SDF, is a linear combination of expenditure variation, and prices adjusted by shares. In empirical examination, I estimate the pricing kernel for a two-sector economy with goods and services. There are two time series factors, expenditure relative to price of services, and price of goods relative to services. This model explains the cross-section variation of expected returns well, outperforms Consumption-CAPM using aggregate consumption quantity growth, and in many cases performs better than Fama-French 5-Factor model.

To explain this new approach, I begin by a simple two-period, two-sector model to show the role of detailed prices in consumer's marginal utility. In this model, consumer preference doesn't have analytical utility function over quantities. Using indirect utility function to describe the consumer, marginal utility of expenditure is a tractable function over prices and expenditure. Crucially, in each sector, impact of price on marginal utility differs from the weight of Consumer Price Index (CPI). CPI understates the price of goods, while overstates for services. As the result, consumption quantity index, expenditure denominated by CPI, cannot measure marginal utility, whereas detailed prices of goods and services provide the precise measure. Formally, I derive the non-parametric pricing kernel in a general dynamic model where a representative consumer consumes multiple types of commodities, holds risky assets and risk-free bond with no binding financial constraint. Consumer's marginal utility is decomposed into expenditure and prices augmented by shares. Notably, this decomposition applies for a wide set of path-independent preference, includes situation where direct utility function is unavailable and the situation where a proper calibrated indirect utility function is unavailable. Specifically, when consumer's basket changes in expenditure, marginal utility has different sensitivities to prices of commodities. This framework yields a remedy for Consumption-CAPM, as CPI cannot summarize impact from detailed prices over marginal utility, directly identifying coefficients for each detailed price elicits more information.

In empirical examination, I estimate a dynamic two-sector model of goods and services, identify the risk prices for expenditure and price of goods in SDF. Estimation has SDF summarized by two time-series, expenditure relative to price of services, and price of goods relative to price of services. As consumer spends a larger share in goods when her expenditure drops, marginal utility of expenditure is more sensitive to price variation in goods than services. As the result, the sensitivity to price of goods is larger than expenditure. When expressing SDF as expenditure and price of goods relative to price of services, the variation of relative price adds to explaining the risk premium.

I estimate the model using General Methods of Moments (GMM). Expenditure and prices are constructed using dis-aggregated expenditures and prices from National Income and Product Accounts (NIPA) during sample period 1965-2019. The testing assets are equity portfolios sorted on size, book-market, profitability, investment, momentum and earning-price ratio. There are three main findings.

Firstly, risk price of expenditure, determined by consumer's relative risk-aversion coefficient, is 28.80. Risk price for price of goods is -71.29, in larger absolute value than expenditure, meaning relative price contributes a considerable component in SDF. All else equal, additional 0.1% independent volatility in price of goods implies 2.82% increase of volatility in SDF annually, while CPI weights suggest increase of volatility as 1.10%. As detailed prices help describe systematic risk and explain risk premium, estimation doesn't have the outsized risk-aversion coefficient. These point estimates are robust to testing assets as Size-BM 25 portfolios and 30 industry portfolios, and different subsamples during 1935-2019.

Secondly, the model produces a smaller pricing error compared to traditional traded factor models. Mean Absolute Pricing Error (MAPE) is 0.39%, while Fama-French 5-Factor model has 0.79%. It also outperforms the standard consumption-based model that utilizes the aggregate consumption quantity growth as the factor, where MAPE is 0.71%. In the true SDF, risk price for price is distinctive in two sectors. On the contrary, aggregate consumption growth is a single time series, forces different prices with the same risk price, which leads to poor empirical performance.

Third, covariance between asset returns and relative price of goods are negative, large, and dispersed. Among testing assets with deeper negative covariance to relative price, expected return is higher. Without breaking into expenditure and prices, covariance to aggregate consumption quantity growth distributes around zero, with no discernible relation to expected return. I estimate risk premium for the two time-series factors in Fama-Macbeth regression. Equity assets have risk exposure to relative price ranging from -7.99 to -3.46, additional one 1 unit of negative beta yields risk premium 1.64% per year.

Decomposing SDF with prices and expenditure is general and flexible. I further estimate a four-sector model with food and non-food within goods and services. As prices are more detailed, it is easier to capture the fluctuation in SDF, risk price of expenditure is 14.70, GMM estimation yields MAPE as 0.18%.

Beyond the discussion of what would be the robust description of consumer preference and stochastic discount factor, this paper presents a surprising finding for the literature of Consumption-CAPM. Aggregate shocks underlying the changes in sector-level prices, also changes the expectation of consumption quantity index growth. In the annual frequency, using the standard 25 Size-BM characteristic portfolios, the consumption quantity index growth minus the full-sample unconditional mean as unexpected innovation, I reproduce the failure of Consumption-CAPM. The point estimate of risk-aversion parameter is 106.49, abnormally large. The MAPE is 9.58%, confirming the old consensus that the covariance to innovation of consumption quantity growth has no explanation power for the stochastic discount factor. If using the same len to examine the two time series factors in my model, the total expenditure relative to price of services, price of goods relative to price of services, which are the building blocks in price-model of generalized Consumption-CAPM. The linear combination of these time series conveys considerable variation in the true SDF. The MAPE is 1.07%, much smaller than the innovation of consumption quantity index growth. The formal estimation admits the possible variation in expectation of consumption (quantity, expenditure and prices), uses the diverse equity portfolios of cross-section anomalies as the testing assets, estimate the standard Consumption-CAPM and the new price-model of generalized Consumption-CAPM. The standard Consumption-CAPM that utilizes the aggregate consumption quantity growth as the factor, has MAPE as 0.71%, while the new model of generalized Consumption-CAPM has MAPE of 0.39%. Forcing the full-sample mean as the constant expected growth, neglects the time-varying expectation of consumption quantity index growth. Literature arrived to the overly pessimistic conclusion, in terms of the failure of consumption-CAPM. These shocks driving the time-varying expectation of consumption quantity index growth are also the forces for the variation in sector-level prices. The new model describes the consumer preference in general situation, uses sector-level prices to construct the sufficient statistic of marginal utility, captures these aggregate shocks that are not directly observed.

My work adds an alternative solution to Consumption-CAPM [1]. Admitting detailed prices

[1] Accumulated solutions come from but are not limited to three main aspects. (1) measure: (Parker and Julliard, 2005; Jagannathan and Wang, 2007) addresses time-aggregation bias in quarterly consumption, improves the model fitness in explaining the cross-section variation of returns. (2) marginal utility: (Lettau and Ludvigson, 2001) suggests scaling consumption in face of unobserved habit, introducing the interaction term between CaY and consumption growth improves the model-fitness in cross-sectional test. (Lochstoer, 2009) uses relative price to infer habit in luxury, conditional volatility of relative price helps explain the time-varying risk premia. (Yogo, 2006) considers durable aside from non-durable consumption; (Papanikolaou, 2011) considers leisure, investment-good-specific shock generates variation in labor supply; (Loualiche et al., 2016) considers variety of consumption products, positive shock to firm entry cost reduces variety growth. (Papanikolaou, 2011; Loualiche et al., 2016) introduce new measure of shocks to improve measuring the SDF. (3) marginal investor: (Malloy et al., 2009) use long-run consumption growth of stockholder to explain the cross-section of returns; (Lettau et al., 2019) use capital (labor) share to indirectly infer stockholder's consumption, risk exposure to capital share yields positive risk premium; (Adrian et al., 2014; He et al., 2017) uses the leverage to indirectly measure financial manager's marginal utility of wealth, risk exposure to book leverage yields positive risk premium, while equity-debt ratio is positively priced across different types of financial assets.

avoids overly large point estimate of risk aversion, and simultaneously achieves small model error in explaining the cross-section of expected returns. Estimated risk-aversion coefficient is much smaller than 153 in (Cochrane, 1996) and 142 in (Yogo, 2006). This point estimate is comparable to the range of 19-23 in (Kroencke, 2017), but the new model only uses the raw aggregate consumption statistics. In (Savov, 2011), measuring quantity using the garbage yields a point estimate 17, but leaves high model error in cross-section regression.

My work adds analysis for multiple consumption sectors in the asset pricing literature, by describing a consumer with general preference. It simplifies the analysis where there is no analytically tractable utility function describing the consumer. Literature typically assume the homothetic preference. For example, (Yogo, 2006; Belo, 2010; Yang, 2011; Eraker et al., 2016) considers the nondurable consumption sector and the durable sector. (Piazzesi et al., 2007) include the housing sector. (Dittmar et al., 2020) include the energy sector. These articles consider the utility function with Constant Elasticity of Substitution (CES) for tractable analysis. Nonetheless, estimation in this paper suggests non-homothetic preference within nondurable sector, marginal utility of expenditure is not separable in prices and expenditure. C-CAPM using quantity index cannot describe the SDF. A small thread of literature considers the non-homothetic preference under special cases. (Ait-Sahalia et al., 2004) considers separable utility function over the nondurable sector and the luxury sector. (Lochstoer, 2009) uses Stone-Geary preference for the basic-good and luxury. (Pakoš, 2011) allows heterogeneous elasticity in CES functional form. This paper uses the general indirect utility function to describe the consumer preference. Analysis applies for aforementioned utility functions, and the non-homothetic preference in (Muellbauer, 1976; Boppart, 2014; Comin et al., 2021) where deriving marginal utility over quantities is difficult. The theoretical argument and the estimation method for SDF in this paper are general, allow for flexible empirical application in an economy with multiple consumption sectors. (Dittmar et al., 2020) pursue a more accurate utility function over detailed quantities (food, clothes, energy, etc). This paper derives consumer's marginal utility in a non-parametric expression of dis-aggregated prices.

This paper works on asset pricing literature that studies the endogenous determination of asset price and spot commodity price. My work clarifies that consumption prices directly affect consumer's marginal utility. This literature provides different explanations for asset pricing outcomes from commodity price: (1) commodity price is informative of fundamental shocks in (Papanikolaou, 2011) and (Johnson, 2011); (2) the equilibrium price reflects the producer's inter-temporal decision, such as the durable price relative to nondurable in (Belo, 2010), relative price of basic-good helps measure consumer's habit in basic-good in (Lochstoer, 2009); (3) nominal rigidity increases the correlation between aggregate dividend flow and firm owner's consumption in (Favilukis and Lin, 2016). This paper uses consumer's prices and expenditure to measure marginal utility, aims to explain cross-section variation of expected return. The theoretical prediction echoes the recent finding about CPI in (Roussanov et al., 2021) and price risk-exposure in (An et al., 2023). (Roussanov et al., 2021) measure fundamental shocks using VAR residual in core-CPI and energy-CPI. Compared with (Roussanov et al., 2021), this work provides a general explanation for observing risk-premium of consumption prices, proposes "getting to the detailed CPI". In estimation of two sectors, I show equity assets have negative risk exposure to relative price of goods, with high risk premium. In estimation of multiple sectors, I found negative risk price in grocery (food-at-home), and positive estimate for dining (food-away).

In the estimation of two-sector economy, sector of goods is relatively necessity commodity compared to services, resembles the basic-good in (Ait-Sahalia et al., 2004; Lochstoer, 2009). Departing from these works, sector of services covers large share in consumption basket, doesn't have the large volatility in quantity change. This paper has a different focus: use comprehensive records of prices to measure consumer's marginal utility. Inheriting the classification of NIPA, this paper separates goods and services. This allows for robust identification in long sample using different groups of testing assets. The estimation method proposed in this paper is simple, easy for extension when more observations of accurate prices are available. In extended model, consumer considers food and non-food as different consumption commodities. The extended estimation of multiple sectors better explains the cross-section of returns, in testing assets constructed based on anomalies.

This paper is organized as follows. Section 2 derives the stochastic discount factor, and decomposes the SDF using consumption prices. Section 3 examines the estimation of pricing kernel in a two-sector economy, analyzes point estimates and model fitness, describes the cross-section of risk exposure. Section 4 discusses why the price-model differs from quantity-model, with further empirical investigation. Section 5 extends the estimation of price-model in an economy with multiple sectors. Section 6 clarifies that consumer's marginal utility works as sufficient statistics of shocks, explains the risk prices, and provides further discussion related with inflation. Section 7 concludes.

2 Theory

This section derives the consumer's marginal utility in expression of prices and expenditure. In an economy with a representative consumer without binding financial constraint, marginal utility is the stochastic discount factor. Subsection 2.1 uses a two-period example to show the role of detailed prices in SDF where consumer has price-habit. Subsection 2.2 provides the formal derivation in dynamic environment where consumer holds risky assets and risk-free bond. Consumer's preference over multiple commodities is described by general indirect utility function. Variation of SDF is a simple linear function of expenditure and prices adjusted by shares.

2.1 Motivating Example

Consumer lives for two periods $t = 0$ and $t = 1$. In period $t = 1$, there are high and down states $\{h, d\}$ in probability $\{\pi_h, 1 - \pi_h\}$. Consumer has consumption basket as $\vec{C} = (C_g, C_s)$ in each period and state. Her utility from quantity of goods C_g and services C_s is $u(\vec{C})$. To maximize life-time utility, consumer decides the optimal consumption allocation,

$$\begin{aligned} \max_{\vec{C}_0, \vec{C}_h, \vec{C}_d} \quad & u(\vec{C}_0) + \beta \cdot [\pi_h \cdot u(\vec{C}_h) + (1 - \pi_h) \cdot u(\vec{C}_d)] \\ s.t. \quad & E_0 + \beta \cdot \pi_h \cdot M_h \cdot W_h + \beta \cdot (1 - \pi_h) \cdot M_d \cdot W_d \leq W_0, \\ & P_{g,0} \cdot C_{g,0} + P_{s,0} \cdot C_{s,0} \leq E_0, \\ & P_{g,z} \cdot C_{g,z} + P_{s,z} \cdot C_{s,z} \leq W_z, \quad z \in \{h, d\}. \end{aligned} \tag{1}$$

The asset price for having 1 dollar in period 0 is normalized as $M_0 = 1$. The price $\beta \cdot \pi_h \cdot M_h$ is the current price for having 1 dollar in period 1 under the high state h. Similarly, $\beta \cdot (1 - \pi_h) \cdot M_d$ for the 1 dollar under down state d. Here, M_z is the **stochastic discount factor** adjusting the subjective discount rate β and the natural probability $(\pi_h, 1 - \pi_h)$ of states $z \in \{h, d\}$. Consumer's financial wealth is W_0 units of dollar in current period.

We use **indirect utility function** $V(\vec{P}, E)$ to summarize the consumer's utility when her (total) expenditure is E given the price of goods as P_g and services P_s,

$$\begin{aligned} V(\vec{P}, E) = \max_{C_g, C_s} \quad & u(C_g, C_s) \\ s.t. \quad & P_g \cdot C_g + P_s \cdot C_s \leq E. \end{aligned} \tag{2}$$

Expenditure E counts all spending in goods and services. The optimization problem (1) has the equivalent problem over expenditure allocation,

$$\begin{aligned} \max_{E_0, E_h, E_d} \quad & V(\vec{P}_0, E_0) + \beta \cdot [\pi_h \cdot V(\vec{P}_h, E_h) + (1 - \pi_h) \cdot V(\vec{P}_d, E_d)] \\ s.t. \quad & E_0 + \beta \cdot \pi_h \cdot M_h \cdot W_h + \beta \cdot (1 - \pi_h) \cdot M_d \cdot W_d \leq W_0, \\ & E_z \leq W_z, \quad z \in \{h, d\}. \end{aligned} \tag{3}$$

Consumer's **marginal utility of expenditure** $\mathcal{D}_E V(\vec{P}, E)$, first-order partial derivative to expenditure E, reveals stochastic discount factor, the state price of financial wealth. For the optimal consumption plan, the first-order-condition for expenditure high state h is stated as,

$$\frac{\mathcal{D}_E V(\vec{P}_h, E_h)}{\mathcal{D}_E V(\vec{P}_0, E_0)} = \frac{M_h}{M_0}. \tag{4}$$

Here, state-price of current-period is normalized $M_0 = 1$. We have the similar equation for the down state d.

In this two-period world, if we know a company stock k gives us the dividend $D_{k,h}$ in high state, $D_{k,d}$ in down state, the fair price for this stock is $P_k^s = \beta \cdot \pi_h \cdot M_h \cdot D_{k,h} + \beta \cdot (1-\pi_h) \cdot M_d \cdot D_{k,d}$. When it is period 1 under the high state, realized return from holding the stock is $R_{k,h} = \frac{D_{k,h}}{P_k^s}$. One can use the expectation notation $\mathbb{E}[\cdot]$ to write the consumer's Euler Equation for holding financial assets,

$$1 = \beta \cdot \mathbb{E}[\frac{M_z}{M_0} \cdot R_{k,z}]. \tag{5}$$

In the special case of risk-free bond, where the bond gives us identical 1 unit of dollar in both the high state and the down state, the Euler Equation is $1 = \beta \cdot \mathbb{E}[\frac{M_z}{M_0}] \cdot R_f$. Taking the difference between two asst pricing equations, removing the subjective discount rate β, we have the Euler Equation of excess return in risky assets.

$$0 = \mathbb{E}[\frac{M_z}{M_0} \cdot R^e_{k,z}]. \tag{6}$$

In this situation of two consumption sectors, if we know the expenditure profile $\{E_0, E_h\}$ and the price profile $\{\vec{P}_0, \vec{P}_h\}$, we can infer the variation of stochastic discount factor $\frac{M_h}{M_0} = \frac{\mathcal{D}_E V(\vec{P}_h, E_h)}{\mathcal{D}_E V(\vec{P}_0, E_0)}$, and similarly for the down state $\frac{M_d}{M_0}$. Expected return $\mathbb{E}[R^e_{k,z}]$ is determined by its covariance with **pricing kernel**, variation of SDF across states $\{\frac{M_z}{M_0}\}_{z \in \{h,d\}}$.

Consider a special case of indirect utility function over the good and service,

$$V(\vec{P}, E) = \frac{1}{1-\gamma} \cdot \frac{E^{1-\gamma}}{P_g^{\overline{\omega}_g \cdot (1-\gamma)} \cdot P_s^{(1-\overline{\omega}_g)\cdot(1-\gamma)}} + \frac{\xi}{\epsilon} \cdot (\frac{P_g}{P_s})^{\epsilon}. \tag{7}$$

The parameter γ describes consumer's elasticity of inter-temporal substitution. Here, it is equivalent with the relative risk-aversion coefficient. The parameter $\overline{\omega}_g$ affects the consumer's expenditure shares in good and service. Importantly, it determines how consumer's marginal utility changes with prices. Price-habit $\frac{\xi}{\epsilon} \cdot (\frac{P_g}{P_s})^{\epsilon}$ allows us describing the additional impact from prices to consumer's feeling. This separates the observed share in basket and the importance of prices in marginal utility.

In this example, consumer preference doesn't have an analytical direct utility function $u(\vec{C})$ over quantities, due to the existence of price-habit. Nonetheless, one can still use the indirect utility function to know consumer's optimal consumption basket given the price vector and the expenditure. Using **Roy's Identity**, consumer's observed share in goods is $\omega_g = \frac{\mathcal{D}_g V(\vec{P}, E) \cdot P_g}{\mathcal{D}_g V(\vec{P}, E) \cdot P_g + \mathcal{D}_s V(\vec{P}, E) \cdot P_s}$. The term $\mathcal{D}_g V(\vec{P}, E)$ denotes partial derivative of utility with respect to price of goods P_g, and similar notation for the services. One can derive the share of service

ω_s in symmetric expression.

$$\begin{aligned} \omega_g =& \overline{\omega}_g - \xi \cdot (\frac{P_g}{P_s})^{\epsilon + \overline{\omega}_g \cdot (1-\gamma)} \cdot (\frac{E}{P_s})^{-(1-\gamma)}, \\ \omega_s =& (1 - \overline{\omega}_g) + \xi \cdot (\frac{P_g}{P_s})^{\epsilon + \overline{\omega}_g \cdot (1-\gamma)} \cdot (\frac{E}{P_s})^{-(1-\gamma)}. \end{aligned} \tag{8}$$

Quantities of goods and services are $C_g = \frac{\omega_g \cdot E}{P_g}$ and $C_s = \frac{\omega_s \cdot E}{P_s}$. So we arrive to consumer's basket without looking at the direct utility function.

This equivalent way in describing consumer brings convenience for asset pricing. In this example, pricing kernel is inferred using the expenditure and prices. From current period to the high state in following period, change in the marginal utility of expenditure is,

$$\frac{M_h}{M_0} = \frac{P_{g,h}^{-\overline{\omega}_g \cdot (1-\gamma)} \cdot P_{s,h}^{-(1-\overline{\omega}_g) \cdot (1-\gamma)} \cdot E_h^{-\gamma}}{P_{g,0}^{-\overline{\omega}_g \cdot (1-\gamma)} \cdot P_{s,0}^{-(1-\overline{\omega}_g) \cdot (1-\gamma)} \cdot E_0^{-\gamma}}. \tag{9}$$

The power coefficient $-\overline{\omega}_g \cdot (1-\gamma)$ describes the change of marginal utility to price of goods P_g, $-(1-\overline{\omega}_g) \cdot (1-\gamma)$ for service P_s, and $-\gamma$ for expenditure E. We observe variations of consumption prices in the real pricing kernel of financial assets. Multiplier $\overline{\omega}_g$ for price of goods is larger than the observed share in goods ω_g, while services has the smaller $1-\overline{\omega}_g$ than observed share $1-\omega_g$. Pricing kernel is relatively more sensitive to variation in price of goods than services, to prediction of consumer's basket. Quantity index, expenditure adjusted by consumer price index $\frac{E}{P_g^{\omega_g} \cdot P_s^{1-\omega_g}}$, cannot describe variation in marginal utility. Detailed prices help accurately measure the marginal utility, hence mitigate the difficulty in using quantity index. Quantities cannot describe variation of marginal utility either. There is no analytical function $u(C_g, C_s)$, so the marginal utility in expression of quantities is also not tractable. If we replace expenditure E with quantities $P_g \cdot C_g + P_s \cdot C_s$, we still need the detailed information of prices in equation (9). On the contrary, using prices and expenditure has the simpler format. In the following subsection, we will see this improvement is not limited to this economy with two commodities, and this special example of indirect utility function.

2.2 Formal Decomposition

2.2.1 Economy Environment and Consumer's Decision

I consider the discrete-time infinite-horizon consumption basket allocation problem of a representative consumer. The set of commodity category is fixed set $\mathcal{J}$. The state of the world is described by the $\{\{z_t\}_{t=0}^{\infty}\}$. Motion of history path is $z^{t+1} = (z^t, z_{t+1})$.

Direct utility function $u(\cdot)$ describes consumer's preference over the consumption basket $(C_1, C_2, \ldots, C_J)$ within each period. Over the life-time, consumer has utility function $\mathcal{U}$ over the consumption stream $\tilde{C}$. This paper assumes the path-independent inter-temporal preference, flows of utility $u_t = u(\vec{C}_t)$ in each period and states are added using constant subjective discount rate β,

$$\mathcal{U}(\tilde{C}) = \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot u(\vec{C}_t)]. \tag{10}$$

The consumer participates in the competitive financial market. The price of financial security[2] is $P^s_{k,t}$, the payout is $D_{k,t}$. In period t, the total risk-free rate for one-period bond is $R_{f,t+1}$. The consumer participates in the Walras commodity market. The spot price of commodity is $P_{j,t}$ per unit. The vector $\vec{\theta}_t$ describes the amounts of financial securities held by the consumer at time point t. Consumer has bond payment as B_t, decides the new holding B_{t+1}. Formally, problem (P.1) describes the consumer's life-time decision of consumption and asset holding (P.1),

$$\begin{aligned}
\overline{U}_0(\vec{\theta}_0) = \quad & \sup_{\tilde{C},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot u(\vec{C}_t)] \\
s.t. \quad & \underbrace{\sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = \sum_j P_{j,t} \cdot C_{j,t} + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}}_{\text{budget constraint}}, \\
& C_{j,t} \geq 0; \quad \underbrace{\sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}} \geq \underline{a}}_{\text{financial constraint}}.
\end{aligned} \tag{P.1}$$

The proper construction of $\underline{a}$ ensures no Ponzi-game and the financial constraint never binds, as the same argument of Chapter 8 in (Ljungqvist and Sargent, 2012).

Definition 1. *Define the* ***competitive equilibrium*** *in an endowment economy of commodity quantity process $\tilde{C}$ as*

1. *Given spot price $\{\vec{P}_t\}_{t=0}^{\infty}$, financial asset price $\{\vec{P}^s_t\}_{t=0}^{\infty}$, consumer makes her optimal expenditure decision and financial portfolio decision, as in problem (P.1).*

2. *Spot commodity markets clear. For each consumption sector j, consumer's demand equals the exogenous supply at all time periods and states, $C^*_{j,t} = C_{j,t}$.*

3. *Financial security j with dividend as revenue of commodity j, has market clearing, $\theta^*_{j,t+1} = 1$.*

[2] I use the sup-script s for the financial security.

4. Redundant security k has market clearing, $\theta^*_{k,t+1} = 0$.

5. One-period risk-free bond has market clearing, $B^*_{t+1} = 0$.

In this equilibrium, one can verify the two facts. (1) Consumer's net income flow is the dividend of (risky) financial security, $\sum_{j\in\mathcal{J}} D_{j,t} = \sum_{j\in\mathcal{J}} P_{j,t} \cdot C_{j,t}$. Because the redundant securities and the bond have zero supply, they don't add to the income flow in equilibrium. (2) Consumer's marginal utility of commodity j is the relative spot price $\frac{P_j}{P_J}$ multiplied by her marginal utility of numeraire commodity J. This is described formally as $\frac{\mathcal{D}_j u(\vec{C}_t)}{\mathcal{D}_J u(\vec{C}_t)} = \frac{P_{j,t}}{P_{J,t}}$ for each period and state.

2.2.2 Stochastic Discount Factor

The indirect utility function $V(\vec{P}, E)$ is defined as

$$\begin{aligned} V(\vec{P}, E) = \max_{\vec{C}} \quad & u(C_1, C_2, \dots, C_J) \\ s.t. \quad & \sum_{j\in\mathcal{J}} P_j \cdot C_j \le E. \end{aligned} \tag{S.1}$$

Here, consumption basket must be non-negative: in each sector j, the quantity C_j can't be negative. This paper only considers the normal situation with all prices are strictly positive in each sector.

The indirect utility function $V(\vec{P}, E)$ represents the equivalent utility from consumption spending E and the price of commodities $\vec{P}$ from basket $\vec{C}$. Lemma 1 says in consumer's problem (P.1), one can replace the consumption basket $u(\vec{C})$ with indirect utility function $V(\vec{P}, E)$ in consumer's life-time utility, the explicit consumption basket $\sum_{j\in\mathcal{J}} P_j \cdot C_j$ with expenditure E in the budge constraint. Simplifying the basket decision using indirect utility function doesn't change the outcome of financial asset decisions.

Lemma 1. *Define the consumer's optimal expenditure problem (P.2) as*

$$\begin{aligned} \overline{V}_0^{\text{New}}(\vec{\theta}_0) \quad &= \quad \sup_{\tilde{E},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot V(\vec{P}_t, E_t)] \\ s.t. \quad & \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = E_t + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \\ & E_t \ge 0; \quad \textit{financial constraint.} \end{aligned} \tag{P.2}$$

Optimization problem (P.2) yields equivalent value as optimization problem (P.1). For each optimal consumption policy C^* in problem (P.1), expenditure E^* such that

$$E^*_t = \sum_{j\in\mathcal{J}} P_{j,t} \cdot C^*_{j,t}, \quad \forall t, z^t \tag{11}$$

is an optimal policy in optimization problem (P.2).

In the appendix, I verify the equivalence between optimal stream of consumption basket and expenditure in the dynamic problem. For the representative consumer, the financial constraint never binds, so its shadow price is zero along the optimal path of expenditure. Consumer's interior decision implies $\nu_t = \mathcal{D}_E V(\vec{P}_t, E_t)$, where ν_t is the shadow price of (t, z) budget constraint in the optimization problem (P.2), after the correction of subjective discount rate and natural probability.

Definition 2. *Define the* ***real stochastic discount factor*** $\tilde{M}$ *as*

$$\tilde{M}(\vec{P}_t, E_t) \equiv \mathcal{D}_E V(\vec{P}_t, E_t) \cdot P_{J,t}. \tag{12}$$

The real stochastic discount factor augments the natural distribution of economic states and subjective discount rate when determining the financial asset price. The indirect utility function $V(\vec{P}_t, E_t)$ is Homogeneous of Degree Zero (H.D.0), while the marginal utility $\mathcal{D}_E V(\vec{P}_t, E_t)$ depends on the choice of numeraire in the economy environment. I include $P_{J,t}$ to focus on the relative term $\frac{P_{J,t}}{E_t}$, and to avoid the distracted discussion related to nominal inflation [3]. Hereafter, I use the notation $\tilde{m} = \log(\tilde{M})$, and $\mathrm{d}\tilde{m}$ for the change of $\tilde{m}$, and similar notations for other time-series [4].

Because consumer is the investor in financial market, her shadow price of budget constraint is the stochastic discount factor. When consumer makes the optimal decision of financial portfolio, the consumption spending in the future is simultaneously determined. The shadow price of consumption expenditure is identical with that of financial wealth. Optimal interior decision gives consumer's Euler Equation in rebalancing portfolio. Corollary 1 provides the formal statement for this multi-sector economy.

Corollary 1. *Given the security k and the security f, the real total return $\tilde{R}_{k,t+1}$ and $\tilde{R}_{f,t+1}$ from time t to future time $t+1$ satisfy*

$$\mathbb{E}_t[\frac{\tilde{M}_{t+1}}{\tilde{M}_t} \cdot (\tilde{R}_{k,t+1} - \tilde{R}_{f,t+1})] = 0. \tag{13}$$

Corollary 1 says for arbitrary financial asset, the expected excess return is determined by covariance between excess return and variation in real stochastic discount factor. This is

[3] After adjustment using price of numeraire, utility is the same, as $V(\frac{1}{P_{J,t}} \cdot \vec{P}_t, \frac{1}{P_{J,t}} \cdot E_t)$. Marginal utility $\mathcal{D}_E V(\frac{1}{P_{J,t}} \cdot \vec{P}_t, \frac{1}{P_{J,t}} \cdot E_t)$ needs the additional term $\frac{1}{P_{J,t}}$, so we add back the $P_{J,t}$.

[4] The upper-case character is for the level of prices and expenditure. The lower-case character is for the log amount. I use the notation $\mathrm{d}p$ for log-change in price of sector j, $\mathrm{d}p_j = \log P_{j,t+1} - \log P_{j,t}$. I use tilde symbol for the log-change adjusted by numeraire, $\mathrm{d}\tilde{p}_j = \mathrm{d}p_j - \mathrm{d}p_J$.

similar with the case of single-sector economy, except that nominal return of asset is adjusted by price of numeraire. When considering excess returns, price of numeraire offsets. This allows the proper inference of real stochastic discount factor $\frac{\tilde{M}_{t+1}}{\tilde{M}_t}$, as proposed in next subsection [5].

In competitive equilibrium where the consumer has optimal decision, marginal utility of expenditure adjusted by price of numeraire, $\mathcal{D}_E V(\vec{P}_t, E_t) \cdot P_{J,t}$, is consumer's marginal utility of numeraire, $\mathcal{D}_J u(\vec{C}_t)$. Under certain situations, the function using quantities $\mathcal{D}_J u(\vec{C}_t)$ is complicate. Marginal utility of expenditure adjusted $\mathcal{D}_E V(\vec{P}_t, E_t) \cdot P_{J,t}$ is Homogeneous of Degree Zero, variation has simple expression over prices and expenditure. Using indirect utility function is convenient for describing the real SDF in empirical estimation. The conclusions about inferring the stochastic discount factor can be extended to an economy with endogenous production of commodities. When the consumer has optimal dynamic decision in expenditure and financial wealth, marginal utility of expenditure can be used for inferring the stochastic discount factor. This identification methodology is similar with (Yogo, 2006), where the details in production are irrelevant for using consumption to explain expected returns. Assumption over producers determines the cash flow in equity asset. For arbitrary financial asset, Corollary 1 explains its expected excess return.

2.2.3 Decomposition of Stochastic Discount Factor

In the motivating example of subsection 2.1, prices and expenditure reveal the real stochastic discount factor. Theorem 1 says one can express change of real stochastic discount factor into changes of prices and expenditure, for general situation of consumer preference. To better separate the level and change, I use **pricing kernel** as the equivalent phrase for change of real stochastic discount factor. **Risk price** describes the coefficient of changes in prices and expenditure, in the pricing kernel.

Theorem 1. *In the economy with consumption sectors $\mathcal{J}$, real pricing kernel, the first-order approximated change in real stochastic discount factor is*

$$\mathrm{d}\tilde{m} = -b_e \cdot (\mathrm{d}e - \mathrm{d}p_J) - \sum_{j \in \mathcal{J}} b_j \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) + o(h). \tag{14}$$

The $\mathrm{d}p_j$ is the first-order difference of log price $p_j = \log(P_j)$ in sector j. The $\mathrm{d}e$ is the first-order difference of log total consumption expenditure $e = \log(E)$. The vector of risk-price $\vec{b}$

[5]This paper is silent toward discussion of nominal inflation as in (Boons et al., 2020; Corhay and Tong, 2021). When there is common change in expenditure and prices, real stochastic discount factor doesn't change.

Although the choice of numeraire doesn't matter for discussing excess return for pairwise financial assets, it affects the definition of risk-free bond. For simplicity, this paper assume coupon of one-period (real) bond be constant across economy states with respect to numeraire. If the one-period bond in this economy has constant payment with respect to consumer price index, the real payment with respect to numeraire would be stochastic, contingent on the states.

is

$$
\begin{aligned}
b_e =&\gamma,\\
b_j =&-(\gamma-1)+\sum_{i\in\mathcal{J}}\eta_{j,i}-\sum_{k\in\mathcal{J}}\omega_k\cdot\sum_{i\in\mathcal{J}}\eta_{k,i}.
\end{aligned}
\tag{15}
$$

with γ as relative risk aversion coefficient $-\frac{\mathcal{H}_{E,E}V(\vec{P},E)\cdot E}{\mathcal{D}_E V(\vec{P},E)}$. The high-order term $o(h)$ is with respect to $h=\max\{\{\mathrm{d}p_j\}_j,\mathrm{d}e\}$.

Equation (14) of decomposition says both the fluctuation in expenditure $\mathrm{d}e$, and prices $\{\mathrm{d}p_j\}_{j\in\mathcal{J}}$ contribute to the variation in stochastic discount factor. If there is no change in prices, the pricing kernel is variation of expenditure multiplied by relative risk aversion coefficient γ. Theorem 1 states the new fact related to prices. Equation (15) of risk price summarizes the asymmetric contribution from price in each sector. When price P_j increases by one unit, real stochastic discount factor changes by $b_j\cdot\omega_j$, share of commodity ω_j in the whole consumption basket, multiplied by the risk price b_j. Across consumption sectors, risk price b_j can be different with each other. Given a small share ω_j, if the risk price b_j has large absolute value, pricing kernel can be sensitive to the variation in price of commodity j. Knowing the risk price vector $\vec{b}$ helps accurately gauge the contribution from prices.

Intuitively, marginal utility is high when prices are high. Consumer adjusts the quantity in each sector with different flexibility, so the impact from price in each sector are asymmetric. When the quantity adjustment is difficult, increase of marginal utility induced by price growth is large, such as the necessity commodity. This seems like that the necessity commodity is prior to other commodities, when consumer allocates quantities. When expenditure is lower, quantity of necessity commodity doesn't decrease in the same amplitude. We observe the additional sensitivity of marginal utility to price of necessity commodity.

Formally, consumer's shares convey the impacts of prices over level of utility. When shares are affected by expenditure, adjusted partial derivatives $\frac{\mathcal{D}_j V(\vec{P},E)}{\sum_{i\in\mathcal{J}}\mathcal{D}_i V(\vec{P},E)}$ has non-zero derivative to expenditure E. When the price vector and expenditure change in the same magnitude, consumption basket maintains the same. Marginal utility of expenditure is the weighted outcome of price-partial derivatives, $\mathcal{D}_E V(\vec{P},E)=-\sum_{j\in\mathcal{J}}\mathcal{D}_j V(\vec{P},E)\cdot\frac{P_j}{E}$. As such, the impacts from prices to marginal utility of expenditure $\mathcal{D}_E V(\vec{P},E)$, are not proportional with their impacts to utility $V(\vec{P},E)$. The share vector $\vec{\omega}$ in consumption basket reflects the impacts from prices to utility $V(\vec{P},E)$. It cannot capture the distinct sensitivity in pricing kernel with respect to prices.

Alternatively, we use consumer's adjustment of basket to interpret the asymmetric risk prices. In equation (15), the risk price for sector j's consumption price b_j has an additional term, the difference between $\sum_{i\in\mathcal{J}}\eta_{j,i}$ and the term weighted by expenditure share $\sum_{k\in\mathcal{J}}\omega_k\cdot\sum_{i\in\mathcal{J}}\eta_{k,i}$. Consumer adjusts shares in consumption basket when there is fluctuation in prices and expenditure. Pairwise difference of share elasticity $\eta_{j,i} - \eta_{k,i}$ describes how shares in sector j and k reacts to P_i (price of sector i). When the price vector and expenditure change in the same magnitude, consumption basket maintains the same. Pair difference $\sum_{i\in\mathcal{J}} \eta_{j,i} - \sum_{i\in\mathcal{J}} \eta_{k,i}$ describes how shares in sector j and k react to the expenditure E. If the composition of consumption basket changes with expenditure, we observe the asymmetric risk price vector $\{b_j\}_{j\in\mathcal{J}}$, and accurate measure of real SDF requires detailed prices.

Above analysis applies for the consumer with any path-independent direct utility function $u(\vec{C})$. One can always derive the implied indirect utility function $V(\vec{P}, E)$, the approximated variation of marginal utility $\mathcal{D}_E V(\vec{P}, E)$, and arrive to equation (14). The CES utility function (over quantities) is a special example of homothetic preference where there is no expenditure-effect in consumer's basket, consumption price of each sector has symmetric risk price $b_j = 1-\gamma$ [6]. This assumption of symmetric risk prices might be an inappropriate description. Theorem 1 provides the general expression for the pricing kernel, avoids the special assumption over risk prices.

As explained by (Cochrane, 1996), expected excess return of financial asset is determined by the covariance between excess return and pricing kernel, and risk price $\vec{b}$ tells us the contribution of each covariance. Corollary 2 formally explains how risk price determines the expected excess return.

Corollary 2. *Given the security k, the expected excess return $R^e_{k,t+1}$ satisfies*

$$\begin{aligned}\mathbb{E}_t[R^e_{k,t+1}] =& b_e \cdot \mathbb{E}_t\left[(\mathrm{d}e_{t+1} - \mathrm{d}p_{J,t+1}) \cdot R^e_{k,t+1}\right] \\ &+ \sum_{j\in\mathcal{J}} b_j \cdot \omega_{j,t} \cdot \mathbb{E}_t\left[(\mathrm{d}p_{j,t+1} - \mathrm{d}p_{J,t+1}) \cdot R^e_{k,t+1}\right].\end{aligned} \tag{18}$$

with excess return as the difference between nominal total return $R_{k,t+1}$ and risk-free rate $R_{f,t+1}$.

Using the pricing kernel with consumption prices, the expected return can be decomposed as covariance with price in each sector j, adjusted by the risk price b_j. In empirical estimation, we want to directly estimate the $\{b_j\}_{j\in\mathcal{J}}$ using detailed prices, to explain the variation of expected return across financial assets.

[6] Consider the special case of two sectors $\mathcal{J} = \{g, s\}$,

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^\rho + C_s^\rho)^{\frac{1-\gamma}{\rho}}. \tag{16}$$

The substitution elasticity ρ describes the sensitivity of $\frac{\omega_g}{\omega_s}$ with respect to price $\frac{P_g}{P_s}$. In this example, pairwise difference of share elasticity $\eta_{g,g} - \eta_{s,g} = \eta_{g,s} - \eta_{s,s} = \frac{\rho}{\rho-1}$. Assuming numeraire $J = s$, the pricing kernel is

$$\mathrm{d}\tilde{m} = -\gamma \cdot (\mathrm{d}e - \mathrm{d}p_J) - \sum_{j\in\mathcal{J}} (1-\gamma) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) + o(h). \tag{17}$$

Section 3 estimates this pricing kernel for an economy with two sectors. Section 4 compares the estimation using quantity. Section 5 extends the estimation for a more generalized environment, where goods and services are separated into food and non-food.

3 Estimation

This section estimates the pricing kernel of financial assets in a two-sector economy with goods and services $\mathcal{J} = \{g, s\}$. National Accounts use this classification since 1930s, provide accurate statistics for these sectors. I choose the service as numeraire. I estimate parameters $\vec{b} = (b_g, b_e)$ in the pricing kernel [7]

$$\mathrm{d}\tilde{m}(\vec{b}) \approx -b_e \cdot (\mathrm{d}e - \mathrm{d}p_s) - b_g \cdot \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s). \tag{20}$$

Information of prices and expenditure are summarized by two time series factors, expenditure relative to price of services $\mathrm{d}e - \mathrm{d}p_s$, price of goods relative to price of services $\mathrm{d}p_g - \mathrm{d}p_s$. All growths and returns are calculated in annual frequency. Moment $g_k(\vec{b})$ is the sample-average error of Euler Equation [8] for holding the risky asset k,

$$g_k(\vec{b}) \equiv \frac{1}{T} \cdot \sum_{t=1}^{T} [1 + \mathrm{d}\tilde{m}_{t+1}(\vec{b})] \cdot R^e_{k,t+1}. \tag{21}$$

I denote $g(\vec{b})$ as the vector of $\{g_k(\vec{b})\}_{k=1}^{K}$. The GMM estimator is the optimal parameter values (b_g, b_e) that minimize the average error of moments weighted by matrix W,

$$\vec{b}^* \equiv \arg\min_{\vec{b}} \quad g(\vec{b})' \cdot W \cdot g(\vec{b}) \tag{22}$$

Estimation have two stages: (1) the first stage assigns equal weight to each asset; (2) the second stage uses variance matrix from the first stage as the efficient weight matrix to minimize the standard error of parameters.

[7] The pricing kernel in the two-sector economy is

$$\mathrm{d}\tilde{m} \approx -b_e \cdot (\mathrm{d}e - \mathrm{d}p_J) - b_g \cdot \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_J) - b_s \cdot \omega_s \cdot (\mathrm{d}p_s - \mathrm{d}p_J). \tag{19}$$

All nominal time series are converted into real amount with respect to numeraire, services, $J = s$. So $\mathrm{d}p_s - \mathrm{d}p_J$ is constant zero. Identification of parameters (b_g, b_e) provide sufficient knowledge of pricing kernel.

[8] In the two-sector economy, the moment is

$$g_k(\vec{b}) = \frac{1}{T} \cdot \sum_{t=1}^{T} [1 - b_e \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_g \cdot \omega_{g,t} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})] \cdot R^e_{k,t+1}.$$

Subsection (3.1) explains the measure of total expenditure and prices for the economy of goods and services, and describes the two time series factors for estimating the pricing kernel. Subsection (3.2) describes the main estimation outcome. Subsection (3.3) provides the estimations using alternative testing assets, definitions of consumption sectors, and different sample periods.

3.1 Data Description

3.1.1 Data Construction

I use annual data during 1964-2019 in Table 2.3.4, Table 2.3.5 from the NIPA to construct the sector-level price and total non-durable consumption expenditure. I consider goods and services within nondurable consumption:

- good: food-at-home, apparel, other non-durable goods. I remove energy from the non-durable good sector referring (Nakamura, 2008).
- service: food-away, recreation, health care, financial service, and other service. I remove public transportation, housing from the service sector, following (Hazell et al., 2020).

Expenditure is measured as the total spending across these sectors. I construct the Fisher index [9] as the sector-level price index. Figure (A.1) visually shows that relative price of goods is stationary after adjustment using price of services.

The equity portfolios are from the Kenneth French data library on the website. I choose the equity portfolios based on main cross-section anomalies documented by literature. In the benchmark testing assets, the 6 different stock characteristics are {Size, Book-to-Market, Profitability, Investment, Momentum, Earning/Price ratio}. For each characteristic, there are 5 single-sorted portfolios. In total, there are 30 testing assets. Mutual fund managers build portfolios referring to cross-section return anomalies, so the testing assets replicate the market practice of professional investors. For example, value funds pool companies with high book-market ratio. I estimate the pricing kernel during the time interval of 1965-2019, when the equity portfolios are all available. Combination of anomalies addresses the critique of linear reformation in testing assets. For example, the profitability portfolios are in smaller correlation with the value portfolios.

[9] NIPA reports the nominal expenditure and price indices for each type of goods and services. In each line (category) of personal consumption expenditure table, price is normalized to 1 in the base year, and real quantity is the expenditure adjusted by the price. In each sector, chained quantity index is the summation of quantities for all categories. Sector-level price index is calculated as expenditure divided by the chained quantity index. Previous theoretical discussion of consumer price index uses the formula of Tornqvist Index, for analytical explanation of quantity index. In historical data, Tornqvist index and Fisher index are similar.

3.1.2 Descriptive Statistic

Panel (A) of Table (1) provides the descriptive statistic for the two time series: the expenditure $\mathrm{d}e - \mathrm{d}p_s$, and the relative price of goods $\mathrm{d}p_g - \mathrm{d}p_s$. The first column shows that the mean annual growth rate of expenditure is 1.27%, mean annual growth in relative price of goods is -1.33%. The second and the third columns show the standard deviation and autocorrelation coefficient in each time series. The AR(1) coefficient for expenditure is 0.36, and 0.47 for relative price of goods .

Panel (B) of Table (1) provides the correlation coefficient between relative price of goods and other consumption outcome. The growth in expenditure has weak positive correlation with relative price of goods, as 0.26. Figure (1) provides the visualization for weak correlation between expenditure and relative price of goods. This reflects the fact: goods and services have different business cycle. Alternatively, Figure (A.2) demonstrates the gap of quantity growths across the two sectors. For comparison with the nondurable consumption in literature, I use the quantity index in the same method with NIPA. In this two-sector economy, we observe the weak negative correlation between quantity index and relative price of goods, as -0.17.

Table (A.2) provides correlation coefficients for other business cycle indices. Relative price of goods is negatively correlated with market excess return. Correlation with aggregate labor input is insignificant. Variation in labor income has high correlation with the expenditure, but the correlation with relative price of goods is a weak positive number 0.21. Unfiltered consumption quantity in (Kroencke, 2017) extracts a latent time series from the raw quantity index provided by NIPA. Relative price has negative correlation -0.31 to unfiltered consumption. I construct the long time series of garbage growth in the spirit of (Savov, 2011), measured using the generation waste. Relative price has inaccurate correlation coefficient -0.06 to this alternative measure of consumption quantity. When breaking the quantity index into expenditure and prices, the time series of relative price provides additional information not captured by other economic indicators.

Pairwise correlation between relative price and prices in detailed categories are provided in appendix Table (A.3). For simplicity, all prices are adjusted by price of services, assuming composite service is well-defined. The Food-at-Home has highest correlation with the relative price of goods, because it is volatile and contributes the main share for sector of goods. Dickey-Fuller test of relative price of goods and other prices in main sectors are provided in Table (A.4) in the appendix. Growth in relative price of goods is stationary, as demonstrated in Table (A.4). During the sample period 1965-2019, average share of goods $\omega_{g,t}$ is 39.6%.

3.2 Main Results

Subsection (3.2.1) reports the point estimates of risk price vector. Subsection (3.2.2) compares the fitness of estimation between the price-model and other asset pricing models. Subsection (3.2.3) reports the estimations using Size-BM 25 portfolios and Industry portfolios.

Subsection (3.2.4) documents the distribution of risk exposure to each time-series factor.

3.2.1 Benchmark Estimation

Table (2) reports the point estimates $\vec{b}^*$ that minimize the weighted error of Euler equation, for the pricing kernel in the economy of goods and services.

Risk price vector $\vec{b}$ reflects how each time series contributes to the variation of stochastic discount factor. For expenditure, the parameter value is $b_e^* = 28.80$. As in equation (15) of Theorem 1, risk price of expenditure b_e is consumer's relative risk aversion parameter γ.

The risk price of good's price is $b_g^* = -71.29$, in large absolute value. Admitting the fluctuation in prices, the component of relative price $b_g \cdot \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s)$ helps explain the cross-section of expected returns. Hypothetically, given larger risk price b_g, negative covariance to price of goods yields higher expected return.

Point estimate b_g^* is largely different with 27.80 from $-(\gamma - 1)$. This means impacts from prices to SDF largely depart from weights in CPI, hence detailed prices in goods and services bring considerable improvement in measuring the pricing kernel. All else equal, additional 0.1% independent volatility in price of goods implies 2.82% (7.129% adjusted by share of goods 39.6%) increase of volatility in SDF annually, while CPI weights suggest increase of volatility as 1.10% (2.780% adjusted by 39.6%).

In this estimation, Mean Absolute Pricing Error (MAPE) is 0.39%. MAPE has a clear interpretation for portfolio's annual excess return,

$$\text{MAPE} = \frac{1}{K}\sum_k \left| \underbrace{\frac{1}{T}\cdot\sum_{t=1}^{T} R^e_{k,t+1}}_{\text{Realized Average Excess Return}} - \underbrace{\left[\frac{1}{T}\cdot\sum_{t=1}^{T} -\mathrm{d}\tilde{m}_{t+1}(\vec{b}^*)\cdot R^e_{k,t+1}\right]}_{\text{Model-Predicted Excess Return}} \right|.$$

Across expected returns of testing assets, average statistic is 7.82% (median is 7.33%). The magnitude of MAPE reflects the amount of potential abnormal return in an investment strategy. If an investor uses a model of pricing kernel $\mathrm{d}\tilde{m}$ to hedge the aggregate risk, excess return in his portfolios would come from the covariation between his portfolio return and the pricing kernel. As we observe small MAPE, estimated pricing kernel explains the cross-sectional variation of expected returns.

For alternative evaluation of model fitness, I report the RMSE (Root Mean Square Error). The statistic $\text{RMSE} = \sqrt{\frac{1}{K}\sum_k \left|\frac{1}{T}\cdot\sum_{t=1}^{T}(1+\mathrm{d}\tilde{m}_{t+1})\cdot R^e_{k,t+1}\right|^2}$ takes care of the dispersed pricing error. In estimation, RMSE is 0.44%. I report the p-value for the J-stat, $\mathcal{J} \equiv T\cdot g_{\mathcal{T}}(\vec{b}^*)'\cdot W^*\cdot g_{\mathcal{T}}(\vec{b}^*)$ in 2nd stage of estimation. The p-value of J-stat is high [10], there exists no concern of over-identification.

[10] If we observe the J-stat to be tiny, that means we don't observe variation in the moment of Euler Equation

3.2.2 Comparison of Models

In this section, the pricing kernel in equation (20) is named as the **Price-Model** of Consumption-CAPM, in abbreviation **P-ND**. Table (4) evaluates if price-model **P-ND** is a successful asset pricing model, by comparing fitness with other models. Table (3) provides the simplified estimation for the formal examination in Table (4). Figure (3) directly demonstrates the model error in scatter plots.

For comparable evaluation of the fitness, I consider four traditional asset pricing models. These asset pricing models are **CAPM** (Capital Asset Pricing Model), **FF-5** (Fama-French 5-Factor Model), and two consumption-based models using quantities, **C-ND** (quantity-model of non-durable), **C-D** (quantity-model augmented by the durable quantity). Here, I explain the construction of other consumption-based models.

Model **C-ND** uses direct utility function to describe the marginal utility. In particular, I assume non-durable composite commodity C_{nd} is well defined, described by the chained-quantity index. Consumer has the CRRA preference for the utility flow in each period, $u(C_{nd}) = \frac{C_{nd}^{1-\gamma}}{1-\gamma}$. The pricing kernel is approximated as

$$\mathrm{d}\tilde{m}_{t+1} \approx -b_c \cdot \mathrm{d}c_{nd,t+1}. \tag{23}$$

The risk price for nondurable consumption (quantity index) is $b_c = \gamma$, the risk aversion coefficient in the traditional consumption-CAPM.

Model **C-D** considers the durable stock affects the utility flow of representative consumer. Construction of durable stock C_{dur} follows (Yogo, 2006). I consider the utility flow is log-separable in durable stock and non-durable quantity $u(C_{nd}, C_{dur}) = \frac{(C_{nd} \cdot C_{dur}^{\chi})^{1-\gamma}}{1-\gamma}$. The pricing kernel is decomposed as

$$\mathrm{d}\tilde{m}_{t+1} \approx -b_{nd} \cdot \mathrm{d}c_{nd,t+1} - b_{dur} \cdot \mathrm{d}c_{dur,t+1}. \tag{24}$$

For comparison with model **C-ND**, I use notation b_{nd}. It is determined by the risk aversion coefficient γ. For the stock of durable consumption good, the risk price is $b_{dur} = -(1-\gamma) \cdot \chi$.

For comparable analysis, model **P-D** (price-model augmented by the durable quantity) considers the durable stock affects the utility flow of representative consumer. This assumption is similar with the two-stage budget system in (Parodi et al., 2020). The durable stock acts as the parameter for the indirect utility function from budget allocation decision in non-durable consumption basket. The utility flow is $u(C_g, C_s, C_{dur}) = u(C_g, C_s) \cdot C_{dur}^{\chi \cdot (1-\gamma)} = V(P_g, P_s, E_{nd}) \cdot C_{dur}^{\chi \cdot (1-\gamma)}$ with E_{nd} for expenditure in nondurable. When inferring the fluctuation

$(1 + \mathrm{d}\tilde{m}_{t+1}) \cdot R^e_{k,t+1}$ across testing assets. This would leads to over-identification of parameter $\vec{b}$. An extreme case would be that the asset returns are highly correlated, $R^e_{k,t+1} \equiv k \cdot R^e_{1,t+1}$, then we in fact use the single moment for asset-type No.1. When using the single moment to identify two parameters b_g and b_e, we have the issue of over-identification.

of marginal utility from non-durable expenditure, I consider the separable effect from change of durable stock,

$$\mathrm{d}\tilde{m} \approx \underbrace{-b_e \cdot (\mathrm{d}e - \mathrm{d}p_s) - b_g \cdot \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s)}_{\text{Durable Stock is fixed}} \quad \underbrace{-b_{dur} \cdot \mathrm{d}c_{dur}}_{\text{Quantity Change of Durable}} \quad . \tag{25}$$

This estimation works as if the durable stock is the state variable when calculating the marginal utility of service. The first term uses the durable stock as the previous observed amount, calculates the variation of marginal utility driven by the variations of the nondurable good and service. I assume the contribution of durable stock is stationary [11].

Approximated stochastic discount factor in equation (20) involves the time-varying consumption basket ω_g. In Table (2) of the two-sector economy, the parameters (b_e, b_g) are identified using the equation (18)

$$\begin{aligned}\mathbb{E}_t[R^e_{k,t+1}] =& b_e \cdot \mathbb{E}_t\left[(\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) \cdot R^e_{k,t+1}\right] \\ &+ b_g \cdot \omega_{g,t} \cdot \mathbb{E}_t\left[(\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1}) \cdot R^e_{k,t+1}\right].\end{aligned} \tag{26}$$

For comparison with asset pricing models in the literature, we begin with the simple linear asset pricing models.

Denote the vector of macro risk factor as $\vec{f}$. Under the simple situation of linear asset pricing model with $\mathrm{d}\tilde{m}_{t+1} = -\vec{b}\cdot\vec{f}_{t+1}$, if the drift term of time-series factors $\mathbb{E}_t[\vec{f}_{t+1}]$ is known, there is the simplified identification with scaling, $\mathbb{E}_t[R^e_{k,t+1}] = \frac{\vec{b}}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} \cdot \mathbb{E}_t\left[(\vec{f}_{t+1} - \mathbb{E}_t[\vec{f}_{t+1}]) \cdot R^e_{k,t+1}\right]$. which is derived from splitting the drift-term and diffusion term in stochastic discount factor $\mathrm{d}\tilde{m}_{t+1}$, $\mathbb{E}_t[R^e_{k,t+1}] = -\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}] \cdot \mathbb{E}_t[R^e_{k,t+1}] - \mathbb{E}_t\left[(\mathrm{d}\tilde{m}_{t+1} - \mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]) \cdot R^e_{k,t+1}\right]$. Empirical asset pricing literature, especially the literature of consumption-based asset pricing models, often implicitly includes such assumption of constant expectation (constant conditional mean) $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$, in pursuit of equivalence between the Fama-Macbeth two-pass regression and the GMM estimation with equal weights. Referring (Cochrane, 1996), I use the risk-free rate as the expected drift in stochastic discount factor [12] $R_{f,t+1} = \mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]$. By assumption of constant expectation and linear asset pricing model, risk-free rate is constant, and directly measured as the sample-average outcome R_f. The asset pricing equation becomes familiar to literature,

$$\mathbb{E}_t[R^e_{k,t+1}] = \frac{\vec{b}}{1+R_f} \cdot \mathbb{E}_t\left[(\vec{f}_{t+1} - \mathbb{E}_t[\vec{f}_{t+1}]) \cdot R^e_{k,t+1}\right] \tag{27}$$

[11] In this example, the utility flow $u(C_g, C_s, C_d) = u(C_g, C_s) \cdot C_d^{\chi\cdot(1-\gamma)}$ obeys the feature of constant contribution from the stock of durable goods. (Yogo, 2006; Eraker et al., 2016) shows the empirical importance in inferring the SDF based on business cycle of durable stock. (Yogo, 2006) considers the CES utility function where the consumer substitutes nondurable with the durable stock. The contribution from durable growth to marginal utility of nondurable is time-varying. The example provided here is a simplified version.

[12] Directly replacing $\frac{\vec{b}}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} = \frac{\vec{b}}{1+\vec{b}\cdot\vec{g}_f}$ would maintain the non-linear function over risk price $\vec{b}$.

with $\vec{b}$ to be estimated. The price-model is nonlinear, with the time-varying expenditure share ω_t contingent on the historical path. For comparison, the model **$\mathbf{P}^L$-ND** is the linear factor model that simplifies the price-model **P-ND** (price-factor model). This linear factor model with relative price is $\mathrm{d}\tilde{m}_{t+1} = -b_{e,L} \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_{p,L} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})$. For model **P-D** , the simplified linear factor model with relative price and durable quantity growth is $\mathrm{d}\tilde{m}_{t+1} = -b_{e,L} \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_{p,L} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1}) - b_{dur,L} \cdot \mathrm{d}c_{dur,t+1}$, denoted as model **$\mathbf{P}^L$-D**.

Table (3) summarizes the model fitness in the simplified estimation. For all asset pricing models, estimation with the additional assumption of constant drift term $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$ generates larger equation error compared to the benchmark estimation of model **P-ND** in Table (2) [13]. Specifically, the consumption-CAPM of quantity index, model **C-ND**, generates huge pricing error, as high as 7.85% annually. This model is rejected by the historical data. In model **P-ND**, expenditure share ω_t serves as the sufficient statistic for the impact of by-sector price over consumer's utility, summarizing the unobserved consumer taste parameter and elasticity matrix. The simplified model **$\mathbf{P}^L$-ND** doesn't include this part, its equation error is 1.15% annually, larger than the 0.39% of benchmark model. However, the simplified model **$\mathbf{P}^L$-ND** has smaller equation error than the traded factor models, and the model with variation of durable stock growth **C-D**. This indicates the variation in the two time series risk factors, expenditure relative to price of services $\mathrm{d}e - \mathrm{d}p_s$, price of goods relative to price of services $\mathrm{d}p_g - \mathrm{d}p_s$ helps convey the information in the true pricing kernel.

Table (4) summarizes the model fitness in the formal estimation. All the traded factor models and the consumption-based models are estimated using equation $\mathbb{E}_t[R^e_{k,t+1}] = -\mathbb{E}_t\left[\mathrm{d}\tilde{m}_{t+1} \cdot R^e_{k,t+1}\right]$, without the additional assumption of constant expectation.

In Table (4), the CAPM model has MAPE as high as 1.58%. Given the failure of CAPM, investors seek abnormal returns from the value strategy and profitability strategy. After adding the value factor and profitability factor, the multi-factor asset pricing models gradually deplete the potential abnormal return in investment practice. In Table (4), the Fama-French 5-factor model has a high MAPE 0.79%. This means that the investor can still build the abnormal return from systematic risk not captured by Fama-French 5-factor model. There are more dispersed pricing errors across testing assets. RMSE is much larger than MAPE in the traded-factor models.

MAPE is 0.71% for the pricing kernel of non-durable quantity index, in model **C-ND**. This comparison shows that the quantity-model has larger model error than 0.39% of price-model. The improvement of model-fitness mainly occurs in the testing assets of Size-BM and the Momentum. Improvement occurs because there is large dispersion of factor-loading to-

[13] Often, empirical asset pricing test uses the demeaned time series risk factors. This implicitly assumes the constant drift term, and estimate it as the realized sample-mean $\vec{g}_f \approx \frac{1}{T} \cdot \sum_{\tau=1}^{T} \vec{f}_{\tau+1}$. The simplified estimation inherits this practice. Alternatively, one can add more equations to directly estimate the unconditional drift $\vec{g}_f$, but the choice of weight in GMM estimation is not determined and invokes further problems.

ward the relative price of goods, inside Size-BM testing assets. The Momentum assets have a large dispersion of factor-loading toward expenditure. Model **C-D** has MAPE as 0.66%, because the durable quantity growth provides additional information for macro-economic states. Model **P-D** allows for a more accurate measure of consumer's marginal utility of non-durable expenditure, by admitting the impact from variation of durable stock growth. The MAPE is 0.27%, smaller than the model **C-D** with non-durable quantity and durable stock.

Figure (2) illustrates the two components in MAPE for the simplified estimation outcome of asset pricing models estimated. The two time series risk factors, expenditure relative to price of services, price of goods relative to price of services, are important for approximating the true pricing kernel. Figure (3) illustrates the two components in MAPE for the six asset pricing models in formal estimation. The price-model **P-ND**, the generalized consumption-CAPM for robust specification of consumer preference, is a successful asset pricing model. As in the figure, model-predicted excess return is close with the expected return.

Accurate estimation of $\vec{b}$ using this equation is difficult, because it requires describing consumer's belief $\mathbb{E}_t[\vec{f}_{t+1}]$ in historical data. The consumption-CAPM of quantity index is drastically affected by the mis-specification of time-invariant expectation. The expected consumption growth correlates to price variation in current period. Admitting the expected consumption growth, has the similar role with selecting the price variation out. Still, the time-varying expectation of consumption expenditure raise challenge for identifying parameters in the model **P-ND**. Estimation has unstable point estimate for risk price in different specifications of testing assets. This paper takes the conservative approach: (1) use the full Euler Equation to estimate the risk price for each time-series in the consumer's marginal utility; (2) account the covariance between pricing kernel and excess returns, based on the accurate point estimates. For an asset pricing model with high accuracy in approximating the stochastic discount factor, the expected return financial asset is supposed to be linear over its covariance with the model-predicted pricing kernel. The post-estimation validation is detailed in sub-section 3.2.3.

3.2.3 Other Testing Assets

Previous discussion demonstrates the price-model has high in-sample fitness of estimation across equity portfolios built from cross-section anomalies. When using traditional testing assets and industry portfolios to estimate the pricing kernel, the point estimates of risk price are similar with Table (2). Across other testing assets, covariance with the benchmark pricing kernel in Table (2) explains the cross-section of expected return. This indicates the nice out-of-sample explanation.

Table (5) lists the point estimate of parameters using 25 portfolios double-sorted based on Size and Book-to-Market, the 30 industry portfolios. Columns **Size-BM 25** and **Industry 30** show the choice of testing assets has small effect over the point estimates [14]. When using

[14] Alternative estimations of quantity model and Fama-French 5-factor model are reported in Table (A.5)

the **Size-BM 25** portfolios, point estimate of b_e and b_g slightly changes to 30.05 and -68.26. When using the industry portfolios, point estimate of b_e and b_g slightly changes to 33.27 and -69.95 [15]. In these two setups, over-identification hypothesis are both rejected. The price-model has high fitness in these alternative estimations.

Figure (4) uses the estimated pricing kernel in Table (2), to account the contribution of covariance with pricing kernel in the expected return, for all testing assets. After being normalized with the volatility of pricing kernel, pairwise covariance is the slope term in univariate regression of excess return to pricing kernel. This β_m coefficient resembles the Market-beta of CAPM in financial news and media. As shown in Figure (4), beta to pricing kernel positively correlates with the sample average return across testing assets.

3.2.4 Cross-section of Risk Exposure

This subsection delineates the distribution of risk exposure, correlation between asset return and time-series factors in pricing kernel. Figure (5) plots $\vec{\beta}$ coefficients from first step in Fama-Macbeth regression [16], for benchmark testing assets **Mix-30**. As in second plot of Figure (5), these testing assets have negative correlation to relative price of goods. When price of goods increases slower than price of services, very likely we would observe positive realized excess return in the same year.

Table (6) shows small and value firms have negative correlation with relative price. The smallest portfolio in testing assets sorted by size, has the largest (negative) risk exposure -7.99, while the growth portfolio sorted by book-market has the risk exposure -3.46. Median statistic is -4.84. The point estimates for β_g are statistically different from zero, except for

and (A.7) in online appendix.

The point estimate of Fama-French 5-factor model changes in different testing assets. The risk price for the size factor b_{Size} has large standard error in **Industry 30** portfolio. The risk price for the value-growth factor b_{BM} is -2.24, with large standard error in the **Size-BM 25** portfolio. However, the point estimate of b_{BM} is -5.86 using the **Industry 30** portfolio, with small standard error. Fama-French 5-factor model is constructed with the equity portfolios based on cross-section anomalies, correlation between traded factors and the testing assets is supposed to be more strong. Nonetheless, when separating these portfolios based on construction, irrelevant traded factors tend to have weak correlation. Subsection 3.3.1 provides more detailed discussion.

[15] In estimating the simplified model with additional assumption of constant drift $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$, the covariance in testing assets $\mathbb{E}_t\left[(\vec{f}_{t+1} - \vec{g}_f) \cdot R^e_{k,t+1}\right]$ is weak, hence the point estimates are unstable. Appendix Table A.10 reports those point estimates. The parameter of price $\hat{b}_{g,L}$ changes from -101.03 in estimation using **Mix-30** portfolios to -79.69 in that of **Industry-30** portfolios. The parameter of expenditure $\hat{b}_{e,L}$ is positive but has large standard error in first-stage estimation.

[16] Fama-Macbeth regression allows for straightforward interpretation of risk-premium and risk exposure, but it requires more assumptions than GMM estimation, one can find more explanation in (Cochrane, 1996). For linear pricing kernel $\mathrm{d}m_{t+1} = -\vec{b} \cdot \vec{f}_{t+1}$, 1st step regression calculates the risk exposure $\vec{\beta}$ from $R^e_{k,t+1} = a_k + \vec{\beta}_k \cdot \vec{f}_{t+1}$, 2nd step computes the slope of sample-average return over risk exposure as $\mathbb{E}_t[R^e_{k,t+1}] = \vec{\beta}_k \cdot \vec{\lambda}$.

several portfolios such as the growth portfolio. Point estimates for risk exposure to expenditure β_e are often inaccurate. Table (A.6) in appendix reports the risk exposure for other testing assets. In **Size-BM 25** portfolios, small and value firms have risk exposure in larger absolute value. Most portfolios in **Industry 30** have negative correlation to relative price except for Mining and Coal industries.

Table (6) reports the time-series average risk-premium in the second step of Fama-Macbeth regression, for benchmark and alternative testing assets [17]. Covariance matrix between expenditure and price augments the risk price vector. They jointly determine risk premium. For each group of testing assets, I report both the risk-premium assuming zero-beta rate identical with risk-free rate, and the risk premium without this assumption. Risk-premium for relative price of goods λ_g is significantly negative. All else equal, additional unit of negative risk exposure β_g yields positive return of 1.64% per year.

Risk premium for relative price of goods is quantitatively similar when using other testing assets. Across **Size-BM 25** portfolios, point estimate of λ_g is -1.56%. Across **Industry 30** portfolios, point estimate is -1.43%. In the **Size-BM 25** portfolios, the dispersion of correlation is highly linear, so the identification is weaker. Risk exposure in **Industry 30** has sufficient variation for identifying risk premium for relative price of goods λ_g, but not so for relative expenditure. Point estimate of risk premium λ_e is inaccurate.

3.3 Supplementary Estimation

Subsection (3.3.1) provides the supplementary estimation for the two-sector model, using alternative testing assets. Subsection (3.3.2) investigates definitions of goods and services. Point estimates and fitness of estimation are similar with the benchmark results. Subsection (3.3.3) estimates the model in different sub-sample periods during 1935-2019. In early sample of 1935-1989, point estimates of risk price vector have the same sign, but in smaller absolute value. The sample of 1950-2004 has the similar estimation outcome with benchmark results.

3.3.1 Testing Assets

Table (7) further investigates estimation inside the benchmark testing assets **Mix 30**. Table (7) estimates the pricing kernel in three subsets of testing assets: (a) 5 Size + 5 BM; (b) 5 Profitability + 5 Investment; (c) 5 Momentum + 5 Earning/Price ratio. In the group of **Profit-IK**, the Fama-French 5 factor model generates the smaller model error. In other

[17] OLS-R^2 in 2nd step is reported. I also consider the GLS regression for each model, using the weight-matrix suggested by (Lewellen et al., 2010), to mitigate the concern of strong correlation within testing assets. GLS-R^2 tends to over-interpret certain subset of testing assets in small total volatility, but ex-ante we don't know the systematic component there. Statistic OLS-R^2 and GLS-R^2 assume the zero-beta-rate is identical with risk-free rate, with no omitted component in pricing kernel. I also include statistic COLS-R^2 and CGLS-R^2, where 2nd step of regression includes the intercept term, assuming all assets have identical constant term in excess return. Appendix Table (A.5) reports these statistics for other asset pricing models.

two groups, **Size-BM** and **MoM-EP**, the price-model has better model fitness in the GMM estimation. The MAPE decreases from 0.55% to 0.33% for the testing assets **Size-BM**. It decreases from 0.55% to 0.36% for the testing assets **MoM-EP**.

Among the two groups **Size-BM** and **MoM-EP**, the point estimate for parameters (b_e, b_g) are similar. But the point estimate of b_g is slightly lower when using the testing assets **Profit-IK**. Joint distribution of return covariance affect the point estimates. As these tests using small group of testing assets, when the variation of risk exposure is weak, point estimates are distorted. This situation is more severe for estimation asset pricing models with multiple factors, such as Fama-French 5-factor model [18]. We observe more stable point estimate for the price-model of consumption CAPM. From the side of pricing kernel, this model captures consumer's marginal utility in the more accurate way. From the side of estimation, testing assets have sufficient variation in risk exposure, so we observe similar points estimates across groups of testing assets.

3.3.2 Alternative Definitions

Table (8) shows alternative definitions of consumption sector have small affects over the point estimate of b_e and b_g. Column **Financial-Service** removes the quasi-durable service categories such as health-care and financial service [19], the point estimates are similar. Column **NIPA-Nondurable** uses the classification of non-durable good and service of NIPA, the point estimate of b_g is quantitatively similar. This definition includes the gasoline (energy) for the non-durable good, housing and public transportation for the service, compared with the benchmark estimation in Table (2). Table (2) uses a more rigorous definition of nondurable consumption sector following the literature.

Table (9) investigates whether the price index construction procedure affects the point estimate of b_e and b_g. When using the first-order difference of Tornqvist Index, the point estimate is quantitatively similar. If I use the average change of category-level price index, the point estimate of expenditure parameter b_e is no longer accurate. Using simply the cross-category average price produces non-negligible approximation error. Construction of Fisher-Index and Tornqvist Index takes the size of consumption categories into consideration, approximation errors are smaller when I use these price indices for the simplified two-sector economy.

3.3.3 Long Sample

Table (10) lists the estimation of risk price and risk premium during the time intervals: (a) 1935-1989; (b) 1950-2004; (c) 1965-2019. I construct these alternative samples of 55 years,

[18] Fama-French 5-factor model has under-identification when the number of testing assets is small. This problem maintains when using **Size-BM 25** and **Industry 30**.

[19] Equity issuance cost affect asset returns from the production side, as shown in (Belo et al., 2019). Variation in price of financial services might capture the fluctuation in financing cost.

with the same length as the benchmark sample. Construction of consumption price indices was substantially revised in the early years of NIPA statistics as documented in (Rippy, 2013), so the sample starts from 1935. For consistent testing assets during the three time blocks, I use the **Size-BM 25** portfolios. Across the three sample periods, all point estimates for the relative price of goods b_g are negative. But the point estimate in the early sample 1935-1989 has much larger absolute value in the first-stage, and drastically changes to smaller value in the second-stage estimation. This shows certain testing assets have high idiosyncratic noise in the early years of NIPA records. In the period 1950-2004 and the period 1965-2019, we observe close point estimates for the risk price b_g.

4 Comparison between Quantity and Prices

In empirical estimation, detailed prices help provides more accurate description of pricing kernel than the quantity index. I use the Tornqvist index to provide the analytical explanation for quantity index. When consumer preference is non-homothetic, quantity index cannot accurately measure the SDF. Subsection 4.1 derives the quantity index for examples of Cobb-Douglas utility function and Constant Elasticity of Substitution. Under these situations of homothetic preference, quantity index describes the pricing kernel. This argument doesn't hold in other general situations. Subsection 4.2 shows in historical data, pricing kernel measured using quantity index leads to over-estimation of consumer's risk-aversion coefficient, and worse explanation of expected returns. Subsection 4.3 shows the difficulty in identifying a suitable utility function over quantities. General indirect utility function is an alternative way to describe consumer's preference when the direct utility function over quantities is not tractable. The non-parametric SDF using prices provides a simpler expression, so it is convenient for empirical estimation.

4.1 Explanation of Quantity Index

Consider the special case where the consumer has utility function as $u(C_g, C_s) = \frac{[C_g^{\omega_g} \cdot C_s^{1-\omega_g}]^{1-\gamma}}{1-\gamma}$, This reads as a monotonic transformation for Cobb-Douglas utility function. Share of goods in the consumption basket is fixed as ω_g. We use the service as the numeraire, the real stochastic discount factor is the marginal utility of service[20]. The pricing kernel is written as

$$\mathrm{d}\tilde{m} = -\gamma \cdot (\mathrm{d}e - \mathrm{d}p_J) - (1-\gamma) \cdot \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_J) - (1-\gamma) \cdot \omega_s \cdot (\mathrm{d}p_s - \mathrm{d}p_J). \tag{29}$$

[20]Marginal utility of service is $\mathcal{D}_{C_s} u(C_g, C_s) = [C_{g,t}^{\omega_g} \cdot C_{s,t}^{1-\omega_g}]^{-\gamma} \cdot C_{g,t}^{\omega_g} \cdot C_{s,t}^{-\omega_g} \cdot (1-\omega_g)$. The variation is

$$\mathrm{d}\tilde{m} \approx (1-\gamma) \cdot \omega_g \cdot \mathrm{d}c_g + [(1-\gamma) \cdot \omega_s - 1] \cdot \mathrm{d}c_s. \tag{28}$$

We can replace quantity of services as $C_s = \frac{(1-\omega_g) \cdot E}{P_s}$, and similarly for the goods $C_g = \frac{\omega_g \cdot E}{P_g}$.

In this example, consumer price index as Tornqvist index [21] provided by NIPA summarizes the contribution of detailed prices,

$$\mathrm{d}\log(\mathbf{P}_{\text{Tornqvist}}) = \omega_g \cdot \mathrm{d}p_g + \omega_s \cdot \mathrm{d}p_s. \tag{30}$$

Replacing variation in consumer price index, the pricing kernel is simplified as, $\mathrm{d}\tilde{m} = -\gamma \cdot (\mathrm{d}e - \mathrm{d}p_J) - (1-\gamma) \cdot [\mathrm{d}\log(\mathbf{P}_{\text{Tornqvist}}) - \mathrm{d}p_J]$. Alternatively, we can defines composite good as $\mathbf{C}_t = C_{g,t}^{\omega_g} \cdot C_{s,t}^{1-\omega_g}$. This quantity index is equivalent with total expenditure adjusted by consumer price index,

$$\mathbf{C}_t = \frac{E_t}{\mathbf{P}_{\text{Tornqvist,t}}}. \tag{31}$$

In this economy of goods and services, quantities in consumption basket is summarized the quantity index, as if we are in a one-sector economy of composite commodity.

When choosing the composite good as numeraire, the pricing kernel is,

$$\mathrm{d}\tilde{\mathbf{m}} = -\gamma \cdot [\mathrm{d}e - \mathrm{d}\log(\mathbf{P}_{\text{Tornqvist}})]. \tag{32}$$

Equivalently, this is standard Consumption-CAPM,

$$\mathrm{d}\tilde{\mathbf{m}} = -\gamma \cdot \mathrm{d}\mathbf{c}. \tag{33}$$

More generally, when consumer preference has Constant Elasticity of Substitution, pricing kernel has the identical expression with equation (32). Consider the utility function over quantities as $u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^\rho + C_s^\rho)^{\frac{1-\gamma}{\rho}}$. The marginal utility of services [22] has approximated variation as,

$$\mathrm{d}\tilde{m} \approx -\gamma \cdot \underbrace{(\omega_g \cdot \mathrm{d}c_g + \omega_s \cdot \mathrm{d}c_s)}_{\text{weighted change in quantities}} - \underbrace{\omega_g \cdot (\rho - 1) \cdot (\mathrm{d}c_g - \mathrm{d}c_s)}_{\text{CPI v.s. } P_s}. \tag{35}$$

[21] The level of Tornqvist index is recovered from the growth of index in each period. Growth of index is using the changes of prices across sectors weighted by the previous expenditure share. In this economy with two sectors, the Tornqvist index is updated period by period as,

$$\mathbf{P}_{\text{Tornqvist},t+1} = \mathbf{P}_{\text{Tornqvist},t} \cdot exp^{[\omega_{g,t} \cdot \mathrm{d}p_{g,t+1} + \omega_{s,t} \cdot \mathrm{d}p_{s,t+1}]}.$$

When the shares are constant $\omega_{j,t} = \omega_j$, consumer price index has the analytical expression after normalization, $\mathbf{P}_{\text{Tornqvist},t} = P_{g,t}^{\omega_g} \cdot P_{s,t}^{1-\omega_g}$.

[22] Cobb-Douglas utility function is a special case of $\rho \to 0$. Marginal utility of services has variation as

$$\mathrm{d}\tilde{m} \approx -(\gamma - 1 + \rho) \cdot \frac{C_g^\rho}{C_g^\rho + C_s^\rho} \cdot \mathrm{d}c_g - [(\gamma - 1 + \rho) \cdot \frac{C_s^\rho}{C_g^\rho + C_s^\rho} - (\rho - 1)] \cdot \mathrm{d}c_s. \tag{34}$$

From proportional marginal utility across goods and services, we know $\frac{\frac{\partial u}{\partial C_g}}{\frac{\partial u}{\partial C_s}} = (\frac{C_g}{C_s})^{\rho-1} = \frac{P_g}{P_s}$. Given equation $(\frac{C_g}{C_s})^\rho = \frac{\omega_g}{\omega_s}$, we replace $\frac{C_g^\rho}{C_g^\rho + C_s^\rho}$ using shares. This gives the simplified result.

we arrive to the identical expression with equation (32) after organization [23] . We obtain the nice identical result because the shares help summarize the ratio of quantities.

CES utility function is a special case where the composition of consumption basket is unaffected by the change in expenditure. Other homothetic preferences also have this feature. We derive the identical pricing kernel with equation (32). One can confirm this prediction using Theorem (1): the indirect utility function of homothetic preference has the identical risk prices b_g and b_s, so we can use variation in Tornqvist price index to summarize variations in prices.

In the economy with good and service sectors, if there is the symmetric risk price $b_g = b_s$ across sectors, quantity index $\frac{E_t}{\mathbf{P}_{\text{Tornqvist,t}}}$ is sufficient statistic [24] for pricing kernel. However, we encounter the trouble when consumer preference doesn't have this special feature.

The motivating example of subsection 2.1 uses a typical indirect utility function in the Price-Independent General Linear preference, to demonstrate quantity index inaccurately measure the pricing kernel. When consumer's price-habit has the parameter $h > 0$, we have the risk price of good's price in larger absolute value, $|b_g| > |b_s|$. Under this circumstance, quantity index cannot describe the pricing kernel. Nonetheless, there is an explicit function of marginal utility over prices and expenditure in subsection 2.1.

Subsection 2.2 derives the marginal utility for the consumer with general preference. Equation (15) in Theorem (1) shows the risk price of consumption prices can be different with each other. Under this situation, composite price index cannot convey the different contribution from each commodity price. If the risk price vector is very asymmetric, when breaking the composite consumption quantity into prices and expenditure, we will obtain more information to accurately describe consumer's marginal utility. This estimation in Table (2) doesn't agree with homothetic preference over goods and services, as $b_g < -(b_e - 1)$. This clarifies why we cannot use quantity index to precisely measure the pricing kernel.

4.2 Estimation using Quantity Index

Column (1) of Table (11) reports the large point estimate of risk-aversion coefficient 106.47. Estimation uses the **Size-BM 25** portfolios for comparison with the literature. The linear approximated stochastic discount factor of quantity index is estimated using the equation

$$\mathbb{E}_t[R^e_{k,t+1}] = \frac{1}{1 + \mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} \cdot \gamma \cdot \mathbb{E}_t\left[(\mathrm{d}\mathbf{c}_{t+1} - g_c) \cdot R^e_{k,t+1}\right]. \tag{36}$$

[23] In the first component, one can substitute $\mathrm{d}c_g = \mathrm{d}e + \mathrm{d}\omega_g - \mathrm{d}p_g$ and similarly for $\mathrm{d}c_s$. Given $\omega_g + \omega_s = 1$, we have $\omega_g \cdot \mathrm{d}\omega_g + \omega_s \cdot \mathrm{d}\omega_s = 0$. The weighted change in quantities collapses as $\mathrm{d}e - (\omega_g \cdot \mathrm{d}p_g + \omega_s \cdot \mathrm{d}p_s)$. In the second component, one can substitute $\mathrm{d}p_g - \mathrm{d}p_s = (\rho - 1) \cdot (\mathrm{d}c_g - \mathrm{d}c_s)$, and arrive to $\omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s) = \mathrm{d}\log(\mathbf{P}_{\text{Tornqvist}}) - \mathrm{d}p_s$. The second component is canceled out when choosing the composite good as numeraire. The variation of real stochastic discount factor is marginal utility of services adjusted by price of composite good, $\mathrm{d}\tilde{\mathbf{m}} = \mathrm{d}\tilde{m} - \mathrm{d}p_s + \mathrm{d}\log(\mathbf{P}_{\text{Tornqvist}})$.

[24] In practice of NIPA, chained real quantity index is very similar with this analytical index.

The expected consumption growth $\mathbb{E}_t[\mathrm{d}\mathbf{c}_{t+1}]$ is assumed to be constant g_c, measured as the full-sample mean growth rate. The mean absolute error MAPE $= \frac{1}{K}\sum_k \left|\frac{1}{T}\cdot\sum_{t=1}^{T} R^e_{k,t+1} - \hat{\lambda}_c \cdot \hat{\beta}_{k,c}\right|$ in equation (36) is 9.53% annually. The cross-section of expected return cannot be explained by the correlation to consumption quantity growth. Using alternative testing assets for estimation doesn't change the large equation error in cross-section of expected returns. When estimation uses **Industry 30** portfolios as testing assets, point estimate of implied risk-aversion coefficient is −64.22, and 46.59 in estimation using **Mix-30** portfolios. Large pricing error in cross-section estimation is named as the failure of Consumption-CAPM [25] in literature since (Mankiw and Shapiro, 1986). This asset pricing model using aggregate consumption quantity index is mis-specified. Stringent assumption over the time-series of consumption quantity index growth, further exacerbates the pricing error.

Under the special case where the composite good is well-defined $\mathbf{C}_t = C_{g,t}^{\omega_g} \cdot C_{s,t}^{1-\omega_g}$, the stochastic discount factor using the quantity index has the explicit nonlinear function form $\tilde{\mathbf{M}} = \mathbf{C}_{t+1}^{-\gamma}$. Column (3) of Table (11) reports estimation of the nonlinear stochastic discount factor using the inter-temporal equation

$$\mathbb{E}_t\left[\left(\frac{\mathbf{C}_{t+1}}{\mathbf{C}_t}\right)^{-\gamma} \cdot R^e_{k,t+1}\right] = 0. \tag{39}$$

The point estimate of risk-aversion parameter is large 66.93 for risky assets in **Size-BM 25** portfolios. The gap of expected return and the covariance to nonlinear pricing kernel is large, MAPE $= \frac{1}{K}\sum_k \left|\frac{1}{T}\cdot\sum_{t=1}^{T} R^e_{k,t+1} - \frac{1}{T}\cdot\sum_{t=1}^{T}[1-(\frac{\mathbf{C}_{t+1}}{\mathbf{C}_t})^{-\gamma}]\cdot R^e_{k,t+1}\right|$ is 4.60% annually. The pricing error using nonlinear stochastic discount factor maintains large. Estimation outcomes are similar when the risky assets are **Industry 30** portfolios or **Mix-30** portfolios of

[25] One can translate the consumer's Euler equation as

$$\mathbb{E}_t[R^e_{k,t+1}] = \frac{1}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} \cdot \gamma \cdot \mathbb{E}_t\left[(\mathrm{d}\mathbf{c}_{t+1} - g_c)\cdot(R^e_{k,t+1} - \mathbb{E}_t[R^e_{k,t+1}])\right]. \tag{37}$$

using the covariance $(\mathrm{d}\mathbf{c}_{t+1} - g_c)\cdot(R^e_{k,t+1} - \mathbb{E}_t[R^e_{k,t+1}]) = \sigma_c \cdot \beta_{k,c}$. The asset pricing equation for financial asset k reads as

$$\mathbb{E}_t[R^e_{k,t+1}] = \beta \cdot \tilde{R}_{f,t+1} \cdot \gamma \cdot \sigma_c \cdot \beta_{k,c}. \tag{38}$$

The consumption-beta $\beta_{k,c}$ is estimated using the time-series regression $R^e_{k,t+1} = a_k + \beta_{k,c}\cdot \mathrm{d}\mathbf{c}_{t+1}$ for asset k. The σ_c is the volatility of consumption growth. Additional identification assumption is added, compared to the estimation specification in Section 3. Constant consumption volatility and expected consumption growth implies constant risk-free rate. The simplified identification has equivalent expression with 2nd step of Fama-Macbeth regression $\mathbb{E}_t[R^e_{k,t+1}] = \lambda_c \cdot \beta_{k,c}$, more risk-aversed consumer or more volatile consumption implies the higher risk premium $\lambda_c = \beta \cdot \tilde{R}_f \cdot \gamma \cdot \sigma_c$ from risk exposure to consumption.

Asset pricing tests in Consumption-CAPM in literature sometimes include the zero-beta rate λ_0 in cross-section regression $\mathbb{E}_t[R^e_{k,t+1}] = \lambda_0 + \lambda_c \cdot \beta_{k,c}$, considers excess return of risky assets have identical premium that are not explained by the consumption beta, eg. the risk-free rate equation might not hold. This alternative estimation still has large mean absolute error 1.29% annually.

representative anomalies. The exact nonlinear stochastic discount factor cannot explain the cross-section of expected return.

Column (4) of Table (11) estimate the linear approximated stochastic discount factor using equation,

$$\mathbb{E}_t \left[(1 + \mathrm{d}\tilde{m}_{t+1}) \cdot R^e_{k,t+1} \right] = 0, \tag{40}$$

compares the pricing kernel measured using the quantity index with the price-model. The model **C-ND** is identified using equations

$$\mathbb{E}_t \left[(1 - b_c \cdot \mathrm{d}\mathbf{c}_{t+1}) \cdot R^e_{k,t+1} \right] = 0 \tag{41}$$

The point estimate is $\hat{b}_c = 50.88$. Column (4) and Column (1) have identical approximated model of consumer's marginal utility. The pricing error is reduced for the statistic reason because estimation doesn't restrict the expected growth in each period. The nonlinear stochastic discount factor $(\frac{\mathbf{C}_{t+1}}{\mathbf{C}_t})^{-\gamma}$ in Column (3) doesn't put this restriction of constant expected growth. The model mis-specification error is slightly less severe than Column (1), but the model error is still large.

In the price-model of generalized consumption-CAPM, parameters are identified by estimating equations for risky assets

$$\mathbb{E}_t \left[[1 - b_e \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_g \cdot \omega_{g,t} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})] \cdot R^e_{k,t+1} \right] = 0, \tag{42}$$

In Column (1), estimation specification replicates the difficulty of consumption-CAPM in the annual frequency. For comparison, Column (2) uses the same lens, estimates the linear factor model with relative expenditure and relative price. This reduced-form model is informative for how the two time series factors convey the variation in stochastic discount factor, and how estimation restriction affects the parameter estimates. The point estimate for relative price $\hat{b}_{p,L}$ is negative and in much larger absolute value, while the point estimate of $\hat{b}_{e,L}$ is a small positive number with large standard error. Estimation of Column (2) has the same assumption of the constant drift term with Column (1). The volatility in the relative price of goods is understated, and similarly for the time series of expenditure. Parameters of abnormally large value are chosen to resolve the high expected return of the small volatility. Risky assets have covariance to the slow-moving component in the expenditure growth. Omitting this component of covariance leads to additional equation error and inaccurate point estimate.

Column (5) of Table (11) avoids the omitted variation in expectation of expenditure growth and price growth, attains the much smaller equation error, and relatively small risk-aversion coefficient. The point estimate $\hat{b}_e = 30.05$, much smaller than the risk aversion parameter $\hat{b}_c = 50.88$ in Column (4). As in the descriptive statistic Table (1), the time series of relative price of goods and the time series of expenditure have weak correlation. Intuitively, when these time series are combined together, the positive risk price b_e and the negative risk price b_g leads to unclear economic interpretation of coefficient for the quantity index b_c [26]. As described in Section 3, the risk price for quantity index $\hat{b}_c$ is interpreted as risk aversion parameter γ in literature. Comparing the point estimates, $\hat{b}_e < \hat{b}_c$, ignoring the heterogeneous consumption sectors leads to overstated risk aversion in empirical estimation.

Figure (A) plots the pairwise covariance between the excess return and the quantity of nondurable. Beta to quantity index surrounds zero, in weak correlation with sample average excess return. This echoes the huge pricing error of model **C-ND** in Table (3), the negative OLS-R^2 in Table (A.5), and the similar estimation outcome of cross-section asset pricing test in (Kroencke, 2017). Here are detailed explanations for this problematic outcome: (1) expenditure and price of goods have different risk prices in the true model of consumer's marginal utility, but quantity index adds assumption in combining these components; (2) the expected growth in nondurable expenditure is time-varying in the historical data, this further exacerbates the weak correlation between quantity-beta and expected return.

Previous estimation for model **C-ND** and model **P-ND** uses Euler Equation for holding the risky assets. Representative consumer is indifferent between the different equity portfolios and the risk-free asset. Estimation uses this fact to identify the consumer preference. Columns (6) to (7) in Table (11) further examine the risk-free rate equation

$$\mathbb{E}_t\left[\beta \cdot (1 + \mathrm{d}\tilde{m}_{t+1}) \cdot \tilde{R}^f_{t+1}\right] = 1. \tag{43}$$

This equation reflects consumer's trade-off between current consumption spending and saving. The constant subjective discount rate β is calibrated as 0.98. In Theorem 1, approximation of pricing kernel only considers the first-order terms. This approximation captures the main component when considering the covariance between pricing kernel and the excess return of risky assets. I use the sample variance of pricing kernel as the approximated second-order term, similar with (Savov, 2011; Kroencke, 2017). The vector of moments has $K+1$ equations: equation (40) for excess return of each risky asset $k \in \{1, \cdots, K\}$, and one equation (44) for the risk-free rate as

$$\frac{1}{T} \cdot \sum_{t=1}^{T} \beta \cdot \tilde{R}^f_{t+1} \cdot \left[1 + \frac{1}{T} \cdot \sum_{t=1}^{T} \mathrm{d}\tilde{m}_{t+1}(\vec{b}) - \frac{1}{2} \cdot \hat{Var}[\mathrm{d}\tilde{m}_{t+1}(\vec{b})]\right] = 1. \tag{44}$$

I maintain the equal weight for moments in the first-stage estimation of GMM. The equation (40) of each risky asset reflects consumer is indifferent to risk-free asset, while equation (44) reflects consumer is indifferent to current consumption and future spending. When including the equation of risk-free asset, the point estimates of risk prices are similar in the price model.

[26] In quarterly frequency, identification of consumption-based asset pricing model is challenged by the seasonality in consumption expenditure and asset returns, the point estimates for parameter b_c are unstable across quarters. Online table (A.13) shows the exacerbated difficulty in quarterly estimation. For price-model, estimated parameter value b_g is much larger in quarterly frequency.

Estimation and discussion of quantity index use the **Size-BM 25** equity portfolios for straight-forward comparison to the literature. The benchmark estimation in Section 3 uses **Mix 30** equity portfolios of representative anomalies, calculates moments for inter-temporal equations of risky asset excess return. Estimated price-model has close point estimates. Hereafter, estimation of consumer preference uses inter-temporal equations of risky assets, for coherent discussion in this paper.

4.3 Quantities for Two Sectors

In Table (11), we observe the improvement from model of quantity index toward the price-model **P-ND**. Estimation suggests consumer has non-homothetic preference over the goods and services. Price of goods has different risk price b_g from $-(b_e - 1)$. Directly estimating the parameter b_g allows for the more accurate measure of pricing kernel. When consumer has non-homothetic preference, it is difficult to derive the marginal utility using detailed quantities. Table (12) conducts empirical examination for whether we can use detailed quantities to describe the pricing kernel.

If we relax the CES preference to the more generalized situation, for example non-separable preference based on (Ait-Sahalia et al., 2004),

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\rho_g} + C_s^{\rho_s})^{\frac{1-\gamma}{\rho_s}},$$

marginal utility of services can be written in shares and quantities as [27],

$$\mathrm{d}\tilde{m} \approx -\frac{\rho_g}{\rho_s} \cdot [\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{1-\omega_g}{\rho_s}} \cdot \mathrm{d}c_g - \{[\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{1-\omega_g}{\rho_s}} + \gamma\} \cdot \mathrm{d}c_s. \quad (46)$$

Table (12) estimates the pricing kernel in linear function of variations in quantities,

$$\mathrm{d}\tilde{m} \approx -\hat{b}_{c_g} \cdot \mathrm{d}c_g - \hat{b}_{c_s} \cdot \mathrm{d}c_s. \quad (47)$$

Parameter b_{c_g} is sensitivity of marginal utility of services $\mathcal{D}_{C_g} u(C_g, C_s)$ with respect to C_g. Estimation of linear pricing kernel gives the rough point estimate of average sensitivity $\hat{b}_{c_g}$. And similarly for the $\hat{b}_{c_s}$.

As shown in the table, when using the quantity variation of goods (services) as the time-series factors to approximate the variation of consumer's marginal utility, the point estimate

[27] Using quantity of goods and that of services, marginal utility of services reads as

$$\mathrm{d}\tilde{m} \approx -\frac{\rho_g}{\rho_s} \cdot (\gamma - 1 + \rho_s) \cdot \frac{C_g^{\rho_g}}{C_g^{\rho_g} + C_s^{\rho_s}} \cdot \mathrm{d}c_g - [(\gamma - 1 + \rho_s) \cdot \frac{C_s^{\rho_s}}{C_g^{\rho_g} + C_s^{\rho_s}} - (\rho_s - 1)] \cdot \mathrm{d}c_s. \quad (45)$$

in the first stage is inaccurate. This specification of direct utility function creates high approximation error of model.

Table (13) investigates whether Stone-Geary preference provides a better description, as specified in (Lochstoer, 2009). The Stone-Geary preference qualitatively captures the essence of non-homothetic preference, but the financial economist meets difficulty in reasonably assuming the structure of consumption-habit. Table (13) shows inaccurate point estimates and worse fitness of estimation.

In the Column **Good**, I assume consumer has the zero habit in the basic-good sector, $u(C_g, C_s) = \frac{[C_g^{\overline{\omega}_g} \cdot (C_s - X_s)^{1-\overline{\omega}_g}]^{1-\gamma}}{1-\gamma}$, decompose the pricing kernel (using service as the numeraire) as

$$\mathrm{d}\tilde{m} \approx -\gamma \cdot \mathrm{d}c_g - [(\gamma - 1) \cdot (1 - \overline{\omega}_g) + 1] \cdot (\mathrm{d}p_g - \mathrm{d}p_s). \tag{48}$$

Negative habit in the services X_s is consistent with the observation of asymmetric risk prices in previous price-model **P-ND**. Given the negative habit $X_s < 0$, observed expenditure share in goods ω_g is larger than the share-parameter $\overline{\omega}_g$. In estimation, I require the point estimate b_{p_g} is smaller than $(b_{c_g} - 1) \cdot (1 - \hat{\omega}_g) + 1$. The share $\hat{\omega}_g$ is calibrated as sample-average 40% for simple estimation. When explicitly estimating the Stone-Geary preference with zero habit in goods, point estimate of risk price b_{c_g} is abnormally large. MAPE is 2.79%. The fitness of estimation is worse than the general price-model **P-ND**.

In the Column **Service** I assume consumer has zero habit in the services, $u(C_g, C_s) = \frac{[(C_g - X_g)^{\overline{\omega}_g} \cdot C_s^{1-\overline{\omega}_g}]^{1-\gamma}}{1-\gamma}$, decompose the pricing kernel (using service as the numeraire) as

$$\mathrm{d}\tilde{m} \approx -\gamma \cdot \mathrm{d}c_s - (1 - \gamma) \cdot \overline{\omega}_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s). \tag{49}$$

Positive habit in the goods X_g is consistent with the observation of risk prices in price-model **P-ND**. Given the positive habit $X_g > 0$, observed expenditure share in goods ω_g is larger than the share-parameter $\overline{\omega}_g$. In estimation, I require the point estimate b_{p_g} is smaller than $(1 - b_{c_s}) \cdot \hat{\omega}_g$. The point estimate for b_{c_s} is 33.79 with considerable precision, but the point estimate for $b_{p_g} = -13.12$ has large standard error. Fitness of estimation is also worse than the general price-model **P-ND**.

Under certain situation, direct utility function might not have the tractable expressions. For example, non-homothetic CES preference in (Comin et al., 2021). The utility $u(C_g, C_s)$ is solution to a non-linear equation of quantities, $1 = C_g^{\rho} \cdot u^{-\rho_g} + C_s^{\rho} \cdot u^{-\rho_s}$. Marginal utility of services is derived from the implicit function theorem,

$$\mathcal{D}_{C_s} u(C_g, C_s) = \frac{u}{C_s} \cdot \frac{\rho \cdot C_s^{\rho} \cdot u^{-\rho_s}}{\rho_g \cdot C_g^{\rho} \cdot u^{-\rho_g} + \rho_s \cdot C_s^{\rho} \cdot u^{-\rho_s}}. \tag{50}$$

There is no analytical expression for the marginal utility of services. Under this situation, estimation of pricing kernel is difficult.

In summary, using detailed quantities helps improve the fitness of estimation, compared with the model **C-ND** using quantity index. Nonetheless, there is larger inaccuracy in estimating the direct utility function over quantities of goods and services. The model **P-ND** considers the general indirect utility function, skips the explicit assumption of function shape and parameters. This allows for the simple empirical estimation.

5 Extension for Multiple Sectors

Zoo of cross-section anomalies emerged in recent decades (Hou et al., 2015; Harvey et al., 2016), especially in the sample period after 1960s. Benchmark testing assets include portfolios of investment, profitability and momentum, to examine whether the asset pricing model can explain the returns of representative anomalies. When considering a more generalized economy with multiple sectors, the extended pricing kernel provides better explanation for testing assets of anomalies. Estimated risk aversion coefficient is smaller. The contribution of detailed prices to pricing kernel is better identified.

Table (14) includes a more detailed consumption basket, by separating the food and non-food categories within goods and services. Estimations relax the definition of consumption sectors gradually. Expenditure and price of food-service, and other detailed product-level data use Table 2.4.4 and Table 2.4.5 provided by NIPA. Overall, estimation with detailed sectors bring small model error, consumption prices have risk price in large absolute value, while expenditure contributes to the pricing kernel in a smaller way.

Column **Food in Good** separates the goods into food-good (food-at-home) and the non-food. I estimate parameters (b_{gf}, b_{gn}, b_e) using the pricing kernel as

$$\begin{aligned} \mathrm{d}\tilde{m} \approx &- b_{gf} \cdot \omega_{gf} \cdot (\mathrm{d}p_{gf} - \mathrm{d}p_s) - b_{gn} \cdot \omega_{gn} \cdot (\mathrm{d}p_{gn} - \mathrm{d}p_s) \\ &- b_e \cdot (\mathrm{d}e - \mathrm{d}p_s). \end{aligned} \tag{51}$$

Estimation of b_{gf} is quantitatively close with b_g in the benchmark estimation of two-sector economy. Due to the inaccurate estimate of b_{gn}, it is difficult to conclude whether the non-food goods are different from the food goods. Construction of total nondurable expenditure and price of services use more dis-aggregated data, b_e has the point estimate 14.13, smaller than the parameter value 28.80 in the benchmark estimation of the two-sector economy.

Column **Food in Service** separates services into food-service (food-away) and non-food. I estimate parameters (b_g, b_{sf}, b_e) in the similar procedure, all nominal time series are adjusted using price of non-food services . Point estimate for parameter b_g is -71.95, similar with the estimation -71.29 in two-sector economy. I observe the risk-price for food-service $b_{sf} = 302.21$. The estimation for parameter b_e is 21.63, smaller than the benchmark estimation in Table (5).

Column **Four** considers all sectors, estimate parameters $(b_{gf}, b_{gn}, b_{sf}, b_e)$ using the pricing kernel as

$$\begin{aligned} \mathrm{d}\tilde{m} \approx &- b_{gf} \cdot \omega_{gf} \cdot (\mathrm{d}p_{gf} - \mathrm{d}p_{sn}) - b_{gn} \cdot \omega_{gn} \cdot (\mathrm{d}p_{gn} - \mathrm{d}p_{sn}) \\ &- b_{sf} \cdot \omega_{sf} \cdot (\mathrm{d}p_{sf} - \mathrm{d}p_{sn}) - b_e \cdot (\mathrm{d}e - \mathrm{d}p_{sn}). \end{aligned} \tag{52}$$

The risk price for food-service is quantitatively close with estimation of three-sectors in **Food in Service**. SDF have different sensitivities to prices of goods and services within food. Food-at-home has risk price b_{gf} as -78.10, but food-service has risk price as 302.37. Estimated risk price for expenditure b_e is 14.70, in a small value. Further, MAPE in estimation of four-sector economy is 0.18%, smaller than all other consumption-based models.

When the consumer decides her basket over multiple consumption sectors, distinguishing the risk price for dis-aggregated prices improves measuring the SDF. Risk price of expenditure is determined by consumer's relative risk-aversion coefficient. We observe smaller point estimate for parameter b_e. The variation of consumption expenditure has small contribution toward the fluctuation in SDF. On the contrary, variations in detailed prices contribute the main fluctuation. We don't see the puzzling high point estimate of risk-aversion in literature.

In Fama-Macbeth regression, after including these detailed prices, cross-section fitness slightly improves, prices have negative risk premiums. Food-good and food-service have positive correlation, while the correlations with non-food good are weak. Covariance matrix of prices and risk price vector together determines the observed risk premium (vector), so there is no simple match between risk price and risk premium.

The extended price-model with four sectors well explains the family of cross-section anomalies documented in (Hou et al., 2015). Table (15) reports the average estimation outcome across 114 anomalies, available during the years 1968-2019. For each anomaly, testing assets are 15 portfolios double sorted on size and the firm characteristics from database **global-q**. Estimation is conducted by group of 15 testing assets, then the average point estimate and the model fitness are reported. Average MAPE of price-model **P-ND** is 0.22%, comparable with 0.24% in augmented q-factor model of (Hou et al., 2021), outperforms 0.73% in the consumption-based quantity model **C-ND**.

6 Discussion

6.1 Sufficient Statistic of Shocks

Systematic risk comes from primitive macroeconomic shocks that cannot be diversified using financial assets. This paper doesn't explicitly measure each shock. These primitive shocks are summarized by the change of consumption expenditure, and the changes of prices in each sector. As in the motivating example of subsection 2.1, consumer fully anticipates prices of next period, chooses optimal allocation of expenditure across time and states. Economists can combine realized prices and expenditure to infer change of consumer's shadow price of financial wealth, the pricing kernel. Theorem (1) constructs **sufficient statistics** for pricing kernel in a generalized economy environment.

Quantitative asset pricing papers used to identify the "primitive macroeconomic shock" empirically, and then explain how shock propagates in the model [28]. Decomposition of consumer's marginal utility works in the similar way here. The fundamental shocks lead to variation in consumer's price vector and total expenditure, and correspondingly the stochastic discount factor $\tilde{M}$. This paper goes directly to empirical inference, skips solving the full quantitative model.

In supplementary estimations, I don't observe discernible change in model-fitness when including supplement proxies for primitive shocks. These checks include (1) each traded factor in Fama-French 5-factor model and momentum factor; (2) fundamental shocks extracted from the capital-good price [29], durable-good price, gasoline-good price and public transportation service price; (3) innovations from economic quantity outcomes: the labor hour in private sector, the landfill garbage, unfiltered consumption; (4) innovations from wealth distribution: nominal wealth of bottom 90% households (deflated using service price), top 10% households, the wealth-share of wealthiest 1% household and 5% [30]. Appendix Table (A.14) reports the GMM estimation outcomes. There is no obvious improvement in the fitness of estimation. For each additional proxy of primitive shock, point estimate of risk price has large standard error. Table (A.13) shows MKT factor and size factor correlate with time-series components in the price-model. When assigning non-zero risk price b_x for included proxy, we observe the adjustment in parameters of price-model. When including the MKT factor, point estimate for price of goods b_g has smaller absolute value. When including the size factor point estimate of parameter b_g has larger absolute value.

The shocks constructed with energy price and public-transportation slightly add to the model-fitness of Fama-Macbeth regression, and the risk-premium is significantly negative. However, the distribution of risk exposure tends to be weak for volatile innovations in energy and transportation prices. Overall, consumption price works well in summarizing the consumer welfare outcome from the primitive shocks.

[28]For example, (Papanikolaou, 2011; Kogan and Papanikolaou, 2014) document the cross-section risk exposure to investment-specific-technology change (IST shock), (Loualiche et al., 2016) extracts the entry-cost shock from firm entry-exit statistics across industries, (Belo et al., 2019) extracts the equity-issuance-cost shock from the debt growth and equity issuance fraction across listed firms, (Dou et al., 2022) extracts the common fund flow shock in financial intermediary. (Roussanov et al., 2021) estimate VAR model for core-inflation and energy-inflation, use the residual of VAR model as inflation shocks.

[29]These estimations use the IST shock constructed in (Papanikolaou, 2011) during the years 1965-2008, and the extended measure of investment-good price shock to 1965-2019, using the price of equipment in capital investment from NIPA Table 114-Line 11-Equipment.

[30]These statistics are available from the website of Emmanuel Saez and Gabriel Zucman. For non-stationary wealth shares, I filter the linear trends.

6.2 Explanation of Risk Price

Consumer preference determines her choice of consumption basket within each period, and how she allocates her financial wealth across different states and periods. Subsection (6.2.1) explains how the pairwise difference in the share elasticity matrix η determines the asymmetric vector of risk price $\vec{b}$. This assumes the representative consumer, provides the qualitative explanation for empirical facts. Subsection (6.2.2) discusses the situation with multiple consumers. Redistribution of consumer welfare helps explain the large point estimates of risk prices.

6.2.1 Consumer preference

Lemma 2 demonstrates how consumption price and total expenditure shape the consumption basket, the shares ω across sectors, when the matrix of share elasticity η has the stable pairwise difference [31].

Lemma 2. *Given consumption sector k and j, change in the relative share $\mathcal{S}_{k,j} = \frac{\omega_k}{\omega_j}$ can be decomposed into the price effect and the income effect,*

$$\begin{aligned} \mathrm{d}s_{k,j} =&(1-\eta_{k,k}+\eta_{j,k})\cdot \mathrm{d}p_k - (1-\eta_{j,j}+\eta_{k,j})\cdot \mathrm{d}p_j - \sum_{i\neq k,j}(\eta_{k,i}-\eta_{j,i})\cdot \mathrm{d}p_i \\ &+\sum_i(\eta_{k,i}-\eta_{j,i})\cdot \mathrm{de} + o(h). \end{aligned} \tag{53}$$

The small character s is the log relative share $s=\log(\mathcal{S})$. The $\mathrm{d}s_{k,j}$ is the log-growth of relative share between sector k and j. The term $o(h)$ is a higher-order term to the change of expenditure and the prices.

Corollary 3 formally states how share elasticity explains the asymmetric risk premium cross consumption sectors. Without the explicit assumption of consumer preference, this corollary provides a robust prediction for risk price of consumption prices .

Corollary 3. *Define the **Engel Slope** for the sector pair (k,j) as sensitivity of relative share $s_{k,j} = \log(\frac{\omega_k}{\omega_j})$ to expenditure,*

$$\mathrm{ES}_{k,j}(\vec{P},E) = \lim_{\mathrm{de}\to 0}\frac{s_{k,j}(p,e+\mathrm{de}) - s_{k,j}(p,e)}{\mathrm{de}}, \tag{54}$$

[31] Demand systems in (Deaton and Muellbauer, 1980; Parodi et al., 2020) use absolute share for identification. In practice, these estimation has stable point estimates and good model fitness in micro-economic literature. Lemma 2 further simplifies the demand system using the pairwise equations of relative share, in the similar fashion. Importantly, Lemma 2 helps us directly read how expenditure changes the shares.

In real stochastic discount factor, the risk price of necessity price P_k is more negative than the luxury price P_j, as necessity sector k inferior to than luxury sector j,

$$b_k - b_j \quad = \quad \mathrm{ES}_{k,j}(\vec{P}, E). \tag{55}$$

When the Engel Slope $\mathrm{ES}_{k,j}$ is in negative value, after an increase in total consumption expenditure $\mathrm{d}e > 0$, consumer allocates a smaller fraction of spending to the necessity sector k, while the expenditure share in the luxury sector j expands. The equation (55) in Corollary 3 says for this necessity sector, we shall witness the consumer charges higher risk compensation for the equity portfolio correlated with the necessity spot price. Intuitively, consumer has more rigid adjustment in quantity of necessity, so the increased price of necessity brings more severe reduction of consumer's utility. Suppose we fix the prices, decrease in total expenditure $\mathrm{d}e < 0$ pushes the consumer to cut the luxury quantity in larger extent than the necessity. On the opposite side, to maintain the same utility, unit increase of necessity price requires the larger compensation of expenditure than the unit increase of luxury price.

In particular, the gap between the risk price coefficient $\{b_j\}_j$ is exactly the marginal effect of expenditure effect (income effect) in the consumption portfolio. We see this simple relationship between the risk-price of consumption sector and the position of sectors along the Engel curve, as stated in Corollary 3.

For the economy with good and service, estimated Engel slope $\mathrm{ES}_{g,s} \approx -0.49$. This estimate is consistent with the estimate of (Boppart, 2014) and (Comin et al., 2021), where the quantitative magnitude of expenditure effect (income effect) is small. However, financial asset returns tend to require large risk compensation from necessity price growth, than this prediction of intra-period consumption basket. If we use the Engel slope from consumer's static decision of basket to back-out the risk-price, implied risk-price for the relative price of goods is $b_g = \omega_s \cdot \mathrm{ES}_{g,s} + 1 - b_e = -28.10$ with additional -0.30 from $\omega_s \cdot \mathrm{ES}_{g,s}$. But in estimation of financial market pricing kernel, we observe the point estimate $b_g = -71.29$ in a much larger absolute value.

For the economy admitting food-service and non-food within service sector, we observe high point estimate for the price of food-service b_{sf}. Estimation using financial asset returns says food-service is superior (luxury) to non-food service. Qualitatively, this is reasonable. However, financial market tends to assign an overly large risk price for the positive correlation with food-service price.

6.2.2 Discussion of Multiple Consumers

In a more generalized economy with multiple consumers, we can express the representative consumer's utility function as a weighted outcome of individual consumers,

$$\begin{aligned} V(\vec{P}, \mathbf{E}; \alpha) \equiv \max_{\{E(n)\}} \quad & \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot V^{(n)}(\vec{P}, E(n)) \\ s.t. \quad & \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} E(n) \leq \mathbf{E}. \end{aligned} \tag{56}$$

The Negishi-weight[32] reflects the difference across individual consumer's marginal utility. Here, I pick consumer (1) as the benchmark consumer construct the Negishi weight, $\alpha(n) = \frac{\mathcal{D}_E V^{(1)}(\vec{P}, E^{(1)})}{\mathcal{D}_E V^{(n)}(\vec{P}, E^{(n)})}$. Theorem (1) decomposes the marginal utility for general situation of consumer preference. Although we don't know the explicit form of representative consumer's indirect utility function $V(\vec{P}, \mathbf{E}; \alpha)$, pricing kernel can be organized in the same way of Theorem (1). Corollary (4) formally states the decomposition results assuming the representative consumer has the fixed Negishi weights α.

Corollary 4. *Given invariant distribution of Negishi-weight $\{\alpha^*(n)\}_n$ along the equilibrium path, the log-change in real marginal utility of expenditure for the representative consumer approximately equals*

$$\mathrm{d}\tilde{m} = -\sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\boldsymbol{e} - \mathrm{d}p_J) + o(h). \tag{57}$$

where α is the artificial Negishi-weight, $\vec{\omega}$ is the aggregate expenditure share, $\boldsymbol{e}$ is the (log) aggregate total consumption expenditure, and the vector $b(\alpha)$ is written with aggregate expenditure share $\vec{\omega}$ and representative consumer's elasticity η

$$\begin{aligned} b_j(\alpha) =& -[\gamma(\alpha) - 1] + \sum_{i=1}^{J} \eta_{j,i}(\alpha) - \sum_{k=1}^{J} \omega_k \cdot \sum_{i=1}^{J} \eta_{k,i}(\alpha), \\ b_e(\alpha) =& \gamma(\alpha). \end{aligned} \tag{58}$$

For consistent notation, I use $b(\alpha)$ for the representative consumer with Negishi weights α. Even if individual consumers have identical indirect utility function, the risk price vector

[32] Some papers use Pareto weight as the alternative name. In the appendix **Explanations for Aggregation** I discuss how to reveal the Negishi weight, and how to recover the effective representative consumer given we observe the equilibrium outcome of consumption distribution, where consumers simultaneously decide the expenditures in the dynamic environment.

$b(\alpha)$ can be different from the prediction of individual consumer's preference, because Negishi weights α work as the additional parameters.

Previously, I assume the Representative Consumer has fixed preference for clear explanation and simple estimation. If the primitive macroeconomic shocks induce both fluctuation of consumption price, and the redistribution of consumption across consumers, we would witness significant distortion of aggregate share elasticity. Limited stock market participation, financial constraints lead to welfare redistribution, because the aggregate shocks are not perfectly shared across consumers. Sensitivity of pricing kernel over prices can depart from the prediction of consumer preference, due to the feature of incomplete financial market. Here I provide the hypothetical decomposition for the quantitative gap between the aggregate-elasticity measured from the financial asset returns and the representative consumer's preference, in the economy with incomplete financial market.

Consumers might have borrowing constraint and transaction cost in equity assets. When these constraints are binding, the fluctuation in α resembles the change of wedge, as opposed to the consumer's counter-factual optimal expenditure. The fluctuation of effective Negishi weights can come from the change in distribution of wedges. Corollary (5) generalizes the the decomposition results, admitting the change in the fluctuation of Negishi-weight distribution $\{\alpha(n)\}_n$.

Corollary 5. *Given the process of effective Negishi-weight distribution $\{\alpha(n)\}_n$ along the equilibrium path, the log-change in real marginal utility of expenditure for the representative consumer approximately equals*

$$\begin{aligned} \mathrm{d}\tilde{m} = & -\sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\boldsymbol{e} - \mathrm{d}p_J) \\ & + \frac{1}{N} \cdot \sum_{n} s(n) \cdot \mathrm{d}\log[\alpha(n)] + o(\hat{h}). \end{aligned} \tag{59}$$

where $\mathrm{d}\alpha$ is the directional derivative of Negishi-weight distribution, $\vec{\omega}$ is the aggregate expenditure share, $\boldsymbol{e}$ is the (log) aggregate total consumption expenditure, and the vector $b(\alpha)$ is defined in the Corollary 4, the expenditure-ratio $s(n)$ is the ratio of consumer-expenditure and aggregate-expenditure $\frac{E^{(n),*}}{\mathbf{E}^*}$ in the equilibrium. The perturbation term is the norm of perturbation term $\hat{h} = \max\{h, \frac{1}{N} \cdot \mathrm{d}\alpha\}$.

When the wedge changes with the aggregate shock, it contributes to the discrepancy between pricing kernel in financial market and the change of representative consumer [33]. Corollary (5) admits the contribution from variation of wedges, for a time-varying representative

[33](Basak and Cuoco, 1998; Chien and Lustig, 2010; Chien et al., 2011) explain the role of wedge in pricing kernel, for an economy with transaction constraint. Equilibrium outcomes from a heterogeneous-agent model can be mapped into this accounting framework. The decomposition helps to read the contribution of prices to pricing kernel, from the channel of welfare redistribution.

consumer. For the observation of excessive risk price, if we observe the change of welfare weights α in correlation with the consumption prices, we would be see the additional contribution from consumption prices. In an economy where unconstrained household [34] is not accurately observed, prices still summarize the unconstrained consumer's marginal utility well. Further, if the expenditure discrepancy between unconstrained consumer and other consumers are negatively correlated with consumption prices, we would observe high sensitivity of pricing kernel over prices.

6.3 Relative Price and Nominal Price

Inflation has always been an important topic in asset pricing. Inheriting the thought of consumption quantity index, traditional asset pricing papers measure nominal inflation as growth in consumption price index, focus on outcomes related to treasury bond. In (Cieslak and Povala, 2015), expectation of (nominal inflation) explains the yield rate of treasury bond. In (Eraker, 2008; Eraker et al., 2016), expectation of nominal inflation is negatively correlated with the expected real quantity growth in consumption, variation of nominal inflation contributes to the fluctuation of SDF when the consumer has Epstein-Zin preference. Further, when the nominal inflation has negative correlation with SDF, nominal yield rate of short-term bond has negative beta to SDF. This helps explain the upward slope in yield curve (Eraker, 2008). Compared with nondurable, durable stock growth has stronger negative correlation with nominal inflation, as documented by (Eraker et al., 2016).

More broadly, when monetary policy is non-neutral for the economy, nominal inflation leads to time-varying risk premium. In literature such as (Drechsler et al., 2018), cash (liquid asset) is required for investment in risky asset. There, monetary expansion invokes positive nominal inflation and lower liquidity premium simultaneously. We observe correlation between nominal inflation and time-varying risk premia, due to the transaction restriction in financial assets. However, variation of nominal inflation doesn't directly add to the variation of SDF. In literature such as (Silva, 2016), expansive monetary policy increases the wealth share of borrowers (risk-tolerant consumers), hence the lower risk premium. Similarly, we don't observe the direct correlation between the nominal inflation and stochastic discount factor.

This paper investigates how detailed consumption prices contribute to fluctuation of stochastic discount factor. Decomposition of marginal utility in this paper extends to New-Keynesian models. When consumer has no binding financial constraints, marginal utility of expenditure is identical with stochastic discount factor, consumer's prices and expenditure summarize the impact of aggregate shocks to pricing kernel.

(Weber, 2014) documents the firms with infrequent price adjustment have greater correlation with Market factor of CAPM, suggesting the higher systematic risk. (Chava et al., 2022) construct the direct measure of (nominal) inflation exposure using the textual data from

[34] For example, wealthy stockholder in (Malloy et al., 2009) and fund manager in (He et al., 2017).

earning calls, they document the negative return in response to earning calls across firms with high inflation exposure. Consumer's utility in (Weber, 2014) can be described by consumer price index, while this paper investigates the asymmetric impacts from prices to marginal utility. In a full New-Keynesian model where producers are heterogeneous in price adjustment across sectors, nominal cash flow has different correlations with prices. This type of model can illustrate why equity returns have dispersed risk exposure to prices. Alternatively, in an extended model of (Corhay and Tong, 2021) where the monetary policy is contingent on detailed consumer's prices, one can observe producer's borrowing has asymmetric reaction, correspondingly. This provide explanation for the cross-sectional variation of risk exposure in nominal dividend flow.

In (Roussanov et al., 2021)[35], when the economy has the negative long-run productivity growth shock, the producer in consumption sector increases the output price. Hence, innovation in long-run productivity growth has negative correlation to the core inflation. Bringing the theoretical insight from (Roussanov et al., 2021) to a economy of heterogeneous goods-services sectors, the asynchronized fluctuation in price of goods and services, can be outcome of producers' different reaction to the long-run productivity shock and short run shock. Due to the different flexibility in price adjustment, long-run and short-run shocks are the underlying aggregate shocks for the variation in sector-level prices. Simple accounting of risk premium can be conducted here. One observes the asymmetric risk price of productivity shocks, due to their impact to the marginal utility of expenditure, via the asymmetric impacts over sector-level prices. Contrary to the traditional wisdom of recursive preference, if the inflation of goods is mainly attributed by the negative long-run productivity shock, consumer in the economy would charge high risk price for the long-run productivity shock.

7 Conclusion

Prices and expenditure describe consumer's marginal utility in an economy with multiple consumption sectors. This paper provides a non-parametric decomposition of stochastic discount factor using variations of detailed prices and expenditure.

In an economy of goods and services, variation of stochastic discount factor is summarized by variations in price of goods, price of services and total expenditure. This price model explains excess return of equity portfolios in U.S. market during 1965-2019. Because consumer has more difficulty in adjusting the share of goods, high price of goods leads to high marginal utility. In empirical examination of this model, price of goods has a negative risk price. Fluctuation of prices contributes to the variation of SDF. Consumer's risk-aversion, the risk price of expenditure, has a small point estimate. Financial assets have strong and dispersed risk exposure to relative price of goods, and it yields negative risk premium.

[35] Their revised version in 2024 has more elaboration of the quantitative model.

The price model directly estimate the risk price for prices and expenditure, while the quantity index puts restriction in traditional Consumption-CAPM. This helps accurately measure the variation of stochastic discount factor, and explain the cross-sectional variation of expected returns. The price model is simple. This provides convenience for empirical estimation, compared to the pricing kernel using detailed quantities.

Measuring the pricing kernel using prices and expenditure is a general approach. Inference method can be flexibly extended to detailed consumption sectors. Including dis-aggregated prices helps capture the systematic risk priced across financial assets. Generally, consumer's marginal utility increases more when there is growth in price of necessity commodities, as such, risk price is more negative.

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A Figure

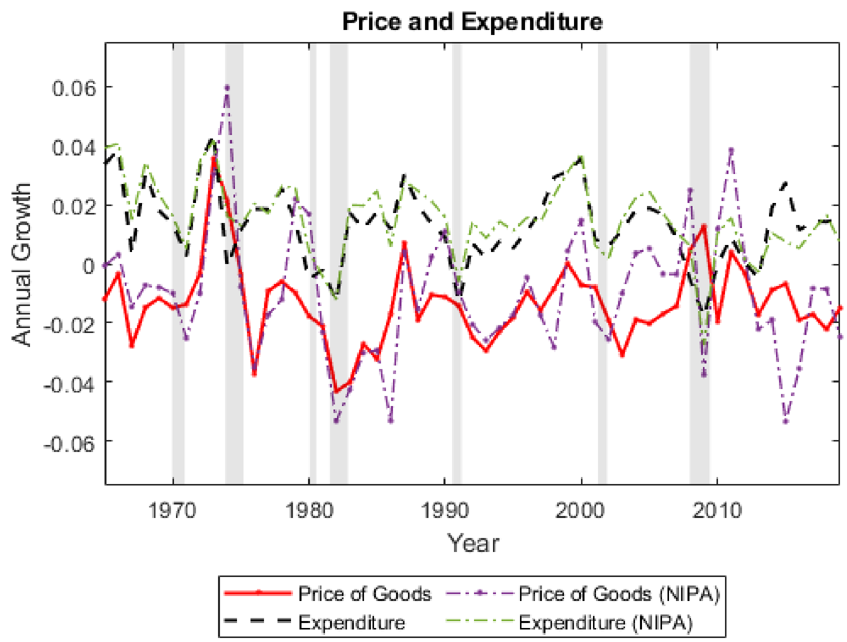

Figure 1: **Time Series of Economic Outcomes**

The X-axis is time-axis. The figure plots the annual log change of price of goods relative to price of services $\mathrm{d}p_g - \mathrm{d}p_s$ and the log change or expenditure deflated by price of services $\mathrm{d}e - \mathrm{d}p_s$. The red thick line plots the price of goods, black dashed line plots the expenditure. All time series are relative to price of services. Definition of sectors described in Section (3). The purple dashed line plots the price, the green dashed line for expenditure, using original definition of nondurable sectors in NIPA.

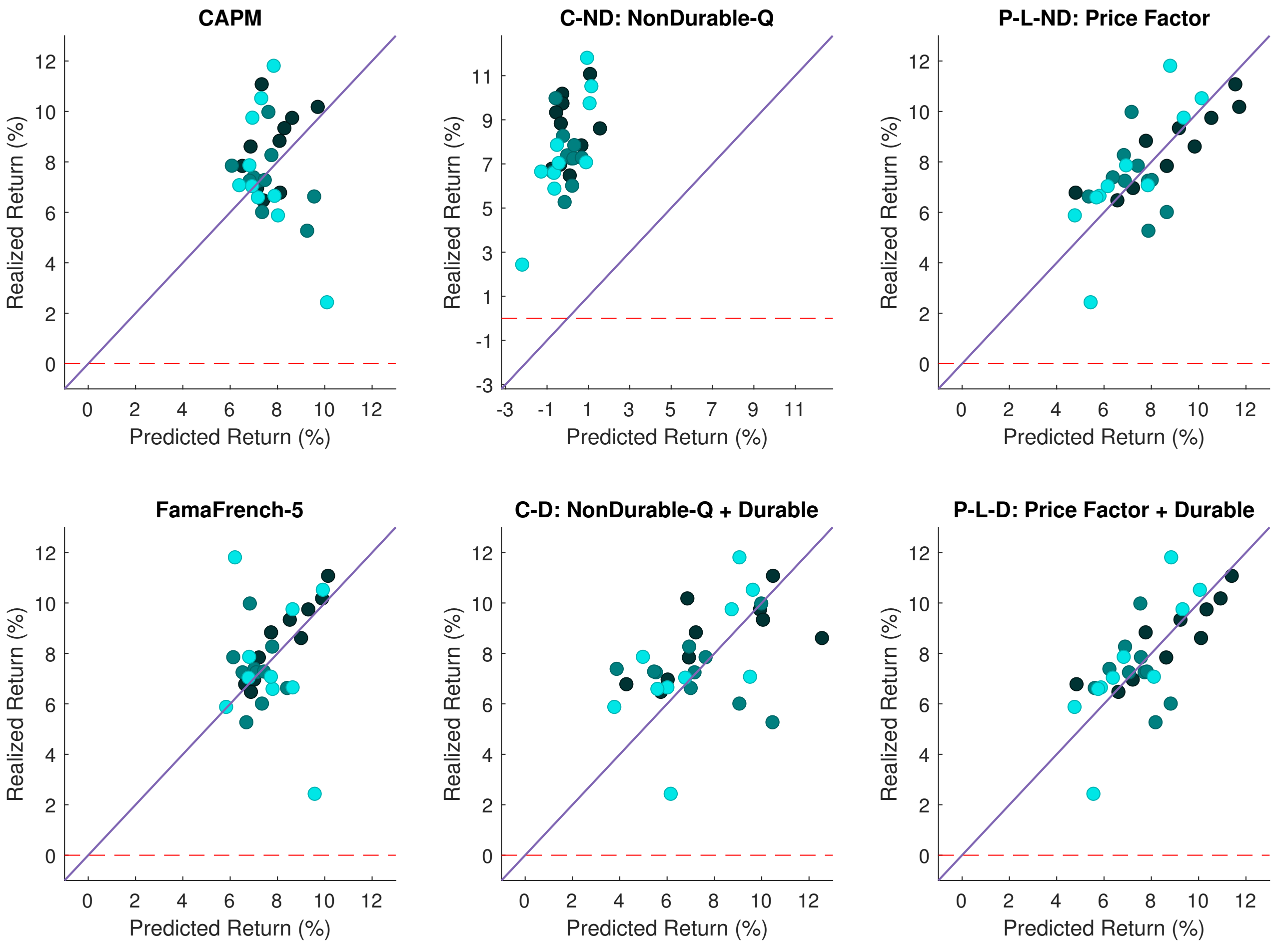

Figure 2: **Fitness of Linear Asset Pricing Models in Simplified Estimation**

The X-axis is the model-predicted excess return of simplified linear factor models, $-\frac{1}{T}\cdot\sum_{t=1}^{T}\frac{\hat{b}}{1+R_f}\cdot[\vec{f}_{t+1}-\frac{1}{T}\cdot\sum_{\tau=1}^{T}\vec{f}_{\tau+1}]\cdot R^e_{k,t+1}$ for each asset k. The Y-axis is the average excess return in sample, $\frac{1}{T}\cdot\sum_{t=1}^{T}R^e_{k,t+1}$. Subfigure **CAPM** uses the Market factor. Subfigure **FamaFrench-5** is the Fama-French 5-Factor Model. Subfigure **C-ND** uses quantity index of nondurable consumption, as the single factor. Subfigure **C-D** includes quantity of durable stock. Subfigure $\mathbf{P}^L$**-ND** uses two time series risk factors, expenditure relative to price of services, price of goods relative to price of services. Simplified linear model $\mathbf{P}^L$**-ND** is $\mathrm{d}\tilde{m}_{t+1} = -b_{e,L}\cdot(\mathrm{d}e_{t+1}-\mathrm{d}p_{s,t+1})-b_{g,L}\cdot(\mathrm{d}p_{g,t+1}-\mathrm{d}p_{s,t+1})$, doesn't include the correction of expenditure share. Subfigure $\mathbf{P}^L$**-D** includes quantity of durable stock. Full description of asset pricing models, construction of durable stock, quantity index and price index are described in Section (3). The dark-green dots are Size-BM portfolios. The green dots are Profitability-Investment portfolios. The light-green dots are Momentum portfolios and Earning/Price portfolios.

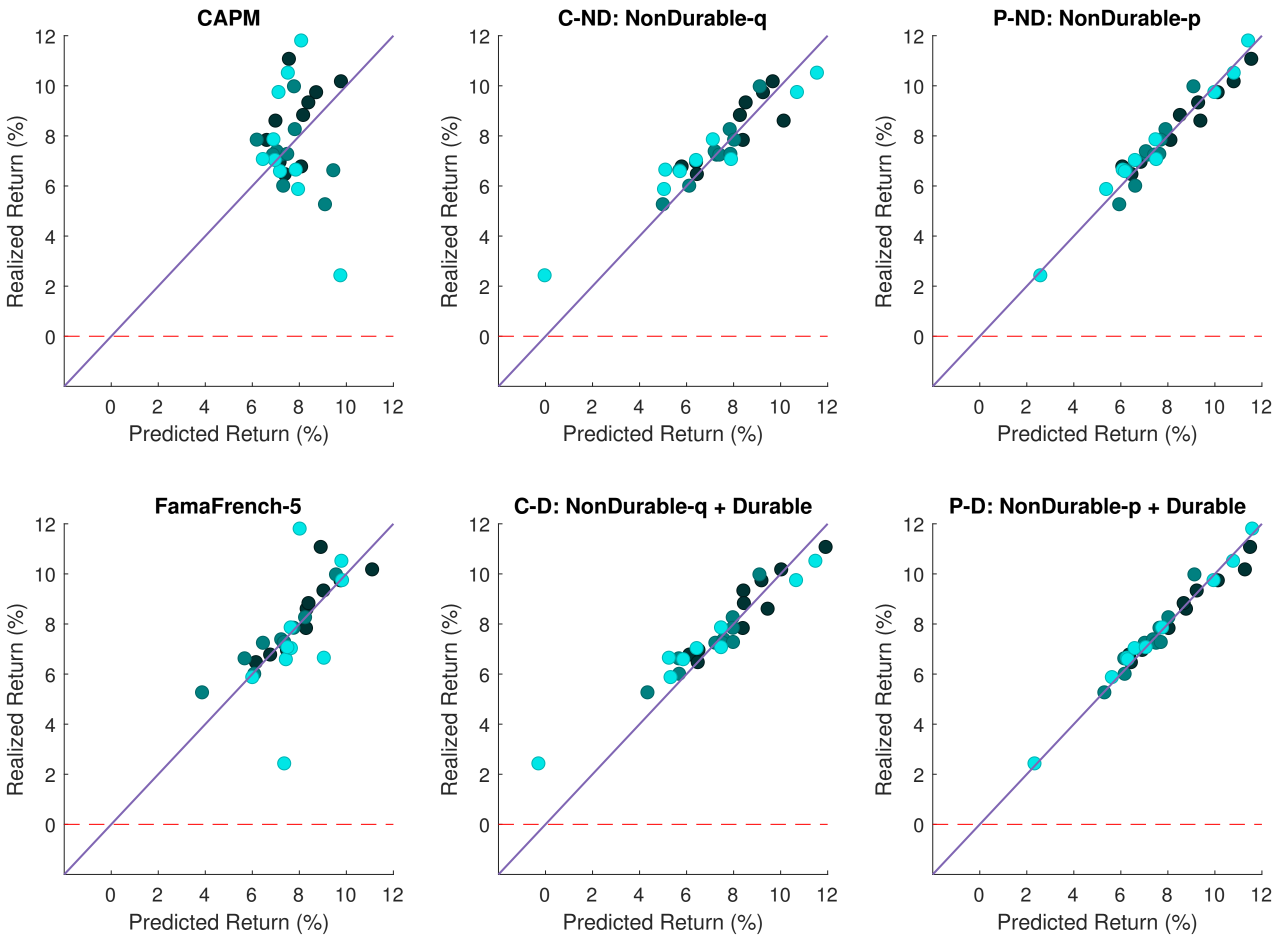

Figure 3: **Fitness of Asset Pricing Models in Formal Estimation**

The X-axis is the model-predicted excess return, $-\frac{1}{T} \cdot \sum_{t=1}^{T} \mathrm{d}\tilde{m}_{t+1} \cdot R^e_{k,t+1}$ for each asset k. The Y-axis is the average excess return in sample, $\frac{1}{T} \cdot \sum_{t=1}^{T} R^e_{k,t+1}$. Subfigure **CAPM** uses the Market factor. Subfigure **FamaFrench-5** is the Fama-French 5-Factor Model. Subfigure **C-ND** uses quantity index of nondurable consumption, as the single factor. Subfigure **C-D** includes quantity of durable stock. Subfigure **P-ND** uses price of goods, price of services, total expenditure in the nondurable consumption sector and expenditure share of goods. Model **P-ND** is $\mathrm{d}\tilde{m}_{t+1} = -b_e \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_g \cdot \omega_{g,t} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})$. Subfigure **P-D** includes quantity of durable stock. Full description of asset pricing models, construction of durable stock, quantity index and price index are described in Section (3). The dark-green dots are Size-BM portfolios. The green dots are Profitability-Investment portfolios. The light-green dots are Momentum portfolios and Earning/Price portfolios.

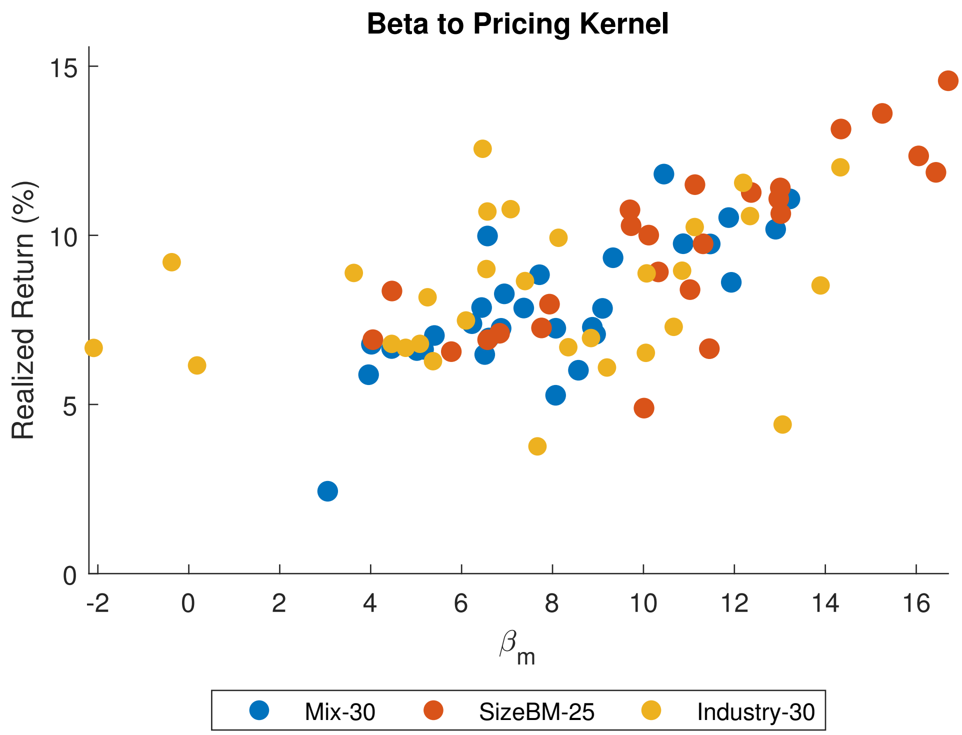

Figure 4: **Financial Asset Returns and Covariance with Pricing Kernel**

The X-axis is slope in univariate regression of excess return to pricing kernel of two-sector economy, for each asset k. Univariate regression is specified as $R^e_{k,t+1} = a_k + \beta_{k,m} \cdot \mathrm{d}m_{t+1}$. Pricing kernel is the benchmark estimation in Table (2). The Y-axis is the average excess return in sample. The **Mix-30** are quintile portfolios sorted by Size, BM, Profitability, Investment, Momentum, Earning/Price. The **SizeBM-25** are 25 portfolios double sorted by Size and BM. The **Industry-30** are industry portfolios.

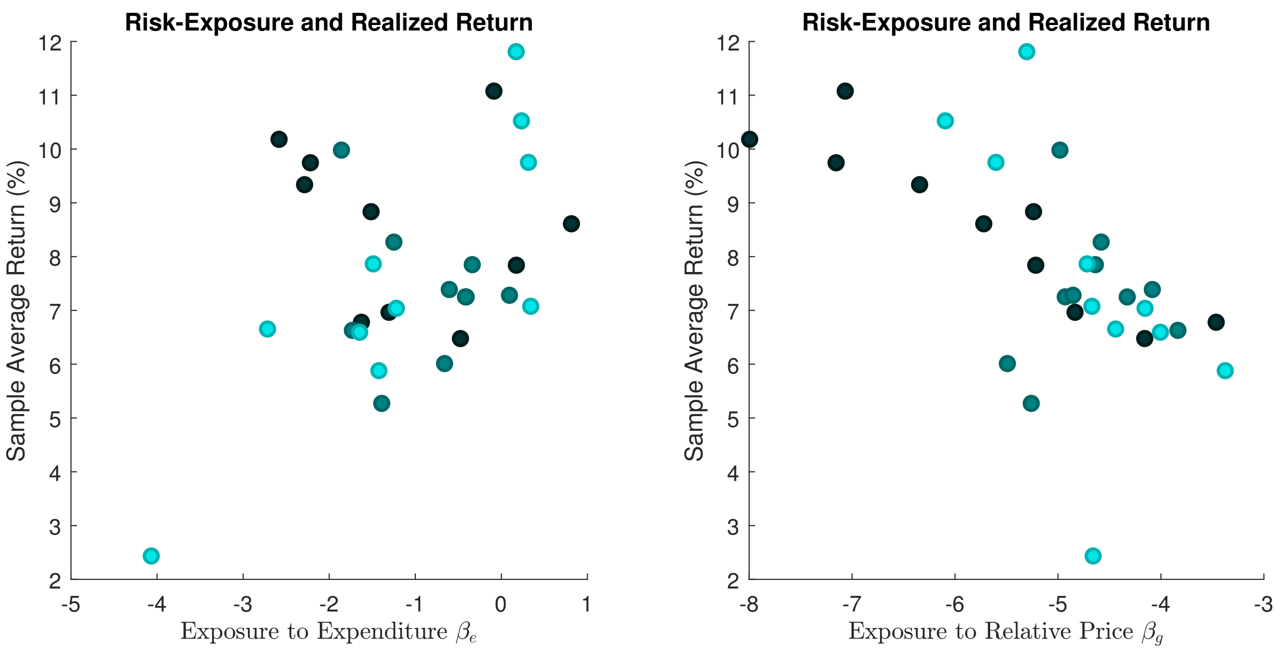

Figure 5: **Risk Exposure to Factors**

The X-axis is joint $\vec{\beta}$ in 1st step of Fama-Macbeth regression. The Y-axis is the average excess return in sample. The dark-green dots are Size-BM portfolios. The green dots are Profitability-Investment portfolios. The light-green dots are Momentum portfolios and Earning/Price portfolios.

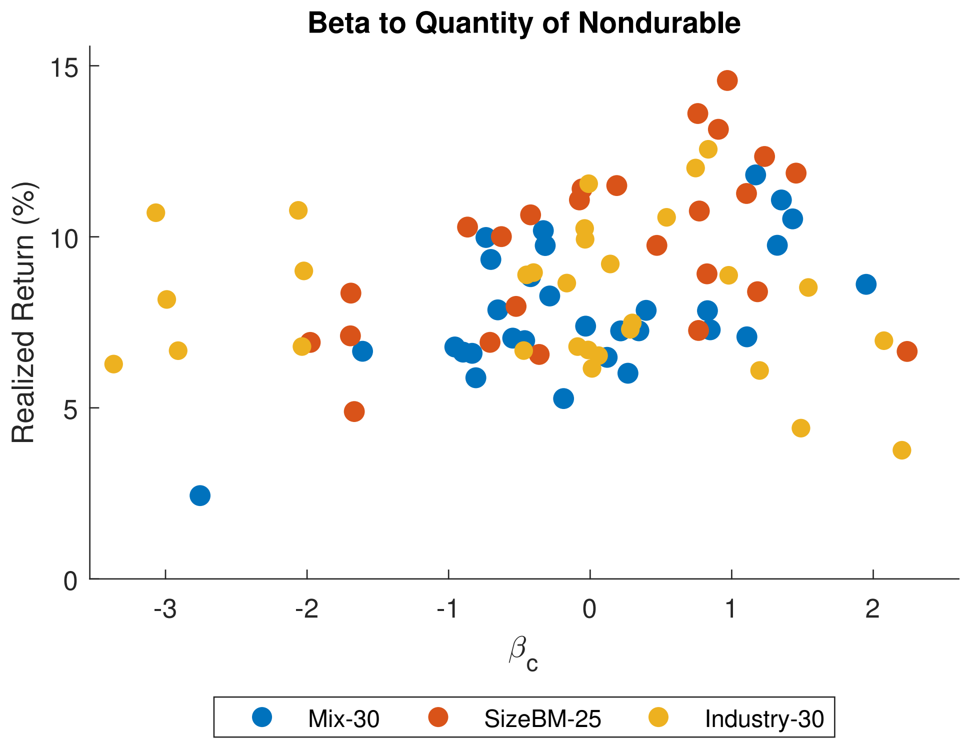

Figure 6: **Financial Asset Returns and Covariance with Quantity of Nondurable**

The X-axis is slope in univariate regression of excess return to quantity index of nondurable consumption, for each asset k. Univariate regression is specified as $R^e_{k,t+1} = a_k + \beta_{k,c} \cdot \mathrm{d}c_{t+1}$. The Y-axis is the average excess return in sample. Other explanations for portfolios are the same with Figure (4).

B Table

Table 1: **Descriptive Statistic**

Time span of sample is during 1965-2019. Annual growth of relative price of goods (price of goods relative to price of services) is denoted as $\mathrm{d}\tilde{p}_g$, growth of expenditure (nondurable expenditure relative to price of services) is denoted as $\mathrm{d}\tilde{e}$, growth of quantity index in nondurable is denoted as $\mathrm{d}c_{nd}$. Panel (A) reports the unconditional mean, standard deviation, and auto-correlation coefficient. Panel (B) reports the correlation between relative price of goods and other consumption outcomes. All the standard errors are Newey-West standard error adjusted with two periods, reported in parenthesis.

Panel (A): Time Series - Statistic			
	Mean(*pct*)	SE(*pct*)	AR(1)
$\mathrm{d}\tilde{e}$	1.27	1.28	0.36
(*s.e.*)	(0.21)	(0.13)	(0.12)
$\mathrm{d}\tilde{p}_g$	-1.33	1.38	0.47
(*s.e.*)	(0.24)	(0.23)	(0.13)
Panel (B): Business Cycle - Correlation			
	$\mathrm{d}\tilde{e}$	$\mathrm{d}c_{nd}$	
$\mathrm{Corr}(z, \mathrm{d}\tilde{p}_g)$	0.26	-0.17	
(*s.e.*)	(0.18)	(0.17)	

Table 2: **Estimation of Pricing Kernel**

This table reports the point estimate of risk price vector $\vec{b}$ in GMM estimation, for pricing kernel of a two-sector economy with goods and services, using annual data during 1965-2019. testing assets are equity portfolios constructed with size, BM ratio, profitability, investment, momentum and earning/price. Panel **Risk Price** reports the vector $\vec{b}$. Panel **Stats** reports the MAPE (Mean Absolute Pricing Error) and J-pval (p-value for the J-stat). In constructing the weight matrix for GMM, **1st-Stage** uses the Identity Matrix, **2nd-Stage** uses the asymptotical variance of 1st-Stage estimation. T-stat is reported in brackets, Newey-West standard error has adjustment for two periods.

	Risk Price	
	1st-Stage	2nd-Stage
b_e	28.80	30.75
$[t]$	[1.95]	[14.08]
b_g	-71.29	-72.26
$[t]$	[-2.31]	[-15.89]
	Stats	
MAPE	0.39	
RMSE	0.44	
J-pval		91.48

Table 3: **Fitness of Linear Asset Pricing Models in Simplified Estimation**

The fitness of linear asset pricing models are reported. Estimation of linear factor model $d\tilde{m}_{t+1} = \vec{b} \cdot \vec{f}_{t+1}$ uses equation $\mathbb{E}_t[R^e_{k,t+1}] = \frac{\vec{b}}{1+\mathbb{E}_t[d\tilde{m}_{t+1}]} \cdot \mathbb{E}_t\left[(\vec{f}_{t+1} - \vec{g}_f) \cdot R^e_{k,t+1}\right]$, assuming risk factors $\vec{f}$ have the constant conditional mean, $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$. GMM estimation outcome is reported. Estimation uses annual data during 1965-2019. Estimation outcome using testing assets of **Mix 30** portfolios is reported. Column **CAPM** uses the Market factor. Column **FF-5** is the Fama-French 5-Factor Model. Column **C-ND** uses quantity index of nondurable consumption, as the single factor. Column **C-D** includes quantity of durable stock. Column **P^L-ND** uses two time series risk factors, expenditure relative to price of services, price of goods relative to price of services. Column **P^L-D** includes quantity of durable stock. Full description of linear asset pricing models, construction of durable stock, quantity index and price index are decribed in Section (3). Simplified linear model **P^L-ND** doesn't include the correction of expenditure share, and similarly for model **P^L-D**. Construction of MAPE (Mean Absolute Pricing Error), RMSE (Root Mean Square Error), and J-pval (p-value for the J-stat) are described in Section (3).

	Specification of Model					
	Traded Factor		Quantity		Price (Linear)	
	CAPM	FF-5	C-ND	C-D	P^L-ND	P^L-D
MAPE	1.67	1.20	7.85	1.68	1.15	1.10
RMSE	2.32	1.96	8.01	2.15	1.43	1.42
J-pval	95.96	97.15	96.97	98.04	98.46	98.21

Table 4: **Fitness of Asset Pricing Models in Formal Estimation**

The fitness of asset pricing models are reported. Estimation of model $\mathrm{d}\tilde{m}_{t+1}$ uses equation $\mathbb{E}_t[R^e_{k,t+1}] = -\mathbb{E}_t\left[\mathrm{d}\tilde{m}_{t+1} \cdot R^e_{k,t+1}\right]$. GMM estimation outcome is reported. Estimation uses annual data during 1965-2019. Column **CAPM** uses the Market factor. Column **FF-5** is the Fama-French 5-Factor Model. Column **C-ND** uses quantity index of nondurable consumption, as the single factor. Column **C-D** includes quantity of durable stock. Column **P-ND** uses price of goods, price of services and total expenditure. Column **P-D** includes quantity of durable stock. Full description of asset pricing models, construction of durable stock, quantity index and price index are decribed in Section (3). Construction of MAPE (Mean Absolute Pricing Error), RMSE (Root Mean Square Error), and J-pval (p-value for the J-stat) are described in Section (3).

	Specification of Model					
	Traded Factor		Quantity		Price	
	CAPM	FF-5	C-ND	C-D	P-ND	P-D
MAPE	1.58	0.79	0.71	0.66	0.39	0.27
RMSE	2.20	1.37	0.87	0.83	0.44	0.36
J-pval	93.38	81.07	96.23	95.12	91.48	92.08

Table 5: **Parameters using Other Testing Assets**

This table reports the point estimate of risk price vector $\vec{b}$ in GMM estimation, for model **P-ND**, using alternative testing assets. Panel (A) reports the vector of risk price $\vec{b}$. Panel (B) reports the statistics for each estimation. 2nd column and 3rd column use 25 portfolios double sorted based on Size and BM ratio. 4th column and the 5th column use 30 industry portfolios. Description of estimation is identical with Table (2), explanation of reported statistics is identical with Table (4).

	Specification of Testing Assets			
	Size-BM 25		Industry 30	
	Panel (A): Risk Price			
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	30.05	33.72	33.27	33.88
$[t]$	[2.61]	[13.06]	[4.38]	[24.98]
b_g	-68.26	-63.83	-69.95	-67.92
$[t]$	[-2.90]	[-11.68]	[-3.04]	[-17.21]
	Panel (B): Stats			
MAPE	0.38		0.84	
RMSE	0.51		0.99	
J-pval		81.48		94.03

Table 6: **Risk Exposure to Factors**

This table reports the point estimate of risk exposure in Fama-Macbeth 2-step regression, $\vec{\beta}$, for different subsets of testing assets in **Mix 30**. T-stat is reported in brackets, Newey-West standard error has adjustment for 2 periods. For each testing asset, sample-average excess return (%) is reported as μ, and volatility is reported as σ.

	Estimation Outcomes in 1st Step				
BM	G	2	3	4	V
β_e	-1.63	-1.30	0.17	0.81	-0.09
[t]	[-0.71]	[-0.64]	[0.08]	[0.36]	[-0.03]
β_g	-3.46	-4.83	-5.22	-5.72	-7.07
[t]	[-1.59]	[-2.51]	[-2.64]	[-2.66]	[-2.76]
μ	6.78	6.97	7.84	8.61	11.08
σ	19.47	16.96	16.37	18.48	20.72
Size	S	2	3	4	B
β_e	-2.58	-2.22	-2.29	-1.51	-0.48
[t]	[-0.77]	[-0.80]	[-0.91]	[-0.66]	[-0.23]
β_g	-7.99	-7.16	-6.34	-5.24	-4.16
[t]	[-2.51]	[-2.73]	[-2.65]	[-2.40]	[-2.08]
μ	10.18	9.75	9.34	8.84	6.48
σ	28.53	22.83	20.69	19.24	17.06
Profit	L	2	3	4	H
β_e	-1.39	-0.66	-0.41	-0.61	-1.25
[t]	[-0.63]	[-0.33]	[-0.19]	[-0.30]	[-0.54]
β_g	-5.26	-5.49	-4.33	-4.08	-4.58
[t]	[-2.50]	[-2.90]	[-2.15]	[-2.10]	[-2.10]
μ	5.27	6.01	7.25	7.39	8.27
σ	23.04	17.51	16.83	16.22	18.40

	Estimation Outcomes in 1st Step				
Invest	L	2	3	4	H
β_e	-1.86	-0.34	-0.42	0.09	-1.73
$[t]$	[-0.75]	[-0.17]	[-0.21]	[0.04]	[-0.69]
β_g	-4.98	-4.64	-4.93	-4.85	-3.84
$[t]$	[-2.10]	[-2.42]	[-2.57]	[-2.36]	[-1.60]
μ	9.98	7.85	7.25	7.28	6.63
σ	19.48	15.07	16.01	17.54	22.97
MoM	L	2	3	4	W
β_e	-4.07	-2.72	-1.65	-1.49	0.17
$[t]$	[-1.82]	[-1.27]	[-0.83]	[-0.70]	[0.06]
β_g	-4.66	-4.44	-4.01	-4.72	-5.30
$[t]$	[-2.19]	[-2.18]	[-2.13]	[-2.33]	[-1.93]
μ	2.43	6.65	6.59	7.86	11.81
σ	26.78	19.84	17.12	16.30	20.17
EP	L	2	3	4	H
β_e	-1.42	-1.22	0.34	0.32	0.23
$[t]$	[-0.68]	[-0.59]	[0.17]	[0.13]	[0.09]
β_g	-3.38	-4.15	-4.67	-5.60	-6.09
$[t]$	[-1.69]	[-2.11]	[-2.52]	[-2.41]	[-2.49]
μ	5.88	7.04	7.08	9.75	10.52
σ	19.41	16.73	16.01	18.11	19.36

Table 6: **Risk Premium**

This table reports the time-series average risk premium $\vec{\lambda}$ in Fama-Macbeth two-step regression, for model **P-ND**, using annual data during 1965-2019. Panel (A) reports the time-series average risk premium in sample. Panel (B) reports the statistics for each estimation. In 2nd, 4th and 6th columns, risk premium is estimated assuming zero-beta rate identical with risk-free rate. Estimation of risk premium is **without** intercept term in 2nd step of regression. In 3rd, 5th and 7th columns, risk premium is estimated **with** intercept term in 2nd step of regression. T-stat is reported in brackets. Calculation of t-stat uses the simple standard error. OLS-R^2 calculates the Fama-Macbeth two-step regression without intercept term in 2nd step, and similarly for GLS-R^2. COLS-R^2 calculates the Fama-Macbeth two-step regression with intercept term in 2nd step, and similarly for CGLS-R^2.

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	Panel (A): Risk Premium					
	without	with	without	with	without	with
λ_e	0.54	0.65	0.38	0.43	-0.06	-0.19
$[t]$	[1.26]	[1.55]	[0.55]	[0.64]	[-0.17]	[-0.59]
λ_g	-1.64	-1.11	-1.56	-1.28	-1.43	-0.20
$[t]$	[-3.91]	[-2.05]	[-4.19]	[-2.50]	[-3.34]	[-0.50]
α	-	2.90	-	1.99	-	6.96
$[t]$	-	[0.93]	-	[0.63]	-	[2.62]
	Panel (B): Stats					
OLS-R^2	0.43		0.63		-1.49	
GLS-R^2	0.15		-0.38		-0.10	
COLS-R^2		0.53		0.67		0.10
CGLS-R^2		0.15		0.01		0.06

Table 7: **Subgroup of Testing Assets**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for model **P-ND**, using different subsets of testing assets within **Mix 30**. Column **Size-BM** uses the size and BM portfolios. Column **Profit-IK** uses the profitability and investment portfolios. Column **MoM-EP** uses the momentum and earning/price portfolios. Other description in Table (5) applies.

	Specification of Testing Assets					
	Size-BM		Profit-IK		MoM-EP	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	25.15	28.65	40.79	42.74	27.12	24.73
$[t]$	[2.05]	[4.57]	[2.74]	[5.41]	[1.34]	[4.21]
b_g	-71.94	-62.63	-62.93	-72.75	-74.44	-81.56
$[t]$	[-3.11]	[-5.97]	[-1.90]	[-5.00]	[-1.97]	[-6.85]
	Panel (B): Stats					
MAPE	0.33		0.36		0.36	
RMSE	0.41		0.42		0.37	
J-pval		25.15		45.57		40.40

Table 8: **Definitions of Sector**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for model **P-ND**, using different definitions of consumption sectors. Column **Financial-Service** considers a narrow definition of nondurable services, where financial service are excluded. Column **NIPA-Nondurable** considers nondurable good sector and service sector, using NIPA definition. Gasoline goods are included. Point estimates of risk price vector, model-fitness statistic in first stage of GMM and J-stat in second stage are reported. All estimations use **Mix 30** portfolios as testing assets. Other description in Table (2) and Table (4) applies.

	Specification of Sector Definition			
	Financial Service		NIPA-Nondurable	
	Panel (A): Risk Price			
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	28.12	26.55	35.77	36.19
$[t]$	[1.70]	[15.25]	[4.78]	[27.71]
b_g	-73.37	-76.57	-71.84	-69.44
$[t]$	[-2.15]	[-13.40]	[-2.31]	[-11.19]
	Panel (B): Stats			
MAPE	0.39		0.40	
RMSE	0.45		0.47	
J-pval		91.40		93.68

Table 9: **Construction of Price Index**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for model **P-ND**, using different definitions of price index. Column **Tornqvist** constructs the Tornqvist index as sector-level price. Column **Average** uses the average change of sub-category price index in each sector. Point estimates of risk price vector, model-fitness statistic in first stage of GMM and J-stat in second stage are reported. All estimations use **Mix 30** portfolios as testing assets. Other description in Table (2) and Table (4) applies.

	Method of Price Calculation			
	Tornqvist		Average	
	Panel (A): Risk Price			
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	26.09	27.20	15.51	17.33
$[t]$	[1.74]	[13.36]	[0.99]	[11.27]
b_g	-70.82	-69.75	-83.72	-78.54
$[t]$	[-2.20]	[-16.42]	[-2.51]	[-23.97]
	Panel (B): Stats			
MAPE	0.37		0.35	
RMSE	0.43		0.44	
J-pval		92.32		93.74

Table 10: **Long-Sample Estimation**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for asset pricing model **P-ND**, using different sample periods during 1935-2019. All estimations use the **Size-BM 25** portfolios as the testing assets.

	Sample Period					
	1935-1989		1950-2004		1965-2019	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	31.56	31.64	35.41	39.59	30.05	33.72
[t]	[3.69]	[26.79]	[3.19]	[12.49]	[2.61]	[13.06]
b_g	-47.41	-45.67	-65.65	-62.79	-68.26	-63.83
[t]	[-2.68]	[-11.06]	[-2.85]	[-13.66]	[-2.90]	[-11.68]
	Panel (B): Stats					
MAPE	0.70		0.32		0.38	
RMSE	0.95		0.38		0.51	
J-pval		82.51		96.93		81.48

Table 11: **Quantity Index**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for different consumption-based models, using testing assets **Size-BM 25**. Columns in **C-ND** use the quantity index in the nondurable consumption sector, as a single factor. Columns in **CRRA** use the nonlinear model of quantity index to estimate equation $\mathbb{E}_t\left[(\frac{\mathbf{C}_{t+1}}{\mathbf{C}_t})^{-\gamma} \cdot R^e_{k,t+1}\right] = 0$. Columns in **P-ND** use the relative price of goods, and expenditure in the nondurable consumption sector. Columns **Restricted Estimation** report estimations assuming risk factors $\vec{f}$ have the constant conditional mean, $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$. Estimations have the same specification with Table (3). Columns **Unrestricted Estimation** report estimations have the same specification with Table (2). Columns **Risk-Free Rate** report estimations including the risk-free asset equation. Panel (A) reports the point estimates of risk price vector in first stage of GMM. T-stat is reported in brackets, Newey-West standard error has adjustment for 2 periods. Panel (B) reports the point estimate of risk-aversion parameter and the standard error. Panel (C) reports the model-fitness statistic in first stage of GMM and J-stat in second stage.

	Specification of Model and Estimation Equation						
	Restricted Estimation		Unrestricted Estimation			Risk-Free Rate	
	C-ND (1)	P-ND (2)	CRRA (3)	C-ND (4)	P-ND (5)	C-ND (6)	P-ND (7)
	Panel (A): Risk Price						
b_c	106.47			50.88		50.95	
$[t]$	[1.98]			[4.74]		[4.74]	
b_e		49.69			30.05		32.34
$[t]$		[0.63]			[2.61]		[2.98]
b_g		-97.99			-68.26		-66.83
$[t]$		[-3.17]			[-2.90]		[-2.90]
	Panel (B): Risk-Aversion						
γ	106.47	49.69	66.93	50.88	30.05	50.95	32.34
$(s.e.)$	(53.89)	(78.25)	(28.13)	(10.74)	(11.51)	(10.75)	(10.86)
	Panel (C): Stats						
MAPE	9.53	1.07	4.60	0.79	0.38	0.76	0.38
RMSE	9.81	1.50	4.74	0.95	0.51	0.93	0.50
J-pval	89.60	93.34	88.22	95.51	81.48	80.64	76.38

Table 12: **Quantities**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for the pricing kernel measured with quantities. Estimation uses the quantity of goods and quantity of services. 1st-stage estimation outcome and 2nd-stage are reported. Point estimates of risk price vector, model-fitness statistic in first stage of GMM and J-stat in second stage are reported. All estimations use **Mix 30** portfolios as testing assets. Other description in Table (2) and Table (4) applies.

	Risk Price	
	1st-Stage	2nd-Stage
b_{c_g}	45.04	37.22
$[t]$	[1.09]	[5.66]
b_{c_s}	6.34	10.61
$[t]$	[0.22]	[2.74]
	GMM Stats	
MAPE	0.53	
RMSE	0.65	
J-pval		91.31

Table 13: **Habit Model**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for Stone-Geary preference. 1st-stage estimation outcome is reported for each specification of zero-habit sector. Column **Good** assumes the consumer has zero habit in goods, **Service** assumes the zero habit in services. Point estimates of risk price vector, model-fitness statistic in first stage of GMM and J-stat in second stage are reported. All estimations use **Mix 30** portfolios as testing assets. Other description in Table (2) and Table (4) applies.

	Zero-Habit Sector	
	Good	Service
b_{c_g}	182.54	
$[t]$	[2.56]	
b_{c_s}		33.79
$[t]$		[2.70]
b_{p_g}	108.92	-13.12
$[t]$	[1.60]	[-0.81]
	GMM Stats	
MAPE	2.91	0.53
RMSE	4.04	0.64
J-pval	95.91	95.73

Table 14: **Detailed Consumption Sectors**

This table reports the point estimate for the risk price vector $\vec{b}$ in GMM estimation, for model **P-ND** with multiple consumption sectors. **Food in Good** considers food-good and non-food within good sector. **Food in Service** considers food-service and non-food within service sector. **Four** considers all four sectors. Point estimates of risk price vector, model-fitness statistic in first stage of GMM and J-stat in second stage are reported. All estimations use **Mix 30** portfolios as testing assets. Other description in Table (2) and Table (4) applies.

	Specification of Consumption Sector		
	Food in Good	Food in Service	Four
b_e	14.13	21.63	14.70
$[t]$	[1.77]	[5.54]	[1.74]
b_g		-71.95	
$[t]$		[-3.58]	
b_{gf}	-51.85		-78.10
$[t]$	[-1.34]		[-2.60]
b_{gn}	-76.62		-88.46
$[t]$	[-1.72]		[-2.44]
b_{sf}		302.21	302.37
$[t]$		[2.11]	[2.02]
		Stats	
MAPE	0.26	0.19	0.18
RMSE	0.31	0.25	0.21
J-pval	88.49	92.02	88.08

Table 15: **Average Fitness of Asset Pricing Models**

The average fitness of asset pricing models are reported. GMM estimation is conducted for 114 groups of testing asset. All statistics are the average outcome across estimations. Estimation uses annual data during 1968-2019. Column **CAPM** uses the Market factor. Column **Q-5** is the augmented q-factor model. Column **P-ND** uses prices of food goods, non-food goods, food services, non-food services and total expenditure. Column **P-D** includes quantity of durable stock. Other descriptions are identical with Table (4).

	Specification of Model					
	Traded Factor		Quantity		Price	
	CAPM	Q-5	C-ND	C-D	P-ND	P-D
MAPE	2.20	0.24	0.73	0.67	0.22	0.21
RMSE	2.74	0.30	0.92	0.86	0.27	0.26
J-pval	38.27	46.45	76.63	73.48	49.62	49.03

Online Appendix

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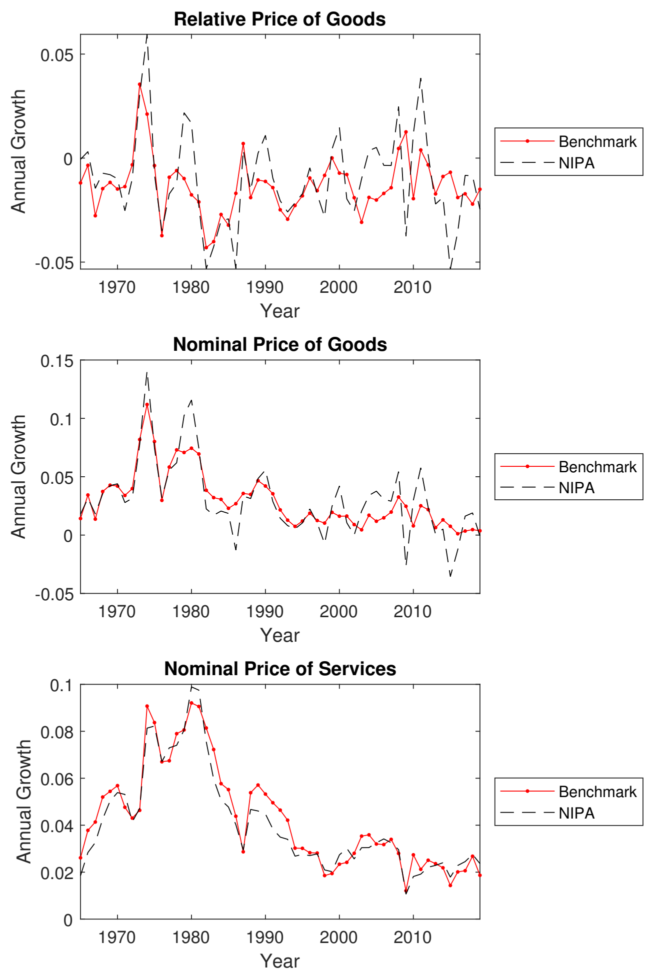

Figure A.1: Plot of Price Indices

The X-axis is time-axis, the first row of figure plots the annual log change of price of goods relative to price of services $\mathrm{d}p_g - \mathrm{d}p_s$. The second row of figure plots the annual log change in nominal price of goods $\mathrm{d}p_g$. The third row of figure plots the annual log change in nominal price of services $\mathrm{d}p_s$. The red thick line plots the price index using the definition of sectors described in Section (3). The dark dashed line plots the price index using the original definition of sectors in NIPA.

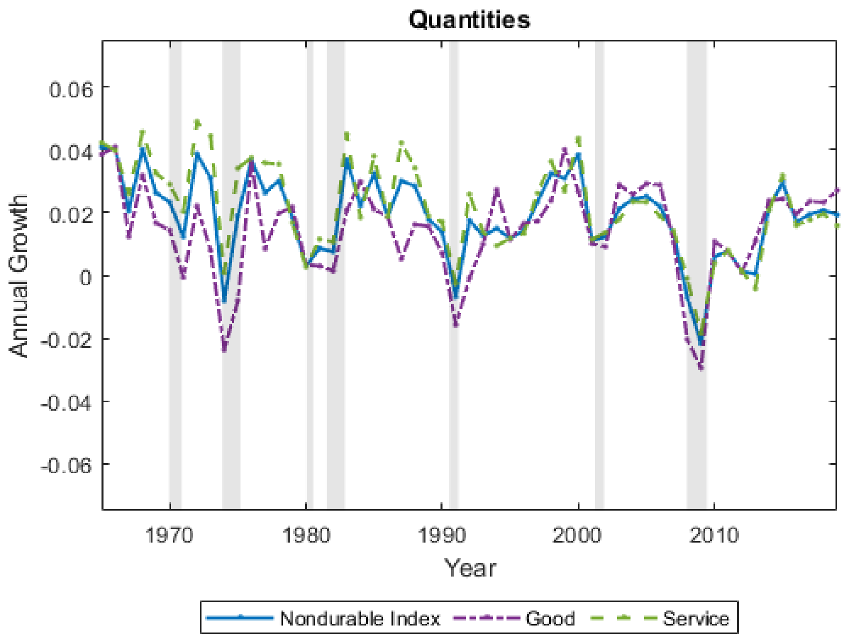

Figure A.2: Time Series of Quantity Outcomes

The X-axis is time-axis. The figure plots the annual log change of quantity indices. The blue thick line plots the quantity index for the whole nondurable consumption $\mathrm{d}c_{nd}$. The purple dashed line plots the quantity for the nondurable goods, the green dashed line for the services. Definition of sectors is described in Section (3).

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Table A.1: Variable Definition

Variable Name	Definition	Time-Span	Data Source
		NIPA	
Non-Durable Expenditure	Current-Price Amount	1965-2019, Annual	Table 2.3.4
Good Price, Service Price	Implied sector-level deflator	same as above	Table 2.3.4, 2.3.5
Food Prices	Line 27-29, Line 83	same as above	Table 2.4.4, 2.4.5
Wage, Labor Hour, Output	Private industries	same as above	Table 6.6, 6.9, 6.1
		Financial Data	
Equity Return	-	1965-2019, Annual	Kenneth French's website
VIX	-	1991-2019, Annual	Compustat CBOE
EP-10Y	1/PE - 10 Year Yield	1965-2019, Annual	Campbell Shiller's website
CaY	-	1965-2017, Annual	Martin Lettau's website

Table A.2: Correlation with Economic Outcomes

This table reports correlation between relative price of goods and other economic outcomes. Market factor is denoted as **MKT**. Growth in GDP (including both the consumption and investment) is denoted as **Output**. Stock market uncertainty index is denoted as **VIX**, available from 1991-2019. Growth in dividend-price ratio from Robert Shiller's website, minus the 10-Year treasury yield rate is denoted as **EP-Y10**. Growth in Consumption-Wealth CaY index from Martin Lettau's website, is denoted as **CaY**. Growth in households labor earning comes from NIPA Table 2.1, denoted as **income**. Growth in per-employee wage comes from NIPA Table 6.6, denoted as **Wage-emp**. Growth in per-hour wage comes from CPS, denoted as **Wage-hour**. Unfiltered consumption quantity, **Unfiltered-C** comes from Tim Kroencke's website. Growth of households garbage, use **LandFill** and **Generation** of waste from EPA website.

Panel (c): Business Cycle - Correlation

	MKT	Hour	Output
Corr(z, d$p_{g/s}$)	-0.35	0.01	0.01
($s.e.$)	(0.15)	(0.19)	(0.21)
	Income	Wage-emp	Wage-hour
Corr(z, d$p_{g/s}$)	0.21	0.43	0.28
($s.e.$)	(0.20)	(0.15)	(0.15)
	VIX	EP-Y10	CAY
Corr(z, d$p_{g/s}$)	0.33	0.22	-0.21
($s.e.$)	(0.18)	(0.10)	(0.20)
	Unfiltered-C	LandFill	Generation
Corr(z, d$p_{g/s}$)	-0.31	-0.00	-0.06
($s.e.$)	(0.14)	(0.17)	(0.16)

Table A.3: Pairwise Correlation in Prices

Time span of sample is during 1965-2019. Correlation coefficient are computed for the price x and the price of goods, all nominal prices are deflated by the price of services. Standard error is in parenthesis.

	Business Cycle - Correlation		
	PCE	Good	Dur Good
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.58	0.55	0.14
$(s.e)$	(0.10)	(0.11)	(0.08)
	Vehicle	Furniture	Rec Vehicle
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.06	0.27	0.12
$(s.e)$	(0.18)	(0.08)	(0.12)
	Other DurGood	(NIPA) NDur Good	Food
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.30	0.61	0.92
$(s.e)$	(0.12)	(0.12)	(0.05)
	Clothes	Gasoline	Other NDurGood
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.32	0.15	0.36
$(s.e)$	(0.09)	(0.15)	(0.13)
	(NIPA) Serv	House	Util
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.10	0.05	-0.04
$(s.e)$	(0.06)	(0.07)	(0.11)
	Health	Transport	Recreation
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	-0.13	0.27	0.23
$(s.e)$	(0.08)	(0.12)	(0.17)
	FoodAway	Finance	Other Service
$\mathrm{Corr}(\mathrm{d}p_{x/s}, \mathrm{d}p_{g/s})$	0.45	-0.23	0.28
$(s.e)$	(0.09)	(0.12)	(0.12)

Table A.4: Dickey-Fuller Test

Time span of sample is during 1965-2019. Dickey-Fuller test is implemented for prices relative to price of services. P-value is reported in brackets.

	Dickey-Fuller Test		
	NDur Good	Wage	Dur Good
$p_{x/s}$	1.00	1.00	0.00
$[p]$	[0.00]	[0.01]	[0.18]

Table A.5: Fitness of Asset Pricing Models, Fama-Macbeth Regression

The fitness of asset pricing models are reported. Estimation outcome of Fama-Macbeth two-step regression using testing assets of **Mix 30** portfolios is reported. Estimation uses annual data during 1965-2019. Column **CAPM** uses the Market factor. Column **FF-5** is the Fama-French 5-Factor Model. Column **C-ND** uses quantity index of nondurable consumption, as the single factor. Column **C-D** includes quantity of durable stock. Column **P-ND** uses price of goods, price of services and total expenditure. Column **P-D** includes quantity of durable stock. Full description of asset pricing models, construction of durable stock, quantity index and price index are decribed in Section (3).OLS-R^2 calculates the Fama-Macbeth two-step regression without intercept term in 2nd step, and similarly for GLS-R^2. COLS-R^2 calculates the Fama-Macbeth two-step regression with intercept term in 2nd step, and similarly for CGLS-R^2.

	Specification of Model					
	Traded Factor		Quantity		Price	
	CAPM	FF-5	C-ND	C-D	P-ND	P-D
OLS-R^2	-0.48	-0.07	-16.76	-0.28	0.43	0.44
GLS-R^2	0.12	0.15	0.13	0.13	0.15	0.15
COLS-R^2	0.04	0.58	0.34	0.38	0.53	0.53
CGLS-R^2	0.12	0.16	0.13	0.13	0.15	0.15

Table A.6: Risk Exposure to Factors, Other Testing Assets

This table reports the point estimate of risk exposure in Fama-Macbeth 2-step regression, $\vec{\beta}$, for different subsets of testing assets in **Size-BM 25**. T-stat is reported in brackets, Newey-West standard error has adjustment for 2 periods. For each testing asset, sample-average excess return (%) is reported as μ, and volatility is reported as σ.

	Estimation Outcomes in 1st Step				
SMALL	BM1	BM2	BM3	BM4	BM5
β_e	-4.11	-2.36	-2.87	-1.40	-1.46
$[t]$	[-1.14]	[-0.69]	[-0.89]	[-0.39]	[-0.37]
β_g	-6.90	-7.71	-8.16	-8.54	-9.41
$[t]$	[-2.01]	[-2.37]	[-2.66]	[-2.51]	[-2.54]
μ	4.89	11.08	10.64	13.60	14.57
σ	36.06	29.54	26.75	25.45	27.86
ME2	BM1	BM2	BM3	BM4	BM5
β_e	-3.70	-2.64	-2.20	-0.74	-0.88
$[t]$	[-1.27]	[-0.95]	[-0.76]	[-0.24]	[-0.28]
β_g	-5.51	-6.58	-8.00	-8.70	-8.66
$[t]$	[-1.99]	[-2.48]	[-2.90]	[-3.02]	[-2.88]
μ	6.91	10.28	11.40	11.86	12.35
σ	27.99	22.72	22.44	21.71	23.23
ME3	BM1	BM2	BM3	BM4	BM5
β_e	-3.13	-2.38	-1.03	-1.55	-0.88
$[t]$	[-1.24]	[-0.90]	[-0.42]	[-0.55]	[-0.29]
β_g	-5.55	-6.82	-6.59	-6.69	-7.97
$[t]$	[-2.31]	[-2.72]	[-2.84]	[-2.51]	[-2.73]
μ	7.10	10.01	9.75	11.50	13.14
σ	24.33	20.94	18.87	21.65	22.66

Estimation Outcomes in 1st Step					
ME4	BM1	BM2	BM3	BM4	BM5
β_e	-1.14	0.14	0.60	-1.16	-1.46
$[t]$	[-0.56]	[0.04]	[0.32]	[-0.41]	[-0.61]
β_g	-5.28	-5.16	-5.04	-4.97	-4.84
$[t]$	[-2.72]	[-1.65]	[-2.85]	[-1.86]	[-2.14]
μ	6.69	8.88	6.09	9.93	8.65
σ	17.36	28.30	16.82	19.90	17.92
BIG	BM1	BM2	BM3	BM4	BM5
β_e	-0.22	1.93	2.16	-1.33	-0.50
$[t]$	[-0.10]	[0.87]	[1.00]	[-0.70]	[-0.19]
β_g	-3.91	-3.59	-3.54	-3.42	-3.22
$[t]$	[-1.82]	[-1.70]	[-1.72]	[-1.90]	[-1.31]
μ	7.48	6.96	3.76	6.67	6.79
σ	22.82	19.62	25.50	17.23	20.22

This table reports the point estimate of risk exposure in Fama-Macbeth 2-step regression, $\vec{\beta}$, for different subsets of testing assets in **Industry 30**. T-stat is reported in brackets, Newey-West standard error has adjustment for 2 periods. For each testing asset, sample-average excess return (%) is reported as μ, and volatility is reported as σ. Full name of industry is listed at the end of the table.

	Estimation Outcomes in 1st Step					
	Food	Beer	Smoke	Games	Books	Hshld
β_e	-1.46	-1.16	0.35	-1.88	-1.62	-1.14
$[t]$	[-0.61]	[-0.41]	[0.11]	[-0.60]	[-0.60]	[-0.56]
β_g	-4.84	-4.97	-2.93	-7.79	-5.65	-5.28
$[t]$	[-2.14]	[-1.86]	[-0.98]	[-2.63]	[-2.21]	[-2.72]
μ	8.65	9.93	12.56	11.55	6.53	6.69
σ	17.92	19.90	22.51	29.84	23.09	17.36
	Clths	Hlth	Chems	Txtls	Cnstr	Steel
β_e	-5.63	-0.87	-3.54	-5.31	-1.02	2.16
$[t]$	[-1.73]	[-0.33]	[-1.69]	[-1.67]	[-0.41]	[1.00]
β_g	-6.74	-2.41	-1.96	-6.05	-6.63	-3.54
$[t]$	[-2.17]	[-0.97]	[-0.99]	[-2.00]	[-2.80]	[-1.72]
μ	10.70	8.89	6.67	8.17	7.30	3.76
σ	29.73	18.93	20.45	29.78	22.45	25.50
	FabPr	ElcEq	Autos	Carry	Mines	Coal
β_e	-0.22	-1.57	-5.61	-0.71	0.23	0.41
$[t]$	[-0.10]	[-0.55]	[-2.01]	[-0.22]	[0.08]	[0.09]
β_g	-3.91	-6.95	-7.00	-7.37	-0.36	0.56
$[t]$	[-1.82]	[-2.54]	[-2.64]	[-2.40]	[-0.13]	[0.12]
μ	7.48	10.24	6.28	10.57	6.15	9.21
σ	22.82	24.05	29.42	26.11	29.08	42.50

	Estimation Outcomes in 1st Step					
	Oil	Util	Telcm	Servs	BusEq	Paper
β_e	1.93	0.60	-0.50	-3.81	0.14	-1.33
$[t]$	[0.87]	[0.32]	[-0.19]	[-1.07]	[0.04]	[-0.70]
β_g	-3.59	-5.04	-3.22	-5.54	-5.16	-3.42
$[t]$	[-1.70]	[-2.85]	[-1.31]	[-1.63]	[-1.65]	[-1.90]
μ	6.96	6.09	6.79	10.77	8.88	6.67
σ	19.62	16.82	20.22	28.81	28.30	17.23
	Trans	Whlsl	Rtail	Meals	Fin	Other
β_e	-3.28	-2.44	-3.93	-2.14	0.39	0.20
$[t]$	[-1.48]	[-0.88]	[-1.41]	[-0.60]	[0.15]	[0.09]
β_g	-5.51	-6.54	-5.99	-7.84	-7.46	-6.61
$[t]$	[-2.62]	[-2.49]	[-2.25]	[-2.32]	[-2.95]	[-3.10]
μ	6.79	8.95	9.00	12.01	8.52	4.41
σ	20.72	24.47	22.76	31.52	21.67	20.53

Note for industry names:

1 Food: Food Products
2 Beer: Beer and Liquor
3 Smoke: Tobacco Products
4 Games: Recreation
5 Books: Printing and Publishing
6 Hshld: Consumer Goods
7 Clths: Apparel
8 Hlth: Healthcare, Medical Equipment, Pharmaceutical Products
9 Chems: Chemicals
10 Txtls: Textiles
11 Cnstr: Construction and Construction Materials
12 Steel: Steel Works Etc
13 FabPr: Fabricated Products and Machinery
14 ElcEq: Electrical Equipment
15 Autos: Automobiles and Trucks
16 Carry: Aircraft, ships, and railroad equipment
17 Mines: Precious Metals, Non-Metallic, and Industrial Metal Mining
18 Coal: Coal
19 Oil: Petroleum and Natural Gas
20 Util: Utilities
21 Telcm: Communication
22 Servs: Personal and Business Services
23 BusEq: Business Equipment
24 Paper: Business Supplies and Shipping Containers
25 Trans: Transportation
26 Whlsl: Wholesale
27 Rtail: Retail
28 Meals: Restaraunts, Hotels, Motels
29 Fin: Banking, Insurance, Real Estate, Trading
30 Other: Everything Else

Table A.4: Subgroup of Testing Assets, Fama-Macbeth Regression

This table reports the time-series average risk premium $\vec{\lambda}$ in Fama-Macbeth two-step regression, for model **P-ND**, using different subsets of testing assets within **Mix 30**. Column **Size-BM** uses the size and BM portfolios. Column **Profit-IK** uses the profitability and investment portfolios. Column **MoM-EP** uses the momentum and earning/price portfolios. Other description in Table (6) applies.

	Specification of Testing Assets					
	Size-BM		Profit-IK		MoM-EP	
	Panel (A): Risk Premium					
	without	with	without	with	without	with
λ_e	0.16	0.12	-0.66	-0.33	0.98	1.24
[t]	[0.29]	[0.22]	[-0.95]	[-0.50]	[2.02]	[2.32]
λ_g	-1.51	-1.03	-1.42	0.39	-1.83	-1.01
[t]	[-3.66]	[-1.75]	[-3.27]	[0.53]	[-4.15]	[-1.39]
α	-	2.85	-	8.87	-	4.25
[t]	-	[0.87]	-	[2.18]	-	[1.14]
	Panel (B): Stats					
OLS-R^2	0.65		-0.48		0.73	
GLS-R^2	0.18		-0.57		0.36	
COLS-R^2		0.85		0.06		0.78
CGLS-R^2		0.25		0.28		0.36

Table A.5: Parameters, Quantity

This table reports the point estimate of the risk price parameter b_c in GMM estimation, for model **C-ND**, using sets of testing assets **Mix 30**, **Size-BM 25**, **Industry 30**, during the time-interval 1965-2019. Panel (A) reports the point estimate. Panel (B) reports statistics of model fitness. Other description of statistics in Table (2) and Table (4) applies.

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_c	51.16	49.27	50.88	46.26	52.64	48.81
$[t]$	[4.31]	[18.94]	[4.74]	[28.43]	[4.23]	[26.17]
	Panel (B): GMM Stats					
MAPE	0.71		0.79		0.98	
RMSE	0.87		0.95		1.36	
J-pval		96.23		95.51		96.78

Table A.6: Subgroup of Testing Assets, Quantity

This table reports the point estimate of the risk price parameter b_c in GMM estimation, for asset-pricing model **C-ND**, using different subsets of testing assets in **Mix 30**. Column **Size-BM** uses the size and BM portfolios. Column **Profit-IK** uses the profitability and investment portfolios. Column **MoM-EP** uses the momentum and earning/price portfolios. Other description in Table (5) applies.

	Specification of Testing Assets					
	Size-BM		Profit-IK		MoM-EP	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_c	51.13	38.18	52.30	53.41	50.26	43.31
$[t]$	[4.32]	[8.99]	[3.96]	[9.56]	[4.56]	[13.63]
	Panel (B): Stats					
MAPE	0.70		0.38		1.04	
RMSE	0.79		0.46		1.18	
J-pval		40.94		79.76		41.53

Table A.7: Parameters, Fama-French 5-Factor Model

This table reports the point estimate of the risk price vector $\vec{b}$ in GMM estimation, for Fama-French 5-Factor Model **FF-5**, using sets of testing assets **Mix 30**, **Size-BM 25**, **Industry 30**, during the time-interval 1965-2019. Panel (A) reports the point estimate. Panel (B) reports statistics of model fitness. Other description of statistics in Table (4) and Table (5) applies.

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_{MKT}	2.38	2.51	2.51	2.65	2.64	2.78
$[t]$	[3.77]	[10.82]	[4.39]	[10.04]	[4.02]	[7.94]
b_{Size}	1.72	1.64	1.28	1.20	0.88	0.68
$[t]$	[2.15]	[5.36]	[1.32]	[2.92]	[0.69]	[1.45]
b_{BM}	-3.44	-3.06	-2.24	-1.82	-5.86	-4.88
$[t]$	[-2.05]	[-4.45]	[-1.07]	[-2.99]	[-2.13]	[-6.31]
b_{Profit}	6.56	6.69	5.79	6.28	5.18	5.30
$[t]$	[4.28]	[11.59]	[2.39]	[9.33]	[2.96]	[10.62]
b_{Invest}	7.42	7.33	6.97	7.37	9.36	8.21
$[t]$	[4.36]	[9.10]	[3.16]	[10.67]	[2.05]	[6.91]
	Panel (B): Stats					
MAPE	0.79		0.65		1.09	
RMSE	1.37		0.81		1.37	
J-pval		81.07		59.85		84.45

Table A.8: Subgroup of Testing Assets, Fama-French 5-Factor Model

This table reports the point estimate of the risk price vector $\vec{b}$ in GMM estimation, for Fama-French 5-Factor Model **FF-5**, using different subsets of testing assets in **Mix 30**. Column **Size-BM** uses the size and BM portfolios. Column **Profit-IK** uses the profitability and investment portfolios. Column **MoM-EP** uses the momentum and earning/price portfolios. Other description in Table (5) applies.

	Specification of Testing Assets					
	Size-BM		Profit-IK		MoM-EP	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_{MKT}	2.27	2.40	2.72	2.55	-0.28	2.01
[t]	[2.76]	[4.48]	[3.24]	[4.43]	[-0.07]	[2.01]
b_{Size}	0.24	0.91	1.22	1.15	10.96	6.00
[t]	[0.19]	[1.29]	[0.55]	[0.73]	[1.02]	[1.76]
b_{BM}	2.89	0.64	-7.79	-7.48	-12.17	-6.16
[t]	[1.01]	[0.40]	[-1.56]	[-2.08]	[-1.00]	[-1.27]
b_{Profit}	1.71	3.79	5.90	5.59	25.31	14.77
[t]	[0.54]	[1.47]	[4.20]	[4.85]	[1.17]	[3.27]
b_{Invest}	-2.14	2.64	12.56	11.90	10.14	9.14
[t]	[-0.44]	[1.19]	[2.18]	[2.94]	[1.21]	[2.83]
	Panel (B): Stats					
MAPE	0.35		0.11		0.52	
RMSE	0.40		0.14		0.65	
J-pval		14.51		91.92		21.61

Table A.9: Parameters, 1935-2019

This table reports the estimation during the time-interval 1935-2019. In the 2nd column and the 3rd column, teststing assets are Size-BM 25 portfolios. In the 4th column and the 5th column, teststing assets are Industry 30 portfolios. Panel (A-B) reports the estimate of GMM estimation. Other description of statistics in Table (2) and Table (4) applies.

	Specification of Testing Assets			
	Size-BM 25		Industry 30	
	Panel (A): Risk Price			
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	31.40	37.30	35.72	32.54
$[t]$	[3.17]	[10.06]	[3.71]	[12.06]
b_g	-68.76	-69.77	-69.03	-66.66
$[t]$	[-3.14]	[-12.89]	[-4.48]	[-19.39]
	Panel (B): GMM Stats			
MAPE	0.58		0.92	
RMSE	0.75		1.25	
J-pval		78.78		84.30

Table A.9: Parameters, No Correction in Share

This table reports the estimation without the correction of expenditure share $\omega_{g,t}$, during the sample period 1935-2019. The model $\mathbf{P}^{L}$**-ND** of $\mathrm{d}\tilde{m}_{t+1} = -b_{e,L} \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_{p,L} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})$ is estimated. Panel (A-B) reports the estimate of GMM estimation. Other description of statistics in Table (2) and Table (4) applies.

	Specification of Testing Assets			
	Size-BM 25		Industry 30	
	Panel (A): Risk Price			
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
$b_{e,L}$	26.93	30.26	31.94	29.44
$[t]$	[2.88]	[8.34]	[3.90]	[11.62]
$b_{p,L}$	-35.65	-34.52	-34.96	-34.49
$[t]$	[-3.27]	[-14.55]	[-4.61]	[-15.62]
	Panel (B): GMM Stats			
MAPE	0.48		0.89	
RMSE	0.63		1.17	
J-pval		62.35		82.10

Table A.10: Parameters, unconditional GMM

This table reports the point estimate for the risk price vector $\vec{b}$ in unconditional GMM estimation of model **$\mathbf{P}^L$-ND** of $\mathrm{d}\tilde{m}_{t+1} = -b_{e,L} \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1}) - b_{p,L} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})$. Estimation uses equation $\mathbb{E}_t[R^e_{k,t+1}] = \frac{1}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} \cdot \mathbb{E}_t\left[[b_{e,L} \cdot (\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1} - g_{\tilde{e}}) + b_{p,L} \cdot (\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1} - g_{\tilde{p}})] \cdot R^e_{k,t+1}\right]$, with $g_{\tilde{e}}$ as the constant expected growth of total expenditure relative to price of services, and $g_{\tilde{p}}$ for the relative price of goods. Panel (A) reports the vector of risk price $\vec{b}$. Panel (B) reports the statistics for each estimation. 2nd column and 3rd column use **Mix 30** portfolios. 4th column and the 5th column use **Size-BM 25** portfolios. 6th column and the 7th column use **Industry 30** portfolios. Other description in Table (3) and Table (5) applies.

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
$b_{e,L}$	59.05	61.13	47.49	49.29	18.17	19.95
$[t]$	[0.75]	[8.24]	[0.63]	[5.38]	[0.55]	[1.35]
$b_{p,L}$	-101.03	-97.56	-93.67	-79.58	-79.69	-81.64
$[t]$	[-2.16]	[-8.98]	[-3.17]	[-9.66]	[-1.89]	[-9.24]
	Panel (B) Stats					
MAPE	1.15		1.07		2.42	
RMSE	1.43		1.50		3.37	
J-pval		98.46		93.34		96.92

Table A.11: Parameters, Quantity, unconditional GMM

This table reports the point estimate for the risk price parameter b_c in unconditional GMM estimation of model **C-ND** of $\mathrm{d}\tilde{m}_{t+1} = -b_c \cdot \mathrm{d}c_{t+1}$. Estimation uses equation $\mathbb{E}_t[R^e_{k,t+1}] = \frac{b_c}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]} \cdot \mathbb{E}_t\left[(\mathrm{d}c_{t+1} - g_c) \cdot R^e_{k,t+1}\right]$, with g_c as the constant expected growth of nondurable consumption compositie quantity index. Panel (A) reports the risk price parameter b_c. Panel (B) reports the statistics for each estimation. 2nd column and 3rd column use **Mix 30** portfolios. 4th column and the 5th column use **Size-BM 25** portfolios. 6th column and the 7th column use **Industry 30** portfolios. Other description in Table (3) and Table (5) applies.

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_c	48.27	50.54	106.47	101.46	-66.54	-60.76
$[t]$	[1.10]	[7.94]	[1.98]	[5.75]	[-1.84]	[-6.64]
	Panel (B): GMM Stats					
MAPE	7.85		9.53		7.96	
RMSE	8.01		9.81		8.35	
J-pval		96.97		89.60		97.30

Table A.12: Seasonality in Quarterly-Growth Estimation: Price

This table reports the point estimate of the risk price vector $\vec{b}$ in GMM estimation, for asset-pricing model **P-ND**, using quarter-to-quarter growth during 1965Q1-2019Q4. Column **Q1** uses the 1st quarter stock return and pricing kernel, and similarly for Columns **Q2-Q4**. All estimations use the **Size-BM 25** portfolios as the testing assets. Sample uses NIPA quarterly statistics. Other description in Table (2) applies.

	Q1	Q2	Q3	Q4
	Panel (A): Risk Price			
b_e	94.76	-143.69	-62.61	158.78
$[t]$	[1.18]	[-0.46]	[-1.17]	[5.98]
b_g	-228.11	-359.54	-524.45	-102.18
$[t]$	[-0.99]	[-1.03]	[-2.98]	[-0.98]
	Panel (B): GMM			
MAPE	0.33	0.30	0.71	0.40
RMSE	0.40	0.38	0.85	0.49
J-pval	79.62	82.12	80.67	79.24

Table A.13: Seasonality in Quarterly-Growth Estimation: Quantity

This table reports the point estimate of the risk price parameter b_c in GMM estimation, for asset-pricing model **C-ND**, using quarter-to-quarter growth during 1965Q1-2019Q4. Pricing kernel is approximated using the consumption quantity index growth in nondurable sector. Other description in Table (A.12) applies.

	Q1	Q2	Q3	Q4
	Panel (A): Risk Price			
b_c	136.63	16.47	74.42	132.82
$[t]$	[1.20]	[0.17]	[2.13]	[4.53]
	Panel (B): GMM			
MAPE	0.35	0.48	0.83	0.39
RMSE	0.42	0.65	1.02	0.47
J-pval	88.64	83.30	88.32	84.52

Table A.14: Estimation with Supplementary Proxy of Shock

This table reports the estimation using the two-sector pricing kernel and supplement proxy for priminitive shocks. The proxies include each traded factor in Fama-French 5-factor model and momentum factor. Risk price for the supplement proxy is reported as b_x. Description of statistics in Table (2) and Table (4) applies.

	Specification of Additional Shock Proxy					
	MKT	Size	Value	Profit	Invest	MoM
b_e	32.15	23.80	31.15	26.21	30.94	27.05
$[t]$	[3.05]	[1.05]	[2.35]	[1.51]	[2.36]	[2.55]
b_g	-58.75	-82.35	-69.70	-73.76	-68.68	-72.72
$[t]$	[-3.70]	[-1.64]	[-2.41]	[-2.23]	[-2.51]	[-2.69]
b_x	0.26	-0.55	-0.40	0.53	-0.53	0.10
$[t]$	[0.38]	[-0.58]	[-0.72]	[0.69]	[-0.53]	[0.18]
MAPE	0.35	0.31	0.28	0.37	0.32	0.38
RMSE	0.41	0.39	0.38	0.43	0.40	0.44
J-pval	88.68	89.82	89.19	88.99	88.90	88.99

The proxies are from time series of prices. These include IST shock constructed in (Papanikolaou,2011) (available during the years 1965-2008), the extended measure of investment-good price shock measured as equipment price, durable-good price (detrended using Hodrick-Prescott filter, or the linear trend is removed), gasoline-good price and public transportation service price. Shock from nominal equipment price is constructed in the similar way with (Papanikolaou,2011) during 1965-2019. Other nominal prices are adjusted using price of services.

	Specification of Additional Shock Proxy					
	Capital-Good		Durable-good		Gasoline	Trans.
	IST	Equipment	HP	Linear		
b_e	32.13	32.34	37.05	34.24	28.21	37.41
[t]	[4.17]	[3.18]	[3.34]	[3.69]	[1.75]	[2.94]
b_g	-55.94	-62.82	-59.89	-63.48	-66.34	-67.11
[t]	[-4.46]	[-3.65]	[-3.60]	[-3.81]	[-3.17]	[-2.76]
b_x	9.16	-6.25	20.40	11.36	-0.91	16.90
[t]	[0.73]	[-0.41]	[0.55]	[0.43]	[-0.33]	[0.56]
MAPE	0.42	0.36	0.34	0.35	0.38	0.36
RMSE	0.51	0.48	0.44	0.46	0.49	0.46
J-pval	92.28	74.36	71.27	75.68	75.38	77.74

The proxies include the labor hour in private sector, the landfill garbage, the unfiltered consumption, nominal wealth of bottom 90% households denoted as **W-90**, top 10% households denoted as **W-10**, the wealth-share of wealthiest 1% household denoted as **ws-1**. Nominal wealth are adjusted using price of services.

	Specification of Additional Shock Proxy					
	Hour	Landfill	Unf-C	W-90	W-10	ws-1
b_e	40.87	33.63	29.85	28.60	28.46	28.40
$[t]$	[3.92]	[3.15]	[1.20]	[1.72]	[1.69]	[1.80]
b_g	-59.33	-65.78	-74.99	-65.20	-62.40	-62.57
$[t]$	[-2.71]	[-3.26]	[-3.95]	[-2.93]	[-3.00]	[-2.07]
b_x	-8.74	-3.70	-1.96	-0.80	-0.02	-2.02
$[t]$	[-0.82]	[-0.40]	[-0.13]	[-0.23]	[-0.00]	[-0.29]
MAPE	0.37	0.35	0.38	0.39	0.39	0.39
RMSE	0.42	0.41	0.44	0.45	0.46	0.46
J-pval	89.70	89.83	89.62	94.34	94.30	94.25

Table A.13: Risk Exposure to Factors, Traded Factors

This table reports the point estimate of risk exposure $\vec{\beta}$ for each traded factor in Fama-French 5-factor model and momentum factor. T-stat is reported in brackets, Newey-West standard error has adjustment for 2 periods. For each testing asset, sample-average excess return (%) is reported as μ, and volatility is reported as σ.

	Estimation Outcomes in 1st Step					
Factor	MKT	Size	BM	Profit	Invest	MoM
β_e	-0.92	-1.60	2.26	-1.04	0.30	3.69
$[t]$	[-0.44]	[-0.83]	[1.38]	[-0.98]	[0.25]	[1.55]
β_g	-4.74	-2.61	-2.40	0.18	-0.48	-1.87
$[t]$	[-2.39]	[-1.42]	[-1.54]	[0.18]	[-0.41]	[-0.83]
R^2	0.15	0.11	0.07	0.02	0.00	0.07
μ	6.94	3.30	4.09	3.38	3.51	8.05
σ	17.57	13.77	14.26	9.15	9.81	18.12

E Proof and Discussion

E.1 Notation

For consistent notation, I use the upper-case character for the nominal price and the nominal expenditure. I use the lower-case character for the log nominal price $p_j = \log(P_j)$, and the log expenditure $e = \log(E)$.

I use notation $\mathcal{D}_j f$ as the derivative of function f with respect to j-th element. I use $\mathcal{D}_{j,i} f$ as the second-order derivatives of function f, where $\mathcal{D}_{j,i} f = \mathcal{D}_i \mathcal{D}_j f$. I denote the matrix of second-order derivatives as $\mathcal{H} f$ where $\mathcal{H}_{j,i} f = \mathcal{D}_{j,i} f$. I denote the first-order difference of variable x as $\mathrm{d}x = x' - x$.

I use the core-IDU function $V^*(p) = V(p, 1)$ to simplify the notation. The budget set is Homogeneous of Degree Zero,

$$\left\{ \vec{C} \in \mathcal{X} | \sum_{j \in \mathcal{J}} (k \cdot P_j) \cdot C_j \leq k \cdot E \right\} = \left\{ \vec{C} \in \mathcal{X} | \sum_{j \in \mathcal{J}} P_j \cdot C_j \leq E \right\}, \quad k > 0.$$

Because the budget set is H.D.0, when the consumption spending is positive, the core-IDU function V^* and the indirect utility function V has relationship as $V(P, E) = V^*(E^{-1} \cdot P)$. Thorough this paper, I require the core-IDU function V^* with continuous third-order derivatives.

E.2 Approximation

E.2.1 Preference

Definition 3. *Define the **relative expenditure share** between the k-th sector and the j-th sector as $\mathcal{S}_{k,j}$,*

$$\mathcal{S}_{k,j} \equiv \frac{\omega_k}{\omega_j}.$$

*Define the **core-IDU** as the value of IDU given price vector P and 1 unit consumption spending E,*

$$V^*(P) \equiv V(P, 1).$$

E.2.2 Relative Share

Lemma 2. *Given consumption sector k and j, change in the relative share $\mathcal{S}_{k,j} = \frac{\omega_k}{\omega_j}$ can be decomposed into the price effect and the income effect,*

$$\begin{aligned} \mathrm{d}s_{k,j} =&(1 - \eta_{k,k} + \eta_{j,k}) \cdot \mathrm{d}p_k - (1 - \eta_{j,j} + \eta_{k,j}) \cdot \mathrm{d}p_j - \sum_{i \neq k,j} (\eta_{k,i} - \eta_{j,i}) \cdot \mathrm{d}p_i \\ &+ \sum_i (\eta_{k,i} - \eta_{j,i}) \cdot \mathrm{d}e + o(h). \end{aligned} \tag{53}$$

The small character s is the log relative share $s = \log(\mathcal{S})$. The $\mathrm{d}s_{k,j}$ is the log-growth of relative share between sector k and j. The term $o(h)$ is a higher-order term to the change of expenditure and the prices.

Proof. By Taylor's Theorem, for $s_{k,j}$ with continuous second-order derivatives in neighborhood of $a = (p, e)$, there exists $\theta \in [0, 1]$ such that

$$s_{k,j}(a + h) - s_{k,j}(a) = \mathcal{D}s_{k,j}(a) \cdot h + \frac{1}{2} \cdot h^T \cdot \mathcal{H}s_{k,j}(a + \theta \cdot h) \cdot h \tag{60}$$

Denote the term $o(h; a) = \frac{1}{2} \cdot h^T \cdot s_{k,j}(a + \theta \cdot h) \cdot h$. The term $o(h; a)$ is higher-order in h in the sense that given the sup-norm $||h|| \equiv \sup_j |h_j|$, $\lim_{||h|| \to 0} \frac{o(h;a)}{||h||} = 0$ for arbitrary a.

Given the optimal consumption bundle is unique, the Roy Identity tells us that **absolute share** ω can be written as

$$\omega_j = \frac{P_j \cdot C_j}{E} = \frac{P_j \cdot \mathcal{D}_j V^*}{\sum_i P_i \cdot \mathcal{D}_i V^*}. \tag{61}$$

Replacing the absolute share ω_k and ω_j, the log of **relative expenditure share** satisfies

$$\begin{aligned} s_{k,j} &= \log(\frac{\omega_k}{\omega_j}) \\ &= \log(P_k) + \log[-\mathcal{D}_k V^*] - \log(P_j) - \log[-\mathcal{D}_j V^*]. \end{aligned} \tag{62}$$

Now I explicitly decompose the term $\mathcal{D}s_{k,j}(a) \cdot h$. Recall the first-order derivative of composition satisfies $\mathcal{D}[\log \circ f(a)] = \frac{\mathcal{D}f(a)}{f(a)}$. Recall $a = (\vec{p}, e)$ and $h = (\vec{p}_B - \vec{p}, e_B - e)$, the term

$\mathcal{D}s_{k,j}(a)\cdot h$ is decomposed as below,

$$\begin{aligned}\mathcal{D}s_{k,j}(a)\cdot h =&(p_{k,B}-p_k)-(p_{j,B}-p_j)\\ &+[\sum_{i=1}^{J}\mathcal{D}_{k,i}V^*\cdot(E^{-1}\cdot P_i)\cdot(p_{i,B}-p_i)\\ &\quad+\sum_{i=1}^{J}\mathcal{D}_{k,i}V^*\cdot(-E^{-1}\cdot P_i)\cdot(e_B-e)]\cdot[\mathcal{D}_kV^*]^{-1}\\ &-[\sum_{i=1}^{J}\mathcal{D}_{j,i}V^*\cdot(E^{-1}\cdot P_i)\cdot(p_{i,B}-p_i)\\ &\quad+\sum_{i=1}^{J}\mathcal{D}_{j,i}V^*\cdot(-E^{-1}\cdot P_i)\cdot(e_B-e)]\cdot[\mathcal{D}_jV^*]^{-1}.\end{aligned}\tag{63}$$

I use the matrix of **share elasticity** $\eta(\vec{P},E)$ at $(\vec{P},E)$. For succinct notation, I use η as the local elasticity. The level of second order derivative $-\frac{\mathcal{D}_{k,i}V^*}{\mathcal{D}_kV^*}\cdot(E^{-1}\cdot P_k)$ changes when there is monotonic transformation of utility function V^*. Using the second order derivative matrix minus removing the weighted average term $\overline{\eta}=\frac{1}{J}\cdot\sum_{i=1}^{J}\omega_i\cdot\sum_{k=1}^{J}\left[-\frac{\mathcal{D}_{k,i}V^*}{\mathcal{D}_kV^*}\cdot(E^{-1}\cdot P_k)\right]$, share elasticity is defined as the normalized outcome,

$$\eta_{k,i}=-\frac{\mathcal{D}_{k,i}V^*}{\mathcal{D}_kV^*}\cdot(E^{-1}\cdot P_k)-\overline{\eta}$$

Intra-period decision of consumption basket only depends on the pairwise difference of elasticity.

Substituting $\eta_{k,i}-\eta_{j,i}=-\frac{\mathcal{D}_{k,i}V^*}{\mathcal{D}_kV^*}\cdot(E^{-1}\cdot P_k)+\frac{\mathcal{D}_{j,i}V^*}{\mathcal{D}_jV^*}\cdot(E^{-1}\cdot P_j)$, the equation (60) is written as

$$\begin{aligned}s_{k,j}(a+h)-s_{k,j}(a)=&(1-\eta_{k,k}+\eta_{j,k})\cdot(p_{k,B}-p_k)-(1-\eta_{j,j}+\eta_{k,j})\cdot(p_{j,B}-p_j)\\ &-\sum_{i\neq k,j}(\eta_{k,i}-\eta_{j,i})\cdot(p_{i,B}-p_i)\\ &+\sum_{i}(\eta_{k,i}-\eta_{j,i})\cdot(e_B-e)\\ &+\frac{1}{2}\cdot h^T\cdot\frac{\mathcal{H}s_{k,j}(a+\theta\cdot h)}{\mathcal{S}_{k,j}(a)}\cdot h\end{aligned}\tag{64}$$

For simple notation, I denote the first-order difference using $\mathrm{d}s_{k,j}=s_{k,j}(a+h)-s_{k,j}(a)$, $\mathrm{d}p_k=p_{k,B}-p_k$ and $\mathrm{d}e=e_B-e$, so the equation (60) reads as equation (53). The Hessian matrix of indirect utility function changes after the monotonic transformation. Notice for describing the change of relative share, only the pairwise difference of share elasticity matters. Absolute level cancels out. □

E.2.3 Dynamic Decision with IDU

Lemma 1. *Define the consumer's optimal expenditure problem (P.2) as*

$$\begin{aligned}
\overline{V}_0^{\text{New}}(\vec{\theta}_0) \quad &= \quad \sup_{\tilde{E},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot V(\vec{P}_t, E_t)] \\
s.t. \qquad & \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = E_t + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \\
& E_t \geq 0; \quad \textit{financial constraint.}
\end{aligned} \tag{P.2}$$

Optimization problem (P.2) yields equivalent value as optimization problem (P.1). For each optimal consumption policy C^* in problem (P.1), expenditure E^* such that

$$E^*_t = \sum_{j\in\mathcal{J}} P_{j,t} \cdot C^*_{j,t}, \quad \forall t, z^t \tag{11}$$

is an optimal policy in optimization problem (P.2).

Proof. Following Theorem 7.6 and Theorem 9.2 in (Stokey,1989), I require Assumption 9.1-9.2 in (Stokey,1989) to ensure the proper measure space and the well-defined optimal consumption plan for the optimization problem (P.1). Similar assumptions are required to ensure the proper measure space and the well-defined optimal expenditure plan for the optimization problem (P.2). Optimization problem (P.1) and (P.2) have identical solution and life-time utility. Verification is slightly lengthy because consumer makes decision for infinite horizon.

Step 1: Construct problem (P.3) with psuedo constraint $\sum_j P_j \cdot C_j \leq E$ in each period and state,

$$\begin{aligned}
\overline{V}_0(\vec{\theta}_0) \quad &= \quad \sup_{\tilde{E},\tilde{C},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot u(\vec{C}_t)] \\
s.t. \qquad & \sum_j P_{j,t} \cdot C_{j,t} \leq E_t, \\
& \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = \sum_j P_{j,t} \cdot C_{j,t} + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \\
& C_{j,t} \geq 0; \quad \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}} \geq \underline{a}.
\end{aligned} \tag{P.3}$$

This convert the consumer's problem into two stages within each period and state: decide the expenditure first, then the basket. I verify Problem (P.2) generates lower value than problem (P.3),

$$\overline{V}_0^{\text{New}}(\vec{\theta}_0) \leq \overline{V}_0(\vec{\theta}_0).$$

To see this is true, I construct the auxiliary consumption bundle $\tilde{C}^{E*}$ implied by the optimal expenditure plan E^* in problem (P.2),

$$C_{j,t}^{E*}(z^t) = \frac{E_t^*(z^t)}{P_{j,t}(z^t)} \cdot \frac{P_{j,t}(z^t) \cdot \mathcal{D}_j V(\vec{P}_t(z^t), E_t^*(z^t))}{\sum_i P_{i,t}(z^t) \cdot \mathcal{D}_i(\vec{P}_t(z^t), E_t^*(z^t))}. \tag{65}$$

By construction,consumption plan C^{E*} depletes the expenditure in each period and state,

$$\sum_{j \in \mathcal{J}} P_{j,t}(z^t) \cdot C_{j,t}^{E*}(z^t) = E_t^*(z^t).$$

Notice that $V[\vec{P}_t(z^t), E_t^*(z^t)] = u(\vec{C}_t^{E*}(z^t))$ at each time-state node, so the objective function value is identical. I construct the investment plan of financial security and risk-free bond exactly the same as the optimal policy in problem (P.2). Therefore, the plan $(\tilde{C}^{E*}, \tilde{E}^*, \tilde{\theta}^{E*}, \tilde{B}^{E*})$ is feasible in Problem (P.3). For arbitrary feasible expenditure plan of problem (P.2), a feasible plan can be constructed in the similar way, so I conclude problem (P.2) generates (weakly) lower value than the problem (P.3).

Recall that Problem (P.3) adds additional constraints to the Problem (P.1), so Problem (P.3) generates (weakly) lower value than Problem (P.1). Overall, I conclude problem (P.2) generates lower value than the problem (P.1).

Step 2: I verify $\overline{U}_0(\vec{\theta}_0) \leq \overline{V}_0(\vec{\theta}_0)$. Construct the implied expenditure plan E^{C*} from the optimal consumption plan C^* in problem (P.1),

$$E_t^{C*}(z^t) = \sum_{j \in \mathcal{J}} P_{j,t}(z^t) \cdot C_{j,t}^*(z^t).$$

Again, the exact inter-temporal budget constraints implies $V[\vec{P}_t(z^t), E_t^{C*}(z^t)] = V^*(E_t^{C*}(z^t)^{-1} \cdot \vec{P}_t(z^t)) = u(\vec{C}_t^*(z^t))$ at each time-state node, so objective function values are identical. A financial portfolio plan $\tilde{\theta}^{C*}$ can be constructed exactly the same as solution in problem (P.2). Therefore, the plan $(\tilde{E}^{C*}, \tilde{\theta}^{C*})$ is feasible in Problem (P.2). Given the enlarged feasible set, I conclude Problem (P.1) generates lower value than the Problem (P.2).

Step 3: Combine step (1) and step (2), we conclude

$$\overline{U}_0(\vec{\theta}_0) = \overline{V}_0(\vec{\theta}_0).$$

Furthermore, for each optimal policy c^* in problem (P.1), E^{C*} such that

$$E_t^{C*}(z^t) = \sum_{j \in \mathcal{J}} P_{j,t}(z^t) \cdot C_{j,t}^*(z^t), \quad , \forall t, z \tag{66}$$

is also an optimal policy in the optimization problem (P.2). To see this is true, recall the optimal value $\overline{U}_0(\vec{\theta}_0)$ is attained by the consumption plan C^*, so E^{C*} attains the optimal value $\overline{V}_0(\vec{\theta}_0)$. In the symmetric argument, for each optimal policy E^* in problem (P.1), C^{E*} constructed as in equation (65) is also an optimal policy in the optimization problem (P.1).

□

E.2.4 Stochastic Discount Factor

Theorem 1. *In the economy with consumption sectors $\mathcal{J}$, real pricing kernel, the first-order approximated change in real stochastic discount factor is*

$$\mathrm{d}\tilde{m} = -b_e \cdot (\mathrm{d}e - \mathrm{d}p_J) - \sum_{j \in \mathcal{J}} b_j \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) + o(h). \tag{14}$$

The $\mathrm{d}p_j$ is the first-order difference of log price $p_j = \log(P_j)$ in sector j. The $\mathrm{d}e$ is the first-order difference of log total consumption expenditure $e = \log(E)$. The vector of risk-price $\vec{b}$ is

$$\begin{aligned} b_e =& \gamma, \\ b_j =& -(\gamma - 1) + \sum_{i \in \mathcal{J}} \eta_{j,i} - \sum_{k \in \mathcal{J}} \omega_k \cdot \sum_{i \in \mathcal{J}} \eta_{k,i}. \end{aligned} \tag{15}$$

with γ as relative risk aversion coefficient $-\frac{\mathcal{H}_{E,E} V(\vec{P},E) \cdot E}{\mathcal{D}_E V(\vec{P},E)}$. The high-order term $o(h)$ is with respect to $h = \max\{\{\mathrm{d}p_j\}_j, \mathrm{d}e\}$.

Proof. By the Taylor's Theorem, in the neighborhood of $a = (\vec{p}, e)$, there exists $\theta \in [0,1]$ such that

$$\tilde{m}(a+h) - \tilde{m}(a) = \mathcal{D}\tilde{m}(a) \cdot h + \frac{1}{2} \cdot h^T \cdot \mathcal{H}\tilde{m}(a + \theta \cdot h) \cdot h. \tag{67}$$

The log of **real Stochastic Discount Factor** $\tilde{M}$ satisfies

$$\tilde{m} = \log[\sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (-E^{-2}) \cdot P_j] + \log(P_J). \tag{68}$$

Now I explicitly decompose the term $\mathcal{D}\tilde{m}(a) \cdot h$. Recall the first-order derivative of composition satisfies $\mathcal{D}[\log \circ f(a)] = \frac{\mathcal{D}f(a)}{f(a)}$. The term $\mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J$ is decomposed as below,

$$\begin{aligned} \mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J = & [\sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (-E^{-2}) \cdot P_j]^{-1} \cdot [\sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (-E^{-2}) \cdot \mathrm{d}p_j \\ & \qquad + \sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (2 \cdot E^{-3}) \cdot P_j \cdot \mathrm{d}e] \\ & + [\sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (-E^{-2}) \cdot P_j]^{-1} \cdot [\sum_{j=1}^{J} \sum_{i=1}^{J} \mathcal{D}_{j,i} V^* \cdot (-E^{-3}) \cdot P_j \cdot \mathrm{d}p_i \\ & \qquad + \sum_{j=1}^{J} \sum_{i=1}^{J} \mathcal{D}_{j,i} V^* \cdot E^{-4} \cdot P_j \cdot P_i \cdot \mathrm{d}e]. \end{aligned} \tag{69}$$

Denote $A = \sum_{j=1}^{J} \mathcal{D}_j V^* \cdot P_j$. Replacing the formulas

$$\begin{aligned}\mathcal{D}_E V(\vec{P}, E) =& \sum_{j=1}^{J} \mathcal{D}_j V^* \cdot (-E^{-2}) \cdot P_j, \\ V_j(\vec{P}, E) =& \mathcal{D}_j V^* \cdot E^{-1}, \\ \omega_k =& \frac{P_k \cdot \mathcal{D}_k V^*}{\sum_{j=1}^{J} \mathcal{D}_j V^* \cdot P_j},\end{aligned}$$

yields the simplified $\mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J$ as below,

$$\begin{aligned}&\mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J \\ &= [A \cdot (-E^{-2})]^{-1} \cdot [\sum_{j=1}^{J} \omega_j \cdot A \cdot (-E^{-2}) \cdot \mathrm{d}p_j + A \cdot (2 \cdot E^{-2}) \cdot \mathrm{d}e] \\ &+ [A \cdot (-E^{-2})]^{-1} \cdot [\sum_{j=1}^{J} \sum_{i=1}^{J} \mathcal{D}_{j,i} V^* \cdot (-E^{-3}) \cdot P_j \cdot P_i \cdot \mathrm{d}p_i \\ &\qquad + \sum_{j=1}^{J} \sum_{i=1}^{J} \mathcal{D}_{j,i} V^* \cdot E^{-3} \cdot P_j \cdot P_i \cdot \frac{\mathrm{d}e}{e}].\end{aligned} \tag{70}$$

We further replace the term with second-order derivatives,

$$\begin{aligned}\mathcal{D}_{j,i} V^* \cdot E^{-3} \cdot P_j \cdot P_i =& \mathcal{D}_i V^* \cdot P_i \cdot \frac{\mathcal{D}_{j,i} V^* \cdot E^{-3} \cdot P_j \cdot P_i}{\mathcal{D}_i V^* \cdot P_i} \\ =& \omega_i \cdot A \cdot (-E^{-2}) \cdot \frac{\mathcal{D}_{j,i} V^* \cdot P_j}{\mathcal{D}_i V^* \cdot E} \\ =& \omega_i \cdot A \cdot (-E^{-2}) \cdot \frac{\mathcal{D}_{i,j} V^* \cdot P_j}{\mathcal{D}_i V^* \cdot E} \\ =& \omega_i \cdot A \cdot (-E^{-2}) \cdot [\overline{\eta} - \eta_{i,j}].\end{aligned}$$

The term $\mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J$ is further simplified as

$$\begin{aligned}\mathcal{D}\tilde{m}(a) \cdot h - \mathrm{d}p_J =& (\sum_{j=1}^{J} \omega_j \cdot \mathrm{d}p_j - \mathrm{d}e) - \mathrm{d}e \\ &+ \sum_{i=1}^{J} \omega_i \cdot \sum_{j=1}^{J} [\overline{\eta} - \eta_{i,j}] \cdot (\mathrm{d}p_i - \mathrm{d}e).\end{aligned} \tag{71}$$

The term $\mathcal{D}\tilde{m}(a) \cdot h$ is further simplified as

$$\begin{aligned}\mathcal{D}\tilde{m}(a) \cdot h =&(\sum_{j=1}^{J} \omega_j \cdot \mathrm{d}p_j - \mathrm{d}e) \\ &- (\mathrm{d}e - \mathrm{d}p_J) + (J \cdot \overline{\eta}) \cdot \sum_{i=1}^{J} \omega_i \cdot (\mathrm{d}p_i - \mathrm{d}e) - \sum_{i=1}^{J} \omega_i \cdot \sum_{j=1}^{J} \eta_{i,j} \cdot (\mathrm{d}p_i - \mathrm{d}e) \\ =&(J \cdot \overline{\eta} + 1) \cdot (\sum_{i=1}^{J} \omega_i \cdot \mathrm{d}p_i - \mathrm{d}e) \\ &- (\mathrm{d}e - \mathrm{d}p_J) - \sum_{i=1}^{J} \omega_i \cdot (\sum_{j=1}^{J} \eta_{i,j} \cdot \mathrm{d}p_i - \sum_{j=1}^{J} \eta_{i,j} \cdot \mathrm{d}e).\end{aligned} \tag{72}$$

Hence, the First-Order Approximated Linear SDF is

$$\mathrm{d}\tilde{m} = -\sum_{j=1}^{J} b_j \cdot \omega_j \cdot \mathrm{d}p_j - b_e \cdot \mathrm{d}e + \mathrm{d}p_J + o(h).$$

has risk price $\vec{b}$ as

$$b_j = -(J \cdot \overline{\eta} + 1) + \sum_{k=1}^{J} \eta_{j,k}, \tag{73}$$

$$b_e = (J \cdot \overline{\eta} + 1) + 1 - \sum_{j=1}^{J} \omega_j \cdot \sum_{k=1}^{J} \eta_{j,k}. \tag{74}$$

By construction, absolute consumption shares add-up to 1,

$$\sum_{j=1}^{J} \omega_j = 1.$$

Therefore, the risk price vector b satisfies

$$\sum_{j=1}^{J} \omega_j \cdot b_j + b_e = 1. \tag{75}$$

Considering the relative change using the deflator P_J, the First-Order Approximated Linear SDF is

$$\mathrm{d}\tilde{m} = -\sum_{j=1}^{J} b_j \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e \cdot (\mathrm{d}e - \mathrm{d}p_J) + o(h).$$

Recall the consumer adjusts shares based on the pairwise difference of share elasticity. The absolute level of elasticity doesn't plays the role.

Given $(\vec{P}, E)$, the relative risk-aversion is $-\frac{\mathcal{H}_{E,E}V(\vec{P},E)\cdot E}{\mathcal{D}_E V(\vec{P},E)}$, denote as γ. This is from the definition of sensitivity of marginal utility with respect to the expenditure. The risk price of expenditure b_e describes the response of marginal utility to expenditure, when the price vector is fixed. As such,

$$b_e = \gamma. \tag{76}$$

The sensitivity of marginal utility of expenditure to each price P_j needs further derivation. Replacing $\gamma = (J \cdot \overline{\eta} + 1) + 1 - \sum_{j=1}^{J} \omega_j \cdot \sum_{k=1}^{J} \eta_{j,k}$ for $J \cdot \overline{\eta} + 1$, the vector of risk price $\vec{b}$ is further simplified as

$$b_j = -(\gamma - 1) + [\sum_{k \in \mathcal{J}} \eta_{j,k} - \sum_{i \in \mathcal{J}} \omega_i \cdot \sum_{k \in \mathcal{J}} \eta_{i,k}]. \tag{77}$$

Alternatively, one can use the H.D.O property $\mathcal{D}_E V(\vec{P}, E) = \sum_{j=1}^{J} \mathcal{D}_j V^*(E^{-1} \cdot \vec{P}) \cdot (-E^{-2}) \cdot P_j$ to verify the identity equation with second order derivatives,

$$-\frac{\mathcal{H}_{E,E}V(\vec{P}, E) \cdot E}{\mathcal{D}_E V(\vec{P}, E)} = 2 - \sum_{j=1}^{J} \omega_j \cdot \sum_{k=1}^{J} \frac{\mathcal{D}_{j,k} V^* \cdot P_k}{\mathcal{D}_j V^* \cdot E}.$$

□

Corollary 3. *Define the* ***Engel Slope*** *for the sector pair (k, j) as sensitivity of relative share $s_{k,j} = \log(\frac{\omega_k}{\omega_j})$ to expenditure,*

$$\mathrm{ES}_{k,j}(\vec{P}, E) = \lim_{\mathrm{de} \to 0} \frac{s_{k,j}(p, e + \mathrm{de}) - s_{k,j}(p, e)}{\mathrm{de}}, \tag{54}$$

In real stochastic discount factor, the risk price of necessity price P_k is more negative than the luxury price P_j, as necessity sector k inferior to than luxury sector j,

$$b_k - b_j = \mathrm{ES}_{k,j}(\vec{P}, E). \tag{55}$$

Proof. Recall the Lemma (2),

$$\mathrm{ES}_{k,j}(\vec{P}, E) = \sum_{i=1}^{J} \eta_{k,i}(\vec{P}, E) - \sum_{i=1}^{J} \eta_{j,i}(\vec{P}, E). \tag{78}$$

Theorem (1) tells us,

$$b_k - b_j = \sum_{i=1}^{J} \eta_{k,i}(\vec{P}, E) - \sum_{i=1}^{J} \eta_{j,i}(\vec{P}, E). \tag{79}$$

So we arrive to $b_k - b_j = \mathrm{ES}_{k,j}(\vec{P}, E)$. □

Corollary 1. *Given the security k and the security f, the real total return $\tilde{R}_{k,t+1}$ and $\tilde{R}_{f,t+1}$ from time t to future time $t+1$ satisfy*

$$\mathbb{E}_t[\frac{\tilde{M}_{t+1}}{\tilde{M}_t}\cdot(\tilde{R}_{k,t+1}-\tilde{R}_{f,t+1})]=0. \tag{13}$$

Proof. I refer the standard argument as in Chapter 13, (Ljungqvist, Sargent, 2012). The Lagrangian for the consumption allocation problem is

$$\begin{aligned} L_0(\vec{\theta}_0,\lambda_0,\nu_0,\nu_0^e)= \quad & \sup_{\tilde{E},\tilde{\theta},\tilde{\nu},\tilde{\nu}^a,\tilde{\nu}^e} \lim_{T\to\infty}\{\beta^{T+1}\cdot\mathbb{E}\big[\sum_k P_{k,T+1}\cdot\mathcal{D}_E V(\vec{P}_T,E_T)\big] \\ & +\mathbb{E}\big[\sum_{t=1}^{T}\beta^t\cdot[V(\vec{P}_t,E_t)+\nu_t\cdot(\text{ budget constraint}) \\ & \quad +\nu_t^a\cdot(\text{bounded total wealth}) \\ & \quad +\nu_t^e\cdot(\text{non-negative spending})]\big]\}. \end{aligned} \tag{L.1}$$

Here, the *budget constraint* reads as

$$\sum_k \theta_{k,t}\cdot(P^s_{k,t}+D_{k,t})+B_t=E_t+\sum_k\theta_{k,t+1}\cdot P^s_{k,t}+\frac{B_{t+1}}{R_{f,t+1}}.$$

The *bounded total wealth constraint* reads as

$$\sum_k\theta_{k,t+1}\cdot P^s_{k,t}+\frac{B_{t+1}}{R_{f,t+1}}\geq \underline{a}.$$

Given $\lim_{E\to 0}\mathcal{D}_E V(\vec{P},E)=-\infty$, the shadow price $\nu_t^e\equiv 0$. Optimal stream of investment amount in k-th financial security $\tilde{\theta_k}$ implies the motion equation of shadow price ν_t

$$\begin{aligned} & \beta^t\cdot\pi(z^t)\cdot[\nu_t(z^t)+\nu_t^a(z^t)]\cdot P^s_{k,t}(z^t) \\ = & \sum_{z_{t+1}|z^t}\beta^{t+1}\cdot\pi(z^{t+1})\cdot\nu_{t+1}(z^{t+1})\cdot[P^s_{k,t+1}(z^{t+1})+D_{k,t+1}(z^{t+1})]. \end{aligned} \tag{80}$$

Optimal stream of expenditure $\tilde{E}$ implies the equality between shadow price ν_t and marginal utility of expenditure,

$$\nu_t(z^t)=\mathcal{D}_E V(\vec{P}_t(z^t),E_t(z^t)). \tag{81}$$

Similarly, at the succeeding time-state z^{t+1}, the FOC of consumption spending also holds

$$\nu_{t+1}(z^{t+1})=\mathcal{D}_E V(\vec{P}_{t+1}(z^{t+1}),E_{t+1}(z^{t+1})).$$

For short notation, I use $\mathcal{D}_E V_t(z^t)$ for $\mathcal{D}_E V(\vec{P}_t(z^t), E_t(z^t))$. Substituting FOCs of consumption spending into the equation of shadow price (80) yields,

$$\begin{aligned} &[\mathcal{D}_E V_t(z^t) + \eta_t^a(z^t)] \cdot P_{k,t}^s(z^t) \\ =& \sum_{z_{t+1}|z^t} \beta \cdot \pi(z_{t+1}|z^t) \cdot \mathcal{D}_E V_{t+1}(z^{t+1}) \cdot [P_{k,t+1}^s(z^{t+1}) + D_{k,t+1}(z^{t+1})]. \end{aligned} \tag{82}$$

If household is unconstrained $\nu_t^a(z^t) = 0$, this equation is

$$1 = \beta \cdot \mathbb{E}[\frac{\mathcal{D}_E V_{t+1}(z^{t+1})}{\mathcal{D}_E V_t(z^t)} \cdot \frac{P_{k,t+1}^s(z^{t+1}) + D_{k,t+1}(z^{t+1})}{P_{k,t}^s(z^t)}|z^t]. \tag{83}$$

Denote the real total return for financial asset k as

$$\tilde{R}_{k,t\to t+1}(z^{t+1}) = \frac{[P_{k,t+1}^s(z^{t+1}) + D_{k,t+1}(z^{t+1})]/P_{J,t+1}(z^{t+1})}{P_{k,t}^s(z^t)/P_{J,t}(z^t)}.$$

I conclude $\mathbb{E}[\beta \cdot \frac{\tilde{M}_{t+1}}{\tilde{M}_t} \cdot \tilde{R}_{k,t\to t+1}|\mathcal{I}_t] = 1$. The similar argument can be constructed for arbitrary finite time-interval (t, t'), hence optimal financial wealth allocation implies the Euler equation for the real total return of financial asset k

$$\mathbb{E}[\beta^{t'-t} \cdot \frac{\tilde{M}_{t'}}{\tilde{M}_t} \cdot \tilde{R}_{k,t\to t'}|\mathcal{I}_t] = 1. \tag{84}$$

Similarly, there exists the Euler equation of financial asset f,

$$\mathbb{E}[\beta^{t'-t} \cdot \frac{\tilde{M}_{t'}}{\tilde{M}_t} \cdot \tilde{R}_{f,t\to t'}|\mathcal{I}_t] = 1. \tag{85}$$

The excess return satisfies

$$\mathbb{E}[\beta^{t'-t} \cdot \frac{\tilde{M}_{t'}}{\tilde{M}_t} \cdot (\tilde{R}_{k,t\to t'} - \tilde{R}_{f,t\to t'})|\mathcal{I}_t] = 0. \tag{86}$$

Removing the constant non-zero term $\beta^{t'-t}$ gives us the Euler equation across financial assets

$$\mathbb{E}[\frac{\tilde{M}_{t'}}{\tilde{M}_t} \cdot (\tilde{R}_{k,t\to t'} - \tilde{R}_{f,t\to t'})|\mathcal{I}_t] = 0. \tag{87}$$

□

Corollary 2. *Given the security k, the expected excess return $R^e_{k,t+1}$ satisfies*

$$\begin{aligned}\mathbb{E}_t[R^e_{k,t+1}] =& b_e \cdot \mathbb{E}_t\left[(\mathrm{d}e_{t+1} - \mathrm{d}p_{J,t+1}) \cdot R^e_{k,t+1}\right] \\ &+ \sum_{j\in\mathcal{J}} b_j \cdot \omega_{j,t} \cdot \mathbb{E}_t\left[(\mathrm{d}p_{j,t+1} - \mathrm{d}p_{J,t+1}) \cdot R^e_{k,t+1}\right].\end{aligned} \tag{18}$$

with excess return as the difference between nominal total return $R_{k,t+1}$ and risk-free rate $R_{f,t+1}$.

Proof. Recall the return spread across pairs of financial assets approximately equals the spread of deflated total return,

$$R_{k,t\to t'} - R_{f,t\to t'} \approx \tilde{R}_{k,t\to t'} - \tilde{R}_{f,t\to t'}, \tag{88}$$

so the equation of real current pricing kernel is written as

$$\mathbb{E}[\frac{\tilde{M}_{t'}}{\tilde{M}_t} \cdot (R_{k,t\to t'} - R_{f,t\to t'})|\mathcal{I}_t] \approx 0. \tag{89}$$

The change in real stochastic discount factor is further approximated as the 1 plus the logarithmic growth, $\frac{\tilde{M}_{t'}}{\tilde{M}_t} \approx 1 + \mathrm{d}\tilde{m}_{t\to t'}$. □

E.3 Aggregation

Here, I explain the price-decomposition of pricing kernel in the heterogeneous-agent economy. In this economy, the consumer n has indirect utility function $V^{(n)}(\vec{P}, E^{(n)})$ for intratemporal consumption decision. Consumer (n) has initial endowment of financial security $\vec{\theta}_0^{(n)}$.

Definition 4. *The price system (P, P^s) and the consumption allocation $\tilde{C}$ constitutes the (Heterogeneous Consumer) Competitive Equilibrium if*

1. *$\tilde{C}^{(n)}$ solves problem*

$$\begin{aligned} \overline{U}_0(\vec{\theta}_0^{(n)}) \quad &= \quad \sup_{\tilde{C},\tilde{\theta}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=1}^{T} \beta^t \cdot u^{(n)}(\vec{C}_t)] \\ s.t. \qquad & \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = \sum_j P_{j,t} \cdot C_{j,t} + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \\ & C_{j,t} \geq 0; \quad \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}} \geq \underline{a}. \end{aligned} \tag{P.1-HA}$$

2. *commodity market (j, t) clears in the demand side $\sum_{n\in\mathcal{N}} C^{(n)}_{j,t} = \mathbf{C}_{j,t}$.*

3. *commodity market clears in the supply side, labor market clear, given the model specification of producers;*

4. *financial security market clears, given the model specification of foreign borrowing and lending.*

At the equilibrium path $\{\vec{P}^*, E^*, \lambda^*\}$, $\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})$ is the marginal utility of consumption expenditure $E^{(n)}$. I choose the consumer (1) as the benchmark consumer for the aggregation analysis. Construct the Negishi-weight $\alpha(1) = 1$, and

$$\alpha^*(n) = \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}. \tag{90}$$

The distribution of expenditure $\{E^{(n),*}\}_{(n)}$ solves the auxiliary-optimization problem,

$$\begin{aligned} V(\vec{P}, \mathbf{E}; \alpha) \equiv \max_{E} \quad & \frac{1}{N} \cdot \sum_{n\in\mathcal{N}} \alpha(n) \cdot V^{(n)}(\vec{P}, E(n)) \\ s.t. \quad & \frac{1}{N} \cdot \sum_{n\in\mathcal{N}} E(n) \leq \mathbf{E}. \end{aligned} \tag{SP.1}$$

I denote the aggregate consumption spending on the equilibrium path as $\mathbf{E}^* = \frac{1}{N} \cdot \sum_{n\in\mathcal{N}} E^{(n),*}$.

Theorem 2. *In the economy where price system* (P, M) *and quantity system* $(\{\tilde{c}^{(n)}\}_{n\in\mathcal{N}}, \{\tilde{\ell}^{(n)}\}_{n\in\mathcal{N}})$ *constitute a Competitive Equilibrium for* N *heterogeneous consumers with preference* $\{\succeq^{(n)}\}$*, there exists a* ***Representative Consumer*** *with preference* $\succeq^{\mathcal{N}}$ *such that*

- *price system* (P, M) *and quantity system* $(\sum_{n\in\mathcal{N}} \tilde{c}^{(n)}, \sum_{n\in\mathcal{N}} \tilde{\ell}^{(n)})$ *constitute a Competitive Equilibrium for* N *homogeneous consumers with preference* $\succeq^{\mathcal{N}}$.

The indirect utility function of the Representative Consumer is $V(\vec{P}, \mathbf{E}; \alpha)$ *with the Negishi weight constructed in equation (90). Along the equilibrium path, the Representative Consumer has identical marginal utility of expenditure with the Benchmark Consumer*

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}), \tag{91}$$

and the absolute expenditure share of artificial consumer $V(\vec{P}^*, \mathbf{E}^*; \alpha)$ *is identical with observed aggregate expenditure share,*

$$\vec{\omega}(\vec{P}^*, \mathbf{E}^*; \alpha) = \sum_{n\in\mathcal{N}} \frac{E^{(n),*}}{\sum_{m\in\mathcal{N}} E^{(m),*}} \cdot \vec{\omega}^{(n)}(\vec{P}^*, E^{(n),*}) \tag{92}$$

The weight α_n reflects the shadow price of consumer (n)'s budget constraint in the Competitive Equilibrium. If we take the aggregation consumption bundle as if the Representative Consumer's choice, the Negishi weight works as if it is the "Taste" of representative consumer over individual consumers. Recall we use the expenditure share over commodities to reveal the single-consumer's preference over consumption bundle. Here, we use the expenditure allocation across consumers to reveal the Representative Consumer's social preference over individual consumers.

By constructing the representative consumers consistent with the aggregate consumption expenditure and the fluctuation of SDF, the representative consumer's indirect utility function also reveals the financial market SDF $\{M_t\}$. Decomposition of indirect utility function in Section (2) is non-parametric, so previous analysis holds in the heterogeneous-consumer economy. This allows the economist to track the marginal utility of investor even if we fail to explicitly identify who is the unconstrained financial market investor.

In the economy with homothetic preference and additive utility flow, where the financial market is complete for the consumer, $\{\alpha^*(n)\}_n$ is invariant along the equilibrium path [36]. In other words, in the economy with perfect risk-sharing and homothetic preference, we have fixed Negishi weight [37]. The numerator $\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})$ in the Negishi weight can't be

[36] Along the equilibrium path, $\{\alpha^*(n)\}_n$ might vary, if the economy has idiosyncratic labor endownment and non-trivial wealth-constraint.

[37] The international finance literature often consider the integrated economy with fixed Negishi weight. I depart from the Constant Social Planner's problem because the consumption allocation is implemented by the financial market and the commodity market. On the equilibrium path, the aggregate wealth allocation might be inconsistent with the Representative Consumer constructed from the intra-temporal consumption allocation.

removed, because we attempt to track the benchmark consumer (1) by constructing a Representative Consumer whose welfare is comparable with benchmark consumer (1). Only in this way, fluctuation of Representative Consumer's welfare is meaningful for tracking the financial market SDF.

I use $\gamma(n)$ to describe the curvature of utility flow in consumer (n)'s life-time utility (EIS parameter in path-independent utility function),

$$\gamma^{(n)} = -\frac{\mathcal{H}_{E,E}V^{(n)}(\vec{P}, E) \cdot E}{\mathcal{D}_E V^{(n)}(\vec{P}, E)}. \tag{93}$$

Definition 5. *Define the* ***real Marginal Utility of Expenditure*** *of consumer* (n) *as* $\tilde{M}(n)$*,*

$$\tilde{M}(\vec{P}_t, E_t; n) \equiv \mathcal{D}_E V^{(n)}(\vec{P}_t, E_t(n)) \cdot P_{J,t}. \tag{94}$$

One can extend the analysis of representative consumer to each consumer (n),

Corollary 6. *First-Order Approximated change of Marginal Utility* $\tilde{M}(n)$ *is*

$$\mathrm{d}\tilde{m}(n) = -\sum_{j=1}^{J} b_j(n) \cdot \omega_j(n) \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(n) \cdot (\mathrm{d}e(n) - \mathrm{d}p_J) + o(h). \tag{95}$$

with high-order term $o(h)$ *for* $h = \max\{\{\mathrm{d}p_j\}_j, \mathrm{d}e(n)\}$*. The coefficient vector* $b(n)$ *is*

$$\begin{aligned} b_j(n) =& -[\gamma(n) - 1] + \sum_{i=1}^{J} \eta_{j,i}(n) - \sum_{k=1}^{J} \omega_k(n) \sum_{i=1}^{J} \eta_{k,i}(n), \\ b_e(n) =& \gamma(n). \end{aligned} \tag{96}$$

Before diving into the generalization of Effective Representative Consumer, I construct the curvature $\gamma(\alpha)$ for the Effective Representative Consumer with weights α. It works as if the Effective Representative Consumer substitutes the utility-flow across time-periods using this curvature $\gamma(\alpha)$,

$$\gamma^{(\alpha)} = -\frac{\mathcal{H}_{E,E}V(\vec{P}, \mathbf{E}; \alpha) \cdot \mathbf{E}}{\mathcal{D}_E V(\vec{P}, \mathbf{E}; \alpha)}. \tag{97}$$

Here, I want to clarify the abuse of math symbols: the sup-script (n) highlights the name of consumer (n), while the sup-script (α) serves as the name of this artificial Representative Consumer. Hereafter, I generalize the analysis of single-consumer's marginal utility for this constructed consumer (α). I have to emphasize the fact, $\gamma^{(\alpha)}$ is endogenously determined by the distribution of risk aversion $\gamma(n)$ and the expenditure across consumers $\frac{E^{(n),*}}{E^*}$ in equilibrium. Lemma E.1 restates this fact. It is the standard result in asset pricing textbook.

Lemma E.1. *For (static) Effective Representative Consumer with the curvature $\gamma(\alpha)$ satisfying equations below*

$$\gamma(\alpha) = \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{E^*} \cdot \frac{1}{\gamma(n)}. \tag{98}$$

This (static) Effective Representative Consumer is dynamically consistent on the equilibrium-path.

The Effective Representative Consumer is ex-post constructed, her inter-temporal consumption expenditure is consistent with the observed aggregate expenditure across time periods: (a) change in marginal utility of expenditure is identical with consumer (1); (b) aggregate consumption basket is the optimal intra-period consumption basket decision.

In Corollary 4, the parameter $\gamma(\alpha)$ is the reciprocal of expenditure-weighted reciprocal risk aversion. The elasticity of relative sector-share $\eta(\alpha)$ is derived similarly with the case of stationary representative consumer, assuming the weight α is fixed. There is no simple analytical expression over the $\eta(\alpha)$ elasticity matrix from the distribution of individual consumer.

In the economy with generalized consumption preference, the representative consumer implied by the competitive equilibrium outcome might depart from the individual consumer, in the sense that the functional form of indirect utility function is different. This result departs from (32), because I don't pursue the similarity in the functional form of utility function. This result departs from (5), because I recover the Representative Consumer using the approach of "revealed social preference". I only require the consistent marginal utility of aggregate spending and the consistent aggregate consumption portfolio. Further, I recover the inter-temporal preference for the Effective Representative Consumer, so that the dynamic of aggregate expenditure is consistent with the marginal utility of hypothetical unconstrained consumer.

The construction of Negishi weight departs from (55) and (9). Both (55) and (9) use the marginal utility of consumption as the denominator of Negishi weight. In (55), the numerator of the Negishi weight is chosen to be consistent with the optimal fiscal tax transfer. In (9), the numerator is the average marginal utility of consumption across consumers. Compared with (55) and (9), I use benchmark consumer's marginal utility as the numerator, this ensures the reverse-engineered representative consumer has identical marginal utility with the financial market investor. To be clear, my construction of Negishi weight is the result implied by consumption allocation in a competitive equilibrium. I don't rely on the government or other legal authority assigning the consumption across consumers directly. Recovering the Effective Representative Consumer is an ex-post accounting exercise. It simplifies the asset-pricing analysis, without touching the exact underlying model. To some extent, the empirical exercise here is similar with the computation technique in the economy with financial market friction (14), where the stochastic weights augment the aggregate consumption quantity in the SDF.

The generalized decomposition of SDF in the Corollary (4) formally verify the equivalence between an economy with heterogeneous-consumers, and an simplified economy with representative consumer. The empirical analysis is under the assumption of the Corollary (4). Although Corollary (4) puts strong assumptions over the distribution of Negishi weight, it serves as the benchmark to compare the true data and hypothetically ideal economy with complete financial market. When consumers have binding borrowing constraints or transaction constraints, effective Negishi weights vary. Corollary (5) extends the analysis for this situation.

E.3.1 Proof for Aggregation

Lemma E.2. *At $(\vec{P}^*, \mathbf{E}^*)$, artificial consumer has marginal utility of expenditure equivalent with the benchmark-consumer* (1)

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}) \tag{99}$$

Proof. By construction of $V(\vec{P}, \mathbf{E}; \alpha)$,

$$\begin{aligned} V(\vec{P}, \mathbf{E}; \alpha) = \max_{s \in \triangle^{N-1}} \quad & \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot V^{(n)}(\vec{P}, s(n) \cdot E) \\ s.t. \quad & \sum_{n \in \mathcal{N}} s(n) \leq 1. \end{aligned} \tag{100}$$

Denote the optimal solution for this static optimization problem satisfies $s^{*static}(n)$.

By the envelope theorem,

$$\begin{aligned} \mathcal{D}_E V(\vec{P}, \mathbf{E}; \alpha) =& \mathcal{D}_E \left[\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot s^{*static}(n) \cdot V^{(n)}(\vec{P}, s^{*static}(n) \cdot E) \right] \\ =& \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot s^{*static}(n) \cdot \mathcal{D}_E V^{(n)}(\vec{P}, s^{*static}(n) \cdot E) \\ =& \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot s^{*static}(n) \cdot \mathcal{D}_E V^{(n)}(\vec{P}, s^{*static}(n) \cdot E) \end{aligned} \tag{101}$$

Effective Negishi-weight α^* is constructed based on the equilibrium allocation $\alpha^*(n) = \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}$. We can see the first fact: $s^*(n) = \frac{E^{(n),*}}{\mathbf{E}^*}$ solves the static problem for $s^{*static}$. We can see the second fact as below,

$$\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} s^*(n) \cdot \alpha^*(n) \cdot \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*}) = [\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} s^*(n)] \cdot \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}). \tag{102}$$

Recall by construction $\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} s^*(n) = 1$, Therefore, we can conclude

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}).$$

□

Lemma E.3. *At $(\vec{P}^*, \mathbf{E}^*)$, absolute expenditure share of artificial consumer $V(\vec{P}^*, \mathbf{E}^*; \alpha)$ is identical with aggregate expenditure share on the equilibrium path,*

$$\vec{\omega}(\vec{P}^*, \mathbf{E}^*; \alpha) = \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \vec{\omega}^{(n)}(\vec{P}^*, E^{(n),*}) \tag{103}$$

Proof. Recall the Roy identity,

$$\omega_j^{(n)}(\vec{P}, E^{(n)}) = -\frac{\mathcal{D}_j V^{(n)}(\vec{P}, E^{(n)})}{\mathcal{D}_E V^{(n)}(\vec{P}, E^{(n)})} \cdot \frac{P_j}{E^{(n)}}. \tag{104}$$

Hence, I can simplify the formula as below

$$\begin{aligned}
&\sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \omega_j^{(n)}(\vec{P}^*, E^{(n),*}) \\
=& -\sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \frac{\mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \frac{P_j^*}{E^{(n),*}} \\
=& -\sum_{n \in \mathcal{N}} \frac{P_j^*}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \frac{\mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}
\end{aligned} \tag{105}$$

By the Roy Identity, absolute expenditure share of artificial consumer $V(\vec{P}^*, \mathbf{E}^*; \alpha)$ is

$$\omega_j(\vec{P}^*, \mathbf{E}^*; \alpha) = -\frac{\mathcal{D}_j V(\vec{P}^*, \mathbf{E}^*; \alpha)}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)} \cdot \frac{P_j}{E^*}. \tag{106}$$

By the envelope theorem,

$$\mathcal{D}_j V(\vec{P}^*, \mathbf{E}^*; \alpha) = \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot \mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*}). \tag{107}$$

I derive the formula $\frac{\alpha(n)}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)}$ as

$$\begin{aligned}
\frac{\alpha(n)}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)} =& \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \frac{1}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)} \\
=& \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \frac{1}{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})} \\
=& \frac{1}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}.
\end{aligned} \tag{108}$$

The first equality comes from the definition of Proper Negeishi weight $\alpha^*(n) = \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}$. The second equality comes from Lemma E.2, $\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})$.

I simplify the formula below

$$
\begin{aligned}
\frac{\mathcal{D}_j V(\vec{P}^*, \mathbf{E}^*; \alpha)}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)} &= \frac{\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha(n) \cdot \mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})} \\
&= \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \frac{1}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*}).
\end{aligned} \tag{109}
$$

The first equality substitutes the formula $\mathcal{D}_j V(\vec{P}^*, \mathbf{E}^*; \alpha)$ with equation (107). The second equality substitutes the formula $\frac{\alpha(n)}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)}$ with equation (108).

Therefore, absolute expenditure share of artificial consumer $V(\vec{P}^*, \mathbf{E}^*; \alpha)$ is simplified as

$$
\omega_j(\vec{P}^*, \mathbf{E}^*; \alpha) = -\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \frac{\mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \frac{P_j}{E^*}. \tag{110}
$$

By construction of E^*,

$$
\sum_{m \in \mathcal{N}} E^{(m),*} = N \cdot E^*,
$$

Replacing $\omega_j^{(n)}(\vec{P}, E^{(n)}) = -\frac{\mathcal{D}_j V^{(n)}(\vec{P}, E^{(n)})}{\mathcal{D}_E V^{(n)}(\vec{P}, E^{(n)})} \cdot \frac{P_j}{E^{(n)}}$, I close the proof with

$$
\begin{aligned}
\omega_j(\vec{P}^*, \mathbf{E}^*; \alpha) &= -\sum_{n \in \mathcal{N}} \frac{P_j^*}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \frac{\mathcal{D}_j V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} \\
&= \sum_{n \in \mathcal{N}} \frac{P_j^*}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \omega_j^{(n)}(\vec{P}^*, E^{(n),*}) \cdot \frac{E^{(n),*}}{P_j^*} \\
&= \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \omega_j^{(n)}(\vec{P}^*, E^{(n),*}).
\end{aligned}
$$

□

Theorem 2. *In the economy where price system (P, M) and quantity system $(\{\tilde{c}^{(n)}\}_{n \in \mathcal{N}}, \{\tilde{\ell}^{(n)}\}_{n \in \mathcal{N}})$ constitute a Competitive Equilibrium for N heterogeneous consumers with preference $\{\succeq^{(n)}\}$, there exists a* ***Representative Consumer*** *with preference $\succeq^{\mathcal{N}}$ such that*

- *price system (P, M) and quantity system $(\sum_{n \in \mathcal{N}} \tilde{c}^{(n)}, \sum_{n \in \mathcal{N}} \tilde{\ell}^{(n)})$ constitute a Competitive Equilibrium for N homogeneous consumers with preference $\succeq^{\mathcal{N}}$.*

The indirect utility function of the Representative Consumer is $V(\vec{P}, \mathbf{E}; \alpha)$ with the Negishi weight constructed in equation (90). Along the equilibrium path, the Representative Consumer has identical marginal utility of expenditure with the Benchmark Consumer

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}), \tag{91}$$

and the absolute expenditure share of artificial consumer $V(\vec{P}^*, \mathbf{E}^*; \alpha)$ is identical with observed aggregate expenditure share,

$$\vec{\omega}(\vec{P}^*, \mathbf{E}^*; \alpha) = \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \vec{\omega}^{(n)}(\vec{P}^*, E^{(n),*}) \tag{92}$$

Proof. The construction of representative consumer is completed after I verify the Lemma E.2 and the Lemma E.3. □

Corollary 6. *First-Order Approximated change of Marginal Utility $\tilde{M}(n)$ is*

$$\mathrm{d}\tilde{m}(n) = -\sum_{j=1}^{J} b_j(n) \cdot \omega_j(n) \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(n) \cdot (\mathrm{de}(n) - \mathrm{d}p_J) + o(h). \tag{95}$$

with high-order term $o(h)$ for $h = \max\{\{\mathrm{d}p_j\}_j, \mathrm{de}(n)\}$. The coefficient vector $b(n)$ is

$$\begin{aligned} b_j(n) =& -[\gamma(n) - 1] + \sum_{i=1}^{J} \eta_{j,i}(n) - \sum_{k=1}^{J} \omega_k(n) \sum_{i=1}^{J} \eta_{k,i}(n), \\ b_e(n) =& \gamma(n). \end{aligned} \tag{96}$$

Proof. We apply the decomposition recipe in Theorem 1, by generalizing γ as consumer (n)'s curvature $\gamma(n)$, ω as consumer (n)'s absolute expenditure share $\omega(n)$, and η as consumer (n)'s intra-temporal elasticity matrix $\eta(n)$. □

Lemma E.1. *For (static) Effective Representative Consumer with the curvature $\gamma(\alpha)$ satisfying equations below*

$$\gamma(\alpha) = \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{E^*} \cdot \frac{1}{\gamma(n)}. \tag{98}$$

This (static) Effective Representative Consumer is dynamically consistent on the equilibrium-path.

Proof. Consider the equation below,

$$\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} f^{(n)}(\frac{M}{\alpha(n)}) = \mathbf{E}. \tag{111}$$

where $f^{(n)}(\lambda)$ is the inverse-function of consumer (n)'s marginal utility given price vector $\vec{P}$,

$$\mathcal{D}_E V^{(n)}(\vec{P}, f^{(n)}(\lambda)) = \lambda. \tag{112}$$

One can derive the fact below,

$$\dot{f}^{(n)}(\lambda) = \frac{1}{\mathcal{H}_{EE} V^{(n)}(\vec{P}, E)}. \tag{113}$$

for $E^{(n)}$ solving the equation $\mathcal{D}_E V^{(n)}(\vec{P}, E^{(n)}) = \lambda$. From the equation (111), we can arrive to the first-order differential with respect to $(\mathrm{d}M, \mathrm{d}\mathbf{E})$,

$$\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \dot{f}^{(n)}(\frac{M}{\alpha(n)}) \cdot \frac{M}{\alpha(n)} \cdot \frac{\mathrm{d}M}{M} = \mathrm{d}\mathbf{E}. \tag{114}$$

Recall the endogenous Negishi weights α^* implied in the equilibrium outcome satisfies the equation below,

$$\alpha^*(n) = \frac{\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})} = \frac{M^*}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}. \tag{115}$$

As such, the consumer (n)'s expenditure $E^{(n),*}$ solves the equation below,

$$\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*}) = \frac{M^*}{\alpha^*(n)}. \tag{116}$$

Therfore, we can replace

$$\dot{f}^{(n)}(\frac{M^*}{\alpha^*(n)}) = \frac{1}{\mathcal{H}_{EE} V^{(n)}(\vec{P}^*, E^{(n),*})}$$

and $\frac{M^*}{\alpha^*(n)} = \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})$, so we end up with

$$\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \frac{1}{\mathcal{H}_{EE} V^{(n)}(\vec{P}^*, E^{(n),*})} \cdot \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*}) \cdot \frac{\mathrm{d}M}{M} = \mathrm{d}\mathbf{E}. \tag{117}$$

After further organization, we find the expenditure-weighted inverse curvature similar with asset-pricing literature,

$$\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \frac{\frac{E^{(n),*}}{\mathbf{E}^*}}{E^{(n),*} \cdot \frac{\mathcal{H}_{EE} V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}} \cdot \frac{\mathrm{d}M^*}{M^*} = \frac{\mathrm{d}\mathbf{E}^*}{\mathbf{E}^*}. \tag{118}$$

Denote the new term as

$$\frac{1}{b_e(\alpha^*)} = \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \frac{\frac{E^{(n),*}}{\mathbf{E}^*}}{E^{(n),*} \cdot \frac{\mathcal{H}_{EE} V^{(n)}(\vec{P}^*, E^{(n),*})}{\mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})}}.$$

The reciprocal of right-hand-side is the standard harmonic risk-aversion in literature,

$$\frac{1}{\gamma(\alpha)} = \frac{1}{\sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\mathbf{E}^*} \cdot \frac{1}{\gamma(n)}}. \tag{119}$$

□

Corollary 4. *Given invariant distribution of Negishi-weight $\{\alpha^*(n)\}_n$ along the equilibrium path, the log-change in real marginal utility of expenditure for the representative consumer approximately equals*

$$\mathrm{d}\tilde{m} = -\sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\boldsymbol{e} - \mathrm{d}p_J) + o(h). \tag{57}$$

where α is the artificial Negishi-weight, $\vec{\omega}$ is the aggregate expenditure share, $\boldsymbol{e}$ is the (log) aggregate total consumption expenditure, and the vector $b(\alpha)$ is written with aggregate expenditure share $\vec{\omega}$ and representative consumer's elasticity η

$$\begin{aligned} b_j(\alpha) =& -[\gamma(\alpha) - 1] + \sum_{i=1}^{J} \eta_{j,i}(\alpha) - \sum_{k=1}^{J} \omega_k \cdot \sum_{i=1}^{J} \eta_{k,i}(\alpha), \\ b_e(\alpha) =& \gamma(\alpha). \end{aligned} \tag{58}$$

Proof. Recall Lemma E.2,

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}).$$

Recall the benchmark consumer's interior expenditure decision implies,

$$\mathcal{D}_E V^{(1)}(\vec{P}^*, E^{(1),*}) = M.$$

Therefore, the financial market SDF $\{M_t(z^t)\}$ can be measured by the marginal utility of constructed aggregate consumer,

$$M_t(z^t) = \mathcal{D}_E V(\vec{P}_t^*(z^t, \{z_{(n)}^t\}), \mathbf{E}_t^*(z^t, \{z_{(n)}^t\}); \alpha)$$

Recall the definition of real SDF,

$$\mathrm{d}\log(\tilde{M}) = \mathrm{d}\log[\mathcal{D}_E V(\alpha) \cdot P_J]. \tag{120}$$

First-order approximation of $\log[\mathcal{D}_E V(\alpha) \cdot P_J]$ is similar with the analysis of representative consumer. From Lemma E.3, it is legitimate to replace the expenditure share implied by the artificial-consumer with

$$\vec{\omega}(\vec{P}^*, \mathbf{E}^*; \alpha) = \sum_{n \in \mathcal{N}} \frac{E^{(n),*}}{\sum_{m \in \mathcal{N}} E^{(m),*}} \cdot \vec{\omega}^{(n)}(\vec{P}^*, E^{(n),*}),$$

at each time-state node along the equilibrium path. So I close the proof. □

Corollary 5. *Given the process of effective Negishi-weight distribution $\{\alpha(n)\}_n$ along the equilibrium path, the log-change in real marginal utility of expenditure for the representative consumer approximately equals*

$$\begin{aligned} \mathrm{d}\tilde{m} = &- \sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\boldsymbol{e} - \mathrm{d}p_J) \\ &+ \frac{1}{N} \cdot \sum_n s(n) \cdot \mathrm{d}\log[\alpha(n)] + o(\hat{h}). \end{aligned} \tag{59}$$

where $\mathrm{d}\alpha$ is the directional derivative of Negishi-weight distribution, $\vec{\omega}$ is the aggregate expenditure share, $\boldsymbol{e}$ is the (log) aggregate total consumption expenditure, and the vector $b(\alpha)$ is defined in the Corollary 4, the expenditure-ratio $s(n)$ is the ratio of consumer-expenditure and aggregate-expenditure $\frac{E^{(n),*}}{\mathbf{E}^*}$ in the equilibrium. The perturbation term is the norm of perturbation term $\hat{h} = \max\{h, \frac{1}{N} \cdot \mathrm{d}\alpha\}$.

Proof. Recall we use the Envelope Theorem in the proof of Aggregation-Lemma E.2,

$$\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha) = \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha^*(n) \cdot \frac{E^{(n),*}}{\mathbf{E}^*} \cdot \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*}) \tag{121}$$

Now we extend the parameter as $\hat{a} = (a, \alpha)$ with $a = (\vec{p}, e)$, the change of parameter as

$$\hat{h} = \max\{\{\mathrm{d}p_j\}_j, \mathrm{d}e, \{\mathrm{d}\alpha_n\}_n\}$$

The term $\mathcal{D}\tilde{m}(\hat{a}) \cdot \hat{h} - \mathrm{d}p_J$ is decomposed as below,

$$\mathcal{D}\tilde{m}(\hat{a}) \cdot \hat{h} - \mathrm{d}p_J = \mathcal{D}_a\tilde{m}(\hat{a}) \cdot h + \mathcal{D}_\alpha\tilde{m}(\hat{a}) \cdot \mathrm{d}\alpha - \mathrm{d}p_J$$

Recall the first-order derivative of composition satisfies

$$\mathcal{D}_{\alpha_n}[\frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha_n \cdot f(a;n)] = \frac{1}{N} \cdot f(a;n).$$

Hence, we arrive to the directional change as

$$\mathcal{D}_{\alpha_n} \log\left[\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)\right] = \frac{1}{\mathcal{D}_E V(\vec{P}^*, \mathbf{E}^*; \alpha)} \cdot \frac{E^{(n),*}}{\mathbf{E}^*} \cdot \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})$$

For succinct notation, I denote $\mathcal{D}_E V(n) = \mathcal{D}_E V^{(n)}(\vec{P}^*, E^{(n),*})$. Therefore, we arrive to the new decomposition of effective RA's marginal utility as below

$$\begin{aligned}
\mathrm{d}\tilde{m} = &- \sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\mathbf{e} - \mathrm{d}p_J) \\
&+ \frac{1}{N} \cdot \sum_n \frac{E^{(n),*}}{\mathbf{E}^*} \cdot \frac{\mathcal{D}_E V(n)}{\mathcal{D}_E V(\alpha)} \cdot \mathrm{d}\alpha(n) + o(\hat{h}).
\end{aligned}$$

Notice $\mathrm{d}\alpha(n)$ is the absolute term of changes, and $\frac{\mathcal{D}_E V(n)}{\mathcal{D}_E V(\alpha)}$ is $\alpha^*(n)$ by construction. Convert the equation into the log-change,

$$\begin{aligned}
\mathrm{d}\tilde{m} = &- \sum_{j=1}^{J} b_j(\alpha) \cdot \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}p_J) - b_e(\alpha) \cdot (\mathrm{d}\mathbf{e} - \mathrm{d}p_J) \\
&+ \frac{1}{N} \cdot \sum_n \frac{E^{(n),*}}{\mathbf{E}^*} \cdot \mathrm{d}\log[\alpha(n)] + o(\hat{h}).
\end{aligned}$$

□