Econometrica Supplementary Material

SUPPLEMENT TO "THE REALIZED LAPLACE TRANSFORM OF VOLATILITY"

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APPENDIX

This appendix consists of a shorter section that describes the added details with regard to the empirical work in the paper and a longer section that presents asymptotic results for the realized Laplace transform for the case in which volatility has a deterministic intraday component.

A.1. *Empirical Documentation*

FOR THE ANALYSIS of the empirical section in the paper as a measure for the unobservable integrated variance, $\int_{t-1}^{t}\sigma_s^2\,ds$, we use truncated variation (TV), originally proposed by Mancini (2001), which we construct in the manner

$$\mathrm{TV}_{[t-1,t]}(\alpha,\varpi)=\sum_{i=[(t-1)/\Delta_n]+1}^{[t/\Delta_n]}|\Delta_i^n X|^2 1_{\{|\Delta_i^n X|\le\alpha\Delta_n^{\varpi}\}},\quad \alpha>0,\varpi\in(0,1/2), \tag{31}$$

where here $\varpi=0.49$ (i.e., very close to $1/2$) and α is $4\times\sqrt{BV}$, where BV denotes the bipower variation of Barndorff-Nielsen and Shephard (2004, 2006) over the time interval $[t-1,t]$.

We next provide details on the calculation of the implied volatility densities in the right panel of Figure 1. We first recall (see, e.g., Barndorff-Nielsen and Shephard (2001) and the references therein) that the generalized-inverse-Gaussian (GIG) distribution that we use in the analysis is positively supported and is controlled by three parameters (ν,δ,γ). If $x\sim GIG(\nu,\delta,\gamma)$, then the density of x is given by

$$\frac{\left(\frac{\gamma}{\delta}\right)^{\nu}}{2K_\nu(\delta\gamma)}x^{\nu-1}\exp\left(-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)\right),\quad x>0, \tag{32}$$

where K_ν is a modified Bessel function of the third kind.

The three-parameter GIG density is fitted to the observed S&P 500 realized Laplace transform as follows. We select three abscissas, $u_1=0.10$, $u_2=4.0$, and $u_3=8.0$, which lie near the origin, in the central part, and in the upper part, respectively, of the effective domain $[0,8]$ of the realized Laplace transform. We then solve the three estimating equations $V_T(X,\Delta_n,u_j)-\mathcal{L}_{\mathrm{GIG}}(u_j|\theta)=0,\ j=1,2,3$, to obtain $\hat{\theta}$, where $\mathcal{L}_{\mathrm{GIG}}(u_j|\theta)$ is the Laplace trans-

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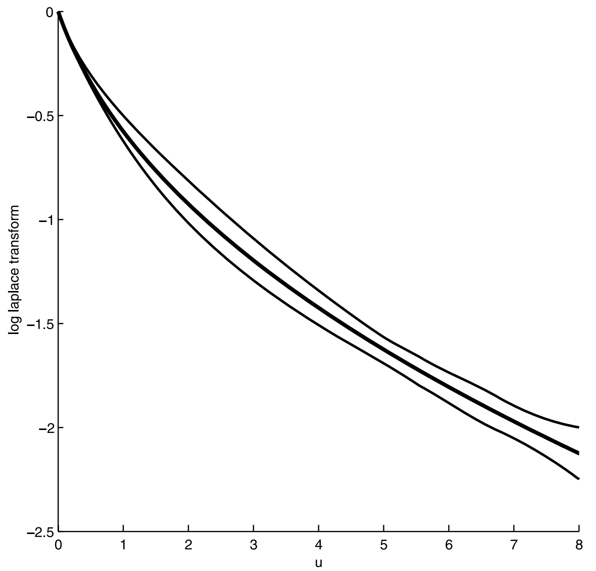

FIGURE 2.—GIG model-implied log-Laplace transforms of the S&P 500 spot variance. The figure shows the implied log-Laplace transform for the spot variance under the generalized-inverse-Gaussian distribution with the data-determined confidence interval for the nonparametric estimate of the log transform.

form of the GIG distribution evaluated at u_j given the 3×1 parameter vector θ. The resulting point estimate remains unchanged for other values of u that lie in the same general regions.

The fit of the GIG is essentially exact since $\mathcal{L}_{\text{GIG}}(u|\hat{\theta})$ and $V_T(X, \Delta_n, u)$ agree to within machine precision over $u \in [0, 8]$. The quality of the fit is evident from Figure 2, which indicates that $\mathcal{L}_{\text{GIG}}(u_j|\hat{\theta})$ goes right through the middle of the 2σ confidence band of Figure 1.

By way of contrast, Figure 3 reveals the poor fit of the gamma distribution, which is the marginal distribution of the affine Cox–Ingersoll–Ross (CIR) model, estimated similarly using two abscissas, $u_1 = 0.10$ and $u_2 = 8.0$. (The gamma distribution is a special case of the GIG distribution with $\delta = 0$ and $\nu > 0$ in (31).)

A.2. *The Case of a Deterministic Intraday Component in Volatility*

It is well recognized that financial volatility has a pronounced deterministic intraday U-shaped pattern; see, for example, Andersen and Bollerslev (1998) for an early account of this phenomenon. When this is the case, it is easy to show that the infill asymptotic result of Theorem 1 remains the same (provided the deterministic pattern is captured by a differentiable function). Therefore, here we look only at the situation when a joint infill and long-span asymptotics

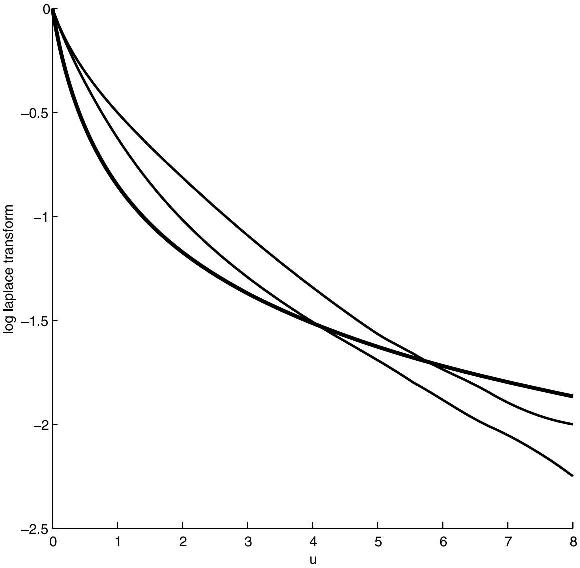

FIGURE 3.—CIR model-implied log-Laplace transform of the S&P 500 spot variance. The figure shows the implied log-Laplace transform for the spot variance under the gamma distribution with the data-determined confidence interval for the nonparametric estimate of the log transform.

is used, that is, the setting of Theorem 2. Also, for simplicity we look only at the case of $k = 0$ and $v = 0$ for $\widehat{\mu}_k(u, v)$, which in this case is simply $\frac{1}{T}V_T(X, \Delta_n, u)$.

To this end, we suppose that the underlying process, which we now denote with $\widetilde{X}$, has the dynamics

$$d\widetilde{X}_t = \alpha_t\,dt + \widetilde{\sigma}_t\,dW_t + \int_{\mathbb{R}} \delta(t-, x)\mu(ds, dx), \tag{33}$$

where $\widetilde{\sigma}_t^2 = f(t - [t]) \times \sigma_t^2$ for some deterministic 0.5-Hölder continuous function f with $f(t) > 0$ and $\int_0^1 f(s)\,ds = 1$; the processes α_t and σ_t, the measure μ, and the stochastic function $\delta(t, x)$ are all defined as in equation (3). In other words, the only change from the original setup is that the stochastic volatility process $\widetilde{\sigma}_t^2$ now has a deterministic component. We think, without loss of generality, that the unit time interval represents a day, so that $f(t)$ captures the *intraday* deterministic pattern of volatility. In this case, the limit of our realized Laplace transform under the joint long-span and infill asymptotics ($T \to \infty$ and $\Delta_n \to 0$) when Assumptions A, B, and C hold is

$$\begin{aligned} &\frac{1}{T}V_T(\widetilde{X}, \Delta_n, u) \xrightarrow{\mathbb{P}} \int_0^1 \mathbb{E}\big(e^{-uf(s)\sigma_s^2}\big)\,ds = \int_0^1 \mathcal{L}_{\sigma^2}(uf(s))\,ds, \\ &\mathcal{L}_{\sigma^2}(u) = \mathbb{E}\big(e^{-u\sigma_t^2}\big), \quad u \geq 0. \end{aligned} \tag{34}$$

In other words, when the volatility has a deterministic intraday pattern, the realized Laplace transform is an estimator for the integrated-over-the-day Laplace transform of volatility.

Further, it is easy to show that under Assumptions A, B, and C, and provided $T \uparrow \infty$ and $\Delta_n \downarrow 0$ with $\sqrt{T}\Delta_n^{1-\beta/2-\iota} \to 0$ for $\iota > 0$ arbitrarily small (and the additional requirement that $f(t)$ is differentiable), we have

$$\sqrt{T}\left(\frac{1}{T}V_T(\widetilde{X}, \Delta_n, u) - \int_0^1 \mathcal{L}_{\sigma^2}(uf(s))\,ds\right) \xrightarrow{\mathcal{L}} \widetilde{\Psi}'(u), \tag{35}$$

where $\widetilde{\Psi}'(u)$ is a Gaussian process with variance–covariance $\sum_{l=-\infty}^{\infty}\mathbb{E}(\widetilde{Z}_t(u) \times \widetilde{Z}_{t-l}(v))$ for

$$\widetilde{Z}_t(u) = \int_{t-1}^{t}\left(e^{-uf(s-[s])\sigma_s^2} - \mathbb{E}\left(e^{-uf(s-[s])\sigma_s^2}\right)\right) ds \quad \text{for} \quad t \in \mathbb{N}.$$

Most of the times our interest is in the properties of σ_t and not $\widetilde{\sigma}_t$, and there is a simple nonparametric procedure to "clean" the intraday component of the volatility that we now present.

Set $\Delta_n = 1/n$ for $n \in N$ and $i_t = t - 1 + i - [i/n]n$ for $t = 1, \dots, T$ and $i = 1, \dots, nT$. We define

$$\begin{aligned}
&\widehat{g}_i = \frac{n}{T}\sum_{t=1}^{T}\left|\Delta_{i_t}^n \widetilde{X}\right|^2 1\left(\left|\Delta_{i_t}^n \widetilde{X}\right| \le \alpha\Delta_n^{\varpi}\right), \quad i = 1, \dots, nT, \\
&\widehat{g} = \frac{1}{n}\sum_{i=1}^{n}\widehat{g}_i, \\
&\widehat{f}_i = \frac{\widehat{g}_i}{\widehat{g}} 1_{\{\widehat{g} \neq 0\}}, \quad i = 1, \dots, nT, \alpha > 0, \varpi \in (0, 1/2).
\end{aligned} \tag{36}$$

Intuitively, $\widehat{g}_i$ is our estimator of the average variance over a particular high-frequency interval of the day and, as a result, note that $\widehat{g}_i = \widehat{g}_j$ for $|i - j| = n$. $\widehat{g}$ is our estimator for the mean of the integrated variance over the day. Thus the ratio $\widehat{f}_i$ is an estimate for the intraday deterministic component of volatility.

We then define our estimator of the empirical Laplace transform of σ_t^2, which "cleans" for the deterministic intraday patterns in volatility as

$$\widehat{V}_T(\widetilde{X}, \Delta_n, u) = \frac{1}{n}\sum_{i=1}^{nT}\cos\left(\sqrt{2un}\,\widehat{f}_i^{-1/2} 1_{\{\widehat{f}_i \neq 0\}}\Delta_i^n \widetilde{X}\right). \tag{37}$$

Intuitively, we rescale the high-frequency increments, corresponding to the time of day they belong to, with our estimate for the deterministic intraday component of volatility. Note that we do not need to make any assumption regarding the possible presence of a deterministic component in the jump compensator, as our realized Laplace transform estimator is robust to jumps.

We show that our time-of-day adjusted $\widehat{V}_T(\widetilde{X}, \Delta_n, u)$ is a consistent estimate of $\mathcal{L}_{\sigma^2}(u)$ (contrast this with the limit in (34)). Our further goal is to quantify the asymptotic effect on $\widehat{V}_T(\widetilde{X}, \Delta_n, u)$ from the cleaning of the deterministic component of volatility, that is, to compare this feasible estimator with the infeasible one

$$V_T(X, \Delta_n, u) = \frac{1}{n} \sum_{i=1}^{nT} \cos(\sqrt{2un}\Delta_i^n X), \tag{38}$$

where the *unobservable* process X (defined on the original probability space) has the dynamics

$$dX_t = \alpha_t \, dt + \sigma_t \, dW_t + \int_{\mathbb{R}} \delta(t-, x)\mu(ds, dx), \tag{39}$$

that is, exactly as the observable process $\widetilde{X}$ but with no intraday deterministic component of volatility. The next theorem makes this comparison and hence characterizes the asymptotic behavior of $\widehat{V}_T(\widetilde{X}, \Delta_n, u)$.

THEOREM 3: *Suppose the observable process $\widetilde{X}$ has dynamics given by* (33) *and X has dynamics given in* (39) (*both defined on the same probability space*). *Assume that Assumptions* A, B, *and* C *hold. Assume further that for any $t \geq 0$ and any $p > 0$*,

$$\mathbb{E}\biggl(|\alpha_t|^p + |\sigma_t|^p + \int_{\mathbb{R}} |\delta(t, x)|^p \nu(x) \, dx + |v_t|^p + |v_t'|^p + \int_{\mathbb{R}} |\delta'(t, x)|^p \underline{\nu}(dx)\biggr) < C, \tag{40}$$

where $C > 0$ is some constant that does not depend on t.

(a) *Then if $T \to \infty$ and $\Delta_n \to 0$ such that $\sqrt{T}\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2} \to 0$ for some arbitrary small $\iota > 0$, we have for any $u \geq 0$*,

$$\sqrt{T}\biggl(\frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u) - \frac{1}{T}V_T(X, \Delta_n, u)\biggr) - \frac{0.5\mathbb{E}(G(u\sigma_t^2))}{\mathbb{E}(\sigma_t^2)} \frac{1}{\sqrt{T}} \int_0^T (\sigma_s^2 - \widetilde{\sigma}_s^2) \, ds \xrightarrow{\mathbb{P}} 0, \tag{41}$$

where we denote the function $G(x) = \sqrt{2}xe^{-x}$.

(b) *In addition, under the same conditions we have*

$$\sqrt{T}\left(\frac{1}{T}V_T(X,\Delta_n,u)-\mathbb{E}\big(e^{-u\sigma_t^2}\big),\frac{1}{T}\int_0^T(\sigma_s^2-\widetilde{\sigma}_s^2)\,ds\right) \tag{42}$$
$$\xrightarrow{\mathcal{L}}\Sigma(u)^{1/2}\times\Xi,$$

where Ξ is a 2×1 standard normal vector and $\Sigma(u)$ is 2×2 matrix of constants given by

$$\Sigma(u)=\mathbb{E}(\mathbf{z}_t(u)\mathbf{z}_t'(u))+\sum_{k=1}^{\infty}\big(\mathbb{E}(\mathbf{z}_t(u)\mathbf{z}_{t+k}'(u))+\mathbb{E}(\mathbf{z}_{t+k}(u)\mathbf{z}_t'(u))\big) \tag{43}$$

for $\mathbf{z}_t(u)=(\int_{t-1}^t(e^{-u\sigma_s^2}-\mathbb{E}(e^{-u\sigma_t^2}))\,ds,\int_{t-1}^t(\sigma_s^2-\widetilde{\sigma}_s^2)\,ds)'$.

(c) *Consistent estimate for $\Sigma(u)$ is given by*

$$\widehat{\Sigma}(u)=\widehat{C}_0(u)+2\sum_{i=1}^{L_T}\omega(i,L_T)\widehat{C}_i(u), \tag{44}$$
$$\widehat{C}_i(u)=\frac{1}{T}\sum_{t=i+1}^{T}(\widehat{\mathbf{z}}_{t-i}(u)\widehat{\mathbf{z}}_t'(u)+\widehat{\mathbf{z}}_t(u)\widehat{\mathbf{z}}_{t-i}'(u)),$$

where for some $\eta>0$ such that $L_TT^{\eta-1/2}\to 0$, $\alpha>0$, and $\varpi\in(0,1/2)$, $\widehat{z}_t(u)$ is defined as

$$\widehat{\mathbf{z}}_t(u)=\begin{pmatrix}\dfrac{1}{n}\displaystyle\sum_{j=tn+1}^{tn+n}\left(\cos\big(\sqrt{2un}\widehat{f}_j^{-1/2}1_{\{\widehat{f}_j\neq 0\}}\Delta_j^n\widetilde{X}\big)-\frac{1}{T}\widehat{V}_T(\widetilde{X},\Delta_n,u)\right)\\ \displaystyle\sum_{j=tn+1}^{tn+n}(\widehat{f}_j^{-1}\wedge T^{\eta}-1)(\Delta_j^n\widetilde{X})^2 1(|\Delta_j^n\widetilde{X}|\le\alpha\Delta_n^{\varpi})\end{pmatrix}.$$

Furthermore, the sequences L_T and $\omega(i,L_T)$ are defined as in Theorem 2 and satisfy the conditions of that theorem.

A consistent estimator for $\mathbb{E}(G(u\sigma_t^2))/\mathbb{E}(\sigma_t^2)$ is given by

$$\frac{1}{nT}\sum_{j=1}^{nT}\frac{(\sqrt{2un}(\widehat{f}_j^{-1/2}\wedge T^{\eta/2})\Delta_j^n\widetilde{X})\sin(\sqrt{2un}(\widehat{f}_j^{-1/2}\wedge T^{\eta/2})\Delta_j^n\widetilde{X})}{\widehat{g}}. \tag{45}$$

Part (a) of the above theorem shows that $\frac{1}{T}\widehat{V}_T(\widetilde{X},\Delta_n,u)$ is a consistent estimator for our object of interest, that is, $\mathbb{E}(e^{-u\sigma_t^2})$. It further characterizes the asymptotic effect of using an estimate from the data for the intraday pattern of volatility on our precision of estimating the Laplace transform of σ_t^2. It is controlled by $\frac{1}{T}\int_0^T(\sigma_s^2 - \widetilde{\sigma}_s^2)\,ds$, which implies the rather intuitive observation that this effect is larger for wider deterministic intraday variations in volatility.

Part (b) of the theorem derives the joint distribution of the error from estimating the intraday pattern and the error associated with the empirical process for estimating the Laplace transform of volatility. Finally, part (c) of the theorem provides an easy to construct feasible estimate for the asymptotic variance–covariance $\Sigma(u)$. This provides a feasible way to quantify the precision of estimating $\mathbb{E}(e^{-u\sigma_t^2})$ using $\frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u)$.

We apply the result of Theorem 3 to the same data set used in the empirical application in the paper, that is, 1-minute level data on the S&P 500 futures index spanning the period January 1, 1990 to December 31, 2008. Our choice for the parameters α and ϖ for the construction of $\widehat{g}_i$ is similar to the values of these parameters that we use for computing the truncated variation estimator $\mathrm{TV}_t(\alpha, \varpi)$ in the paper: $\alpha = 4\sqrt{BV}$ and $\varpi = 0.49$. Figure 4 shows the effect of cleaning the possible presence of a diurnal volatility pattern on estimating the Laplace transform of volatility. It compares our original estimate $\frac{1}{T}V(\widetilde{X}, \Delta_n, u)$ with the one corrected for the deterministic pattern, that is, $\frac{1}{T}\widehat{V}(\widetilde{X}, \Delta_n, u)$.[1] As seen from the figure, the effect from cleaning for the deterministic pattern is relatively small, especially when compared with the wedge between the Laplace transform of spot and integrated volatility.

PROOF OF THEOREM 3: As in the proof of Theorems 1 and 2, in the proof of Theorem 3, C denotes a positive constant that does not depend on T and Δ_n, and further can change from line to line. We also use the shorthand $\mathbb{E}_{i-1}^n$ for $\mathbb{E}(\cdot|\mathcal{F}_{(i-1)\Delta_n})$. We start with some preliminary results that we need for the proof.

Preliminary results. We start by introducing the auxiliary estimators for the intraday average variances:

$$\tilde{g}_i = \frac{n}{T}\sum_{t=1}^{T}\widetilde{\sigma}^2_{(i_t-1)\Delta_n}\left|\Delta_{i_t}^n W\right|^2, \quad i = 1, \ldots, nT; \quad \widetilde{g} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{g}_i; \tag{46}$$

$$\widetilde{f}_i = \frac{\widetilde{g}_i}{\widetilde{g}}1_{\{\widetilde{g}\neq 0\}}, \quad i = 1, \ldots, nT.$$

These estimators are formed the same way as $\widehat{g}_i$ with the only difference that we use $\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W$ in their construction instead of the observable truncated increment $\Delta_i^n\widetilde{X}1_{\{|\Delta_i^n\widetilde{X}|\leq\alpha\Delta_n^{\varpi}\}}$. Intuitively, the truncation makes the effect of the jumps on $\widehat{g}_i$ negligible, and hence $\widehat{g}_i$ and $\widetilde{g}_i$ are close, as we show now.

[1] Note that due to the possible presence of an intraday deterministic component of volatility, we denote the observable process as $\widetilde{X}$ and not X. Of course, $\widetilde{X}$ and X coincide when $f(t) \equiv 1$.

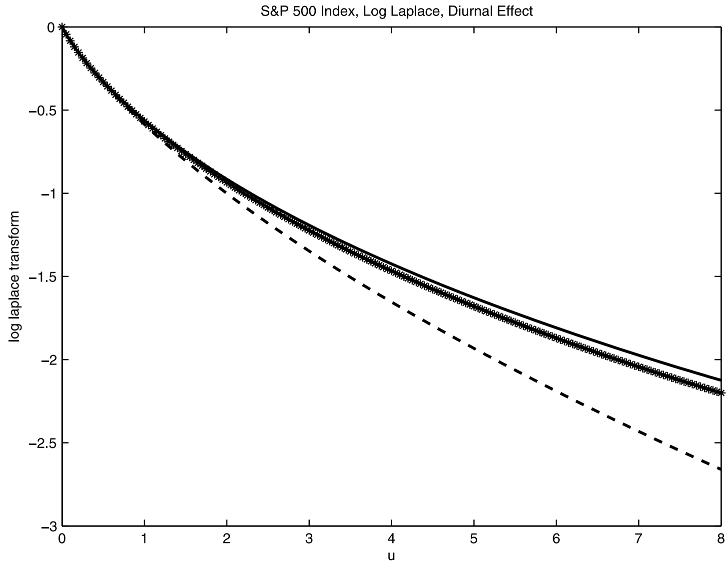

FIGURE 4.—Observed log-Laplace transforms with and without cleaning for an intraday deterministic volatility component. The log-Laplace transforms are estimated using 1-minute S&P 500 stock index data for 1990–2008. The solid line corresponds to $\frac{1}{T}V_T(\widetilde{X}, \Delta_n, u)$ (original estimate in Figure 1); the bold (heavy) line corresponds to the estimator $\frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u)$ introduced here that cleans the deterministic component of volatility; the dashed line corresponds to the empirical Laplace transform of the daily truncated variance.

We can make the decomposition

$$(\Delta_i^n \widetilde{X})^2 1(|\Delta_i^n \widetilde{X}| \le \alpha \Delta_n^{\varpi}) - \widetilde{\sigma}_{(i-1)\Delta_n}^2 (\Delta_i^n W)^2 = \sum_{j=1}^4 \varepsilon_i(j), \tag{47}$$

$$i = 1, \dots, nT,$$

$$\varepsilon_i(1) = \left[(\Delta_i^n \widetilde{X})^2 - \left(\widetilde{\sigma}_{(i-1)\Delta_n} \Delta_i^n W + \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} \delta(s-, x)\mu(ds, dx) \right)^2 \right] \times 1(|\Delta_i^n \widetilde{X}| \le \alpha \Delta_n^{\varpi}),$$

$$\varepsilon_i(2) = -\left(\widetilde{\sigma}_{(i-1)\Delta_n} \Delta_i^n W\right)^2 1(|\Delta_i^n \widetilde{X}| > \alpha \Delta_n^{\varpi}),$$

$$\varepsilon_i(3) = \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} \delta(s-, x)\mu(ds, dx) \right)^2 1(|\Delta_i^n \widetilde{X}| \le \alpha \Delta_n^{\varpi}),$$

$$\varepsilon_i(4) = 2\widetilde{\sigma}_{(i-1)\Delta_n} \Delta_i^n W \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\mathbb{R}} \delta(s-, x)\mu(ds, dx) 1(|\Delta_i^n \widetilde{X}| \le \alpha \Delta_n^{\varpi}).$$

Using Hölder's inequality, the Burkholder–Davis–Gundy inequality, and Assumption B for the process σ_t, as well as the smoothness property of $f(t)$, we have for any $p \in [1, 2]$,

$$\mathbb{E}^n_{i-1}|\varepsilon_i(1)|^p \le C_{(i-1)\Delta_n}\Delta_n^{3p/2}, \quad i, = 1, \dots, nT, \tag{48}$$

where the constant $C_{(i-1)\Delta_n}$ is adapted to $\mathcal{F}_{(i-1)\Delta_n}$ and all its (positive) powers are integrable.

Next, Hölder's inequality implies

$$\mathbb{E}^n_{i-1}|\varepsilon_i(2)|^p \le C_{(i-1)\Delta_n}\Delta_n^{p+(1-\beta\omega)-\iota}, \quad i = 1, \dots, nT, \tag{49}$$

where β is defined in Assumption A, $\iota > 0$ is arbitrarily small, and $C_{(i-1)\Delta_n}$ is defined as above.

For $\varepsilon_i(3)$, we trivially have for any $p \in [1, 2]$,

$$\mathbb{E}^n_{i-1}|\varepsilon_i(3)|^p \le C_{(i-1)\Delta_n}\Delta_n^{1+(2p-\beta)\omega-\iota}, \quad i = 1, \dots, nT, \tag{50}$$

where $\iota > 0$ is arbitrarily small and $C_{(i-1)\Delta_n}$ is as defined above.

Finally, we obviously have $|\varepsilon_i(4)| \le |\varepsilon_i(2)| + |\varepsilon_i(3)|$ and so the above bounds can be used to bound $\mathbb{E}^n_{i-1}|\varepsilon_i(3)|^p$ for any $p \in [1, 2]$.

Combining the above bounds and using successive conditioning and Hölder's inequality (together with the integrability condition (40)), we have

$$\begin{aligned}\mathbb{E}|\widehat{g}_i - \widetilde{g}_i| \le C\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2} \quad \text{and} \quad \mathbb{E}|\widehat{g} - \widetilde{g}| \le C\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2},\\ i = 1, \dots, n, \forall \iota > 0,\end{aligned} \tag{51}$$

and

$$\begin{aligned}\mathbb{E}|\widehat{g}_i - \widetilde{g}_i|^2 \le C\Delta_n^{[(4-2\beta)\varpi-\iota]\wedge 1} \quad \text{and} \quad \mathbb{E}|\widehat{g} - \widetilde{g}|^2 \le C\Delta_n^{[(4-2\beta)\varpi-\iota]\wedge 1},\\ i = 1, \dots, n, \forall \iota > 0.\end{aligned} \tag{52}$$

Parts (a) *and* (b). We first make the decomposition

$$\frac{1}{nT}\sum_{i=1}^{nT}\cos\bigl(\sqrt{2un}\widehat{f}_i^{-1/2}1_{\{\widehat{f}_i\neq 0\}}\Delta_i^n\widetilde{X}\bigr) - \mathbb{E}\bigl[e^{-u\sigma_t^2}\bigr] = \sum_{i=1}^{5} A_i, \tag{53}$$

$$\begin{aligned}A_1 &= \frac{1}{nT}\sum_{i=1}^{nT}\cos\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr) - \mathbb{E}\bigl[e^{-u\sigma_t^2}\bigr],\\ A_2 &= \frac{1}{nT}\sum_{i=1}^{nT}\bigl\{\cos\bigl(\sqrt{2un}\widetilde{f}_i^{-1/2}1_{\{\widetilde{f}_i\neq 0\}}f_{i-[i/n]n}^{1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\\ &\qquad - \cos\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\bigr\}1_{\{B_i\}},\end{aligned} \tag{54}$$

$$A_3 = \frac{1}{nT}\sum_{i=1}^{nT}\big\{\cos\big(\sqrt{2un}\widetilde{f}_i^{-1/2}1_{\{\widetilde{f}_i\neq 0\}}f_{i-[i/n]n}^{1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big) - \cos\big(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big)\big\}1_{\{B_i^c\}},$$

$$A_4 = \frac{1}{nT}\sum_{i=1}^{nT}\big\{\cos\big(\sqrt{2un}\widehat{f}_i^{-1/2}1_{\{\widehat{f}_i\neq 0\}}\Delta_i^n \widetilde{X}\big) - \cos\big(\sqrt{2un}\widetilde{f}_i^{-1/2}1_{\{\widetilde{f}_i\neq 0\}}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big)\big\}1_{\{B_i\cup C_i\}},$$

$$A_5 = \frac{1}{nT}\sum_{i=1}^{nT}\big\{\cos\big(\sqrt{2un}\widehat{f}_i^{-1/2}1_{\{\widehat{f}_i\neq 0\}}\Delta_i^n \widetilde{X}\big) - \cos\big(\sqrt{2un}\widetilde{f}_i^{-1/2}1_{\{\widetilde{f}_i\neq 0\}}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big)\big\}1_{\{B_i^c\cap C_i^c\}},$$

where the sets B_i and C_i are defined as

$$B_i = \big\{\widetilde{g}_i \geq (1+\tau)f_{i-[i/n]n}\mathbb{E}(\sigma_t^2) \cup \widetilde{g}_i \leq (1-\tau)f_{i-[i/n]n}\mathbb{E}(\sigma_t^2) \cup \widetilde{g} \geq (1+\tau)\mathbb{E}(\sigma_t^2) \cup \widetilde{g} \leq (1-\tau)\mathbb{E}(\sigma_t^2)\big\},$$

$$C_i = \big\{\widehat{g}_i \geq (1+\tau)f_{i-[i/n]n}\mathbb{E}(\sigma_t^2) \cup \widehat{g}_i \leq (1-\tau)f_{i-[i/n]n}\mathbb{E}(\sigma_t^2) \cup \widehat{g} \geq (1+\tau)\mathbb{E}(\sigma_t^2) \cup \widehat{g} \leq (1-\tau)\mathbb{E}(\sigma_t^2)\big\}$$

for $i = 1, \dots, nT$ and some constant $\tau \in (0,1)$.

From the proof of Theorem 2, the first component, A_1, is the leading term of $\frac{1}{T}V_T(X, \Delta_n, u) - \mathbb{E}[e^{-u\sigma_t^2}]$. The other components in the above decomposition are due to the cleaning for the diurnal pattern (and the presence of jumps and a drift term in the price increments as well as the time variation in the volatility). The main difficulty in the proof of parts (a) and (b) of the theorem comes from the fact that $\widehat{f}_i$ and $\widetilde{f}_i$ use information from the whole time span $[0, T]$, and further are not bounded from below and above. In the rest of the proof, we further decompose each of the terms in (54) so as to extract the leading components in the asymptotic expansion of $\frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u) - \mathbb{E}[e^{-u\sigma_t^2}]$ and bound the asymptotically negligible parts.

We start with A_3. Using a second-order Taylor expansion of the function $h(x, y) = \cos(a\sqrt{y/x})$ with $a = \sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W$, $x = \widetilde{g}_i$, and $y = \widetilde{g}$ around $(f_{i-[i/n]n}\mathbb{E}(\sigma_t^2), \mathbb{E}(\sigma_t^2))$ (note that on the set B_i^c, $\widetilde{g}_i$ is strictly positive and $\widetilde{g}$ is strictly positive and bounded), we can decompose A_3 as $A_3 = \sum_{j=1}^6 A_3(j)$, where

$$A_3(1) = \frac{0.5\mu}{n}\sum_{i=1}^{n}\frac{\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)} - 0.5\mu\frac{\widetilde{g} - \mathbb{E}(\sigma_t^2)}{\mathbb{E}(\sigma_t^2)}, \tag{55}$$

$$
\begin{aligned}
A_3(2) &= \frac{0.5}{nT}\sum_{i=1}^{nT}\bigl\{\sin\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W-\mu\bigr\}\\
&\quad\times\left(\frac{\widetilde{g}_i-f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}\right),\\
A_3(3) &= \frac{-0.5}{nT}\left(\frac{\widetilde{g}-\mathbb{E}(\sigma_t^2)}{\mathbb{E}(\sigma_t^2)}\right)\sum_{i=1}^{nT}\bigl\{\sin\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\\
&\quad\times\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W-\mu\bigr\},\\
A_3(4) &= \frac{-0.5}{nT}\sum_{i=1}^{nT}\bigl\{\sin\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr\}\\
&\quad\times\left(\frac{\widetilde{g}_i-f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}\right)1_{\{B_i\}},\\
A_3(5) &= \frac{0.5}{nT}\left(\frac{\widetilde{g}-\mathbb{E}(\sigma_t^2)}{\mathbb{E}(\sigma_t^2)}\right)\sum_{i=1}^{nT}\bigl\{\sin\bigl(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr)\\
&\quad\times\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\bigr\}1_{\{B_i\}},\\
A_3(6) &= \frac{1}{nT}\sum_{i=1}^{nT}\bigl\{H_{11}\bigl(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W;\tilde{\tilde{g}}_i,\tilde{\tilde{g}}\bigr)\bigl(\tilde{g}_i-f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\bigr)^2\\
&\quad+H_{12}\bigl(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W;\tilde{\tilde{g}}_i,\tilde{\tilde{g}}\bigr)\\
&\quad\times\bigl(\tilde{g}_i-f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\bigr)(\widetilde{g}-\mathbb{E}(\sigma_t^2))\\
&\quad+H_{22}\bigl(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W;\tilde{\tilde{g}}_i,\tilde{\tilde{g}}\bigr)(\widetilde{g}-\mathbb{E}(\sigma_t^2))^2\bigr\}1_{\{B_i^c\}},
\end{aligned}
$$

where we denote $\mu=\mathbb{E}(G(u\sigma_t^2))$ (recall $G(x)=\sqrt{2}xe^{-x}$), $\tilde{\tilde{g}}_i$ is between $\tilde{g}_i$ and $f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)$, $\tilde{\tilde{g}}$ is between $\tilde{g}$ and $\mathbb{E}(\sigma_t^2)$ (and is different for $i=1,\dots,nT$), and $\tilde{\tilde{f}}_i=\tilde{\tilde{g}}_i/\tilde{\tilde{g}}$, and finally

$$
\begin{aligned}
(56)\qquad H_{11}(a;x,y) &= -\frac{1}{4}\cos\left(a\sqrt{\frac{y}{x}}\right)\frac{a^2y}{x^3}-\frac{3}{4}\sin\left(a\sqrt{\frac{y}{x}}\right)\frac{ay^{1/2}}{x^{5/2}},\\
H_{22}(a;x,y) &= -\frac{1}{4}\cos\left(a\sqrt{\frac{y}{x}}\right)\frac{a^2}{xy}+\frac{1}{4}\sin\left(a\sqrt{\frac{y}{x}}\right)\frac{a}{x^{1/2}y^{3/2}},
\end{aligned}
$$

$$H_{12}(a; x, y) = \frac{1}{4}\cos\left(a\sqrt{\frac{y}{x}}\right)\frac{a^2}{x^2} + \frac{1}{4}\sin\left(a\sqrt{\frac{y}{x}}\right)\frac{a}{x^{3/2}y^{1/2}}.$$

For $A_3(1)$, using the definition of $\widetilde{g}_i$ and $\widetilde{g}$, we have, further,

$$\begin{aligned} A_3(1) &= \frac{0.5\mu}{\mathbb{E}(\sigma_t^2)} \times \frac{1}{nT}\sum_{i=1}^{nT}\big(\sigma^2_{(i-1)\Delta_n} - \widetilde{\sigma}^2_{(i-1)\Delta_n}\big)n(\Delta_i^n W)^2 \\ &= \frac{0.5\mu}{\mathbb{E}(\sigma_t^2)} \times \big(A_3^{(a)}(1) + A_3^{(b)}(1) + A_3^{(c)}(1)\big), \end{aligned} \tag{57}$$

$$\begin{aligned} A_3^{(a)}(1) &= \frac{1}{T}\sum_{t=1}^{T}\int_{t-1}^{t}(\sigma_s^2 - \widetilde{\sigma}_s^2)\,ds, \\ A_3^{(b)}(1) &= \frac{1}{nT}\sum_{i=1}^{nT}\big(\sigma^2_{(i-1)\Delta_n} - \widetilde{\sigma}^2_{(i-1)\Delta_n}\big) - \frac{1}{T}\sum_{t=1}^{T}\int_{t-1}^{t}(\sigma_s^2 - \widetilde{\sigma}_s^2)\,ds, \\ A_3^{(c)}(1) &= \frac{1}{nT}\sum_{i=1}^{nT}\big(\sigma^2_{(i-1)\Delta_n} - \widetilde{\sigma}^2_{(i-1)\Delta_n}\big)\big(n(\Delta_i^n W)^2 - 1\big). \end{aligned} \tag{58}$$

Then using Assumption B and the fact that $f(t)$ is 0.5-Hölder continuous, we have

$$\mathbb{E}\big|A_3^{(b)}(1)\big| \le C\sqrt{\Delta_n}, \tag{59}$$

and, further, for the martingale process we have

$$\mathbb{E}\big|A_3^{(c)}(1)\big| \le \frac{C\sqrt{\Delta_n}}{\sqrt{T}}. \tag{60}$$

Turning to A_1, we can decompose it as $A_1 = A_1(1) + A_1(2)$, where

$$\begin{aligned} A_1(1) &= \frac{1}{T}\sum_{t=1}^{T}\left(\int_{t-1}^{t} e^{-u\sigma_s^2}\,ds - \mathbb{E}\big[e^{-u\sigma_t^2}\big]\right), \\ A_1(2) &= \frac{1}{T}\sum_{i=1}^{nT}\left(\Delta_n\cos\big(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big) - \int_{(i-1)\Delta_n}^{i\Delta_n} e^{-u\sigma_s^2}\,ds\right). \end{aligned} \tag{61}$$

Using the proof of Theorems 1 and 2, we have that

$$A_1(2) = O_p\left(\sqrt{\frac{\Delta_n}{T}} + \Delta_n\right). \tag{62}$$

Then using the stationarity, ergodicity, and mixing conditions, we have

$$\sqrt{T}\big(A_1(1), A_3^{(a)}(1)\big) \xrightarrow{\mathcal{L}} \Sigma(u)^{1/2} \times \Xi. \tag{63}$$

From the proof of Theorem 2, the difference between $\frac{1}{T}V_T(X, \Delta_n, u) - \mathbb{E}[e^{-u\sigma_t^2}]$ and the term A_1 is $o_p(1/\sqrt{T})$. Therefore, the above result shows (42) in Theorem 3.

Since $\widetilde{g}_i = \widetilde{g}_j$ for $|i-j| = n$, we can rewrite $A_3(2)$ as

$$\begin{aligned} A_3(2) = \frac{0.5}{n}\sum_{i=1}^{n}\Bigg\{\frac{1}{T}\sum_{t=1}^{T}\big(\sin\big(\sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W\big) \\ \times \sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W - \mu\big)\Bigg\}\Bigg\{\frac{\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}\Bigg\}. \end{aligned} \tag{64}$$

Using Assumption C, the fact that $G(x)$ is bounded, and Lemma VIII.3.102 in Jacod and Shiryaev (2003), we have (recall the definition of the constant μ above)

$$\begin{aligned} &\mathbb{E}_{i-1}^n\big(\sin\big(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big)\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W - G\big(u\sigma_{(i-1)\Delta_n}^2\big)\big) = 0, \\ &\quad i = 1, \dots, nT, \\ &\mathbb{E}\big|\mathbb{E}_{j-1}^n\big(G\big(u\sigma_{(i-1)\Delta_n}^2\big) - \mu\big)\big| \le C\big(\alpha_{(i-j)/n}^{\text{mix}}\big)^{1-\iota}, \\ &\quad j, i = 1, \dots, nT, j \le i, \ \iota > 0 \text{ arbitrarily small.} \end{aligned} \tag{65}$$

Therefore

$$\begin{aligned} &\mathbb{E}\Bigg(\frac{1}{T}\sum_{t=1}^{T}\big(\sin\big(\sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W\big)\sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W - \mu\big)\Bigg)^2 \\ &\quad \le \frac{C}{T}\int_0^\infty (\alpha_s^{\text{mix}})^{1-\iota}\,ds. \end{aligned} \tag{66}$$

Similar analysis shows

$$\begin{aligned} &\mathbb{E}\Bigg(\frac{1}{nT}\sum_{i=1}^{nT}\big(\sin\big(\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big)\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W - \mu\big)\Bigg)^2 \\ &\quad \le \frac{C}{T}\int_0^\infty (\alpha_s^{\text{mix}})^{1-\iota}\,ds, \\ &\mathbb{E}\big(\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big)^2 \le \frac{C}{T}\int_0^\infty (\alpha_s^{\text{mix}})^{1-\iota}\,ds, \quad i = 1, \dots, n, \end{aligned} \tag{67}$$

$$\mathbb{E}(\widetilde{g} - \mathbb{E}(\sigma_t^2))^2 \leq \frac{C}{T}\int_0^\infty (\alpha_s^{\mathrm{mix}})^{1-\iota}\, ds + C\Delta_n,$$

where for the last bound we made use of the fact that $f(t)$ is a 0.5-Hölder continuous function. Using Chebychev's inequality and the above results, we also easily get

$$\mathbb{P}(B_i) \leq C\mathbb{E}\big(\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big)^2 + C\mathbb{E}(\widetilde{g} - \mathbb{E}(\sigma_t^2))^2 \leq \left(\frac{C}{T} + C\Delta_n\right). \tag{68}$$

The bounds in (66) and (67) and an application of Cauchy–Schwarz inequality give

$$\mathbb{E}|A_3(2) + A_3(3)| \leq \frac{C}{T} + \frac{C\sqrt{\Delta_n}}{\sqrt{T}}. \tag{69}$$

Turning to $A_3(4)$, we first can decompose it as

$$\begin{aligned} A_3^a(4) &= \frac{-0.5}{n}\sum_{i=1}^n \left(\frac{1}{T}\sum_{t=1}^T \big(\sin\big(\sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W\big)\right. \\ &\qquad \left. \times \sqrt{2un}\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W - \mu\big)\right)\left(\frac{\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}\right)1_{\{B_i\}}, \\ A_3^b(4) &= \frac{-0.5\mu}{n}\sum_{i=1}^n \left(\frac{\widetilde{g}_i - f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}{f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)}\right)1_{\{B_i\}}. \end{aligned} \tag{70}$$

Then we can use the results in (66) and (67) (and Chebychev's inequality for $A_3^b(4)$) to conclude

$$\mathbb{E}|A_3(4)| \leq \frac{C}{T} + \frac{C\sqrt{\Delta_n}}{\sqrt{T}}. \tag{71}$$

Similar analysis can be used to show

$$\mathbb{E}|A_3(5)| \leq \frac{C}{T} + \frac{C\sqrt{\Delta_n}}{\sqrt{T}}. \tag{72}$$

Turning to $A_3(6)$, first using the fact that on the set B_i^c, $\widetilde{g}_i$ is bounded from below and $\widetilde{g}$ is bounded from below and above, we have

$$\begin{aligned} &\big|H_{11}\big(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W; \tilde{\tilde{g}}_i, \tilde{\tilde{g}}\big)\big| + \big|H_{22}\big(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W; \tilde{\tilde{g}}_i, \tilde{\tilde{g}}\big)\big| \\ &\quad + \big|H_{12}\big(\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W; \tilde{\tilde{g}}_i, \tilde{\tilde{g}}\big)\big| \\ &\leq \big|\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big|^2 \vee \big|\sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big|, \quad i = 1, \dots, nT. \end{aligned} \tag{73}$$

Then combining this with the above bounds in (66) and (67) and using the integrability condition in (40) together with Hölder's inequality we get

$$\mathbb{E}|A_3(6)| \le \left(\frac{C}{T} + C\Delta_n\right)^{1-\iota}, \quad \iota > 0 \text{ arbitrarily small.} \tag{74}$$

We continue next with A_2 and A_4. We can use the trivial inequalities

$$\begin{aligned}
\mathbb{P}\big(\widehat{g}_i \le (1-\tau) f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big) &\le \mathbb{P}\big(\widetilde{g}_i \le (1-\tau/2) f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big) \\
&\quad + \mathbb{P}\big(|\widehat{g}_i - \widetilde{g}_i| \ge \tau f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)/2\big), \\
\mathbb{P}\big(\widehat{g}_i \ge (1+\tau) f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big) &\le \mathbb{P}\big(\widetilde{g}_i \ge (1+\tau/2) f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)\big) \\
&\quad + \mathbb{P}\big(|\widehat{g}_i - \widetilde{g}_i| \ge \tau f_{i-[i/n]n}\mathbb{E}(\sigma_t^2)/2\big), \\
\mathbb{P}(\widehat{g} \le (1-\tau)\mathbb{E}(\sigma_t^2)) &\le \mathbb{P}(\widetilde{g} \le (1-\tau/2)\mathbb{E}(\sigma_t^2)) \\
&\quad + \mathbb{P}(|\widehat{g} - \widetilde{g}| \ge \tau\mathbb{E}(\sigma_t^2)/2), \\
\mathbb{P}(\widehat{g} \ge (1+\tau)\mathbb{E}(\sigma_t^2)) &\le \mathbb{P}(\widetilde{g} \ge (1+\tau/2)\mathbb{E}(\sigma_t^2)) \\
&\quad + \mathbb{P}(|\widehat{g} - \widetilde{g}| \ge \tau\mathbb{E}(\sigma_t^2)/2),
\end{aligned} \tag{75}$$

and the bound for $\mathbb{P}(B_i)$ derived in (68), together with the first absolute-moment restrictions for the differences $\widehat{g}_i - \widetilde{g}_i$ and $\widehat{g} - \widetilde{g}$ in (51), to get

$$\mathbb{E}(|A_2| + |A_4|) \le \frac{C}{T} + C\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2} \quad \forall \iota > 0. \tag{76}$$

We are left with A_5. First, using the definition of the set $B_i^c \cap C_i^c$ and a first-order Taylor expansion of the function $h(x,y) = \frac{x}{y}$, we have for $i = 1, \ldots, nT$,

$$\begin{aligned}
&\big|\cos\big(\sqrt{2un}\widehat{f}_i^{-1/2}\Delta_i^n \widetilde{X}\big) - \cos\big(\sqrt{2un}\widetilde{f}_i^{-1/2}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big)\big| 1_{\{B_i^c \cap C_i^c\}} \\
&\quad \le C\big|\sqrt{2un}\Delta_i^n \widetilde{X} - \sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big|^{\beta+\iota} \\
&\qquad + C\big|\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big| |\widehat{g}_i - \widetilde{g}_i| 1_{\{B_i^c \cap C_i^c\}} \\
&\qquad + C\big|\sqrt{2un}\sigma_{(i-1)\Delta_n}\Delta_i^n W\big| |\widehat{g} - \widetilde{g}| 1_{\{B_i^c \cap C_i^c\}} \quad \forall \iota \in (0, 1-\beta].
\end{aligned} \tag{77}$$

Using this inequality, we can bound $|A_5| \le C\sum_{j=1}^5 A_5(j)$, where

$$A_5(1) = \frac{1}{nT}\sum_{i=1}^{nT}\big|\sqrt{2un}\Delta_i^n \widetilde{X} - \sqrt{2un}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\big|^{\beta+\iota}, \tag{78}$$

(79) $$A_5(2) = \sqrt{2u}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{T}\sum_{t=1}^{T}\sqrt{n}\left|\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W\right| - \sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t|\right\} \times |\widehat{g}_i - \widetilde{g}_i| 1_{\{B_i^c \cap C_i^c\}},$$

(80) $$A_5(3) = \sqrt{2u}\sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t|\frac{1}{n}\sum_{i=1}^{n}|\widehat{g}_i - \widetilde{g}_i| 1_{\{B_i^c \cap C_i^c\}},$$

(81) $$A_5(4) = \sqrt{2u}\left\{\frac{1}{nT}\sum_{i=1}^{nT}\sqrt{n}\left|\sigma_{(i-1)\Delta_n}\Delta_i^n W\right| - \sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t|\right\} \times |\widehat{g} - \widetilde{g}| 1_{\{|\widehat{g}-\widetilde{g}| \le 4\tau\mathbb{E}(\sigma_t^2)\}},$$

(82) $$A_5(5) = \sqrt{2u}\sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t||\widehat{g} - \widetilde{g}| 1_{\{|\widehat{g}-\widetilde{g}| \le 4\tau\mathbb{E}(\sigma_t^2)\}}.$$

First, it is easy to show that

(83) $$\mathbb{E}\left|\sqrt{n}\Delta_i^n\widetilde{X} - \sqrt{n}\widetilde{\sigma}_{(i-1)\Delta_n}\Delta_i^n W\right|^{\beta+\iota} \le C\Delta_n^{1-\beta/2-\iota/2} \quad \forall \iota \in (0, 1-\beta],$$

and, therefore,

(84) $$\mathbb{E}(A_5(1)) \le C\Delta_n^{1-\beta/2-\iota/2} \quad \forall \iota \in (0, 1-\beta].$$

For $A_5(3)$ and $A_5(5)$, we can use (51) to get

(85) $$\mathbb{E}(A_5(3) + A_5(5)) \le C\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2} \quad \forall \iota > 0.$$

For $A_5(2)$ and $A_5(4)$, we can derive a bound on $\mathbb{E}(\frac{1}{T}\sum_{t=1}^{T}\sqrt{n}|\sigma_{(i_t-1)\Delta_n}\Delta_{i_t}^n W| - \sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t|)^2$ for $i = 1, \dots, n$ and $\mathbb{E}(\frac{1}{nT}\sum_{i=1}^{nT}\sqrt{n}|\sigma_{(i-1)\Delta_n}\Delta_i^n W| - \sqrt{\frac{2}{\pi}}\mathbb{E}|\sigma_t|)^2$ exactly as in (66) (using the integrability conditions on σ_t of the theorem and Assumption C), and then apply Cauchy–Schwarz inequality and (51) to get

(86) $$\mathbb{E}\big(|A_5(2)| + |A_5(4)|\big) \le C\Delta_n^{[(1-\beta/2)\varpi-\iota]\wedge 1/4}/\sqrt{T} \quad \forall \iota > 0.$$

Therefore, overall we have the bound

(87) $$\mathbb{E}|A_5| \le C\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2} + C\Delta_n^{[(1-\beta/2)\varpi-\iota]\wedge 1/4}/\sqrt{T} \quad \forall \iota > 0.$$

Combining all of the above bounds, we get that

$$\mathbb{E}\left|\frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u) - \mathbb{E}\big[e^{-u\sigma_t^2}\big] - A_1(1) - A_3^{(a)}(1)\right| \le C\left(\frac{1}{T^{1-\iota}} + \frac{\Delta_n^{[(1-\beta/2)\varpi-\iota]\wedge 1/4}}{\sqrt{T}} + \Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2}\right)$$

for $\iota > 0$ arbitrarily small. This together with (63) establishes the results in (41) and (42) in parts (a) and (b) of the theorem.

Part (c). We first show that $\widehat{\Sigma}(u)$ is consistent for $\Sigma(u)$ under the conditions of the theorem. Using the assumptions of the theorem and Proposition 1 in Andrews (1991), we have

$$C_0(u) + 2\sum_{i=1}^{L_T} \omega(i, L_T) C_i(u) \xrightarrow{\mathbb{P}} \Sigma(u), \tag{88}$$

$$C_i(u) = \frac{1}{T}\sum_{t=i+1}^{T} (\mathbf{z}_{t-i}(u)\mathbf{z}_t'(u) + \mathbf{z}_t(u)\mathbf{z}_{t-i}'(u)),$$

where $z_t(u)$ is defined in part (b) of the theorem. Therefore, we are left with bounding the difference $\widehat{\Sigma}(u) - (C_0(u) + 2\sum_{i=1}^{L_T} \omega(i, L_T) C_i(u))$. For this we use the bound

$$\|\mathbf{z}_t(u) - \widehat{\mathbf{z}}_t(u)\| \le C \sum_{j=1}^{6} |\widetilde{z}_t^{(j)}|, \tag{89}$$

where

$$\widetilde{z}_t^{(1)} = \frac{1}{n}\sum_{j=tn+1}^{tn+n} \cos\bigl(\sqrt{2un}\widehat{f}_j^{-1/2} 1_{\{\widehat{f}_j \neq 0\}} \Delta_j^n \widetilde{X}\bigr) - \int_t^{t+1} e^{-u\sigma_s^2}\, ds, \tag{90}$$

$$\widetilde{z}_t^{(2)} = \frac{1}{T}\widehat{V}_T(\widetilde{X}, \Delta_n, u) - \mathbb{E}\bigl(e^{-u\sigma_t^2}\bigr), \tag{91}$$

$$\widetilde{z}_t^{(3)} = \sum_{j=tn+1}^{tn+n} (\widehat{f}_j^{-1} \wedge T^{\eta} - 1)\left[(\Delta_j^n \widetilde{X})^2 1(|\Delta_j^n \widetilde{X}| \le \alpha \Delta_n^{\varpi}) - \int_{(j-1)\Delta_n}^{j\Delta_n} \widetilde{\sigma}_s^2\, ds\right], \tag{92}$$

$$\widetilde{z}_t^{(4)} = \sum_{j=tn+1}^{tn+n} \bigl(\widehat{f}_j^{-1} \wedge T^{\eta} - f_{j-[j/n]n}^{-1}\bigr) 1_{\{B_j^c \cap C_j^c\}} \int_{(j-1)\Delta_n}^{j\Delta_n} \widetilde{\sigma}_s^2\, ds, \tag{93}$$

$$\widetilde{z}_t^{(5)} = \sum_{j=tn+1}^{tn+n} \bigl(\widehat{f}_j^{-1} \wedge T^{\eta} - f_{j-[j/n]n}^{-1}\bigr) 1_{\{B_j \cup C_j\}} \int_{(j-1)\Delta_n}^{j\Delta_n} \widetilde{\sigma}_s^2\, ds, \tag{94}$$

$$\widetilde{z}_t^{(6)} = \sum_{j=tn+1}^{tn+n} \int_{(j-1)\Delta_n}^{j\Delta_n} \bigl(f_{j-[j/n]n}^{-1} \widetilde{\sigma}_s^2 - \sigma_s^2\bigr)\, ds. \tag{95}$$

In what follows we bound the second-order moments of each of the terms $\widetilde{z}_t^{(j)}$. From the proof of parts (a) and (b) of the theorem, using the boundedness of $\widetilde{z}_t^{(1)}$ and $\widetilde{z}_t^{(2)}$ as well as the relative speed condition between T and Δ_n of the theorem, we have

$$\mathbb{E}\big|\widetilde{z}_t^{(1)}+\widetilde{z}_t^{(2)}\big|^2\le\frac{C}{\sqrt{T}}. \tag{96}$$

For $\widetilde{z}_t^{(3)}$, we have

$$\mathbb{E}\big|\widetilde{z}_t^{(3)}\big|^2\le CT^{2\eta}\mathbb{E}\left(\sum_{j=tn+1}^{tn+n}\left|(\Delta_j^n\widetilde{X})^2 1(|\Delta_j^n\widetilde{X}|\le\alpha\Delta_n^{\varpi})-\int_{(j-1)\Delta_n}^{j\Delta_n}\widetilde{\sigma}_s^2\,ds\right|\right)^2. \tag{97}$$

Then for $i\neq j$, using successive conditioning, the decomposition in (47) above, and Hölder's inequality together with the integrability conditions in (40), we get

$$\begin{aligned}\mathbb{E}\Bigg\{&\left|(\Delta_i^n\widetilde{X})^2 1(|\Delta_i^n\widetilde{X}|\le\alpha\Delta_n^{\varpi})-\int_{(i-1)\Delta_n}^{i\Delta_n}\widetilde{\sigma}_s^2\,ds\right|\\&\times\left|(\Delta_j^n\widetilde{X})^2 1(|\Delta_j^n\widetilde{X}|\le\alpha\Delta_n^{\varpi})-\int_{(j-1)\Delta_n}^{j\Delta_n}\widetilde{\sigma}_s^2\,ds\right|\Bigg\}\le C\Delta_n^{2+[(4-2\beta)\varpi-\iota]\wedge 1}\end{aligned} \tag{98}$$

for $\iota>0$ arbitrarily small. Similar calculations give

$$\mathbb{E}\left|(\Delta_i^n\widetilde{X})^2 1(|\Delta_i^n\widetilde{X}|\le\alpha\Delta_n^{\varpi})-\int_{(i-1)\Delta_n}^{i\Delta_n}\widetilde{\sigma}_s^2\,ds\right|^2\le C\Delta_n^{1+(4-\beta)\varpi-\iota}\quad\forall\iota>0. \tag{99}$$

Combining these inequalities, we get

$$\mathbb{E}\big|\widetilde{z}_t^{(3)}\big|^2\le CT^{2\eta}\Delta_n^{[(4-2\beta)\varpi-\iota]\wedge 1}\quad\forall\iota>0. \tag{100}$$

Turning to $\widetilde{z}_t^{(4)}$, using the definition of the sets B_i and C_i, as well as first-order Taylor expansion, we have

$$\begin{aligned}\big|\widetilde{z}_t^{(4)}\big|\le C\sum_{j=tn+1}^{tn+n}\Bigg\{&\big[|\widehat{g}_j-\widetilde{g}_j|+|\widehat{g}-\widetilde{g}|+\big|\widetilde{g}_j-f_{j-[j/n]n}\mathbb{E}(\sigma_t^2)\big|\\&+|\widetilde{g}-\mathbb{E}(\sigma_t^2)|\big]1_{\{B_j^c\cap C_j^c\}}\int_{(j-1)\Delta_n}^{j\Delta_n}\widetilde{\sigma}_s^2\,ds\Bigg\}.\end{aligned} \tag{101}$$

Using the bounds in (52) and Hölder's inequality, as well as the integrability conditions in (40), we get

$$\mathbb{E}\big|\widetilde{z}_t^{(4)}\big|^2\le C\left(\frac{1}{T}+\Delta_n^{[(4-2\beta)\varpi-\iota]\wedge 1}\right)^{1-\iota}\quad\forall\iota>0. \tag{102}$$

Turning to $\widetilde{z}_t^{(5)}$, using the definition of the sets B_i and C_i as well as the trivial bound in (75), we get

$$\mathbb{E}\big|\widetilde{z}_t^{(5)}\big|^2 \le CT^{2\eta}\left(\frac{1}{T} + \Delta_n^{[(4-2\beta)\varpi-\iota]\wedge 1}\right)^{1-\iota} \quad \forall \iota > 0. \tag{103}$$

Finally, for $\widetilde{z}_t^{(6)}$ we can write, using the 0.5-Hölder continuity of the function f,

$$\mathbb{E}\big|\widetilde{z}_t^{(6)}\big|^2 \le C\Delta_n. \tag{104}$$

Using the above bounds, the square integrability of $\mathbf{z}_t(u)$ (which follows from the integrability conditions of the theorem), an application of Cauchy–Schwarz inequality, and the relative speed conditions between L_T, T, and Δ_n in the theorem, we get

$$\left\|\widehat{\Sigma}(u) - \left(C_0(u) + 2\sum_{i=1}^{L_T}\omega(i, L_T)C_i(u)\right)\right\| \le CL_T T^{\eta-1/2}. \tag{105}$$

This result combined with (88) proves the consistency of $\widehat{\Sigma}(u)$.

Finally, we prove

$$\frac{1}{nT}\sum_{j=1}^{nT}\frac{(\sqrt{2un}(\widehat{f}_j^{-1/2}\wedge T^{\eta/2})\Delta_j^n\widetilde{X})\sin(\sqrt{2un}(\widehat{f}_j^{-1/2}\wedge T^{\eta/2})\Delta_j^n\widetilde{X})}{\widehat{g}} \xrightarrow{\mathbb{P}} \frac{\mathbb{E}(G(u\sigma_t^2))}{\mathbb{E}(\sigma_t^2)}. \tag{106}$$

First, from (67) and (52), we have $\widehat{g} \xrightarrow{\mathbb{P}} \mathbb{E}(\sigma_t^2)$. Hence we only need to show

$$\frac{1}{nT}\sum_{j=1}^{nT}\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big)\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big) \xrightarrow{\mathbb{P}} \mathbb{E}(G(u\sigma_t^2)). \tag{107}$$

By a law of large numbers, we have

$$\frac{1}{nT}\sum_{j=1}^{nT}\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^n W\big)\sin\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^n W\big) \xrightarrow{\mathbb{P}} \mathbb{E}(G(u\sigma_t^2)), \tag{108}$$

and further we can make the decomposition

$$\begin{aligned}(109)\qquad & \frac{1}{nT}\sum_{j=1}^{nT}\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big)\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big)\\ &\quad -\frac{1}{nT}\sum_{j=1}^{nT}\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\sin\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &=\frac{1}{nT}\sum_{j=1}^{nT}\big(\zeta_j^{(1)}+\zeta_j^{(2)}+\zeta_j^{(3)}\big)\end{aligned}$$

for

$$\begin{aligned}(110)\qquad \zeta_j^{(1)} &= \big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big)\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\Delta_j^n\widetilde{X}\big)\\ &\quad -\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &\quad \times\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big),\end{aligned}$$

$$\begin{aligned}(111)\qquad \zeta_j^{(2)} &= \big\{\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &\quad \times\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &\quad -\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\sin\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\big\}1_{\{B_j^c\cap C_j^c\}},\end{aligned}$$

$$\begin{aligned}(112)\qquad \zeta_j^{(3)} &= \big\{\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &\quad \times\sin\big(\sqrt{2un}\big(\widehat{f}_j^{-1/2}\wedge T^{\eta/2}\big)\widetilde{\sigma}_{(j-1)\Delta_n}\Delta_j^nW\big)\\ &\quad -\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\sin\big(\sqrt{2un}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big)\big\}1_{\{B_j\cup C_j\}}.\end{aligned}$$

For $\zeta_j^{(1)}$, using the result in (83), we have

$$(113)\qquad \mathbb{E}\big|\zeta_j^{(1)}\big|\le T^{\eta}\sqrt{\Delta_n}.$$

For $\zeta_j^{(2)}$, we can use the bounds in (51), use the integrability condition in (40), and apply Hölder's inequality to get

$$\begin{aligned}(114)\qquad \mathbb{E}\big|\zeta_j^{(2)}\big| &\le C\mathbb{E}\big\{\big[|\widehat{g}_j-\widetilde{g}_j|+|\widehat{g}-\widetilde{g}|+\big|\widetilde{g}_j-f_{j-[j/n]n}\mathbb{E}(\sigma_t^2)\big|+|\widetilde{g}-\mathbb{E}(\sigma_t^2)|\big]\\ &\quad \times 1_{\{B_j^c\cap C_j^c\}}\big[\big|\sqrt{n}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big|\vee\big|\sqrt{n}\sigma_{(j-1)\Delta_n}\Delta_j^nW\big|^2\big]\big\}\\ &\le C\bigg(\frac{1}{\sqrt{T}}+\Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2}\bigg)^{1-\iota}\qquad \forall\iota>0.\end{aligned}$$

For $\zeta_j^{(3)}$, we can use Chebychev's inequality and proceed as above to get

$$\mathbb{E}\left|\zeta_j^{(3)}\right| \le CT^{\eta/2}\left(\frac{1}{\sqrt{T}} + \Delta_n^{[(2-\beta)\varpi-\iota]\wedge 1/2}\right)^{1-\iota} \quad \forall \iota > 0. \tag{115}$$

Taking into account the restriction on η in the theorem, we altogether get that $\frac{1}{nT}\sum_{j=1}^{nT}(\zeta_j^{(1)} + \zeta_j^{(2)} + \zeta_j^{(3)})$ is asymptotically negligible and hence we are done. *Q.E.D.*

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