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SUPPLEMENT TO “WEAKLY BELIEF-FREE EQUILIBRIA IN REPEATED GAMES WITH PRIVATE MONITORING” (*Econometrica*, Vol. 79, No. 3, May 2011, 877–892)

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IN THIS SUPPLEMENT, I formally show that the weakly belief-free equilibria identified in Section 4 lie above the Pareto frontier of the belief-free equilibrium payoff set. The proof is based on the characterization theorem by Ely, Horner, and Olszewski (2005) (referred to as EHO hereafter).

To explain their characterization of the belief-free equilibrium payoffs, I first introduce the notion of *regime* $\mathcal{A}$ and an associated value $M_i^{\mathcal{A}}$. Using these concepts, I then find an upper bound for the belief-free equilibrium payoffs. A regime $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a product of nonempty subsets of the stage game action sets $\mathcal{A}_i \subset A_i$, $\mathcal{A}_i \neq \emptyset$, $i = 1, 2$. In each period of a belief-free equilibrium, players typically have multiple best-reply actions and they are played with positive probabilities. A regime corresponds to the set of such actions. For each regime $\mathcal{A}$, define a number

$$M_i^{\mathcal{A}} = \sup v_i$$

such that for some mixed action α_{-i} whose support is $\mathcal{A}_{-i}$ and $x_i : \mathcal{A}_{-i} \times \Omega_{-i} \to \mathbf{R}_+$,

$$v_i \geq g(a_i, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | a_i, a_{-i}) \alpha_{-i}(a_{-i})$$

for all a_i with equality if $a_i \in \mathcal{A}_i$, where $p_{-i}(\omega_{-i}|a_i, a_{-i})$ is the marginal distribution of ω_{-i} given action profile (a_i, a_{-i}). Intuitively, the positive number x_i represents the reduction in player i's future payoffs. Note that a belief-free equilibrium has the property that player i's payoff is solely determined by the opponent's strategy. This is why the reduction in i's future payoffs, x_i, depends on the opponent's action and signal (a_{-i}, ω_{-i}). Note also that the opponent's action a_{-i} is restricted to the component $\mathcal{A}_{-i}$ of the current regime $\mathcal{A} = \mathcal{A}_i \times \mathcal{A}_{-i}$. The above set of inequalities ensures that player i's best reply actions in the current period correspond to set $\mathcal{A}_i$, a component of the regime $\mathcal{A} = \mathcal{A}_i \times \mathcal{A}_{-i}$. Hence, the value $M_i^{\mathcal{A}}$ is closely related to the best belief-free payoff when the current regime is $\mathcal{A}$ (a more precise explanation will be given below).

Now let V^* be the limit set of belief-free equilibrium payoffs when $\delta \to 1$. EHO provided an explicit formula to compute V^*. For our purpose here, I only sketch the relevant part of their characterization to obtain a bound for V^*. In Section 4.1, EHO partitioned all games into three classes: the positive, the negative, and the abnormal cases (for our purpose here, we do not need to know

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their definitions). Their Proposition 6 shows that the abnormal case obtains *only if* one of the players has a dominant action in the stage game that yields the same payoff against all actions of the other player. Clearly, this is not the case in our example with the prisoner's dilemma stage game, so our example is in either the positive or the negative case.[1] If it is in the negative case, EHO's Proposition 5 shows that the only belief-free equilibrium is the repetition of the stage game Nash equilibrium, yielding (0, 0) in our example.

If our example is in the positive case, Proposition 5 in EHO implies that the limit set of belief-free equilibrium payoffs can be calculated as

$$V^* = \bigcup_p \prod_{i=1,2} \Big[\sum_{\mathcal{A}} p(\mathcal{A}) m_i^{\mathcal{A}}, \sum_{\mathcal{A}} p(\mathcal{A}) M_i^{\mathcal{A}} \Big], \tag{S1}$$

where $m_i^{\mathcal{A}}$ is some number (for our purpose here, we do not need to know its definition) and p is a probability distribution over regimes $\mathcal{A}$. The union is taken with respect to all probability distributions p such that the intervals in formula (S1) are well defined (i.e., $\sum_{\mathcal{A}} p(\mathcal{A}) m_i^{\mathcal{A}} \leq \sum_{\mathcal{A}} p(\mathcal{A}) M_i^{\mathcal{A}}$, $i = 1, 2$). The point to note is that V^* is a union of product sets (rectangles), and the efficient point (upper-right corner) of each rectangle is a convex combination of $(M_1^{\mathcal{A}}, M_2^{\mathcal{A}})$.

The characterization (S1) of V^* implies, in the positive case, the belief-free equilibrium payoffs satisfy the bound

$$(v_1, v_2) \in V^* \quad \Longrightarrow \quad v_1 + v_2 \leq \max_{\mathcal{A}} M_1^{\mathcal{A}} + M_2^{\mathcal{A}}, \tag{S2}$$

where maximum is taken over all possible regimes (i.e., for all $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ such that $\mathcal{A}_i \subset A_i$, $\mathcal{A}_i \neq \emptyset$, $i = 1, 2$).

In what follows, I estimate $M_1^{\mathcal{A}} + M_2^{\mathcal{A}}$ for each regime $\mathcal{A}$. In our example, $A_i = \{C, D\}$, so that $\mathcal{A}_i = \{C\}, \{D\}$, or $\{C, D\}$. Before examining each regime, I first derive some general results. Consider a regime $\mathcal{A}$ where $C \in \mathcal{A}_i$. In this case, the incentive constraint in the definition of $M_i^{\mathcal{A}}$ reduces to

$$v_i = g(C, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | C, a_{-i}) \alpha_{-i}(a_{-i}) \tag{S3}$$

$$\geq g(D, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | D, a_{-i}) \alpha_{-i}(a_{-i}). \tag{S4}$$

[1] With some calculation, we can determine which case applies to our example, but this is not necessary to derive our upper bound payoff.

This inequality (S4) can be rearranged as

$$\text{(S5)}\qquad \sum_{a_{-i},\omega_{-i}} x_i(a_{-i},\omega_{-i})p_{-i}(\omega_{-i}|C,a_{-i})\left(\frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}-1\right)\alpha_{-i}(a_{-i}) \geq g(D,\alpha_{-i})-g(C,\alpha_{-i}).$$

Now let

$$L^* = \max_{\omega_{-i},a_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$$

be the maximum likelihood ratio to detect player i's deviation from C to D. The inequality (S5) and $L^*-1>0$ imply[2]

$$\sum_{a_{-i},\omega_{-i}} x_i(a_{-i},\omega_{-i})p_{-i}(\omega_{-i}|C,a_{-i})\alpha_{-i}(a_{-i}) \geq \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1}.$$

Plugging this into the definition (S3) of v_i, we obtain

$$v_i \leq g(C,\alpha_{-i}) - \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1}.$$

This is essentially the formula identified by Abreu, Milgrom, and Pearce (1991). The reason for welfare loss (the second term on the right hand side) is that players are sometimes punished simultaneously in belief-free equilibria. Recall that $M_i^{\mathcal{A}}$ is obtained as the supremum of v_i with respect to x_i and α_{-i} whose support is $\mathcal{A}_{-i}$. (Note that the right hand side of the above inequality, in contrast, does not depend on x_i.) Hence, we have

$$\text{(S6)}\qquad M_i^{\mathcal{A}} \leq \sup g(C,\alpha_{-i}) - \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1},$$

where the supremum is taken *over all* α_{-i} *whose support is* $\mathcal{A}_{-i}$.

Now we calculate the maximum likelihood ratio L^* and determine the right hand side of the inequality (S6). In our example, when $a_{-i}=C$, $\max_{\omega_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$ is equal to (as our example is symmetric, consider $-i=2$ without loss of generality)

$$\frac{p_2(\omega_2=B|D,C)}{p_2(\omega_2=B|C,C)} = \frac{\frac{1}{2}+\frac{1}{8}}{1/3} = \frac{15}{8}.$$

[2]Note that as long as player i's action affects the distribution of the opponent's signal (which is certainly the case in our example), there must be some ω_{-i} which becomes more likely when player i deviates from C to D. Hence, we have $L^*>1$.

When $a_{-i} = D$, $\max_{\omega_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$ is equal to

$$\frac{p_2(\omega_2 = B|D, D)}{p_2(\omega_2 = B|C, D)} = \frac{2/5 + 1/5}{1/4 + 1/8} = \frac{8}{5}.$$

As the former is larger, we conclude $L^* = \frac{15}{8}$. Plugging this into (S6), we obtain the following upper bounds of $M_i^{\mathcal{A}}$.

(a) When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{C\}$,

$$\begin{aligned} M_i^{\mathcal{A}} &\leq g(C, C) - \frac{g(D, C) - g(C, C)}{\frac{15}{8} - 1} \\ &= 1 - \frac{1/2}{\frac{15}{8} - 1} = \frac{3}{7}. \end{aligned}$$

(b) When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{D\}$,

$$\begin{aligned} M_i^{\mathcal{A}} &\leq g(C, D) - \frac{g(D, D) - g(C, D)}{\frac{15}{8} - 1} \\ &= -\frac{1}{6} - \frac{1/6}{\frac{15}{8} - 1} = -\frac{5}{14}. \end{aligned}$$

(c) When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{C, D\}$, the larger upper bound in the above two cases applies, so that we have

$$M_i^{\mathcal{A}} \leq \frac{3}{7}.$$

Given those bounds, we are ready to estimate $M_1^{\mathcal{A}} + M_2^{\mathcal{A}}$ for each regime $\mathcal{A}$.

Case (i), *where* $C \in \mathcal{A}_i$ *for* $i = 1, 2$: The above analysis (cases (a) and (c)) shows

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} \leq \frac{6}{7}.$$

Case (ii), *where* $C \in \mathcal{A}_i$ *and* $\mathcal{A}_{-i} = \{D\}$: Our case (b) shows $M_i^{\mathcal{A}} \leq -\frac{5}{14}$. In contrast, $M_{-i}^{\mathcal{A}}$ is simply achieved by $x_{-i} \equiv 0$ (as D is the dominant strategy in the stage game) so that

$$M_{-i}^{\mathcal{A}} = \sup_{\alpha_i} g(D, \alpha_i) = g(D, C) = \frac{3}{2}.$$

Hence, we have

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} \le \frac{3}{2} - \frac{5}{14} = \frac{8}{7}.$$

Case (iii), *where* $\mathcal{A} = \{D\} \times \{D\}$: Since D is the dominant action in the stage game, $M_i^{\mathcal{A}}$ is achieved by $x_i \equiv 0$. Moreover, the opponent's action is restricted to $\mathcal{A}_{-i} = \{D\}$, so that we have $M_i^{\mathcal{A}} = g(D, D) = 0$. Hence,

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} = 0.$$

Let me summarize our discussion above. If our example is in the negative case as defined by EHO, the only belief-free equilibrium payoff is $(0, 0)$. Otherwise, our example is in the positive case, where the sum of belief-free equilibrium payoffs $v_1 + v_2$ (in the limit as $\delta \to 1$) is bounded above by the maximum of the upper bounds found in Cases (i)–(iii), which is equal to $\frac{8}{7}$. Altogether, those results show that any limit belief-free equilibrium payoff profile (as $\delta \to 1$) $(v_1, v_2) \in V^*$ satisfies $v_1 + v_2 \le \frac{8}{7}$.

To complete our argument, I now examine the belief-free equilibrium payoffs for a fixed discount factor $\delta < 1$. Let $V(\delta)$ be the set of belief-free equilibrium payoff profiles for discount factor $\delta < 1$. The standard argument[3] shows that this is monotone increasing in δ (i.e., $V(\delta) \subset V(\delta')$ if $\delta < \delta'$). Hence, we have $V(\delta) \subset V^*$, so that for any discount factor δ, all belief-free equilibrium payoffs $(v_1, v_2) \in V(\delta)$ satisfy $v_1 + v_2 \le \frac{8}{7}$. Now recall that in our example, our one-period memory transition rule is an equilibrium if $\delta \ge 0.98954$, with reduced game given by

(S7)

	C	D
C	x, x	α, β
D	β, α	y, y

Numerical computation shows $x, y, \alpha, \beta > 0.6$ for $\delta \ge 0.98954$. Hence, the total payoff *in any entry* in our reduced game payoff table (S7) exceeds 1.2, which

[3]The proof is as follows. Suppose we terminate the repeated game under $\delta' > \delta$ randomly in each period with probability $1 - \frac{\delta}{\delta'}$ and start a new game (and repeat this procedure). In this way, we can decompose the repeated game under δ' into a series of randomly terminated repeated games, each of which has effective discount factor equal to $\delta' \times \frac{\delta}{\delta'} = \delta$. Hence, any equilibrium (average) payoff under δ can also be achieved under $\delta' > \delta$. This argument presupposes that public randomization is available (to terminate the game). Even without public randomization, however, our conclusion $V(\delta) \subset V^*$ also holds, because (i) the set of belief-free payoff profiles $V(\delta)$ is smaller without public randomization and (ii) the *same* limit payoff set V^* obtains with or without public randomization (see Ely, Horner, and Olszewski (2004), the online appendix to EHO).

is larger than the upper bound for the total payoffs associated with the belief-free equilibria, $\frac{8}{7}$. This implies that all of our equilibria lie above the Pareto frontier of the belief-free equilibrium payoff set.

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