Risk for Price: Using Generalized Demand System for Asset Pricing

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Consumption-CAPM

- Consumption quantity fails to explain asset returns

- Small volatility of consumption v.s. equity premium
- (Mehra and Prescott, 1985; Hansen and Singleton, 1983)
  - empirical: garbage (Savov, 2011), noise (Kroencke, 2017), non-marketable goods (Belo et al, 2021)

- **Cross-section**: covariance with consumption can't explain the returns
- (Mankiw and Shapiro, 1986)
  - supplementary to nondurable (Yogo, 2006)

- Old puzzle is unsolved

Price for Consumption-CAPM

Observation

- Consumption prices + expenditure ⇒ consumer's utility from basket

Solution

- Detailed price improves measuring stochastic discount factor (SDF)

⇒ Decompose consumer's marginal utility into prices

New Finding: Price Explains Returns

- Use **detailed price** to describe SDF
  - 2 sectors within consumption ⇒ expenditure, prices (goods, services)
  - Estimate consumer's Euler Equation of asset holding
- Smaller pricing error across equity portfolios: $7.85\% \Rightarrow 0.39\%$
  - Testing assets: size, book-market, profitability, investment, momentum, earning-price

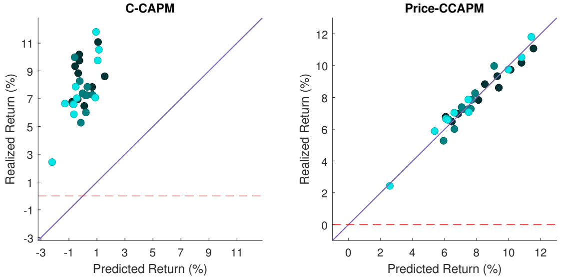

Figure 1: Fitness of Asset Pricing Models

Solution using Detailed Prices

Theory

- Use **indirect utility function** to describe consumer preference Example:IDU
- SDF ⇒ prices and expenditure
- Decomposition of SDF is **general**

- Composition of consumption basket changes with expenditure
  ⇒ Weights of price in SDF deviate CPI
  ⇒ Consumption-CAPM cannot describe SDF
  ⇒ Detailed price improves measuring SDF

Estimation

- Inference implementation is simple
- Flexible application for economy of multiple sectors

Estimation Outcome

- Economy with goods and services, pricing kernel is

$$\mathrm{d}\tilde{m}_{t+1} \approx -b_e \cdot \underbrace{(\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1})}_{\mathrm{d}\tilde{e}\text{, Expenditure adjusted by Price of Services}} - b_g \cdot \omega_{g,t} \cdot \underbrace{(\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})}_{\mathrm{d}\tilde{p}_g\text{, Relative Price of Goods}}, \quad (1)$$

- Small risk-aversion coefficient
  - Expenditure has risk price $\hat{b}_e = 28.80$
- Prices contribute to risk premium
  - Price of goods has risk price $\hat{b}_g = -71.29$
- Cross-section of expected returns
  - High explanation: small MAE 0.39%
- Extended estimation of 4 sectors: Food and non-food within goods and services
  - Smaller risk-aversion $\hat{b}_e = 14.70$.
  - Model fitness is improved to 0.18%.

Difference to Literature

- C-CAPM with heterogeneous commodities
  - (Piazzesi et al., 2007; Dittmar et al., 2020);
  - Durable (Yogo, 2006; Gomes et al., 2009; Belo, 2010; Yang, 2011; Eraker et al., 2016);
  - No suitable quantity index: (Ait-Sahalia et al., 2004; Lochstoer, 2009; Pakoš, 2011)

  This paper: (1) accurate measure of SDF using dis-aggregated prices; (2) approximation is robust to multiple families of utility function

- Asset pricing of commodity price
  - Consumer's price: (Lochstoer, 2009; Roussanov et al., 2021);
  - Other price: (Belo, 2010; Papanikolaou, 2011; Favilukis and Lin, 2016)
- Measuring systematic risk
  - Equity issuance cost shock (Belo et al., 2019), capital share risk (Lettau et al., 2019), firm entry-cost shock (Loualiche et al., 2016), fund flow (Dou et al., 2022)

  This paper: impact of shocks over consumer's marginal utility ⇒ summarized by prices

Guideline

Economy Environment

- Dynamic endowment economy with stream of consumption $\tilde{C} = \{\tilde{C}_j\}_{j \in \mathcal{J}}$
- Commodity market: sector j has price P_j
- Financial market: risky securities and risk-free bond

- Representative consumer decides
  - consumption basket $\vec{C}_t$
  - risky securities $\vec{\theta}_{t+1}$ and risk-free bond B_{t+1}

Competitive Equilibrium

Consumer's Preference

- **Indirect utility function** $V(\vec{P}, E)$ over price $\vec{P}$ and expenditure E is

$$V(\vec{P}, E) = \max_{\vec{C}} \underbrace{u(C_1, C_2, \ldots, C_J)}_{\text{direct utility function over quantities}} \quad s.t. \quad \sum_{j \in \mathcal{J}} P_j \cdot C_j \leq E. \tag{2}$$

- Impact of price over consumer's utility
  - $u(\vec{C}) \overset{\vec{P}}{\Rightarrow}$ optimal $\vec{C}^* \Rightarrow$ utility
  - ✔ $V(\vec{P}, E) \Rightarrow$ utility

- **Sufficient Statistic**: consumption price $\vec{P}$ and expenditure E describe consumer's utility.

Return to Preview

Equivalent Problem with Expenditure

- Consumer maximizes the life-time utility with consumption basket $\vec{C}$

$$\begin{aligned} &\sup_{\tilde{C},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=0}^{T} \beta^t \cdot u(\vec{C}_t)] \\ s.t. \quad & \text{Budget Constraint with } \sum_{j\in\mathcal{J}} P_{j,t} \cdot C_{j,t} \text{ and holding of financial assets } \vec{\theta}_{t+1}, B_{t+1}, \\ & \text{Other Constraints.} \end{aligned} \tag{3}$$

- Given commodity price $\vec{P} \Rightarrow$ equivalent optimization problem of expenditure E

$$\begin{aligned} &\sup_{\tilde{E},\tilde{\theta},\tilde{B}} \quad \lim_{T\to\infty} \mathbb{E}[\sum_{t=0}^{T} \beta^t \cdot V(\vec{P}_t, E_t)] \\ s.t. \quad & \text{Budget Constraint with } E_t \text{ and holding of financial assets } \vec{\theta}_{t+1}, B_{t+1}, \\ & \text{Other Constraints.} \end{aligned} \tag{4}$$

Dynamic Decision

Euler Equation

- Consumer's marginal utility of expenditure equals shadow price of budget constraint.

Definition (SDF)

Define the **real stochastic discount factor** $\tilde{M}$ as

$$\tilde{M}(\vec{P}_t, E_t) := \underbrace{\mathcal{D}_E V(\vec{P}_t, E_t)}_{\text{Marginal Utility of Expenditure}} \cdot \mathbf{P}_t. \tag{5}$$

where $\mathbf{P}_t$ is the consumer price index.

- Expected excess return is determined by the covariance to variation in real SDF.

Price-Model of Consumption-CAPM

Theorem (Decomposition of SDF)

In the economy with consumption sectors $\mathcal{J}$, the first-order approximated change in real stochastic discount factor $\mathrm{d}\tilde{m} = \log(\frac{\tilde{M}_{t+1}}{\tilde{M}_t})$ *is*

$$\mathrm{d}\tilde{m} = -\underbrace{b_e}_{\text{Risk Price of Expenditure}} \cdot \mathrm{d}\tilde{e} - \sum_{j \in \mathcal{J}} \underbrace{b_j}_{\text{Risk Price of Price } P_j} \cdot \omega_j \cdot \mathrm{d}\tilde{p}_j + o(h). \tag{6}$$

with high-order term $o(h)$. The risk price vector $\vec{b}$ is

$$b_e = \gamma; \quad b_j = -\gamma + \sum_{i \in \mathcal{J}} \eta_{j,i} - \sum_{k \in \mathcal{J}} \omega_k \cdot \sum_{i \in \mathcal{J}} \eta_{k,i}. \tag{7}$$

Notations

- $\mathrm{d}\tilde{p}_j$ is change in price P_j adjusted by P_J, $\mathrm{d}\tilde{e}$ for real expenditure.

Explanation of Asymmetric Risk Price

- General situation: expenditure changes composition in consumption basket

- Decreased expenditure
  $\Rightarrow$ share of necessity commodity in consumption basket goes up

- Asymmetric risk price

$$b_n - b_\ell = \underbrace{\sum_{i \in \mathcal{J}} \eta_{n,i} - \sum_{i \in \mathcal{J}} \eta_{\ell,i}}_{\text{Relative share } \frac{\omega_n}{\omega_\ell} \text{ w.r.t Expenditure}} . \tag{8}$$

Engel Slope | Example | Sketch-Marginal Utility | Representative Consumer

- High price of necessity commodity
  $\Rightarrow$ consumer's marginal utility increases more

Cross-section of Returns

Corollary (Euler Equation with Price)

For security k, the excess return $R^e_{k,t+1}$ satisfies

$$\mathbb{E}_t[R^e_{k,t+1}] \approx \quad b_e \cdot \mathbb{E}_t\left[\mathrm{d}\tilde{e}_{t+1} \cdot R^e_{k,t+1}\right] + \sum_{j \in \mathcal{J}} b_j \cdot \omega_{j,t} \cdot \mathbb{E}_t\left[\mathrm{d}\tilde{p}_{j,t+1} \cdot R^e_{k,t+1}\right]. \tag{9}$$

- Expected excess return of financial assets is determined by the covariance between excess return and consumption prices.
- Risk price $\vec{b}$ determines the contribution of each covariance term.
  - Explicitly estimate b_j for price of commodity j.

Guideline

Estimation in a Two-sector Economy

- Economy with goods and services, set of sector is $\mathcal{J} = \{g, s\}$.
- The pricing kernel is approximated as

$$\mathrm{d}\tilde{m}_{t+1} \approx -\, b_e \cdot \underbrace{(\mathrm{d}e_{t+1} - \mathrm{d}p_{s,t+1})}_{\mathrm{d}\tilde{e},\ \text{Expenditure adjusted by Price of Services}} - b_g \cdot \omega_{g,t} \cdot \underbrace{(\mathrm{d}p_{g,t+1} - \mathrm{d}p_{s,t+1})}_{\mathrm{d}\tilde{p}_g,\ \text{Relative Price of Goods}}, \tag{10}$$

- Sample moment of Euler Equation in risky asset k is

$$g_{\mathcal{T},k} = \mathbb{E}_{\mathcal{T}}[R^e_{k,t+1} + \mathrm{d}\tilde{m}_{t+1}(\vec{b}) \cdot R^e_{k,t+1}] \tag{11}$$

- GMM estimates parameters $\vec{b} = (b_e, b_g)$.

Data Description

- Main Data: NIPA Table 2.3.4, Table 2.3.5, 1964-2019 Annual
- Consumption sectors:
  - good: food grocery, apparel, other non-durable goods
  - service: food-away, recreation, health care, financial service, and other service
- Price index: price implied by chained quantity index (Fisher Index)
- Financial assets: 30 portfolios sorted by Size, Book-Market, Profitability, Investment, Momentum, Earning-price ratio.

Time-series Factors in Pricing Kernel

- Relative price of goods has weak correlation to consumption expenditure

Table 1: **Descriptive Statistic**

Panel (A): Time Series - Statistic			
	Mean(pct)	SE(pct)	AR(1)
$\mathrm{d}\tilde{e}$	1.27	1.28	0.36
$(s.e.)$	(0.21)	(0.13)	(0.12)
$\mathrm{d}\tilde{p}_g$	-1.33	1.38	0.47
$(s.e.)$	(0.24)	(0.23)	(0.13)
Panel (B): Correlation			
	$\mathrm{d}\tilde{e}$	$\mathrm{d}c_{nd}$	
Corr$(z, \mathrm{d}\tilde{p}_g)$	0.26	-0.17	
$(s.e.)$	(0.18)	(0.17)	

Plot

Estimation Outcome

Table 2: **Estimation of Pricing Kernel**

		Risk Price
Expenditure	b_e	**28.80**
	$[t]$	[1.95]
Price(Goods)	b_g	**-71.29**
	$[t]$	[-2.31]
	MAE(%)	**0.39**
	RMSE(%)	0.44
	J-pval	91.48

t-stat in bracket.

- Asset-pricing equation for expected return

$$\mathbb{E}_t[R^e_{k,t+1}] \approx \quad b_e \cdot \mathbb{E}_t\left[\mathrm{d}\tilde{e}_{t+1} \cdot R^e_{k,t+1}\right] + b_g \cdot \omega_{g,t} \cdot \mathbb{E}_t\left[\mathrm{d}\tilde{p}_{g,t+1} \cdot R^e_{k,t+1}\right]. \tag{12}$$

MAE

Other Asset Pricing Models

- **CAPM**, excess return of market portfolio
- **FF-5**, Fama-French 5-factor model

- **C-ND**, C-CAPM with nondurable quantity (index)

$$\mathrm{d}\tilde{m}_{t+1} \approx -b_c \cdot \mathrm{d}c_{nd,t+1}. \tag{13}$$

- **C-D**, nondurable quantity + durable stock

$$\mathrm{d}\tilde{m}_{t+1} \approx -b_{nd} \cdot \mathrm{d}c_{nd,t+1} \underbrace{-b_{dur} \cdot \mathrm{d}c_{dur,t+1}}_{\text{Quantity Change of Durable}}. \tag{14}$$

- **P-ND**, Price-CCAPM in previous estimation
- **P-D**, durable stock affects marginal utility of non-durable expenditure,

$$\mathrm{d}\tilde{m} \approx -b_e \cdot \mathrm{d}\tilde{e} - b_g \cdot \omega_g \cdot \mathrm{d}\tilde{p}_g - b_{dur} \cdot \mathrm{d}c_{dur}. \tag{15}$$

  - Simplified linear model P^L-**ND**, P^L-**D**: no time-varying share ω_g.

Durable

Fitness of Models

- Fitness of model estimation is improved when we use model **P-ND**.

Table 3: **Fitness of Asset Pricing Models**

	Traded-Factors		Quantity		Price (Linear)		Price	
	CAPM	FF-5	C-ND	C-D	P^L-ND	P^L-D	**P-ND**	P-D
MAE(%)	1.67	1.20	**7.85**	1.68	1.15	1.10	**0.39**	0.27
RMSE(%)	2.32	1.96	8.01	2.15	1.43	1.42	0.44	0.36

- Consumption-CAPM in literature: Simplified Estimation
- Relax Assumption in Formal Estimation: Formal Estimation

Fitness of Models

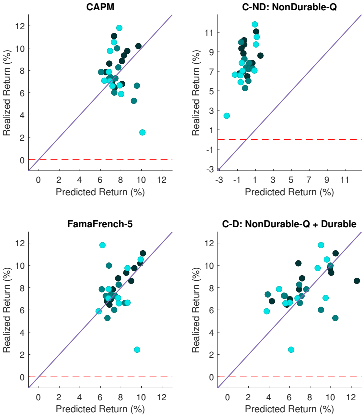

Figure 2: Fitness of Asset Pricing Models

X-axis is Model-Predicted Excess Return. Y-axis is Realized Average Excess Return.

Formal Estimation Summary

Robustness Check

- Alternative testing assets
  - Size-BM 25
  - Industry 30
- Definition of price
  - Share-weighted price index
  - Simple-average price index
- Classification of consumption sector

- Long sample during 1935-2019 Subsample
- Sample including 2021-2022 Covid

Guideline

Comparing Quantity and Prices

- Detailed prices help accurately measure the consumer's marginal utility
  - General description of consumer preference
  - Asymmetric risk prices
- Estimation of parameterized consumer preference
  - Quantity index (special case of homothetic preference)
    - Improper weights assumed for detailed prices
  - Quantity of goods and quantity of services (non-homothetic preference)
    - Stone-Geary Preference has inconsistent point estimate
    - Direct utility function is not tractable

Consumption-CAPM is for Special Situation

- Analytical Example: Cobb-Douglas utility function

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\omega_g} \cdot C_s^{1-\omega_g})^{1-\gamma}, \tag{16}$$

- Composite commodity is identical with quantity index,

$$\mathbf{C} = C_g^{\omega_g} \cdot C_s^{1-\omega_g} = \frac{E}{\underbrace{P_g^{\omega_g} \cdot P_s^{1-\omega_g}}_{\text{Consumer Price Index } \mathbf{P}}}. \tag{17}$$

- Consumption-CAPM using (Tornqvist) quantity index,

$$\mathrm{d}\tilde{m} = -\gamma \cdot \mathrm{d}\mathbf{c}. \tag{18}$$

- Equivalently,

$$\mathrm{d}\tilde{m} = -\gamma \cdot [\mathrm{d}e - \sum_{j \in \mathcal{J}} \omega_j \cdot \mathrm{d}p_j]. \tag{19}$$

- Equivalence holds for CES and other homothetic preference.

Comparison with Quantity Index

Table 4: **Quantity Index**

	C-ND	P-ND
b_c	51.16	-
$[t]$	[4.31]	-
b_e	-	28.80
$[t]$	-	[1.95]
b_g	-	-71.29
$[t]$	-	[-2.31]
MAE(%)	0.71	0.39
RMSE(%)	0.87	0.44
J-pval	96.23	91.48

- Model **C-ND** with quantity index

$$\mathrm{d}\tilde{m} = -\gamma \cdot \mathrm{d}\mathbf{c}. \tag{20}$$

Risk price b_c (risk-aversion γ) is estimated as 51.16.

- Model **P-ND** with price

$$\mathrm{d}\tilde{m} = -b_e \cdot \mathrm{d}\tilde{e} - b_g \cdot \omega_g \cdot \mathrm{d}\tilde{p}_g. \tag{21}$$

Risk price b_e (risk-aversion γ) is estimated as 28.80.

- Model **C-ND** $\Rightarrow$ **P-ND**

$$\mathrm{d}\mathbf{c} \approx (\mathrm{d}e - \mathrm{d}p_s) - \omega_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s). \tag{22}$$

- Fitness is improved

Fisher index | C-CAPM with large γ | Seasonality | Comparing Weights

Using Quantities to Describe Marginal Utility

- Describe consumer's marginal utility using quantities. Plot
- Example: non-separable preference that generalizes (Ait-Sahalia et al., 2004).

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\rho_g} + C_s^{\rho_s})^{\frac{1-\gamma}{\rho_s}}, \tag{23}$$

- $\rho_g > \rho_s$, larger share of goods in low-income state.
- Marginal utility of services is not a simple linear expression using quantities

$$\mathrm{d}\tilde{m}^s \approx -\frac{\rho_g}{\rho_s} \cdot [\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{\omega_s}{\rho_s}} \cdot \mathrm{d}c_g - \{[\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{\omega_s}{\rho_s}} + \gamma\} \cdot \mathrm{d}c_s. \tag{24}$$

Estimation using Quantities is Inaccurate

- Approximate linear pricing kernel with quantities of Goods & Services

$$d\tilde{m} \approx -b_{c_g} \cdot dc_g - b_{c_s} \cdot dc_s. \tag{25}$$

- Inaccurate point estimate in first stage estimation,

Table 5: **Quantities**

	Risk Price	
	1st-Stage	2nd-Stage
b_{c_g}	45.04	37.22
$[t]$	[1.09]	[5.66]
b_{c_s}	6.34	10.61
$[t]$	[0.22]	[2.74]
MAE(%)	0.53	
RMSE(%)	0.65	
J-pval		91.31

Stone-Geary Preference

Table 6: **Habit Model**

	Zero-Habit Sector	
	(1)	(2)
b_{c_g}	182.54	
$[t]$	[2.56]	
b_{c_s}		33.79
$[t]$		[2.70]
b_{p_g}	108.92	-13.12
$[t]$	[1.60]	[-0.81]
	GMM Stats	
MAPE	2.91	0.53
RMSE	4.04	0.64
J-pval	95.91	95.73

- Column (2): Zero-Habit in the sector of services, positive habit X_s in the sector of goods

$$u(C_g, C_s) = \frac{[(C_g - X_g)^{\overline{\omega}_g} \cdot C_s^{1-\overline{\omega}_g}]^{1-\gamma}}{1-\gamma} \tag{26}$$

- pricing kernel is

$$\mathrm{d}\tilde{m} \approx -\gamma \cdot \mathrm{d}c_s - (1-\gamma) \cdot \overline{\omega}_g \cdot (\mathrm{d}p_g - \mathrm{d}p_s). \tag{27}$$

- Inaccurate point estimate of parameters

- Column (1): Alternative specification

$$u(C_g, C_s) = \frac{[C_g^{\overline{\omega}_g} \cdot (C_s - X_s)^{1-\overline{\omega}_g}]^{1-\gamma}}{1-\gamma} \tag{28}$$

- Abnormally large point estimate b_{c_g} for γ

Other examples

- Other examples of non-homothetic preference
  - (Muellbauer, 1976): expenditure changes consumption basket when there is price-habit,

$$V(\vec{P}, E) = \frac{1}{1-\gamma} \cdot [\frac{E}{v(\vec{P})}]^{1-\gamma} + \hat{h}(\vec{P}). \tag{29}$$

with $v(\vec{P}) = P_g^{\overline{\omega}g} \cdot P_s^{1-\overline{\omega}g}$ and price-habit $\hat{h}(\vec{P}) = \frac{\xi}{\epsilon} \cdot (\frac{P_g}{P_s})^{\epsilon}$.

  - (Comin et al., 2021): quantities contribute to utility differently,

$$1 = C_g^{\rho} \cdot u^{-\rho g} + C_s^{\rho} \cdot u^{-\rho s}.$$

utility $u(C_g, C_s)$ is solution to a non-linear equation of quantities, generalized CES.

- Marginal utility of services is not a tractable function over quantities.

- Price-model allows for the flexible application for economy of heterogeneous sectors

Extension to Food-Sectors

Guideline

Summary

- This paper uses detailed price to describes consumer's marginal utility
  - decomposition uses general indirect utility function
  - suits for multiple types of consumer preference
- Estimation in an economy of goods and services
  - new pricing kernel explains the cross-section of expected return
  - price of goods has negative risk price
  - strong correlation between equity return and relative price

- Detailed consumption prices help measure SDF
  - theoretical prediction: price of necessity commodity has more negative risk price
  - empirical examination: asymmetric risk prices for different sectors

Reference

[1] Yacine Ait-Sahalia, Jonathan A Parker, and Motohiro Yogo. Luxury goods and the equity premium. *The Journal of Finance*, 59(6):2959–3004, 2004.

[2] Frederico Belo. Production-based measures of risk for asset pricing. *Journal of Monetary Economics*, 57(2):146–163, 2010.

[3] Frederico Belo, Xiaoji Lin, and Fan Yang. External equity financing shocks, financial flows, and asset prices. *The Review of Financial Studies*, 32(9): 3500–3543, 2019.

[4] Diego Comin, Danial Lashkari, and Martí Mestieri. Structural change with long-run income and price effects. *Econometrica*, 89(1):311–374, 2021.

[5] Robert F Dittmar, Christian Schlag, and Julian Thimme. Non-substitutable consumption growth risk. *Available at SSRN 3289249*, 2020.

[6] Winston Wei Dou, Leonid Kogan, and Wei Wu. Common fund flows: Flow hedging and factor pricing. Technical report, National Bureau of Economic Research, 2022.

[7] Bjørn Eraker, Ivan Shaliastovich, and Wenyu Wang. Durable goods, inflation risk, and equilibrium asset prices. *The Review of Financial Studies*, 29 (1):193–231, 2016.

[8] Jack Favilukis and Xiaoji Lin. Does wage rigidity make firms riskier? evidence from long-horizon return predictability. *Journal of Monetary Economics*, 78:80–95, 2016.

[9] Joao F Gomes, Leonid Kogan, and Motohiro Yogo. Durability of output and expected stock returns. *Journal of Political Economy*, 117(5):941–986, 2009.

[10] Lars Peter Hansen and Kenneth J Singleton. Stochastic consumption, risk aversion, and the temporal behavior of asset returns. *Journal of political economy*, 91(2):249–265, 1983.

[11] Tim A Kroencke. Asset pricing without garbage. *The Journal of Finance*, 72(1):47–98, 2017.

[12] Martin Lettau, Sydney C Ludvigson, and Sai Ma. Capital share risk in us asset pricing. *The Journal of Finance*, 74(4):1753–1792, 2019.

[13] Lars A Lochstoer. Expected returns and the business cycle: Heterogeneous goods and time-varying risk aversion. *The Review of Financial Studies*, 22 (12):5251–5294, 2009.

[14] Erik Loualiche et al. Asset pricing with entry and imperfect competition. *Journal of Finance, forthcoming*, 2016.

[15] N Gregory Mankiw and Matthew D Shapiro. Risk and return: Consumption beta versus market beta. *The Review of Economics and Statistics*, pages 452–459, 1986.

[16] Rajnish Mehra and Edward C Prescott. The equity premium: A puzzle. *Journal of monetary Economics*, 15(2):145–161, 1985.

[17] John Muellbauer. Community preferences and the representative consumer. *Econometrica: Journal of the Econometric Society*, pages 979–999, 1976.

[18] Michal Pakoš. Estimating intertemporal and intratemporal substitutions when both income and substitution effects are present: the role of durable goods. *Journal of Business & Economic Statistics*, 29(3):439–454, 2011.

[19] Dimitris Papanikolaou. Investment shocks and asset prices. *Journal of Political Economy*, 119(4):639–685, 2011.

[20] Monika Piazzesi, Martin Schneider, and Selale Tuzel. Housing, consumption and asset pricing. *Journal of Financial Economics*, 83(3):531–569, 2007.

[21] Nikolai L Roussanov, Yang Liu, and Xiang Fang. Getting to the core: Inflation risks within and across asset classes. *Jacobs Levy Equity Management Center for Quantitative Financial Research Paper*, 2021.

[22] Alexi Savov. Asset pricing with garbage. *The Journal of Finance*, 66(1):177–201, 2011.

[23] Wei Yang. Long-run risk in durable consumption. *Journal of Financial Economics*, 102(1):45–61, 2011.

[24] Motohiro Yogo. A consumption-based explanation of expected stock returns. *The Journal of Finance*, 61(2):539–580, 2006.

Special Case

- Zero price-habit $\hat{h}(\vec{P}) = 0$, the indirect utility function is

$$V(P_g, P_s, E) = \frac{1}{1-\gamma} \cdot [\frac{E}{P_g^{\overline{\omega}_g} \cdot P_s^{1-\overline{\omega}_g}}]^{1-\gamma} \tag{30}$$

$\Rightarrow$ utility function is

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot [C_g^{\overline{\omega}_g} \cdot C_s^{1-\overline{\omega}_g}]^{1-\gamma}. \tag{31}$$

return

Calculating Example

- Calibration:
  - tomorrow: boom and down states $\{h, d\}$
  - identical expenditure, prices are different
  - today: observed share is $\omega_g = 0.40$
  - boom state: $P_{g,h} = 1$ and $P_{s,h} = 1$
  - down state: $P_{g,d} = 1.02$ and $P_{s,d} = 0.9869$
- Identical Consumer Price Index,

$$\mathbf{P}_d = \mathbf{P}_h = 1. \tag{32}$$

- **Identical** quantity index,

$$\mathbf{C}_d = \mathbf{C}_h. \tag{33}$$

Compare the Marginal Utility

- **High** price of **goods** in down state, low price of services
- **High** marginal utility in down state

$$\underbrace{(P_{g,d}^{\overline{\omega}_g} \cdot P_{s,d}^{1-\overline{\omega}_g})}_{\text{High}}{}^{-(1-\gamma)} \cdot E^{-\gamma} > \underbrace{(P_{g,h}^{\overline{\omega}_g} \cdot P_{s,h}^{1-\overline{\omega}_g})}_{\text{Low}}{}^{-(1-\gamma)} \cdot E^{-\gamma}. \tag{34}$$

- High stochastic discount factor $M_d > M_h$. SDF
- $\gamma = 10$, $\overline{\omega}_g - \omega_g = 0.2 \Rightarrow \log(\frac{M_d}{M_h}) \approx 6.8\%$.
  - Comparing the stochastic discount factor, $\frac{M_d}{M_h} = (\frac{P_{g,d}/P_{g,h}}{P_{s,d}/P_{s,h}})^{-(1-\gamma)\cdot(\overline{\omega}_g - \omega_g)}$.

Caveat in Quantity Index

- **Identical** quantity index $\mathbf{C}_d = \mathbf{C}_h$
- **Different** stochastic discount factor $M_d > M_h$
  - high price of goods $\Rightarrow$ high stochastic discount factor

- **Detailed prices** provide the **accurate measure for SDF**

Competitive Equilibrium

- Consumer has optimal decision
  - given commodity price $\vec{P}$ and security prices
  - chooses optimal stream of basket $\tilde{C}$ and financial asset positions $\{\tilde{\theta}, \tilde{B}\}$.
- Commodity markets clear
  - consumer's demand equals the exogenous supply in each sector j.
- Financial asset markets clear
  - zero supply and demand in risk-free bond;
  - consumer owns all share of risky securities.

Return to Model Env.

Consumer Problem with DU

- Consumer maximizes the life-time utility with consumption basket $\vec{C}$

$$\overline{U}_0(\vec{\theta}_0) = \sup_{\tilde{C},\tilde{\theta},\tilde{B}} \lim_{T\to\infty} \mathbb{E}[\sum_{t=0}^{T} \beta^t \cdot u(\vec{C}_t)]$$

$$s.t. \quad \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = \sum_j P_{j,t} \cdot C_{j,t} + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \quad \text{(P-DU)}$$

$$C_{j,t} \geq 0; \quad \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}} \geq \underline{a}.$$

Notations

- Commodity price P_j and consumption quantity C_j
- Price P^s_k and payout D_k for financial security k
- Risk-free rate R_f

Consumer Problem with IDU

- Consumer maximizes the life-time utility with consumption expenditure E

$$\overline{V}_0^{\text{New}}(\vec{\theta}_0) = \sup_{\tilde{E},\tilde{\theta},\tilde{B}} \lim_{T\to\infty} \mathbb{E}[\sum_{t=0}^{T} \beta^t \cdot V(\vec{P}_t, E_t)]$$

$$s.t. \quad \sum_k \theta_{k,t} \cdot (P^s_{k,t} + D_{k,t}) + B_t = E_t + \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}}, \quad \text{(P-IDU)}$$

$$E_t \geq 0; \quad \sum_k \theta_{k,t+1} \cdot P^s_{k,t} + \frac{B_{t+1}}{R_{f,t+1}} \geq \underline{a}.$$

Equivalent Dynamic Problem

Lemma (Equivalence)

Optimization problem of quantities (P-DU) yields equivalent value as the optimization problem of expenditure (P-IDU). For each optimal policy C^* in problem (P-DU), E^* such that

$$E_t^* = \sum_{j \in \mathcal{J}} P_{j,t} \cdot C_{j,t}^*, \quad \forall t, z^t$$

is an optimal policy in the optimization problem (P-IDU).

return

Decomposition (a)

- Roy Identity (Shephard's lemma)

$$\omega_j = -\frac{\mathcal{D}_j V(\vec{P}, E) \cdot P_j}{\mathcal{D}_E V(\vec{P}, E) \cdot E}.$$

- $\mathcal{D}_j V(\vec{P}, E)$ is the first-order partial derivative to price P_j.

return

Decomposition (b)

- Indirect Utility Function is **H.D.0** (Homogeneous of Degree Zero)

$$\mathcal{D}_E V(\vec{P}, E) \cdot E = -\sum_{j \in \mathcal{J}} \mathcal{D}_j V(\vec{P}, E) \cdot P_j.$$

- Replace the right-hand-side
  $\Rightarrow$ Marginal Utility of Expenditure for utility-flow is decomposed as

$$\mathrm{d}\log \mathcal{D}_E V(\vec{P}, E) = \sum_{j \in \mathcal{J}} \omega_j \cdot (\mathrm{d}p_j - \mathrm{d}e) + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}} \omega_k \cdot \left[\frac{\mathcal{D}_{k,j} V(\vec{P}, E)}{\mathcal{D}_k V(\vec{P}, E)} \cdot \frac{P_j}{E}\right] \cdot (\mathrm{d}p_j - \mathrm{d}e) + o(h).$$

return

Risk Price for Expenditure

- Risk price for total consumption expenditure,

$$b_e = \underbrace{\gamma}_{\text{Relative Risk-aversion Coefficient}}. \quad (35)$$

- Expenditure share ω captures the quantitative importance of sector.

$$b_e = -\sum_{j \in \mathcal{J}} \omega_j \cdot \underbrace{b_j}_{\text{Risk Price for Price } P_j}. \quad (36)$$

  - Same change in price $\vec{P}$ and expenditure $E \Rightarrow$ utility is the same.

Shares in Consumption Basket

- Composition of consumption basket: $\omega_j = \frac{P_j \cdot C_j}{E}$, for each sector j
- Share elasticity $\Rightarrow$ adjustment of shares to prices and expenditure

Lemma

Given consumption sectors n and ℓ, change in the relative share $\mathcal{S}_{n,\ell} = \frac{\omega_n}{\omega_\ell}$ can be decomposed into the price effect and the expenditure effect,

$$\begin{aligned} \mathrm{d}s_{n,\ell} =& (1 - \eta_{n,n} + \eta_{\ell,n}) \cdot \mathrm{d}p_n - (1 - \eta_{\ell,\ell} + \eta_{n,\ell}) \cdot \mathrm{d}p_\ell - \sum_{i \neq n,\ell} (\eta_{n,i} - \eta_{\ell,i}) \cdot \mathrm{d}p_i \\ & + \underbrace{\sum_{i \in \mathcal{J}} (\eta_{n,i} - \eta_{\ell,i}) \cdot \mathrm{d}e}_{\text{expenditure effect}} + o(h). \end{aligned} \quad (37)$$

The $\mathrm{d}s_{n,\ell}$ is the log-growth of relative share between sector n and ℓ. The term $o(h)$ is a higher-order term.

Special Situation of Symmetric Risk Price

- Example with Constant Elasticity of Substitution

$$u(\vec{C}) = \frac{1}{1-\gamma} \cdot (C_1^{\rho} + C_2^{\rho} \cdots + C_J^{\rho})^{\frac{1-\gamma}{\rho}}, \tag{38}$$

- No expenditure-effect in the relative share $\mathcal{S}_{k,j} = \frac{\omega_k}{\omega_j}$ for all pairs (k, j),

$$\mathrm{d}s_{k,j} = \frac{\rho}{\rho - 1} \cdot \mathrm{d}p_k - \frac{\rho}{\rho - 1} \cdot \mathrm{d}p_j, \tag{39}$$

- Matrix of share elasticity,

$$\eta = (\gamma + \frac{1}{\rho - 1}) \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{40}$$

return

Special Situation of Symmetric Risk Price

- Example with Constant Elasticity of Substitution,

$$u(\vec{C}) = \frac{1}{1-\gamma} \cdot (C_1^\rho + C_2^\rho \cdots + C_J^\rho)^{\frac{1-\gamma}{\rho}}. \tag{41}$$

- Use the CPI as price of numeraire
- Symmetric risk price across commodities $b_j = \gamma$,

$$\mathrm{d}\tilde{m} = -\gamma \cdot [\mathrm{d}e - \underbrace{\sum_{j \in \mathcal{J}} \omega_j \cdot \mathrm{d}p_j}_{\text{variation in CPI}}] \tag{42}$$

- As if we consider the single-sector economy with composite commodity $(\sum_{j \in \mathcal{J}} C_j^\rho)^{\frac{1}{\rho}}$

return

Using Quantities to Describe Marginal Utility

- It is difficult to describe consumer's marginal utility using quantities. Plot
- Example: non-separable preference similar with (1).

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\rho_g} + C_s^{\rho_s})^{\frac{1-\gamma}{\rho_s}}, \quad (43)$$

$\rho_g > \rho_s$: larger share of goods in low-income state.

- Marginal utility of services: no simple linear expression using quantities

$$\mathrm{d}\tilde{m}^s \approx -\frac{\rho_g}{\rho_s} \cdot [\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{\omega_s}{\rho_s}} \cdot \mathrm{d}c_g - \{[\gamma - (\rho_s - 1)] \cdot \frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{\omega_s}{\rho_s}} + \gamma\} \cdot \mathrm{d}c_s. \quad (44)$$

- $\frac{C_g^{\rho_g}}{C_g^{\rho_g} + C_s^{\rho_s}}$ is reduced as expression of shares $\frac{\frac{\omega_g}{\rho_g}}{\frac{\omega_g}{\rho_g} + \frac{\omega_s}{\rho_s}}$.

Derive Marginal Utility using Quantities: CES

- Example: Constant Elasticity of Substitution (CES).

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\rho} + C_s^{\rho})^{\frac{1-\gamma}{\rho}}, \tag{45}$$

- Marginal utility of quantity in services,

$$\mathrm{d}\tilde{m}^s \approx -\gamma \cdot \underbrace{(\omega_g \cdot \mathrm{d}c_g + \omega_s \cdot \mathrm{d}c_s)}_{\text{weighted change in quantities}} - \underbrace{\omega_g \cdot (\rho - 1) \cdot (\mathrm{d}c_g - \mathrm{d}c_s)}_{\text{CPI v.s. } P_s}. \tag{46}$$

- Substitute $C_g = \frac{\omega_g \cdot E}{P_g}$, the real pricing kernel (numeraire price as CPI) is,

$$\mathrm{d}\tilde{m} = -\,\gamma \cdot [\mathrm{d}e - \mathrm{d}\log(\mathbf{P})]. \tag{47}$$

Equivalent Pricing Kernel using Quantities

- Analytical Example: Cobb-Douglas utility function

$$u(C_g, C_s) = \frac{1}{1-\gamma} \cdot (C_g^{\omega_g} \cdot C_s^{1-\omega_g})^{1-\gamma}, \tag{48}$$

- Composite commodity is,

$$\mathbf{C} = C_g^{\omega_g} \cdot C_s^{1-\omega_g}. \tag{49}$$

- Consumption-CAPM,

$$\mathrm{d}\tilde{m} = -\gamma \cdot \mathrm{d}\mathbf{c}. \tag{50}$$

- Equivalent pricing kernel using quantities,

$$\mathrm{d}\tilde{m} = -\gamma \cdot [\sum_{j \in \mathcal{J}} \omega_j \cdot \mathrm{d}c_j]. \tag{51}$$

- Other homothetic preference: pricing kernel has the same approximated variation

Chained quantity index

- Chained quantity index is similar with the (Tornqvist) quantity index.
- Change of chained quantity index is

$$\frac{E_{g,t+1}\cdot\frac{P_{g,t_0}}{P_{g,t+1}}+E_{s,t+1}\cdot\frac{P_{s,t_0}}{P_{s,t+1}}}{E_{g,t}\cdot\frac{P_{g,t_0}}{P_{g,t}}+E_{s,t}\cdot\frac{P_{s,t_0}}{P_{s,t}}}=\sum_{j\in\{g,s\}}\frac{E_{j,t}\cdot\frac{P_{j,t_0}}{P_{j,t}}}{E_{g,t}\cdot\frac{P_{g,t_0}}{P_{g,t}}+E_{s,t}\cdot\frac{P_{s,t_0}}{P_{s,t}}}\cdot\underbrace{\frac{E_{j,t+1}/P_{j,t+1}}{E_{j,t}/P_{j,t}}}_{\text{variation of quantities}} \tag{52}$$

  Prices are normalized as 1 in bench-year t_0.
- Weight for quantities,
  - Chained quantity index: price-adjusted expenditure $\frac{E_{j,t}\cdot\frac{P_{j,t_0}}{P_{j,t}}}{E_{g,t}\cdot\frac{P_{g,t_0}}{P_{g,t}}+E_{s,t}\cdot\frac{P_{s,t_0}}{P_{s,t}}}$
  - (Tornqvist) quantity index: nominal expenditure $\frac{E_{j,t}}{E_{g,t}+E_{s,t}}$.
- Chained quantity index: easy comparison to bench-year t_0.

Return to Example · Return to Tornqvist index

Indirect Utility Function - Durable

- suppose the durable stock K affects the utility flow

$$u = u(\vec{C}, K).$$

the indirect utility function is

$$V(\vec{P}, E; K) = \max_{\vec{C} \in \mathcal{X}} \quad u(C_1, C_2, \ldots, C_I; K)$$
$$s.t. \quad \sum_{i \in \mathcal{I}} P_i \cdot C_i \leq E.$$

Marginal utility of nondurable expenditure changes with the state variable of durable stock K.

return

Time-series Factors in Pricing Kernel

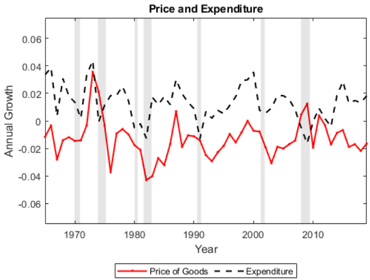

Figure 3: Time Series of Economic Outcomes

Price of goods and (total) expenditure are adjusted by price of services.

Return to Description

Estimation Outcome

Table 2: Estimation of Pricing Kernel

	Subgroups of Testing Assets			**ALL**
	Size-BM	Profit-IK	MoM-EP	**Mix-30**
	Risk Price			
b_e	25.15	40.79	27.12	**28.80**
$[t]$	[2.05]	[2.74]	[1.34]	[1.95]
b_g	-71.94	-62.93	-74.44	**-71.29**
$[t]$	[-3.11]	[-1.90]	[-1.97]	[-2.31]
MAE(%)	0.33	0.36	0.36	**0.39**
RMSE(%)	0.41	0.42	0.37	0.44
J-pval	25.15	45.57	40.40	91.48

t-stat in bracket.

Return to Robustness Estimation

Fitness of Estimation

- Evaluation of model fitness
  - MAE (Mean Absolute Error),

$$\text{MAE} = \frac{1}{K}\sum_k \left| \underbrace{\frac{1}{T}\cdot\sum_{t=1}^{T} R^e_{k,t+1}}_{\text{Realized Average Excess Return}} - \underbrace{\left[\frac{1}{T}\cdot\sum_{t=1}^{T} -\mathrm{d}\tilde{m}_{t+1}(\vec{b}^*)\cdot R^e_{k,t+1}\right]}_{\text{Model-Predicted Excess Return}} \right|. \quad (53)$$

  - RMSE (Root Mean Square Error)

$$\text{RMSE} = \sqrt{\frac{1}{K}\sum_k \left| \frac{1}{T}\cdot\sum_{t=1}^{T}(1+\mathrm{d}\tilde{m}^*_{t+1})\cdot R^e_{k,t+1} \right|^2}. \quad (54)$$

Return to Estimation Outcome Return to Comparison

Weights of Prices in SDF

- Price of goods: SDF 101% (CPI 40%)

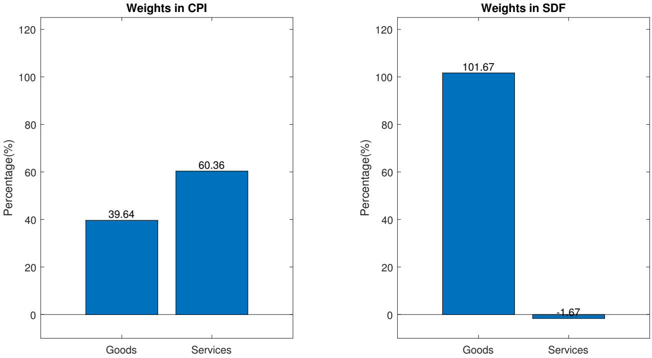

Figure 4: Weights of Prices

Time-series average weights during 1965-2019.

Robust Estimation

- Estimation using **Size-BM 25** and **Industry 30**
  - Point estimates are similar
  - Fitness is good

Table 7: **Estimation using Other Testing Assets**

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	28.80	30.75	30.05	33.72	33.27	33.88
$[t]$	[1.95]	[14.08]	[2.61]	[13.06]	[4.38]	[24.98]
b_g	-71.29	-72.26	-68.26	-63.83	-69.95	-67.92
$[t]$	[-2.31]	[-15.89]	[-2.90]	[-11.68]	[-3.04]	[-17.21]
MAE(%)	0.39		0.38		0.84	
RMSE(%)	0.44		0.51		0.99	
J-pval		91.48		81.48		94.03

FF-5 Subset Summary

Robust Estimation when using Size-BM 25

- Estimation using **Size-BM 25**
  - Point estimates are similar
  - model **P-ND** has small error

Table 8: **Estimation Outcome using Quantity Index**

	C-ND	P-ND
b_c	50.88	-
$[t]$	[4.74]	-
b_e	-	30.05
$[t]$	-	[2.61]
b_g	-	-68.26
$[t]$	-	[-2.90]
MAE(%)	0.79	0.38
RMSE(%)	0.95	0.51
J-pval	95.51	81.48

Mix-30

Estimation of Quarterly Frequency

- Estimation using consumption data of quarterly frequency
  - seasonality exacerbates the weak correlation

Estimation Outcome using Quantity Index

	Quarter-1	Quarter-2	Quarter-3	Quarter-4
	Panel (A): Risk Price			
b_c	136.63	16.47	74.42	132.82
$[t]$	[1.20]	[0.17]	[2.13]	[4.53]
	Panel (B): Stats			
MAE(%)	0.35	0.48	0.83	0.39
RMSE(%)	0.42	0.65	1.02	0.47
J-pval	88.64	83.30	88.32	84.52

Return is quarterly frequency.

Annual

Estimation Outcome: Other Sample Periods

Table 9: **Fitness of Asset Pricing Models: 1935-2019**

	Sample Period					
	1935-1989		1950-2004		1965-2019	
	Panel (A): Risk Price					
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_e	31.56	31.64	35.41	39.59	30.05	33.72
$[t]$	[3.69]	[26.79]	[3.19]	[12.49]	[2.61]	[13.06]
b_g	-47.41	-45.67	-65.65	-62.79	-68.26	-63.83
$[t]$	[-2.68]	[-11.06]	[-2.85]	[-13.66]	[-2.90]	[-11.68]
	Panel (B): Stats					
MAE(%)	0.70		0.32		0.38	
RMSE(%)	0.95		0.38		0.51	
J-pval		82.51		96.93		81.48

Return to Robustness Estimation

Estimation Outcome: Covid-period included

Table 10: **Fitness of Asset Pricing Models: 1965-2022**

	Specification of Model					
	Traded Factor		Quantity		Price	
	CAPM	FF-5	C-ND	C-D	P-ND	P-D
MAE(%)	1.39	0.62	1.27	0.46	0.54	0.19
RMSE(%)	1.98	1.14	1.53	0.66	0.71	0.29
J-pval	90.70	76.76	95.34	92.88	89.06	92.60

Return to Robustness Estimation

Formal Estimation: Fitness of Models

- Risk price $\vec{b}$ in nonlinear model is identified using equation general expectation process

$$\mathbb{E}_t[R^e_{k,t+1}] = -\mathbb{E}_t\left[\mathrm{d}\tilde{m}_{t+1}(\vec{b}) \cdot R^e_{k,t+1}\right]. \tag{55}$$

- Both the time-varying expectation and unexpected innovation contribute,

$$\mathbb{E}_t[R^e_{k,t+1}] = -\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}] \cdot \mathbb{E}_t[R^e_{k,t+1}] - \mathbb{E}_t\left[(\mathrm{d}\tilde{m}_{t+1} - \mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]) \cdot R^e_{k,t+1}\right]. \tag{56}$$

Table 11: **Fitness of Asset Pricing Models**

	Traded-Factors		Quantity		Price	
	CAPM	FF-5	C-ND	C-D	**P-ND**	P-D
MAE(%)	1.58	0.79	0.71	0.66	**0.39**	0.27
RMSE(%)	2.20	1.37	0.87	0.83	0.44	0.36

Return to Estimation Summary

Simplified Estimation: Linear Factor Models

- Risk price $\vec{b}$ in linear model $\mathrm{d}\tilde{m}_{t+1} = -\vec{b} \cdot \vec{f}_{t+1}$ is identified using equation

$$\mathbb{E}_t[R^e_{k,t+1}] = \frac{\vec{b}}{1+\tilde{R}_f} \cdot \mathbb{E}_t \left[(\vec{f}_{t+1} - \underbrace{\mathbb{E}_t[\vec{f}_{t+1}]}_{\text{Assumed to be Constant}}) \cdot R^e_{k,t+1} \right]. \quad (57)$$

- Demeaned factors $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$: covariance of slow-moving component $\mathbb{E}_t[\vec{f}_{t+1}]$ is not considered
  - $\frac{1}{1+\mathbb{E}_t[\mathrm{d}\tilde{m}_{t+1}]}$ measured using the gross risk-free rate $\tilde{R}_f$.

- Model **C-ND** has large MAE 7.85%: slow-moving component $\mathbb{E}_t[\mathrm{d}c_{t+1}]$ exacerbates the failure.
- Model P^L-**ND** has MAE 1.15%
  - analogous linear model of price factor (simplified version from Model **P-ND**) $\mathrm{d}\tilde{m} \approx -b_{e,L} \cdot \mathrm{d}\tilde{e} - b_{g,L} \cdot \mathrm{d}\tilde{p}_g$.

	Specification of Model					
	Traded Factor		Quantity		Price (Linear)	
	CAPM	FF-5	C-ND	C-D	P^L-ND	P^L-D
MAE (%)	1.67	1.20	**7.85**	1.68	1.15	1.10
RMSE (%)	2.32	1.96	8.01	2.15	1.43	1.42

Return to Estimation Summary Return to Robustness Estimation

Simplified Estimation: Linear Factor Models

- Risk price $\vec{b}$ in linear model $\mathrm{d}\tilde{m}_{t+1} = -\vec{b} \cdot \vec{f}_{t+1}$ is identified using equation

$$\mathbb{E}_t[R^e_{k,t+1}] = \frac{\vec{b}}{1+\tilde{R}_f} \cdot \mathbb{E}_t\left[(\vec{f}_{t+1} - \underbrace{\mathbb{E}_t[\vec{f}_{t+1}]}_{\text{Assumed to be Constant}}) \cdot R^e_{k,t+1} \right]. \tag{58}$$

- Demeaned factors $\mathbb{E}_t[\vec{f}_{t+1}] \equiv \vec{g}_f$: covariance of slow-moving component $\mathbb{E}_t[\vec{f}_{t+1}]$ is not considered

- Testing assets: **Size-BM 25** portfolios.
- Model **C-ND** has large MAE 9.53%: slow-moving component $\mathbb{E}_t[\mathrm{d}c_{t+1}]$ exacerbates the failure.
- Model P^L-**ND** has MAE 1.07%
  - analogous linear model of price factor (simplified version from Model **P-ND**) $\mathrm{d}\tilde{m} \approx -b_{e,L} \cdot \mathrm{d}\tilde{e} - b_{g,L} \cdot \mathrm{d}\tilde{p}_g$.

	Specification of Model					
	Traded Factor		Quantity		Price (Linear)	
	CAPM	FF-5	C-ND	C-D	P^L-ND	P^L-D
MAPE	2.48	1.05	**9.53**	1.79	1.07	1.13
RMSE	3.27	1.36	9.81	2.27	1.50	1.46

Return to Estimation Summary

Comparison with Quantity Index, Large γ

Table 12: **Quantity Index, Simplified Estimation**

	C-ND	P^L-ND
b_c	106.47	-
$[t]$	[1.98]	-
$b_{e,L}$	-	47.49
$[t]$	-	[0.63]
$b_{p,L}$	-	-93.67
$[t]$	-	[-3.17]
MAE(%)	**9.53**	1.07
RMSE(%)	9.81	1.50
J-pval	89.60	93.34

Return to Formal Estimation

- Model **C-ND** with quantity index

$$\mathrm{d}\tilde{m} = -\gamma \cdot \mathrm{d}\mathbf{c}. \tag{59}$$

  Risk price b_c (risk-aversion γ) is estimated as 51.16.

- Linear Model P^L-**ND** with price factor

$$\mathrm{d}\tilde{m} = -b_{e,L} \cdot \mathrm{d}\tilde{e} - b_{g,L} \cdot \mathrm{d}\tilde{p}_g. \tag{60}$$

- testing assets: Size-BM25
  - Benchmark: Mix-30 of anomalies
- estimation assumes constant expected growth

Comparison: other models

Table 13: **Risk Price, Fama-French 5-Factor Model**

	Specification of Testing Assets					
	Mix 30		Size-BM 25		Industry 30	
	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage	1st-Stage	2nd-Stage
b_{MKT}	2.38	2.51	2.51	2.65	2.64	2.78
$[t]$	[3.77]	[10.82]	[4.39]	[10.04]	[4.02]	[7.94]
b_{Size}	1.72	1.64	1.28	1.20	0.88	0.68
$[t]$	[2.15]	[5.36]	[1.32]	[2.92]	[0.69]	[1.45]
b_{BM}	-3.44	-3.06	-2.24	-1.82	-5.86	-4.88
$[t]$	[-2.05]	[-4.45]	[-1.07]	[-2.99]	[-2.13]	[-6.31]
b_{Profit}	6.56	6.69	5.79	6.28	5.18	5.30
$[t]$	[4.28]	[11.59]	[2.39]	[9.33]	[2.96]	[10.62]
b_{Invest}	7.42	7.33	6.97	7.37	9.36	8.21
$[t]$	[4.36]	[9.10]	[3.16]	[10.67]	[2.05]	[6.91]
MAE(%)	0.79		0.65		1.09	
RMSE(%)	1.37		0.81		1.37	
J-pval		81.07		59.85		84.45

P-ND

Sufficient Statistic for Systematic Risk

- Multiple fundamental shocks ⇒ fluctuation in prices and expenditure
- Sufficient statistic ⇒ small improvement when supplementing a proxy of shock,

$$\mathrm{d}\tilde{m} \approx -b_e \cdot \mathrm{d}\tilde{e} - b_g \cdot \omega_g \cdot \mathrm{d}\tilde{p}_g - b_x \cdot \underbrace{x}_{\text{Shock proxy}} . \tag{61}$$

Table 14: **Estimation with Supplementary Proxy of Shock**

	Specification of Additional Shock Proxy					
	MKT	Size	Value	Profit	Invest	MoM
b_e	32.15	23.80	31.15	26.21	30.94	27.05
$[t]$	[3.05]	[1.05]	[2.35]	[1.51]	[2.36]	[2.55]
b_g	-58.75	-82.35	-69.70	-73.76	-68.68	-72.72
$[t]$	[-3.70]	[-1.64]	[-2.41]	[-2.23]	[-2.51]	[-2.69]
b_x	0.26	-0.55	-0.40	0.53	-0.53	0.10
$[t]$	[0.38]	[-0.58]	[-0.72]	[0.69]	[-0.53]	[0.18]
MAE(%)	0.35	0.31	0.28	0.37	0.32	0.38
RMSE(%)	0.41	0.39	0.38	0.43	0.40	0.44
J-pval	88.68	89.82	89.19	88.99	88.90	88.99

IST

Shock extracted from Prices

- Investment-Specific Technology shock from (Papanikolaou,2011): 1965-2008
- Other proxies: 1965-2019

Table 15: **Estimation with Supplementary Proxy of Shock**

	Specification of Additional Shock Proxy					
	Price				Quantity	
	IST	Equipment	Durable	Energy	Hour	Unf-C
b_e	32.13	32.34	34.24	28.21	40.87	29.85
$[t]$	[4.17]	[3.18]	[3.69]	[1.75]	[3.92]	[1.20]
b_g	-55.94	-62.82	-63.48	-66.34	-59.33	-74.99
$[t]$	[-4.46]	[-3.65]	[-3.81]	[-3.17]	[-2.71]	[-3.95]
b_x	9.16	-6.25	11.36	-0.91	-8.74	-1.96
$[t]$	[0.73]	[-0.41]	[0.43]	[-0.33]	[-0.82]	[-0.13]
MAPE	0.42	0.36	0.35	0.38	0.37	0.38
RMSE	0.51	0.48	0.46	0.49	0.42	0.44
J-pval	92.28	74.36	75.68	75.38	89.70	89.62

Note: **Unf-C** is for Unfiltered consumption quantity (index).

Traded Factor

Sectors within Consumption

- Quantity of goods & quantity of services: correlation is high, but not synchronized

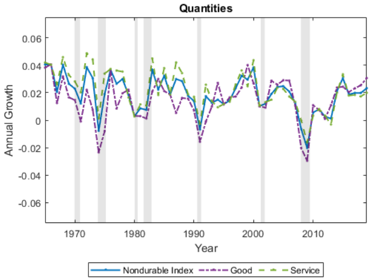

Figure 5: Time Series of Quantity Outcomes.

Return to Estimation

Food within Consumption Sectors

- Food-category and non-food behave differently.

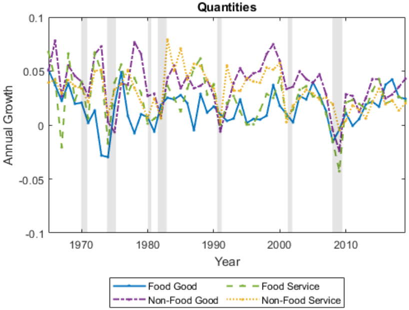

Figure 8(a): Quantities.

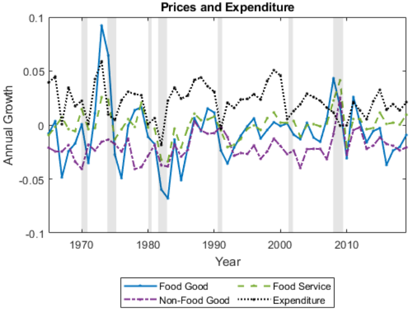

Figure 8(b): Prices and Expenditure.

Return to Estimation

Food within Consumption Sectors

	Descriptive Statistic		
	Mean(pct)	SE(pct)	AR(1)
$de - \mathrm{d}p_{sn}$	2.17	1.51	0.27
$(s.e.)$	(0.23)	(0.16)	(0.13)
$\mathrm{d}p_{gf/sn}$	-0.76	2.72	0.39
$(s.e.)$	(0.44)	(0.48)	(0.11)
$\mathrm{d}p_{gn/sn}$	-2.03	1.20	0.29
$(s.e.)$	(0.20)	(0.17)	(0.12)
$\mathrm{d}p_{sf/sn}$	0.02	1.32	0.25
$(s.e.)$	(0.20)	(0.20)	(0.16)

Return to Estimation

Food within Consumption Sectors

	Correlation		
	$\mathrm{d}p_{gf/sn}$	$\mathrm{d}p_{gn/sn}$	$\mathrm{d}p_{sf/sn}$
Corr$(de - \mathrm{d}p_{sn}, z)$	0.41	0.06	0.34
$(s.e.)$	(0.12)	(0.16)	(0.16)
Corr$(\mathrm{d}p_{gf/sn}, z)$		0.32	0.74
$(s.e.)$		(0.12)	(0.07)
Corr$(\mathrm{d}p_{gn/sn}, z)$			0.51
$(s.e.)$			(0.16)

Return to Estimation

Cross-section of Risk Exposure

- Fama-Macbeth Regression using time-series factors $\vec{f}_{t+1} = (\mathrm{d}\tilde{e}_{t+1}, \mathrm{d}\tilde{p}_{g,t+1})$
  - 1st step: $R^e_{k,t+1} = a_k + \vec{\beta}_k \cdot \vec{f}_{t+1}$
  - 2nd step: $\mathbb{E}_t[R^e_{k,t+1}] = \vec{\beta}_k \cdot \vec{\lambda}$

- Model **P-ND** has dispersed $\vec{\beta}$ in 1st step of Fama-Macbeth regression.

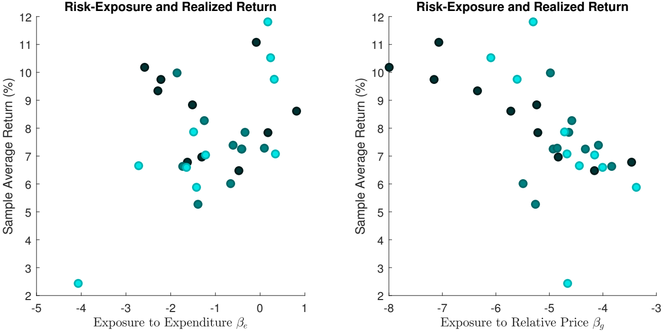

Figure 6: Risk Exposure to Time-series Factors

Cross-section of Risk Exposure

- Value and small firms have larger risk exposure to relative price of goods.

Table 16: **Distribution of Risk Exposure**

	Estimation Outcomes in 1st Step				
BM	Growth	2	3	4	Value
β_e	-1.63	-1.30	0.17	0.81	-0.09
$[t]$	[-0.71]	[-0.64]	[0.08]	[0.36]	[-0.03]
β_g	**-3.46**	-4.83	-5.22	-5.72	**-7.07**
$[t]$	[-1.59]	[-2.51]	[-2.64]	[-2.66]	[-2.76]
μ	6.78	6.97	7.84	8.61	11.08
σ	19.47	16.96	16.37	18.48	20.72
Size	Small	2	3	4	Big
β_e	-2.58	-2.22	-2.29	-1.51	-0.48
$[t]$	[-0.77]	[-0.80]	[-0.91]	[-0.66]	[-0.23]
β_g	**-7.99**	-7.16	-6.34	-5.24	**-4.16**
$[t]$	[-2.51]	[-2.73]	[-2.65]	[-2.40]	[-2.08]
μ	10.18	9.75	9.34	8.84	6.48
σ	28.53	22.83	20.69	19.24	17.06

Industry30

Cross-section of Risk Exposure: Industry portfolios

- Service such as Meals (Restaurant) and Games (Recreation) have larger risk exposure to relative price of goods.
- Merchandise commodities with weaker risk exposure.

Table 17: **Distribution of Risk Exposure**

	Estimation Outcomes in 1st Step					
	Meals	Games	Fin	Carry	Autos	ElcEq
β_e	-2.14	-1.88	0.39	-0.71	-5.61	-1.57
$[t]$	[-0.60]	[-0.60]	[0.15]	[-0.22]	[-2.01]	[-0.55]
β_g	-7.84	-7.79	-7.46	-7.37	-7.00	-6.95
$[t]$	[-2.32]	[-2.63]	[-2.95]	[-2.40]	[-2.64]	[-2.54]
	Beer	Food	FabPr	Oil	Steel	Paper
β_e	-1.16	-1.46	-0.22	1.93	2.16	-1.33
$[t]$	[-0.41]	[-0.61]	[-0.10]	[0.87]	[1.00]	[-0.70]
β_g	-4.97	-4.84	-3.91	-3.59	-3.54	-3.42
$[t]$	[-1.86]	[-2.14]	[-1.82]	[-1.70]	[-1.72]	[-1.90]

return

Inferred Risk Premium

- 2nd step estimation: negative risk premium $\lambda_g = -1.64\%$.

Table 18: **Risk Premium**

	Risk Premium	
λ_e	0.54	0.65
$[t]$	[1.26]	[1.55]
λ_g	**-1.64**	-1.11
$[t]$	[-3.91]	[-2.05]
α	-	2.90
$[t]$	-	[0.93]
OLS-R^2	0.43	
GLS-R^2	0.15	
COLS-R^2		0.53
CGLS-R^2		0.15

t-stat in bracket.

Asymmetric Risk Exposure

- Spread portfolio return correlates with systematic risk measured by price-model.
- Example: anomalies of Momentum

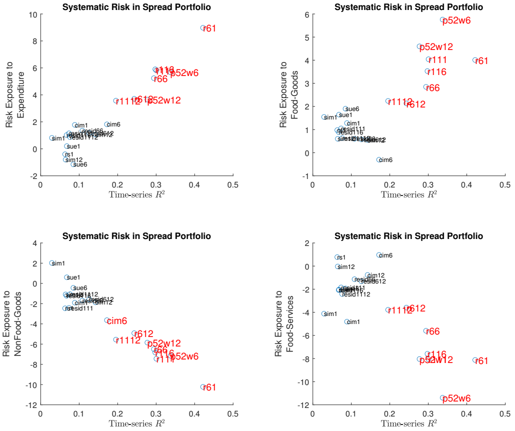

Figure 7: Estimation Outcome for Spread Portfolio

X-axis reports R^2 for regression $R^s_{k,t+1} = a_k + \vec{\beta}_k \cdot \vec{f}_{t+1}$. Y-axis reports $\vec{\beta}_k$.

Infer SDF with Aggregate Outcome

- **Sufficient Statistic**: aggregate consumption outcome describes SDF heterogeneous-consumer economy given the complete financial market.
  - aggregate share $\vec{\omega}$
  - aggregate expenditure $\mathbf{E}$

$\Rightarrow$ Reconstruct the effective representative consumer.

Representative Consumer in Generalized Economy

- Multiple consumers with preference $V(\vec{P}, E)$.
- In equilibrium, we observe the consumer's expenditure distribution $\{E^{(n),*}\}$.
- Equilibrium-implied Negishi Weight (Welfare Weight) is constructed period-by-period as $\alpha^*(n) = \frac{\mathcal{D}_e V(\vec{P}, E^{(1),*})}{\mathcal{D}_e V(\vec{P}, E^{(n),*})}$. with consumer (1) as the unconstrained financial market investor.
- Construct the representative consumer's IDU implied by the equilibrium,

$$
\begin{aligned}
V(\vec{P}, \mathbf{E}; \alpha^*) \equiv \max_{E} \quad & \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} \alpha^*(n) \cdot V(\vec{P}, E(n)) \\
s.t. \quad & \frac{1}{N} \cdot \sum_{n \in \mathcal{N}} E(n) \leq \mathbf{E}.
\end{aligned}
\tag{62}
$$

- Stationary welfare weights $\alpha^* \Rightarrow$ Time-invariant representative consumer
- Change of individual consumer's marginal utility is identical with representative consumer.
- Decomposition of SDF uses $V(\vec{P}, \mathbf{E}; \alpha^*)$. return

Representative Consumer: Analytical Example

- Individual consumer has identical indirect utility function,

$$V(\vec{P}, E(n)) = \frac{1}{1-\gamma} \cdot [\frac{E(n)}{v(\vec{P})}]^{1-\gamma} + \hat{h}(\vec{P}). \quad (63)$$

- Stationary welfare weights $\{\alpha^*(n)\}_n$
- Representative consumer has different preference

$$V(\vec{P}, \mathbf{E}; \alpha^*) = \frac{1}{1-\gamma} \cdot [\frac{\mathbf{E}}{v(\vec{P})}]^{1-\gamma} + \frac{1}{\Phi(\alpha^*)} \cdot \hat{h}(\vec{P}). \quad (64)$$

with multiplier coefficient as

$$\Phi(\alpha^*) = [\sum_{n\in\mathcal{N}} \alpha^*(n)^{\frac{1}{\gamma}}]^{\gamma} \cdot \sum_{n\in\mathcal{N}} \frac{1}{\alpha^*(n)}.$$

- Price-CCAPM: SDF is derived using $V(\vec{P}, \mathbf{E}; \alpha^*)$ return
- Caveat: we cannot use per-capita expenditure $\mathbf{E}$ and individual consumer's function to calculate the SDF.
- Special case of $\hat{h}(\vec{P}) = 0$: collective preference identical with individual

Guideline

Pricing Kernel in a Four-sector Economy

- Price-CCAPM can be extended for multiple sectors.
  - Detailed prices better capture the risk exposure across equity assets.

- 4 sectors: food goods, non-food goods, food services, non-food services
  - Product-level data: NIPA Table 2.4.4, 2.4.5.
  - Estimates $(b_{gf}, b_{gn}, b_{sf}, b_e)$ in extended pricing kernel,

$$\begin{aligned} \mathrm{d}\tilde{m} \approx & - b_{gf} \cdot \omega_{gf} \cdot \underbrace{(\mathrm{d}p_{gf} - \mathrm{d}p_{sn})}_{\text{Food Goods}} - b_{gn} \cdot \omega_{gn} \cdot \underbrace{(\mathrm{d}p_{gn} - \mathrm{d}p_{sn})}_{\text{Non-Food Goods}} \\ & - b_{sf} \cdot \omega_{sf} \cdot \underbrace{(\mathrm{d}p_{sf} - \mathrm{d}p_{sn})}_{\text{Food Services}} - b_e \cdot (\mathrm{d}e - \mathrm{d}p_{sn}). \end{aligned} \tag{65}$$

with non-food services as the numeraire.

Estimation in a Four-sector Economy

Table 19: **Detailed Consumption Sectors**

		Risk Price
Expenditure	b_e	14.70
	$[t]$	[1.74]
Prices:		
Food Goods	b_{gf}	-78.10
	$[t]$	[-2.60]
Non-Food Goods	b_{gn}	-88.46
	$[t]$	[-2.44]
Food Services	b_{sf}	302.37
	$[t]$	[2.02]
	MAE(%)	0.18
	RMSE(%)	0.21
	J-pval	88.08

Plot

- Estimated risk-aversion is 14.70
  - Prices ⇒ variation in SDF
- Goods: similar risk price.
- Food goods and services
  - Grocery is necessity.
  - Dining service is luxury.
- Fitness of estimation is improved.

Return to Summary

Explanation of Zoo of Anomalies

- Post 1960s: zoo of cross-section anomalies
- Estimation using 114 groups of anomaly portfolios during 1968-2019
- Price-CCAPM provides explanation for most of groups

Table 20: **Average Fitness of Asset Pricing Models**

	Traded Factor		Quantity		Prices	
	CAPM	Q-5	C-ND	C-D	**P-ND**	P-D
(Average) MAE(%)	2.20	0.24	0.73	0.67	**0.22**	0.21
(Average) RMSE(%)	2.74	0.30	0.92	0.86	0.27	0.26

Return to Summary

What determines Asymmetric Risk Price?

- **Asymmetric** risk price $\Rightarrow$ Price-CCAPM works better than CCAPM
- What explains (observed) asymmetric risk price?
- Consumer preference: share elasticity
- Welfare-redistribution and time-varying consumer Skip to Aggregation
- Classical asset pricing theories
  - Limited stock market participation
  - Epstein-Zin preference and long-run-risk

Infer SDF with Aggregate Outcome

- Generalization: observed representative consumer is time-varying, when financial market is incomplete due to borrowing constraints or transaction restriction.
- **Fundamental Shocks**:
  - $\rightarrow$ the fluctuation of consumption price is observed,
  - $\rightarrow$ the welfare redistribution across consumers simultaneously occurs.
- Time-varying representative consumer $\Rightarrow$ excessive risk price in consumption prices.

Time-varying Representative Consumer

- **Intuition**: decomposing the variation from $(\vec{P}, \mathbf{E})$ and the welfare weights α^*.
  - High fitness in estimation suggests high correlation between prices $\vec{P}$ and welfare weights α^*.

Corollary (Time-varying Representative Consumer's SDF)

Given the effective Negishi-weight distribution $\{\alpha(n)\}_n$ along the equilibrium path, the change in real marginal utility of expenditure for the representative consumer approximately equals

$$\mathrm{d}\tilde{m} = \underbrace{-\sum_{j\in\mathcal{J}} b_j(\alpha)\cdot\omega_j\cdot(\mathrm{d}p_j - \mathrm{d}p_J)}_{\text{Direct Channel}} - b_e(\alpha)\cdot(\mathrm{d}\mathbf{e} - \mathrm{d}p_J) + \underbrace{\frac{1}{N}\cdot\sum_n s(n)\cdot\mathrm{d}\log[\alpha(n)]}_{\text{Indirect Channel}} + o(\hat{h}). \tag{66}$$

where $\mathrm{d}\alpha$ is the directional derivative of welfare weight, $\vec{\omega}$ is the aggregate expenditure share, $\mathbf{e}$ is the (log) aggregate total consumption expenditure, and the vector $b(\alpha)$ is in similar construction with stationary representative consumer. The expenditure-ratio $s(n)$ is the ratio of consumer (n)'s -expenditure and aggregate-expenditure.

Explanation from Classical Asset Pricing Theories

- Limited stock market participation
  - Fitness improvement: high prices also increases stockholder's marginal utility
  - Point estimates (NIPA): b_e is over-estimated, b_g is **under-estimated**.

  $\Rightarrow$ Empirical challenge in observing the unconstrained consumer.

- Path-dependent preference and long-run-risk
  - Point estimates: high price of goods predicts low quantities growth in the long-run $\Rightarrow$ large $|b_g|$.

  $\Rightarrow$ No direct empirical evidence.