Econometrica Supplementary Material

SUPPLEMENT TO "EFFICIENCY WITH ENDOGENOUS POPULATION GROWTH": TECHNICAL APPENDIX

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In this appendix we provide additional details on some of the ideas developed in our paper. We first give the formal proof that $\mathcal{A}$-efficiency is generically nonempty. The second section extends the notions of efficient fertility developed in the paper for the discrete case to environments in which fertility is a continuous choice variable. It also provides a proof of the first welfare theorem in the continuous choice case. The third section gives an explicit example of an economy with negative externalities. We show that a negative externality can lead to too many people in equilibrium in the $\mathcal{A}$-sense and also show how to decentralize $\mathcal{A}$-efficient allocations through Pigouvian tax systems.

S1. GENERIC NONEMPTINESS OF $\mathbb{A}$

HERE WE PROVE Result 3(b) formally. Suppose that there are multiple solutions to the maximization problem

$$\max_{\{f,x\}} \sum_{i\in\mathcal{P}_0} \alpha_i u_i(f,x) \quad \text{s.t.} \quad \{f,x\}\in A$$

for some choice of α_i with $\alpha_i > 0$ for all $i\in\mathcal{P}_0$. Define $U(f,x;\alpha) = \sum_{i\in\mathcal{P}_0}\alpha_i \times u_i(f,x)$ and choose, arbitrarily, one solution (f^*,x^*). For $i\in\mathcal{P}_0$, define

$$v_i(f^i,x^i) = u_i(f^i,x^i) - \varepsilon_i \frac{d((f^i,x^i),(f^{i*},x^{i*}))}{1+d((f^i,x^i),(f^{i*},x^{i*}))},$$

for any $\varepsilon_i > 0$, where $d(\cdot,\cdot)$ is the appropriate metric. Then $|v_i(f^i,x^i) - u_i(f^i,x^i)| < \varepsilon_i$ for all i and all (f^i,x^i), and $v_i(f^{i*},x^{i*}) = u_i(f^{i*},x^{i*})$.

Define

$$V(f,x) = \sum_{i\in\mathcal{P}_0}\alpha_i v_i(f,x)$$

and note that

$$|V(f,x) - U(f,x;\alpha)| \le \sum_{i\in\mathcal{P}_0}\varepsilon_i.$$

Finally note that $V(f^*,x^*) = U(f^*,x^*;\alpha)$ and that $V(f,x) < U(f,x;\alpha)$ for all $(f,x)\ne(f^*,x^*)$. Thus, it follows that the problem

$$\max_{\{f,x\}} V(f,x) \quad \text{s.t.} \quad \{f,x\}\in A$$

has a unique solution, (f^*, x^*), and because of this, it follows that (f^*, x^*) is $\mathcal{A}$-efficient for the economy with period 0 utility functions given by v_i instead of u_i and nothing else changed.

S2. NONINTEGER FERTILITY

In this section of the appendix, we present a version of our environment that can be used to show that the equilibrium allocation in the Barro and Becker (BB) model is efficient. There are two reasons why the BB model does not quite fit into the framework of Sections 2 and 3 in our paper. First, in the BB model, there is no integer constraint, that is, the number of children can be anything in $\mathbb{R}_+$, while our framework constrains children to be natural numbers. Second, the BB model imposes symmetry, that is, only allocations where all siblings do the same thing are considered feasible, while we do not impose symmetry.

S2.1. *Notation and Definitions*

Assume a finite number of dynasties, each associated with one dynastic head $i \in \mathcal{P}_0 = \{1, \dots, N\}$. The maximal number of children per person is $\bar{f}$. Let $\mathcal{F} = [0, \bar{f}]$. Then we can define the set of potential people recursively as $\mathcal{P}_1 = \mathcal{P}_0 \times \mathcal{F}$ and $\mathcal{P}_t = \mathcal{P}_{t-1} \times \mathcal{F}$. As before, let $\mathcal{P} = \bigcup_t \mathcal{P}_t$ be the set of all potential people in this economy. A person $i \in \mathcal{P}_t$ can be written as $i^t = (i^{t-1}, i_t)$, where i^{t-1} is i^t's parent and i_t specifies i^t's position in the sibling order. The measure of children actually born to person i is $f(i) \in \mathbb{R}$. Then $\bar{f} - f(i)$ is the measure of potential children of person i who are not born. To simplify what follows, we will assume that the children that are born have indexes $[0, f(i)]$. Then $i_t \leq f(i^{t-1})$ means that i^t is born and $i_t > f(i^{t-1})$ implies that i^t is not born.

We assume that there are k goods available in each period. There is one representative firm, which behaves competitively. The technology is characterized by a production set $Y \subset \mathbb{R}^{k\infty}$. In other words, an element of the production set is an infinite sequence of k-tuples that describes feasible input/output combinations. Note that goods are interpreted in a broad sense here, they can include labor, leisure, and capital. An element of the production set will be denoted by $y \in Y$. We can write $y = \{y_t\}_{t=0}^{\infty}$, where $y_t = (y_t^1, \dots, y_t^k)$ is the projection of the production plan onto time t. Every person $j \in \mathcal{P}$ receives an endowment $e(j) \in \mathbb{R}_+^k$ if born.

An allocation is a fertility, a consumption, and a production plan, (f, x, y), where $(f, x) = \{f(i), x(i)\}_{i \in \mathcal{P}}$. Define a consumption set $Z \subset \mathcal{F} \times \mathbb{R}^k$. We require that $(f(i), x(i)) \in Z$ if i is born.

As in the paper, for each $i \in \mathcal{P}_0$, we define D_i to be the set of potential descendants of i, including i. Furthermore, we define $I(f)$ to be the set of people that are alive in an allocation with the fertility plan f. Then we define $I(f_i) = I(f) \cap D_i$ to be the set of people of dynasty i alive under the dynastic fertility plan f_i. Finally, let $I_t(f_i) = I(f_i) \cap \mathcal{P}_t$ denote the set of descendants of i that are alive at date t under allocation (f, x, y).

DEFINITION S1: An allocation is feasible if:
1. $(f(i), x(i)) \in Z$ for almost every $i \in I(f)$[1];
2. $\sum_{i\in\mathcal{P}_0}(\int_{I_t(f_i)} x_j\,dj + \int_{I_t(f_i)} c(f_j)\,dj) \leq \sum_{i\in\mathcal{P}_0}\int_{I_t(f_i)} e_j\,dj + y_t\ \forall t \geq 1$;
3. $y \in Y$.

Note that this formulation allows for different children of the same parent to be treated differently. This is more general than the Barro–Becker formulation where all children of the same parent receive the same allocation by assumption.

For simplicity, assume the firm is owned only by members of the initial generation. Let ψ_i be the fraction of the firm that belongs to $i \in \mathcal{P}_0$. Then, profits earned by dynasty i can be written as $\Pi_i = \psi_i \sum_t p_t y_t$.

DEFINITION S2: An allocation $(f, x, y) = (\{(f_j, x_j)\}_{j\in\mathcal{P}}, y)$ is $\mathcal{P}$-efficient if it is feasible and there is no other feasible allocation $(\hat{f}, \hat{x}, \hat{y})$ such that:
1. $u_j(\hat{f}, \hat{x}) \geq u_j(f, x)$ for almost every $j \in P$;
2. there exists $S \subset \mathcal{P}$ with positive measure such that $u_j(\hat{f}, \hat{x}) > u_j(f, x)$ for all $j \in S$.

DEFINITION S3: Given (p, y), a dynastic allocation for dynasty i $(f_i, x_i) = \{f(j), x(j)\}_{j\in D_i}$ is said to be dynastically $\mathcal{P}$-maximizing if $(f(j), x(j)) \in Z$ for almost every $j \in I(f_i)$ and $\sum_t p_t \sum_{j\in I_t(f_i)}[x(j)+c(f(j))] \leq \sum_t p_t \sum_{j\in I_t(f_i)} e(j) + \psi_i \sum_t p_t y_t$ and if $\nexists(\hat{f}_i, \hat{x}_i)$ such that:
1. $(f(j), x(j)) \in Z$ for almost every $j \in I(\hat{f}_i)$;
2. $u_j(\hat{f}_i, \hat{x}_i) \geq u_j(f_i, x_i)$ for almost every $j \in D_i$;
3. there exists $S \subset D_i$ with positive measure such that $u_j(\hat{f}_i, \hat{x}_i) > u_j(f_i, x_i)$ for all $j \in S$;
4. $\sum_t p_t \int_{I_t(\hat{f}_i)}(\hat{x}(j) + c(\hat{f}(j)))\,dj \leq \Pi_i + \sum_t p_t \int_{I_t(\hat{f}_i)} e(j)\,dj$.

Next we define the analogue of a competitive equilibrium among the dynasties in the partition.

DEFINITION S4: The allocation (f^*, x^*, y^*) and prices p^* is a dynastic $\mathcal{P}$-equilibrium if:

[1]Note that because fertility is a continuous variable, it is possible to change assignments of endowments, consumption, fertility, and so forth for measure zero sets of children (for periods beyond period 0) without affecting aggregate resources and/or utilities. Thus, all statements are "almost everywhere." Formally, we use the counting measure on period 0 individuals and Lebesgue measure for all other periods.

1. for all dynasties i, given (p^*, y^*), (f_i^*, x_i^*) is dynastically $\mathcal{P}$-maximizing;
2. (f^*, x^*, y^*) is feasible;
3. Given p^*, y^* maximizes profits; that is, $p^* y \leq p^* y^*$ $\forall y \in Y$.

The definitions of $\mathcal{A}$-efficient allocations, dynastically $\mathcal{A}$-maximizing decisions, and dynastic $\mathcal{A}$-equilibria are similar, and can be obtained by replacing $I(f)$ by $I(f) \cap I(\hat{f})$.

S2.2. *First Welfare Theorem and Proof*

ASSUMPTION S1—No Negative Externalities: *We assume that u_j is monotone increasing in $x_{j'}$, that is, each agent is weakly better off when consumption is increased for a set of agents of positive measure. Thus, there are no negative external effects in consumption.*

ASSUMPTION S2—Positive Externalities only Within a Dynasty: *For all $i \in \mathcal{P}_0$, we assume that if (f, x, y) and $(\hat{f}, \hat{x}, \hat{y})$ are two allocations such that $(f(j), x(j)) = (\hat{f}(j), \hat{x}(j))$ for almost every $j \in D_i$, then $u_j(f, x) = u_j(\hat{f}, \hat{x})$ for almost every $j \in D_i$.*

LEMMA S1: *Assume that u_i is strictly increasing in own consumption for all $i \in \mathcal{P}_0$. Let (f_i^*, x_i^*) be dynastically $\mathcal{P}$-maximizing for dynasty D_i, given prices p and production y. Then $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for all $j \in D_i$ implies that $\sum_t p_t \int_{I_t(f_i)} (x(j) + c(f(j)))\, dj \geq \Pi_i^* + \sum_t p_t \int_{I_t(f_i)} e_j\, dj$.*

PROOF: Lemma S1 will be proved by contradiction. Suppose not. Then there exists a (f_i, x_i) such that $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for almost all $j \in D_i$ and $\sum_t p_t \int_{I_t(f_i)} (x(j) + c(f(j)))\, dj < \Pi_i^* + \sum_t p_t \int_{I_t(f_i)} e_j\, dj$. Then construct a new dynastic allocation $(\tilde{f}_i, \tilde{x}_i)$ as follows: $(\tilde{f}_i, \tilde{x}_i) = (f_i, x_i + \epsilon)$ for $i \in \mathcal{P}_0 \cap D_i$ and $(\tilde{f}_i, \tilde{x}_i) = (f_i, x_i)$ for all other $j \in D_i$. Then $\exists \epsilon > 0$ such that the dynastic allocation $(\tilde{f}_i, \tilde{x}_i)$ does not violate the dynastic budget constraint. Moreover, by Assumption S1 $(\tilde{f}_i, \tilde{x}_i)$ is weakly preferred over (f_i, x_i) by all j in the dynasty and, hence, also over (f_i^*, x_i^*). Finally, by strict monotonicity, $u_i(\tilde{f}_i, \tilde{x}_i) > u_i(f_i^*, x_i^*)$, but this contradicts the assumption that (f_i^*, x_i^*) was dynastically $\mathcal{P}$-maximizing. *Q.E.D.*

PROPOSITION S1: *Suppose $u_i(x_i, f_i)$ is strictly increasing in own consumption for all $i \in \mathcal{P}_0$. If (p^*, f^*, x^*, y^*) is a $\mathcal{P}$-dynastic Walrasian equilibrium, then $\sum_t [p_t y_t^* + \int_{\mathcal{P}_t \cap I(f)} p_t e(j)\, dj] < \infty$ and (f^*, x^*, y^*) is $\mathcal{P}$-efficient.*

PROOF: First, note that because $u_i(x_i, f_i)$ is strictly monotone in own consumption, for all $i \in \mathcal{P}_0$, for the given allocation to be a dynastic $\mathcal{P}$

equilibrium, (f_i, x_i) must be dynastically $\mathcal{P}$-maximizing and, hence, $\Pi_i + \sum_t p_t \int_{I_t(f_i)} e(j)\,dj < \infty$ for all i. Summing over all dynasties and substituting in the definition of profits Π_i gives $\sum_t [p_t y_t^* + (\int_{\mathcal{P}_t \cap I(f)} e(j))\,dj] < \infty$, which proves the first part.

Suppose now that (f^*, x^*, y^*, p^*) is a dynastic $\mathcal{P}$-equilibrium, and by way of contradiction, assume that it is not $\mathcal{P}$-efficient. Then there exists an alternative feasible allocation (f, x, y) that is $\mathcal{P}$-superior to (f^*, x^*, y^*). That is, $u_j(f_i, x_i) \geq u_j(f_i^*, x_i^*)$ for almost all $j \in P$ and there exists $S \subset P$ such that for all $j \in S$, $u_j(f_i, x_i) > u_j(f_i^*, x_i^*)$. For some i, it must be that $S \cap D_i$ has strictly positive measure. Then, for this dynasty i, because (f_i^*, x_i^*) is dynastically $\mathcal{P}$-maximizing and because there are no external effects across dynasties (Assumption S2), it must be that (f_i, x_i) was not affordable, that is,

$$\sum_t p_t^* \int_{I_t(f_i)} \big(x(j) + c(f(j))\big)\,dj > \Pi_i^* + \sum_t p_t^* \int_{I_t(f_i)} e(j)\,dj.$$

Moreover, by Lemma S1, we know that for all other dynasties,

$$\sum_t p_t^* \int_{I_t(f_i)} \big(x(j) + c(f(j))\big)\,dj \geq \Pi_i^* + \sum_t p_t^* \int_{I_t(f_i)} e(j)\,dj$$

must hold. Summing up over all dynasties, we get

$$\text{(S1)} \qquad \sum_t p_t^* \int_{I_t(f)} \big(x(j) + c(f(j))\big)\,dj > \sum_t p_t^* \left[y_t^* + \int_{I_t(f)} e(j)\,dj \right].$$

Because the right-hand side is finite, the strict inequality is preserved. Profit maximization implies that $p^* y^* \geq p^* y$ for all other production plans $y \in Y$. Using this, we can rewrite (S1) as

$$\text{(S2)} \qquad \sum_t p_t^* \int_{I_t(f)} \big(x(j) + c(f(j))\big)\,dj > \sum_t p_t^* \left[y_t + \int_{I_t(f)} e(j)\,dj \right].$$

Finally, feasibility of (f, x, y) implies that

$$\int_{I_t(f)} \big(x(j) + c(f(j))\big)\,dj \leq y_t + \int_{I_t(f)} e(j)\,dj \quad \text{for all } t.$$

Multiplying the above by p_t^* and summing up over all t gives

$$\sum_t p_t^* \int_{I_t(f)} \big(x(j) + c(f(j))\big)\,dj \leq \sum_t p_t^* \left[y_t + \int_{I_t(f)} e(j)\,dj \right].$$

However, this contradicts (S2). This completes the proof. *Q.E.D.*

The proof that $\mathcal{A}$-dynastic Walrasian equilibrium allocations are $\mathcal{A}$-efficient is similar and is omitted.

S3. POLLUTION

In this section, we outline the details of the example discussed in Section 5.2 of the main paper. The example features a negative external effect across agents. We characterize the set of symmetric $\mathcal{A}$-efficient and symmetric $\mathcal{P}$-efficient allocations, and show how to implement them using Pigouvian taxes. We find that typically both a pollution tax and a "child" tax are necessary to implement efficient allocations.

Assume that there is a continuum of agents in period 1, indexed by $i \in [0, 1]$. Each period 1 agent has a unit endowment of the consumption good, $e_1(i) = 1$ for all i. This endowment is divided between own consumption, $c_1(i)$, and child rearing. The cost of rearing n children is θn. The jth child of the ith period 1 agent is denoted by (i, j). Agents in period 1 are altruistic toward their own children as in Barro and Becker (1989) and have utilities given by $u(c(i)) + \beta n_i^\alpha \int_0^{n_i} V(i, j)\, dj$, where $V(i, j)$ is the utility received by child (i, j). We assume that u satisfies the Inada condition, $u'(0) = \infty$ and that $\alpha \in (-1, 0)$ which guarantees that fertility is concave in n.

Each agent in period 2 can sell labor that can be transformed into a consumption good according to the linear technology $c = l$. The utility of the agent is $V(i, j) = v(c_2(i, j), C) - l(i, j)$, where $c_2(i, j)$ is consumption of individual (i, j), $C = \int_0^1 \int_0^{n_i} c_2(i, j)\, dj\, di$ is aggregate production in the economy, and $l(i, j)$ is the amount of time that (i, j) works. We abstract from constraints on leisure and assume only that there is disutility from work for simplicity.

Finally, feasibility requires

$$\int_0^1 \int_0^{n_i} c_2(i, j)\, dj\, di = \int_0^1 \int_0^{n_i} l(i, j)\, dj\, di,$$

and

$$\int_0^1 \theta n_i\, di + \int_0^1 c_1(i)\, di = \int_0^1 e_1(i)\, di = 1.$$

We assume that the economy lasts for only two periods. Note that this is a simplified version of a Barro–Becker economy, where if $v_2 = 0$, the equilibrium is efficient as in Theorem 2 in the paper. The effect of pollution is captured by assuming that $v_2 < 0$. For simplicity, we assume that there is no pollution in the first period.

Define $\bar{n} = \int_0^1 n_i\, di$ as the number of people born in period 1 and alive in period 2.

S3.1. *Characterization of Symmetric, Efficient Allocations*

CLAIM 1: *If $v(c, C)$ is strictly concave in c for each C, then in every $\mathcal{P}$-efficient and in every $\mathcal{A}$-efficient allocation, the second period consumptions are equal across agents, $c(i, j) = c$ for all (i, j), where c must satisfy*

$$v_1(c, \bar{n}c) + \bar{n}v_2(c, \bar{n}c) = 1. \tag{S3}$$

PROOF: First, from Proposition 1 in the paper, all $\mathcal{P}$-efficient and $\mathcal{A}$-efficient allocations are Pareto optimal given the set of people. As is standard with quasilinear preferences, without limits on labor supply, a necessary condition for Pareto optimality is that the allocation solves a planner's problem with equal weights (or the agent with the lowest weight will work an infinite amount of hours). The allocation of labor effort across the agents is ambiguous, each giving the planner the same utility (but corresponding to different optima among the agents). Because of this, it follows that, given n_i, second period consumption must solve

$$\max \int_0^1 \int_0^{n_i} \left[v\left(c_2(i, j), \int_0^1 \int_0^{n_i} c_2(i, j) \right) - l(i, j) \right] dj\, di$$
$$\text{s.t.} \quad \int_0^1 \int_0^{n_i} c_2(i, j)\, dj\, di \leq \int_0^1 \int_0^{n_i} l(i, j)\, dj\, di.$$

Because $v(c, C)$ is strictly concave in c for each given value of C, it follows that the optimal choice of $c(i, j)$ is a constant, $c(i, j) = c$ for some c. (If not, the planner's utility can be increased by giving each agent the average c because this does not change C or the aggregate labor supply required.) Thus, the problem can be rewritten as

$$\max_c \bar{n}v(c, \bar{n}c) - \bar{n}c.$$

The result follows directly from this. *Q.E.D.*

For symmetric, efficient allocations, it follows that $l(i, j) = c_2$ as well and so, in this case, this equation can also be written as

$$v_1(l, nl) + nv_2(l, nl) = 1. \tag{S4}$$

ASSUMPTION S3: *Assume that there is a unique solution to* (S4) *for every n.*

Under Assumption S3, equation (S4) implicitly defines a relationship between l and n, call this $l(n)$, that is, $l(n)$ is the efficient labor supply, given population size n, in any efficient, symmetric allocation. Then the utility of a person alive in period 2 at the symmetric efficient labor choice, given n, is given by

(S5) $$V(n) \equiv v(l(n), nl(n)) - l(n).$$

LEMMA S2: $\frac{\partial V}{\partial n} < 0$.

PROOF: We have $\frac{\partial V}{\partial n} = v_1 l'(n) + v_2[l(n) + nl'(n)] - l'(n) = l'(n)[v_1 + nv_2 - 1] + l(n)v_2 = l(n)v_2$, where the third equality uses (S4). However, this last expression is negative because $v_2 < 0$ by assumption and $l(n) > 0$ from feasibility. *Q.E.D.*

Next, we want to discern what restrictions efficiency places on n in symmetric, efficient allocations. The conclusions from the following discussion are summarized in Proposition S2. Define $U(n)$ to be the utility of the typical parent in a symmetric efficient allocation in which (S3) is satisfied in the second period. That is,

$$\begin{aligned} U(n) &= u(e_1 - \theta n) + \beta n^{\alpha+1} V(n) \\ &= u(e_1 - \theta n) + \beta n^{\alpha} \int_0^n (v(c, nc) - c)\, dj. \end{aligned}$$

ASSUMPTION S4: *Utility $U(n)$ is strictly concave in n and $U'(0) > 0$.*

Note that the feasible choices for n in symmetric allocations are bounded above by $n = e_1/\theta$ and here $U'(e_1/\theta) < 0$. Define $n_{\mathcal{A}}$ to be the unique value of n that maximizes $U(n)$. Since $U'(0) > 0$, it follows that $n_{\mathcal{A}} > 0$ and so we can use first order conditions to characterize $n_{\mathcal{A}}$. Then $n_{\mathcal{A}}$ satisfies

(S6) $$-\theta u'(e_1 - \theta n_{\mathcal{A}}) + \beta(\alpha + 1) n_{\mathcal{A}}^{\alpha} V(n_{\mathcal{A}}) + \beta n_{\mathcal{A}}^{\alpha+1} V'(n_{\mathcal{A}}) = 0.$$

It follows from Lemma S2 that $n_{\mathcal{A}}^{\alpha+1} V'(n_{\mathcal{A}}) < 0$. Finally, it must also be true that $V(n_{\mathcal{A}}) > 0$ because, if this does not hold, U is maximized at $n_{\mathcal{A}} = 0$. To see that the allocation characterized by (S4) and (S6) is $\mathcal{A}$-efficient, note that any potentially superior allocation must involve the same $n_{\mathcal{A}}$, c_1, and c_2, because otherwise the period 1 agents would be strictly worse off. That leaves only rearrangements of labor among period 2 agents, which would immediately make some period 2 agents strictly worse off.

We now show that any allocation with $n < n_{\mathcal{A}}$ can be dominated in the $\mathcal{A}$ and the $\mathcal{P}$ sense. Suppose $n < n_{\mathcal{A}}$ and consider the following change in plan:

1. Increase n to $n + \Delta n$.
2. Have $c(i, j) = \hat{c}$ for all i and all $0 \leq j \leq n + \Delta n$, where $\hat{c}$ is the solution to $v_1(\hat{c}, (n + \Delta n)\hat{c}) + (n + \Delta n) v_2(\hat{c}, (n + \Delta n)\hat{c}) = 1$, that is, $\hat{c}$ is the new optimal consumption level for all agents in the second period given that n has been increased to $n + \Delta n$.

3. Let $l(i,j) = \hat{l}$ for all i and for all $0 \le j \le n$, where $\hat{l}$ is defined by $v(\hat{c}, (n+\Delta n)\hat{c}) - \hat{l} = v(c, nc) - c$. That is, $\hat{l}$ is just enough extra leisure so that all children in period 2 are indifferent to this change.
4. Let $l(i,j) = \tilde{l}$ for all i and for all $n \le j \le n + \Delta n$, where $\tilde{l}$ is defined by feasibility $n\hat{l} + \Delta n\tilde{l} = (n+\Delta n)\hat{c}$.

Note that this plan has children (i,j) with $n \le j \le n+\Delta n$ working more than those children (i,j) with $0 \le j \le n$ (to compensate them for the loss in utility they experience from the increased population).

Finally, it remains to be shown that the new allocation is strictly better for the typical parent. The utility received by the typical parent from the new allocation is given by

$$\begin{aligned} W(\Delta n) &= u(e_1 - \theta n - \theta\Delta n) \\ &\quad + \beta(n+\Delta n)^{\alpha} \int_0^{n+\Delta n} \big[v(\hat{c}, (n+\Delta n)\hat{c}) - l(i,j)\big]\, dj \\ &= u(e_1 - \theta n - \theta\Delta n) + \beta(n+\Delta n)^{\alpha}\bigg[(n+\Delta n)v(\hat{c}, (n+\Delta n)\hat{c}) \\ &\quad - \int_0^{n+\Delta n} [l(i,j)]\, dj\bigg] \\ &= u(e_1 - \theta n - \theta\Delta n) \\ &\quad + \beta(n+\Delta n)^{\alpha}\big[(n+\Delta n)v(\hat{c}, (n+\Delta n)\hat{c}) - (n+\Delta n)\hat{c}\big] \\ &= u(e_1 - \theta n - \theta\Delta n) + \beta(n+\Delta n)^{\alpha+1}\big[v(\hat{c}, (n+\Delta n)\hat{c}) - \hat{c}\big] \\ &= u(e_1 - \theta n - \theta\Delta n) + \beta(n+\Delta n)^{\alpha+1}[V(n+\Delta n)]. \end{aligned}$$

Differentiating this with respect to Δn, we obtain

$$\begin{aligned} W'(\Delta n) &= -\theta u'(e_1 - (n+\Delta n)\theta) + \beta(\alpha+1)(n+\Delta n)^{\alpha}[V(n+\Delta n)] \\ &\quad + \beta(n+\Delta n)^{\alpha+1}[V'(n+\Delta n)]. \end{aligned}$$

Evaluating this at $\Delta n = 0$, we obtain

$$W'(0) = -\theta u'(e_1 - n\theta) + \beta(\alpha+1)n^{\alpha}V(n) + \beta n^{\alpha+1}V'(n).$$

This is positive for all $n < n_{\mathcal{A}}$. Thus, by construction, this change improves the welfare of the parent and leaves indifferent all children (i,j) with $0 \le j \le n$.

The change in utility of the children (i,j) with $n \le j \le n + \Delta n$ is given by $v(\hat{c}, n\hat{c}) - \tilde{l}$. By continuity, this is approximately equal to $V(n)$ when Δn is sufficiently small. Thus, this gives a Pareto improvement anytime $n < n_{\mathcal{A}}$ and $V(n) > 0$.

Moreover, it follows that if $n > n_{\mathcal{A}}$, the symmetric allocation that satisfies (S3) is not $\mathcal{A}$-efficient. To see this, simply note that by reducing n to $n_{\mathcal{A}}$ and using the corresponding allocation that satisfies (S3) improves the welfare of the surviving period 2 agents (because $V'(n) < 0$) and improves the welfare of the period 1 agents as well (because U is maximized at $n = n_{\mathcal{A}}$).

The following proposition summarizes this discussion.

PROPOSITION S2: *In every symmetric $\mathcal{P}$-efficient allocation we have*

$$\begin{aligned} &n \geq n_{\mathcal{A}}, \\ &c_2(i,j) = c_2 \quad \textit{satisfies} \quad v_1(c_2, nc_2) + nv_2(c_2, nc_2) = 1, \\ &l(i,j) = l = c_2, \\ &c_1(i) = c_1 = 1 - \theta n, \\ &V(n) \geq 0. \end{aligned}$$

Also, there is a unique symmetric $\mathcal{A}$-efficient allocation with $n = n_{\mathcal{A}}$, as defined by (S6).

S3.2. *Implementation of Efficient Allocations*

Next, we characterize equilibrium choices of fertility and consumption by private agents acting in a decentralized way in markets. Because there is no way to transfer goods physically across the periods and because all dynasties are identical, we can, without loss of generality, assume that bequests are not allowed.

In the market equilibrium allocation with a tax of τ_c on the production of c_2 we have that each period 2 child takes as given C, τ_c, and T_2, the transfer received from the government. Thus, a period 2 agent solves

$$\max_{\{c_2, l\}} v(c_2, C) - l \quad \text{s.t.} \quad (1+\tau_c)c_2 = l + T_2.$$

Let $\tilde{c}_2(\tau_c, C, T_2)$ denote the individual's solution to this problem. Note that it does not depend on the period 1 choice of n due to the assumed linearity of the period 2 production function. In equilibrium, $\tilde{C} = \tilde{n}\tilde{c}_2$ and so the first order condition (FOC) is

$$\text{(S7)} \qquad v_1(\tilde{c}_2, \tilde{n}\tilde{c}_2) = (1+\tau_c).$$

If $\tau_c = 0$, we have the standard, fixed population result that each agent sets his own marginal rate of substitution equal to the price ratio, ignoring the external effect on the other agents. This induces too much output in the equilibrium and, hence, it follows that the undistorted market equilibrium quantities (i.e., with $\tau_c = 0$) are neither $\mathcal{P}$- nor $\mathcal{A}$-efficient.

In period 1, the parent maximizes utility knowing what will happen in period 2. However, each period 1 parent views that their own choice of n will have no effect on $\tilde{C}$. Thus, a parent solves

$$\max_{\{c_1,n\}} u(c_1) + \beta n^{\alpha+1}[v(\tilde{c}_2, C) - \tilde{c}_2]$$
$$\text{s.t.} \quad c_1 + (\theta + \tau_n)n = 1 + T_1.$$

The FOC imply that

$$(\theta + \tau_n)u'(\tilde{c}_1)/\beta = (\alpha + 1)\tilde{n}^{\alpha}[v(\tilde{c}_2, \tilde{n}\tilde{c}_2) - \tilde{c}_2]. \tag{S8}$$

Comparing these conditions with those derived above for symmetric, efficient allocations, we see that $\tau_c = -\tilde{n}v_2(\tilde{c}_2, \tilde{n}\tilde{c}_2)$ is required (to see this, compare (S7) with (S4)). This is standard: each agent must be induced to set his consumption so that his marginal rate of substitution equals the marginal social cost, not the marginal private cost.

It also follows that privately chosen fertility, n, will generally not be $\mathcal{A}$-efficient unless an additional fertility tax is set correctly. For any given n, $n \geq n_{\mathcal{A}}$, equation (S8) defines the unique level of τ_n necessary to implement n as an equilibrium choice (in conjunction with $\tau_c = -nv_2(c_2, nc_2)$). For example, to implement $n_{\mathcal{A}}$ as an equilibrium outcome, it is necessary that $\tau_c = -n_{\mathcal{A}}v_2(l^*(n_{\mathcal{A}}), n_{\mathcal{A}}l^*(n_{\mathcal{A}}))$ and $\tau_n = -\beta n_{\mathcal{A}}^{\alpha+1}\tilde{c}_2 v_2(\tilde{c}_2, \hat{n}\tilde{c}_2)/u'(\tilde{c}_1)$ per child (this follows from (S6) and (S8)). Note that $\tau_n > 0$. This (τ_c, τ_n) decentralizes the unique symmetric $\mathcal{A}$-efficient allocation and, hence, it follows that the resulting allocation is both $\mathcal{P}$- and $\mathcal{A}$-efficient. Whether a given symmetric, $\mathcal{P}$-efficient allocation will require nontrivial taxation also follows from these results. Examination of (S8) shows that implementation with $\tau_n = 0$ is equivalent to

$$\theta u'(e_1 - \theta n)/\beta = (\alpha + 1)n^{\alpha}\big[v(l^*(n), nl^*(n)) - l^*(n)\big].$$

Typically, there will be only one such n. Note, however, that for large n, one would typically expect $\tau_n < 0$ to be required for implementation. The following proposition summarizes this discussion:

PROPOSITION S3: *The unique symmetric, $\mathcal{A}$-efficient allocation can be implemented with positive taxes on both the number of children and second period consumption. Most symmetric, $\mathcal{P}$-efficient allocations require nonzero taxes on children (but they could be negative) and all require a positive tax on second period consumption.*

As argued above, with only $\tau_n = 0$, the resulting equilibrium will not be $\mathcal{A}$-efficient. In other words, standard Pigouvian taxes are not sufficient to implement $\mathcal{A}$-efficient allocations. A fertility tax is needed in addition. However, are equilibrium allocations with $\tau_n = 0$ (and τ_c set as described above) $\mathcal{P}$-efficient?

The answer is typically yes. The lack of a fertility tax will lead to overproduction of children from the perspective of period 1 agents, but there is no allocation that is superior in the $\mathcal{P}$-sense, because it would necessarily require less children to be born, which would make those children strictly worse off. Note that this logic would be different if the externality was positive. Then an equilibrium with $\tau_n = 0$ would lead to too low fertility and a $\mathcal{P}$-superior allocation could easily be constructed.

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REFERENCES

BARRO, R. J., AND G. S. BECKER (1989): "Fertility Choice in a Model of Economic Growth," *Econometrica*, 57, 481–501. [6]

JONES, L. E., AND R. E. MANUELLI (1990): "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy*, 98, 1008–1038.