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SUPPLEMENT TO "AMBIGUITY IN THE SMALL AND IN THE LARGE"

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APPENDIX S.A: LOCALLY LIPSCHITZ PREFERENCES

WE CONSIDER A PREFERENCE $\succcurlyeq$ that admits a monotonic, continuous, normalized, Bernoullian representation (I, u), and introduce a novel axiom that is equivalent to the assertion that I is locally Lipschitz.[1] Recall that $x_h \in X$ denotes the certainty equivalent of act $h \in \mathcal{F}$.

AXIOM 1—Locally Bounded Improvements: *For every $h \in \mathcal{F}^{\text{int}}$, there are $y \in X$ and $g \in \mathcal{F}$ with $g(s) \succ h(s)$ for all s such that, for all $(h^n) \subset \mathcal{F}$ and $(\lambda^n) \subset [0, 1]$ with $h^n \to h$ and $\lambda^n \downarrow 0$,*

$$\lambda^n g + (1 - \lambda^n) h^n \prec \lambda^n y + (1 - \lambda^n) x_{h^n} \quad \textit{eventually}.$$

To gain intuition, focus on the constant sequence with $h^n = h$. Since preferences are Bernoullian, the individual's evaluation of $\lambda y + (1 - \lambda) x_h$ changes linearly with λ. On the other hand, her evaluation of $\lambda g + (1 - \lambda) h$ may improve in arbitrary nonlinear (though continuous) ways as λ increases from 0 to 1 (recall that g is pointwise preferred to h). The axiom states that when λ is close to 0, this improvement is comparable to the *linear* change in preference that applies to $\lambda y + (1 - \lambda) x_h$ (which may still be very rapid, if y is much preferred to x_h). Hence, it imposes a bound on the instantaneous rate of change in preferences as a function of λ. Furthermore, this bound is required to be uniform in a neighborhood of h.

PROPOSITION S1: *Let $\succcurlyeq$ be a preference that admits a monotonic, continuous, Bernoullian, normalized representation (I, u). Then $\succcurlyeq$ satisfies Axiom 1 if and only if I is locally Lipschitz in the interior of its domain.*

PROOF: *If*. Functionally, the displayed equation in Axiom 1 is equivalent to

$$\begin{aligned} \text{(S1)} \qquad & I\big(\lambda^n\big[u \circ g - u \circ h^n\big] + u \circ h^n\big) \\ & \quad = I\big(\lambda^n u \circ g + \big(1 - \lambda^n\big) u \circ h^n\big) < I\big(\lambda^n u(y) + \big(1 - \lambda^n\big) u\big(x^n\big)\big) \\ & \quad = \lambda^n u(y) + \big(1 - \lambda^n\big) u\big(x^n\big) = \lambda^n\big[u(y) - I\big(u \circ h^n\big)\big] + I\big(u \circ h^n\big). \end{aligned}$$

[1]That is, for every $a \in \operatorname{int} B_0(\Sigma, u(X))$, there are $\varepsilon > 0$ and $L > 0$ such that $|I(b) - I(c)| \leq L\|b - c\|$ for all $b, c \in B_0(\Sigma, u(X))$ with $\|b - a\| < \varepsilon$ and $\|c - a\| < \varepsilon$.

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Notice that the second equality uses the assumption that I is normalized. Since $u \circ h^n \to u \circ h$ in the sup norm, for every $\varepsilon \in (0, \min_s[u(g(s)) - u(h(s))])$ and for n large enough, $\max_s |u(h(s)) - u(h^n(s))| < \min_s[u(g(s)) - u(h(s))] - \varepsilon$, so that, for every s, $u(h^n(s)) = u(h(s)) + [u(h^n(s)) - u(h(s))] < u(h(s)) + \min_{s'}[u(g(s')) - u(h(s'))] - \varepsilon \leq u(h(s)) + u(g(s)) - u(h(s)) - \varepsilon = u(g(s)) - \varepsilon$. In other words, $u(g(s)) - u(h^n(s)) > \varepsilon$ for all s and all n large enough. Moreover, for n large enough, $\lambda^n \varepsilon + h^n \in B_0(\Sigma, u(X))$. Since I is monotonic, rearranging terms yields

$$\frac{I(\lambda^n \varepsilon + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I\big(u \circ h^n\big) \quad \text{eventually.}$$

Again because $u \circ h^n \to u \circ h$, eventually $I(u \circ h^n) \geq I(u \circ h) - \varepsilon$, so finally

$$\frac{I(\lambda^n \varepsilon + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) + \varepsilon \quad \text{eventually.}$$

This implies that for a suitable $\varepsilon > 0$, $I^\circ(u \circ h; \varepsilon) \leq u(y) - I(u \circ h) + \varepsilon < \infty$.

To sum up, for every h such that $u \circ h \in \operatorname{int} B_0(\Sigma, u(X))$, there are $\varepsilon > 0$ and $y \in X$ such that $I^\circ(u \circ h; \varepsilon) \leq u(y) - I(u \circ h) + \varepsilon < \infty$. Since I is monotonic, by Proposition 4 in Rockafellar (1980), I is directionally Lipschitzian; by Theorem 3 therein, the Clarke–Rockafeller derivative of I in the direction a at $u \circ h$, denoted $I^\uparrow(u \circ h; a)$, equals $\liminf_{b \to a} I^\circ(u \circ h; b)$. Since $I^\circ(u \circ h; \cdot)$ is monotonic because I is, this implies that, for all a such that $a(s) < \varepsilon$, $I^\uparrow(u \circ h; a) \leq I^\circ(u \circ h; \varepsilon) < \infty$. Therefore, the constant function 0 is in the interior of $\{a : I^\uparrow(u \circ h; a) < \infty\}$. Again by Theorem 3 in Rockafellar (1980), this implies that I is directionally Lipschitz with respect to the vector 0; as noted on page 267 therein, it is "an easy fact to verify" that this is equivalent to the assertion that I is locally Lipschitz at $u \circ h$.

Only if. Conversely, suppose I is Lipschitz near $u \circ h$. Since h is interior, I is monotonic and normalized, and $I^\circ(u \circ h; \cdot)$ is continuous, there is $\varepsilon > 0$ such that $I^\circ(u \circ h; \varepsilon) < u(y) - I(u \circ h) - \varepsilon$ for some $y \in X$. Then, for all $(h^n) \to h$ and $(\lambda^n) \downarrow 0$, eventually

$$\begin{aligned} &\frac{I(\lambda^n[\varepsilon + u \circ h^n] + (1 - \lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} \\ &\quad = \frac{I(\lambda^n \varepsilon + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) - \varepsilon. \end{aligned}$$

Now choose n large enough so that $\max_s |u(h(s)) - u(h^n(s))| < \frac{\varepsilon}{2}$. Then a fortiori, for every s, $u(h(s)) - u(h^n(s)) < \frac{\varepsilon}{2}$, that is, $u(h(s)) < u(h^n(s)) + \frac{\varepsilon}{2}$ and, therefore, $u(h(s)) + \frac{\varepsilon}{2} < u(h^n(s)) + \varepsilon$. Because h is interior, there is $\delta \in (0, \frac{\varepsilon}{2}]$ such that $u \circ h + \delta = u \circ g$ for some $g \in \mathcal{F}$; for such g, the above argument implies that $u(g(s)) < u(h^n(s)) + \varepsilon$ for all s, and of course $g(s) \succ h(s)$ for all s. By monotonicity, conclude that, for all n sufficiently large,

$$\frac{I(\lambda^n u \circ g + (1-\lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) - \varepsilon.$$

Finally, by choosing n large enough, we can ensure that $I(u \circ h^n) < I(u \circ h) + \varepsilon$ and, therefore,

$$\frac{I(\lambda^n u \circ g + (1-\lambda^n) u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I\big(u \circ h^n\big).$$

Rearranging terms yields Eq. (S1), so the axiom holds. *Q.E.D.*

APPENDIX S.B: NICE MBL PREFERENCES

PROPOSITION S2: *A monotonic, isotone, and concave function $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$ (for some interval Γ) is nice everywhere in the interior of its domain.*

PROOF: Recall that a monotone concave I is locally Lipschitz; furthermore, ∂I coincides with the superdifferential of I (e.g., Rockafellar (1980, p. 278)) and it is monotone in the sense that

$$\text{(S2)} \qquad \forall c, c' \in \operatorname{int} B_0(\Sigma, \Gamma), Q \in \partial I(c), Q' \in \partial I\big(c'\big), \quad Q\big(c-c'\big) \leq Q'\big(c-c'\big).$$ [2]

Fix $c' \in \operatorname{int} B_0(\Sigma, \Gamma)$ and suppose that $Q_0 \in \partial I(c')$. Then, for every $c \in \operatorname{int} B_0(\Sigma, \Gamma)$ and every $Q \in \partial I(c)$, $Q(c - c') \leq 0$. Since c' is interior, the set $\hat{\Gamma} = \Gamma \cap \{\gamma \in \mathbb{R} : \gamma > c'(s)\ \forall s\}$ is nonempty. Moreover, for any $\gamma \in \hat{\Gamma}$ and for all $Q \in \partial I(1_S\gamma)$, $Q(1_S\gamma - c') \leq 0$. But since $\gamma - c'(s) > 0$ for all s and since I is monotonic, this requires that $\partial I(1_S\gamma) = \{Q_0\}$ for all $\gamma \in \hat{\Gamma}$.

In particular, pick $\alpha, \beta \in \hat{\Gamma}$ with $\alpha > \beta$. Since I is isotone, $I(1_S\alpha) > I(1_S\beta)$. By the mean-value theorem (Lebourg (1979)), there must be $\mu \in (0, 1)$ and $Q \in \partial I(\mu 1_S\alpha + (1-\mu)1_S\beta) = \partial I([\mu\alpha + (1-\mu)\beta]1_S)$ such that $I(1_S\alpha) - I(1_S\beta) = Q(1_S\alpha - 1_S\beta) = Q(1_S)(\alpha - \beta)$. But $\mu\alpha + (1-\mu)\beta \in \hat{\Gamma}$, so $Q = Q_0$, and, therefore, $I(1_S\alpha) = I(1_S\beta)$—a contradiction. Therefore, I must be nice at c. *Q.E.D.*

We now provide an axiom for MBL preferences that ensures niceness. There are obvious similarities with Axiom 1.

[2]Since ∂I is the superdifferential of I, $Q(c' - c) \geq I(c') - I(c)$ and $Q'(c - c') \geq I(c) - I(c')$. Summing these inequalities yields the inequality in the text.

AXIOM 2—Nonnegligible Worsenings at h: *There are* $y \in X$ *with* $y \prec h$ *and* $g \in \mathcal{F}$ *with* $g(s) \prec h(s)$ *for all* s *such that, for all* $(h^n) \subset \mathcal{F}$ *and* $(\lambda^n) \subset [0,1]$ *with* $h^n \to h$ *and* $\lambda^n \downarrow 0$,

$$\lambda^n g + (1-\lambda^n) h^n \prec \lambda^n y + (1-\lambda^n) x_{h^n} \quad \textit{eventually}.$$

This axiom rules out the possibility that preferences may be "flat" when moving from h toward pointwise less desirable acts g. We argue as for Axiom 1: the individual's evaluation of $\lambda y + (1-\lambda)x_h$ changes linearly with λ, whereas her evaluation of $\lambda g + (1-\lambda)h$ may worsen in arbitrary nonlinear ways as λ increases from 0 to 1. Axiom 2 states that when λ is close to 0, this worsening is comparable to the *linear* decrease in preference that applies to $\lambda y + (1-\lambda)x_h$ (which may still be very slow, if y is almost as good as x_h).

Mas-Colell (1977) characterized preferences over consumption bundles (i.e., on $\mathbb{R}^n_+$) represented by a (locally) Lipschitz and regular utility function; his notion of regularity is related to niceness (cf. Mas-Colell (1977, p. 1411)); for instance, if utility is continuously differentiable, the requirement is that its gradient be nonvanishing on $\mathbb{R}^n_{++}$. Mas-Colell's axiom is not directly related to ours.

PROPOSITION S3: *Let* $\succcurlyeq$ *be an MBL preference with representation* (I, u), *and assume that* I *is normalized. Then* $\succcurlyeq$ *satisfies Axiom 2 at* $h \in \mathcal{F}^{\text{int}}$ *if and only if* I *is nice at* $u \circ h$.

PROOF: *If*. As in the proof of Proposition S1, for $g, y, (h^n), (\lambda^n)$ as in the axiom,

$$\begin{aligned} &I\big(\lambda^n[u \circ g - u \circ h^n] + u \circ h^n\big) \\ &\quad < \lambda^n\big[u(y) - I\big(u \circ h^n\big)\big] + I\big(u \circ h^n\big) \quad \text{eventually}. \end{aligned}$$

For n large, $\|u \circ h^n - u \circ h\| < 1$ and, therefore, $u(h^n(s)) - u(g(s)) = [u(h^n(s)) - u(h(s))] + u(h(s)) - u(g(s)) < 1 + \max_s[u(h(s)) - u(g(s))] \equiv \delta$. Since $h(s) \succ g(s)$ for all s, $\delta > 0$. Furthermore, as $n \to \infty$, eventually $\lambda^n(-\delta) + u \circ h^n \in B_0(\Sigma, u(X))$ and so, by monotonicity of I,

$$I\big(\lambda^n(-\delta) + u \circ h^n\big) < \lambda^n\big[u(y) - I\big(u \circ h^n\big)\big] + I\big(u \circ h^n\big) \quad \text{eventually}.$$

Rearranging gives

$$\frac{I(\lambda^n(-\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I\big(u \circ h^n\big) \quad \text{eventually}.$$

Since $h^n \to h$ and I is continuous, for every $\varepsilon > 0$, eventually $I(u \circ h^n) \geq I(u \circ h) - \varepsilon$ and so

$$\frac{I(\lambda^n(-\delta) + u \circ h^n) - I(u \circ h^n)}{\lambda^n} < u(y) - I(u \circ h) + \varepsilon \quad \text{eventually}.$$

Therefore, $I^0(u \circ h; -\delta) \le u(y) - I(u \circ h) + \varepsilon$. Since this is true for all $\varepsilon > 0$, then $I^0(u \circ h; -\delta) \le u(y) - I(u \circ h) < 0$ as $y \prec h$. But since $I^0(u \circ h; -\delta) = \max_{Q \in \partial I(u \circ h)}(-\delta)Q(S) = -\delta \min_{Q \in \partial I(u \circ h)} Q(S)$ and every $Q \in \partial I(u \circ h)$ is a positive measure because I is monotonic, the zero measure Q_0 cannot belong to $\partial I(u \circ h)$.

Only if. Conversely, suppose I is nice at $u \circ h$. Since h is interior, there is $\delta > 0$ such that $u \circ h - \delta = u \circ g$ for some $g \in \mathcal{F}^{\text{int}}$. Since $Q_0 \notin \partial I(u \circ h)$ and I is monotonic, $I^0(u \circ h; -\frac{1}{2}\delta) < 0$. Hence, for all sequences $\lambda^n \to 0$ and $h^n \to h$ (acts), and for all $\varepsilon \in (0, -I^0(u \circ h; -\frac{1}{2}\delta))$, eventually

$$\frac{I\left(\lambda^n\left(-\frac{1}{2}\delta\right) + u \circ h^n\right) - I(u \circ h^n)}{\lambda^n} < -\varepsilon.$$

In particular, find $y \in X$ such that $y \prec h$ and $I(u \circ h) - u(y) < -\frac{1}{2}I^0(u \circ h; -\frac{1}{2}\delta)$, which is possible because h is interior. Add $-\frac{1}{2}I^0(u \circ h; -\frac{1}{2}\delta)$ on both sides of this inequality to conclude that $I(u \circ h) - u(y) - \frac{1}{2}I^0(u \circ h; -\frac{1}{2}\delta) < -I^0(u \circ h; -\frac{1}{2}\delta)$ and so eventually

$$\frac{I\left(\lambda^n\left(-\frac{1}{2}\delta\right) + u \circ h^n\right) - I(u \circ h^n)}{\lambda^n} \\ < u(y) - I(u \circ h) + \frac{1}{2}I^0\left(u \circ h; -\frac{1}{2}\delta\right).$$

Also, for n large, $I(u(h^n)) \le I(u(h)) - \frac{1}{2}I^0(u \circ h; -\frac{1}{2}\delta)$; conclude that, eventually,

$$\frac{I\left(\lambda^n\left(-\frac{1}{2}\delta\right) + u \circ h^n\right) - I(u \circ h^n)}{\lambda^n} < u(y) - I\big(u \circ h^n\big).$$

Rewriting yields

$$I\left(\lambda^n\left[-\frac{1}{2}\delta + u \circ h^n\right] + \big(1 - \lambda^n\big)u \circ h^n\right) \\ < \lambda^n\big[u(y) - I\big(u \circ h^n\big)\big] + I\big(u \circ h^n\big) \quad \text{eventually.}$$

Finally, if n is large enough, $\|u \circ h^n - u \circ h\| < \frac{1}{2}\delta$, so for all s, $-\frac{1}{2}\delta + u(h^n(s)) = -\frac{1}{2}\delta + u(h(s)) + [u(h^n(s)) - u(h(s))] > -\delta + u(h(s)) = u(g(s))$. Hence, finally, monotonicity implies

$$I\big(\lambda^n u \circ g + \big(1 - \lambda^n\big)u \circ h^n\big) \\ < \lambda^n u(y) - \big(1 - \lambda^n\big)I\big(u \circ h^n\big) \quad \text{eventually,}$$

as required. *Q.E.D.*

APPENDIX S.C: CALCULATIONS FOR EXAMPLE 4

Since I is continuously differentiable, it is strictly differentiable; see Clarke (1983, Corollary to Proposition 2.2.1). In particular, for all $e \in B_0(\Sigma)$, $h^n \to h$ and $\lambda^n \downarrow 0$, $(\lambda^n)^{-1}[I(\lambda^n e + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)] \to \nabla I(h) \cdot e$. Hence, if $\nabla I(h) \cdot f > \nabla I(h) \cdot g$, then for all sequences $\lambda^n \downarrow 0$ and $h^n \downarrow 0$, eventually $(\lambda^n)^{-1}[I(\lambda^n f + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)] > (\lambda^n)^{-1}[I(\lambda^n g + (1-\lambda^n)h^n) - I((1-\lambda^n)h^n)]$, so Eq. (7) will hold for n large: hence, in this case $f \succcurlyeq_h^* g$. This is, in particular, the case if $h_1 > h_2 \geq 0$.

To analyze Cases 2 and 3 of the example, note first that, for any pair $f, g \in \mathcal{F}$, using the formula for the difference of two cubes, $f \succcurlyeq g$ iff

$$\text{(S3)} \qquad \sum_{i=1,2} [P^i \cdot (f-g)]\big[(P^i \cdot f)^2 + (P^i \cdot g)^2 + (P^i \cdot f)(P^i \cdot g)\big] \geq 0.$$

Now consider ε, f, g, f_ε, and g_ε as in the main text. The rankings $\lambda^n f_\varepsilon + (1-\lambda^n)h^n \succcurlyeq \lambda^n g_\varepsilon + (1-\lambda^n)h^n$ and $\lambda^n f_\varepsilon + (1-\lambda^n)k^n \succcurlyeq \lambda^n g_\varepsilon + (1-\lambda^n)k^n$ are then equivalent to

$$\text{(S4)} \qquad \sum_{i=1,2} P^i \cdot \lambda^n[1+2\varepsilon, -1+2\varepsilon] \times \big\{\big[P^i \cdot \lambda^n[3+\varepsilon, 1+\varepsilon] + \gamma\big]^2 + \big[P^i \cdot \lambda^n[2-\varepsilon, 2-\varepsilon] + \gamma\big]^2 + \big[P^i \cdot \lambda^n[3+\varepsilon, 1+\varepsilon] + \gamma\big]\big[P^i \cdot \lambda^n[2-\varepsilon, 2-\varepsilon] + \gamma\big]\big\} \geq 0,$$

$$\text{(S5)} \qquad \sum_{i=1,2} P^i \cdot \lambda^n[1+2\varepsilon, -1+2\varepsilon] \times \big\{\big[P^i \cdot \lambda^n[2+\varepsilon, 2+\varepsilon] + \gamma\big]^2 + \big[P^i \cdot \lambda^n[1-\varepsilon, 3-\varepsilon] + \gamma\big]^2 + \big[P^i \cdot \lambda^n[2+\varepsilon, 2+\varepsilon] + \gamma\big]\big[P^i \cdot \lambda^n[1-\varepsilon, 3-\varepsilon] + \gamma\big]\big\} \geq 0.$$

In Case 3 ($\gamma = 0$), divide Eqs. (S4) and (S5) by $(\lambda^n)^3$, and set $\varepsilon = 0$ to obtain the conditions

$$(2p-1)\big[(1+2p)^2 + 4 + 2(1+2p)\big] + (1-2p)\big[\big(1+2(1-p)\big)^2 + 4 + 2\big(1+2(1-p)\big)\big] \geq 0,$$

$$(2p-1)\big[4 + \big(1+2(1-p)\big)^2 + 2\big(1+2(1-p)\big)\big] + (1-2p)\big[4 + (1+2p)^2 + 2(1+2p)\big] \geq 0;$$

by inspection, the left-hand side (l.h.s.) of the second inequality is the negative of the l.h.s. of the first. Furthermore, the l.h.s. of the first condition equals

$(2p-1)[(1+2p)^2-(1+2(1-p))^2+4(2p-1)]>0$, because $p>\frac{1}{2}$. Therefore, for any n, when $\varepsilon=0$, Eq. (S4) holds as a strict inequality, whereas the inequality in Eq. (S5) fails. Hence, the same is true for any n when ε is positive but small. Thus. $f_\varepsilon \not\succcurlyeq_h^* g_\varepsilon$ for any $\varepsilon \geq 0$ if $h=[0,0]$.

In Case 2 ($\gamma>0$), first take $\varepsilon=0$. We claim that Eqs. (S4) and (S5) can both hold only if they are, in fact, equalities. To see this, note that $P^1\cdot[\alpha,\beta]=P^2\cdot[\beta,\alpha]$ for any $\alpha,\beta\in\mathbb{R}$; hence, when $\varepsilon=0$ and $h=[\gamma,\gamma]$, the l.h.s. of Eq. (S5) can be rewritten as

$$\sum_{i=1,2} P^{3-i}\cdot\lambda^n[-1,1]\big\{\big[P^{3-i}\cdot\lambda^n[2,2]+\gamma\big]^2+\big[P^{3-i}\cdot\lambda^n[3,1]+\gamma\big]^2$$
$$+\big[P^{3-i}\cdot\lambda^n[2,2]+\gamma\big]\big[P^{3-i}\cdot\lambda^n[3,1]+\gamma\big]\big\}.$$

It is apparent that this is the negative of the l.h.s. of Eq. (S4) when $\varepsilon=0$ and $h=[\gamma,\gamma]$, except that we first use P^2 and then P^1, rather than the opposite as in Eq. (S4). This proves the claim.

Next, we claim that Eq. (S4) holds as a strict inequality, which proves the assertion in the text that $f\not\succcurlyeq_h^* g$. Since $p>\frac{1}{2}$ and $\gamma>0$, the first and third terms in braces are strictly greater for $i=1$ than for $i=2$. Since $P^2\cdot[1,-1]=-P^1\cdot[1,1]$, the l.h.s. of Eq. (S4) is the difference of these terms that is multiplied by $P^1\cdot\lambda^n[1,-1]>0$ and, hence, it is strictly positive.

Finally, if $\varepsilon>0$ and since $h=[\gamma,\gamma]$, we have $\nabla I(h)\cdot(f+\varepsilon)=\nabla I(h)\cdot f+\nabla I(h)\cdot\varepsilon=\nabla I(h)\cdot g+\nabla I(h)\cdot\varepsilon>\nabla I(h)\cdot g-\nabla I(h)\cdot\varepsilon=\nabla I(h)\cdot(g-\varepsilon)$, which, as noted above, implies that $f_\varepsilon\succcurlyeq_h^* g_\varepsilon$.

As noted in footnote 11 in the main paper, here $\partial I(0)$ contains *only* the zero vector. However, consider the monotonic, locally Lipschitz functional $J:\mathbb{R}^2\to\mathbb{R}$ given by $J(h)=\min(I(h),h_1+I(h))$. Then $J(h)=I(h)$ for $h\in\mathbb{R}^2$ with $h_1\geq 0$, and $\partial J(0)=\{[\gamma,0]:\gamma\in[0,1]\}$ (Clarke (1983, Theorem 2.5.1)). Since all mixtures in Eq. (8) are nonnegative when $h\in\mathbb{R}^2_+$ and $\varepsilon<1$, even if g is replaced with $g-\varepsilon$ (cf. the definition of k^n), the analysis in Example 4 applies verbatim to J. In particular, for all $\varepsilon\in[0,1)$, now $f+\varepsilon\succ_{C(0)} g-\varepsilon$, but $f+\varepsilon\not\succcurlyeq_0^* g-\varepsilon$ (the argument in the second paragraph of Example 4 does not apply because J is not (continuously) differentiable at 0).

APPENDIX S.D: RELEVANT PRIORS: A BEHAVIORAL TEST

We conclude by showing that, given an interior act h, whether a probability $P\in \mathrm{ba}_1(\Sigma)$ belongs to the set $C(h)$ can be ascertained without invoking Theorems 6 or 7; indeed, using only the DM's preferences. For the result, we need a notion of lower certainty equivalent of an act f for the incomplete, discontinuous preference $\succcurlyeq_h^*$ (cf. the definition of $C^*(f)$ in GMM, p. 158).

DEFINITION S1: For any act $f\in\mathcal{F}$, a *local lower certainty equivalent* of f at $h\in\mathcal{F}^{\text{int}}$ is a prize $\underline{x}_{f,h}\in X$ such that, for all $y\in X$, $y\prec\underline{x}_{f,h}$ implies $f\succcurlyeq_h^* y$ and $y\succ\underline{x}_{f,h}$ implies $f\not\succcurlyeq_h^* y$.

Furthermore, fix $P \in \mathrm{ba}_1(\Sigma)$ and $f \in \mathcal{F}$, and suppose that $f = \sum_{i=1}^n x_i 1_{E_i}$ for a collection of distinct prizes $x_1, \dots, x_n$ and a measurable partition $E_1, \dots, E_n$ of S. Then define

$$x_{P,f} \equiv P(E_1)x_1 + \cdots + P(E_n)x_n.$$

That is, $x_{P,f} \in X$ is a mixture of the prizes $x_1, \dots, x_n$ delivered by f, with weights given by the probabilities that P assigns to each event $E_1, \dots, E_n$. We then have the following corollary.

COROLLARY S4: *For any $P \in \mathrm{ba}_1(\Sigma)$ and $h \in \mathcal{F}^{\mathrm{int}}$ such that I is nice at $u \circ h$, $P \in C(h)$ if and only if, for all $f \in \mathcal{F}^{\mathrm{int}}$, $\underline{x}_{f,h} \preccurlyeq x_{P,f}$.*

PROOF: We show that $u(\underline{x}_{f,h}) = \min_{P \in C(h)} P(u \circ f)$; thus, the condition in the corollary states that P satisfies $P(u \circ f) \geq \min_{P' \in C(h)} P'(u \circ f)$ for all interior f, so $P(a) \geq \min_{P' \in C(h)} P(a)$ by linearity for all $a \in B_0(\Sigma)$, and $P \in C(h)$ then follows from standard arguments.

If $\underline{x}_{f,h}$ is as in Definition S1, then $\min_{P \in C(h)} P(u \circ f) \geq u(y)$ for all $y \prec \underline{x}_{f,h}$ by (i) in Theorem 6, and so $\min_{P \in C(h)} P(u \circ f) \geq u(\underline{x}_{f,h})$. Conversely, for every y with $u(y) < \min_{P \in C(h)} P(u \circ f)$, there are $\varepsilon > 0$, $y' \in X$, and $f' \in \mathcal{F}$ with $u(y') = u(y) + \varepsilon$, $u \circ f' = u \circ f - \varepsilon$, and $u(y') \leq \min_{P \in C(h)} P(u \circ f')$; then, by (ii) in Theorem 7, since (f, y) is a spread of (f', y'), $f \succcurlyeq_h^* y$. This implies that $y \preccurlyeq \underline{x}_{f,h}$. Hence, $\min_{P \in C(h)} P(u \circ f) \leq u(\underline{x}_{f,h})$ as well. *Q.E.D.*

APPENDIX S.E: ADDITIONAL PROPERTIES OF $\succcurlyeq_h^*$

In addition to agreeing with $\succcurlyeq$ on X, provided $\partial I(u \circ h) \neq \{Q_0\}$, $\succcurlyeq_h^*$ satisfies the following additional properties.

LEMMA S5: *The preference $\succcurlyeq_h^*$ is a monotonic, independent preorder.*

PROOF: Monotonicity and reflexivity are immediate from monotonicity of $\succcurlyeq$. Transitivity is immediate from the definition of $\succcurlyeq_h^*$ and transitivity of $\succcurlyeq$. It remains to be shown that $\succcurlyeq_h^*$ is independent; that is, for all $k \in \mathcal{F}$ and $\mu \in (0, 1]$, $f \succcurlyeq_h^* g$ iff $\mu f + (1-\mu)k \succcurlyeq_h^* \mu g + (1-\mu)k$. Note that

$$\begin{aligned}
&\lambda^n\big[\mu f + (1-\mu)k\big] + \big(1-\lambda^n\big)h^n \\
&\quad = \big(\lambda^n \mu\big) f + \big[1 - \big(\lambda^n \mu\big)\big]\left\{\frac{\lambda^n(1-\mu)}{1-(\lambda^n\mu)}k + \frac{1-\lambda^n}{1-(\lambda^n\mu)}h^n\right\} \\
&\quad \equiv \bar{\lambda}^n f + \big(1-\bar{\lambda}^n\big)\bar{h}^n
\end{aligned}$$

with $(\bar{\lambda}^n) \downarrow 0$ and $(\bar{h}^n) \to h$, and similarly for g. Hence, if $f \succcurlyeq_h^* g$, then eventually $\bar{\lambda}^n f + (1-\bar{\lambda}^n)\bar{h}^n \succcurlyeq \bar{\lambda}^n g + (1-\bar{\lambda}^n)\bar{h}^n$; repeating the argument for all $(\lambda^n), (h^n)$ implies that $\mu f + (1-\mu)k \succcurlyeq_h^* \mu g + (1-\mu)k$. Conversely, if $\mu f + (1-\mu)k \succcurlyeq_h^* \mu g + (1-\mu)k$, define $\tilde{\lambda}^n$ and $\tilde{h}^n$ so that

$$\tilde{\lambda}^n\big[\mu f + (1-\mu)k\big] + \big(1-\tilde{\lambda}^n\big)\tilde{h}^n = \lambda^n f + \big(1-\lambda^n\big)h^n:$$

this requires $\tilde{\lambda}^n = \frac{\lambda^n}{\mu}$, which is in $[0,1]$ for n large and converges to zero as $n \to \infty$, and

$$u \circ \tilde{h}^n = \frac{(1-\lambda^n)u \circ h^n - \tilde{\lambda}^n(1-\mu)u \circ k}{1-\tilde{\lambda}^n},$$

which is in $B_0(\Sigma, u(X))$ for n large (recall that h is interior) and indeed such that $\tilde{h}^n \to h$. Note that $\tilde{\lambda}^n$ and $\tilde{h}^n$ do not depend on f. Again, for n large, $\tilde{\lambda}^n[\mu f + (1-\mu)k] + (1-\tilde{\lambda}^n)\tilde{h}^n \succcurlyeq \tilde{\lambda}^n[\mu g + (1-\mu)k] + (1-\tilde{\lambda}^n)\tilde{h}^n$ and, therefore, by construction, $\lambda^n f + (1-\lambda^n)h^n \succcurlyeq \lambda^n g + (1-\lambda^n)h^n$ and so, repeating for all sequences, $f \succcurlyeq_h^* g$. *Q.E.D.*

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