Econometrica Supplementary Material

SUPPLEMENT TO "GENERICITY AND ROBUSTNESS OF FULL SURPLUS EXTRACTION"

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THIS SUPPLEMENT PROVIDES technical results and proofs.

S.1. PROOF OF LEMMA 8

LEMMA 8: *Let w be a CM mechanism defined on Θ^*. Then,*
(a) *for every $i \in I$, $U_i(\theta_i|w)$ is a continuous function in θ_i;*
(b) *for every $\varepsilon, \varepsilon' > 0$, $\{\mu \in \mathcal{P} : \mu_i[\Omega_i(\varepsilon|w)] > 1 - \varepsilon', \forall i \in I\}$ is open.*

S.1.1. *Proof of Lemma 8(a)*

It suffices to prove that $U_i(\theta_i|q^*, m^*)$ is a continuous function in θ_i, because the continuity of $U_i(\theta_i|q^*, m^*)$ and $w_i(v_{-i}(\theta_{-i}))$ implies the continuity of $U_i(\theta_i|w)$.

First, we show that $u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*)$ is continuous in θ for every $i \in I$. Note that

$$u_i\big(\theta_i, \theta_{-i}|\theta_i, q^*, m^*\big) = \begin{cases} v_i(\theta_i) - \max\limits_{j\neq i} v_j(\theta_j) & \text{if } v_i(\theta_i) > \max\limits_{j\neq i} v_j(\theta_j); \\ 0, & \text{if } v_i(\theta_i) \le \max\limits_{j\neq i} v_j(\theta_j), \end{cases}$$

which implies that

$$u_i\big(\theta_i, \theta_{-i}|\theta_i, q^*, m^*\big) = \max\Big\{0, v_i(\theta_i) - \max_{j\neq i} v_j(\theta_j)\Big\}, \quad \forall \theta \in \Theta^*.$$

Since v_j is continuous for every $j \in I$, $u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*)$ is continuous in θ.

Second, for every $(i, \theta, \theta') \in I \times \Theta^* \times \Theta^*$, we show that

$$\big|u_i\big(\theta_i, \theta_{-i}|\theta_i, q^*, m^*\big) - u_i\big(\theta'_i, \theta_{-i}|\theta'_i, q^*, m^*\big)\big| \le \big|v_i(\theta_i) - v_i\big(\theta'_i\big)\big|. \tag{S.1}$$

There are three cases to check: (i) Bidder i wins the object under both (θ_i, θ_{-i}) and (θ'_i, θ_{-i}), then,

$$u_i\big(\theta_i, \theta_{-i}|\theta_i, q^*, m^*\big) = v_i(\theta_i) - \max_{j\neq i} v_j(\theta_j);$$

$$u_i\big(\theta'_i, \theta_{-i}|\theta'_i, q^*, m^*\big) = v_i\big(\theta'_i\big) - \max_{j\neq i} v_j(\theta_j).$$

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Thus, (S.1) holds. (ii) Bidder i loses the object under both (θ_i, θ_{-i}) and (θ'_i, θ_{-i}); then,

$$u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) = u_i(\theta'_i, \theta_{-i}|\theta'_i, q^*, m^*) = 0,$$

that is, (S.1) holds. (iii) Bidder i wins the object under (θ_i, θ_{-i}) and loses the object under (θ'_i, θ_{-i}) (the case is similar if we switch θ'_i and θ_i); then,

$$u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) = v_i(\theta_i) - \max_{j\neq i} v_j(\theta_j);$$

$$u_i(\theta'_i, \theta_{-i}|\theta'_i, q^*, m^*) = 0 \quad \text{and} \quad \max_{j\neq i} v_j(\theta_j) \geq v_i(\theta'_i).$$

Thus,

$$\begin{aligned}\left|u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) - u_i(\theta'_i, \theta_{-i}|\theta'_i, q^*, m^*)\right| &= v_i(\theta_i) - \max_{j\neq i} v_j(\theta_j) \\ &\leq v_i(\theta_i) - v_i(\theta'_i),\end{aligned}$$

that is, (S.1) holds.

Third, we prove that $U_i(\theta_i|q^*, m^*)$ is a continuous function in θ_i. Note that, for any θ_i and θ'_i in Θ^*_i,

$$\begin{aligned}&\left|U^*_i(\theta_i|q^*, m^*) - U^*_i(\theta'_i|q^*, m^*)\right| \\ &\quad\leq \left|U^*_i(\theta_i|q^*, m^*) - \int_{\Theta^*_{-i}} u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) b_i(\theta'_i)[d\theta_{-i}]\right| \\ &\qquad + \left|\int_{\Theta^*_{-i}} u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) b_i(\theta'_i)[d\theta_{-i}] - U^*_i(\theta'_i|q^*, m^*)\right|.\end{aligned} \tag{S.2}$$

For any θ_i, consider the function defined as follows:

$$\Lambda : \Delta(\Theta^*_{-i}) \to \mathbb{R},$$

$$\Lambda(\mu_{-i}) = \int_{\Theta^*_{-i}} u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*)\mu_{-i}[d\theta_{-i}], \quad \forall \mu_{-i} \in \Delta(\Theta^*_{-i}).$$

Since $u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*)$ is continuous in θ, it is bounded on Θ^*. As a result, Λ is continuous. Note that $b_i(\theta_i), b_i(\theta'_i) \in \Delta(\Theta^*_{-i})$, and $\lim_{\theta'_i\to\theta_i} b_i(\theta'_i) = b_i(\theta_i)$. Consequently,

$$\begin{aligned}&\lim_{\theta'_i\to\theta_i}\left|U^*_i(\theta_i|q^*, m^*) - \int_{\Theta^*_{-i}} u_i(\theta_i, \theta_{-i}|\theta_i, q^*, m^*) b_i(\theta'_i)[d\theta_{-i}]\right| \\ &\quad = \lim_{\theta'_i\to\theta_i}\left|\Lambda(b_i(\theta_i)) - \Lambda(b_i(\theta'_i))\right| = 0.\end{aligned} \tag{S.3}$$

Furthermore,

$$\text{(S.4)}\qquad \lim_{\theta_i'\to\theta_i}\left|\int_{\Theta^*_{-i}} u_i(\theta_i,\theta_{-i}|\theta_i,q^*,m^*)b_i(\theta_i')[d\theta_{-i}] - U_i^*(\theta_i'|q^*,m^*)\right|$$
$$\le \lim_{\theta_i'\to\theta_i}\int_{\Theta^*_{-i}}\left|u_i(\theta_i,\theta_{-i}|\theta_i,q^*,m^*) - u_i(\theta_i',\theta_{-i}|\theta_i',q^*,m^*)\right|b_i(\theta_i')[d\theta_{-i}]$$
$$\le \lim_{\theta_i'\to\theta_i}\int_{\Theta^*_{-i}}\left|v_i(\theta_i)-v_i(\theta_i')\right|b_i(\theta_i')[d\theta_{-i}] = \lim_{\theta_i'\to\theta_i}\left|v_i(\theta_i)-v_i(\theta_i')\right| = 0,$$

where the second inequality follows from (S.1) and the last equality follows from the continuity of v_i.

Finally, (S.2)–(S.4) imply that

$$\lim_{\theta_i'\to\theta_i}\left|U_i^*(\theta_i|q^*,m^*) - U_i^*(\theta_i'|q^*,m^*)\right| = 0.$$

Therefore, $U_i(\theta_i|q^*,m^*)$ is a continuous function in θ_i.

S.1.2. *Proof of Lemma 8(b)*

Fix any μ such that $\mu_i[\Omega_i(\varepsilon|w)] > 1-\varepsilon'$ for every $i\in I$. Since μ is a finite Borel measure on the compact metric space Θ^*, it is tight (Aliprantis and Border (2006, 12.7 Theorem)). For every $i\in I$, since $\mu_i[\Omega_i(\varepsilon|w)] > 1-\varepsilon'$, there exists a compact set $E_i\subset\Omega_i(\varepsilon|w)$ such that $\mu_i[E_i] > 1-\varepsilon'$. We thus have $\min_{i\in I}\min_{\theta_i\in E_i} U_i(\theta_i|w) > 0$ and $\max_{i\in I}\max_{\theta_i\in E_i} U_i(\theta_i|q,m) < \varepsilon$, because $E_i\subset\Omega_i(\varepsilon|w)$. Hence, there exists $\zeta > 0$ such that

$$\text{(S.5)}\qquad 0<\zeta<\min_{i\in I}\min_{\theta_i\in E_i} U_i(\theta_i|q,m) \le \max_{i\in I}\max_{\theta_i\in E_i} U_i(\theta_i|q,m) < \varepsilon-\zeta<\varepsilon.$$

By Lemma 8(a) and the compactness of Θ_i^*, $U_i(\theta_i|w)$ is uniformly continuous in θ_i. As a result, there exists $\alpha > 0$ such that

$$\text{(S.6)}\qquad d_i(\theta_i,\theta_i') < \alpha \quad\Rightarrow\quad \left|U_i(\theta_i|w) - U_i(\theta_i'|w)\right| < \frac{\zeta}{2},\quad \forall i\in I.$$

Second, define $\beta = \frac{\min\{\alpha,\min_{i\in I}\mu_i(E_i)-(1-\varepsilon')\}}{2}$, which implies $\beta<\alpha$ and $\min_{i\in I}\mu_i(E_i) - \beta > 1-\varepsilon'$. We show that every μ' with $d_{\mathcal{P}}(\mu,\mu') < \beta$ satisfies $\mu_i'[\Omega_i(\varepsilon|w)] > 1-\varepsilon'$, which proves Lemma 8(b).

If $d_{\mathcal{P}}(\mu,\mu') < \beta$, we have

$$\text{(S.7)}\qquad \min_{i\in I}\mu_i'(E_i^\beta) > \min_{i\in I}\mu_i(E_i) - \beta > 1-\varepsilon'.$$

For any $i \in I$ and any $\theta_i' \in E_i^\beta$, there exists some $\theta_i \in E_i$, such that $d_i(\theta_i, \theta_i') < \beta < \alpha$. Then, by (S.5) and (S.6), we have

$$0 < \frac{\zeta}{2} < U_i(\theta_i'|w) < \varepsilon - \frac{\zeta}{2},$$

which implies $E_i^\beta \subset \Omega_i(\varepsilon|w)$. As a result, we have

$$\min_{i \in I} \mu_i'(\Omega_i(\varepsilon|w)) \geq \min_{i \in I} \mu_i'(E_i^\beta) > 1 - \varepsilon',$$

where the first inequality follows from $E_i^\beta \subset \Omega_i(\varepsilon|w)$; the second inequality follows from (S.7).

S.2. VIRTUAL BAYESIAN IMPLEMENTATION

We adapt the notation and definitions in Duggan (1997) to the auction setup in Chen and Xiong (2013). Throughout the section, we fix a prior μ with support Θ^μ. For simplicity, we write Θ for Θ^μ. Recall that $V = [0, 1]^n$ is the set of value profiles and the first-order belief of θ_i is a probability distribution $b_i^1(\theta_i) \in \Delta(V)$, defined as

$$b_i^1(\theta_i)(\overline{V}) = b_i(\theta_i)\{\theta_{-i} \in \Theta_{-i} : (v_i(\theta_i), v_{-i}(\theta_{-i})) \in \overline{V}\},$$
$$\forall \text{Borel set } \overline{V} \subset V.$$

Furthermore, define $b_i^{1,-i} \in \Delta(V_{-i})$ as

$$b_i^{1,-i}(\theta_i)(\overline{V}_{-i}) \equiv b_i(\theta_i)(\{\theta_{-i} \in \Theta_{-i} : v_{-i}(\theta_{-i}) \in \overline{V}_{-i}\}),$$
$$\forall \text{Borel set } \overline{V}_{-i} \subset V_{-i}.$$

Clearly, $(v_i(\theta_i), b_i^{1,-i}(\theta_i)) = (v_i(\theta_i'), b_i^{1,-i}(\theta_i'))$ implies that $b_i^1(\theta_i) = b_i^1(\theta_i')$.

Let $X = (\{0, 1\} \times \mathbb{R})^n$ be the set of outcomes. For each $x = (\overline{q}, \overline{m}) = (\overline{q}_i, \overline{m}_i)_{i \in I} \in X$, $\overline{q}_i = 1$ (resp. $\overline{q}_i = 0$) means "the object is (resp. is not) allocated to agent i" and $\overline{m}_i$ specifies the payment of agent i. We require $\sum_i \overline{q}_i \leq 1$, because only one object is for sale. Define the ex post utility of agent i at $\theta \in \Theta$ as

$$u_i(x|\theta) \equiv v_i(\theta_i)\overline{q}_i - \overline{m}_i \quad \text{for each } x = (\overline{q}_i, \overline{m}_i)_{i \in I}.$$

A *social choice function* f is a measurable function from Θ to $\Delta(X)$. Clearly, each direct mechanism (q, m) defined in Chen and Xiong (2013) identifies a social choice function $f^{(q,m)}$ such that, for each $\theta \in \Theta$,

$$\text{(S.8)} \qquad f^{(q,m)}(\theta)\big[\big((1, m_i(\theta)), (0, m_j(\theta))\big)_{j \neq i}\big] = q_i(\theta), \quad \forall i \in I;$$
$$f^{(q,m)}(\theta)\big[(0, m_j(\theta))_{j \in I}\big] = 1 - \sum_{j \in I} q_j(\theta).$$

DEFINITION S.1: A social choice function $f : \Theta \to \Delta(X)$ is first-order measurable if, for each $\theta, \theta' \in \Theta$, $b_i^1(\theta_i) = b_i^1(\theta_i')$ for all i implies that $f(\theta) = f(\theta')$.

We say f is value-measurable if $v_i(\theta_i) = v_i(\theta_i')$ for all i implies that $f(\theta) = f(\theta')$. Clearly, any value-measurable f is first-order measurable. By Duggan (1997, Proposition 4 and Theorem 2), any value-measurable f that satisfies IC can be virtually Bayesian implemented. Theorem S.1 below extends this result: any first-order measurable f that satisfies IC can be virtually Bayesian implemented.

Formally, let $(A = \prod_{i\in I} A_i, g : A \to \Delta(X))$ be a mechanism where A_i is the message space for agent i. A pure strategy of player i is a function $\sigma_i : \Theta_i \to A_i$. A strategy profile $\sigma = (\sigma_i)_{i\in I}$ is a BNE in (A, g) if

$$\sigma_i(\theta_i) \in \arg\max_{a_i \in A_i} \int_{\Theta_{-i}} \int_X u_i(x|\theta) \times g\big(a_i, \sigma_{-i}(\theta_{-i})\big)[dx] b_i(\theta_i)[d\theta_{-i}], \quad \forall \theta_i \in \Theta_i, \forall i \in I.$$

Each social choice function f identifies a direct mechanism $(A = \prod_{i\in I} \Theta_i, f : \prod_{i\in I} \Theta_i \to \Delta(X))$. We say f is (*Bayesian*) *Incentive Compatible (IC)* if truthful reporting (i.e., $\sigma_i(\theta_i) \equiv \theta_i$) is a BNE in the direct mechanism $(A = \prod_{i\in I} \Theta_i, f : \Theta \to \Delta(X))$.

DEFINITION S.2—Duggan (1997): A social choice function f is virtually Bayesian implementable on Θ with prior μ if, for any $\varepsilon > 0$, there are a social choice function h and a mechanism (A, g) such that

(1) $$\sup_{\theta\in\Theta, Y\subset X \text{ is measurable}} \big|h(\theta)(Y) - f(\theta)(Y)\big| < \varepsilon;$$

(2) for any BNE σ in (A, g), $\mu(\{\theta \in \Theta : g \circ \sigma(\theta) = h(\theta)\}) = 1$.

That is, f is virtually Bayesian implementable if, for any $\varepsilon > 0$, there exist a social choice function h and a mechanism (A, g) such that h is ε-close to f in the sense of (1) and every BNE in (A, g) induces the outcome of h with μ-probability 1.

PROPOSITION S.1: *There is a mechanism (A, g) on Θ with $A_i = V_i \times \Delta(V_{-i})$ such that the unique BNE in (A, g) on Θ is $\sigma_i(\theta_i) = (v_i(\theta_i), b_i^{1,-i}(\theta_i))$ for every $(i, \theta) \in I \times \Theta$.*

Proposition S.1 is a counterpart of Duggan (1997, Proposition 4). As in Duggan (1997), we construct a scoring-rule game (i.e., (A, g)) such that truthfully reporting the first-order beliefs is the unique BNE. The proof is relegated to Section S.2.1.

With Proposition S.1, we are ready to present the main result of the section, that is, Theorem S.1, which implies that every first-order IC mechanism employed in Chen and Xiong (2013) is virtually Bayesian implementable.

THEOREM S.1: *If a social choice function $f:\Theta \to \Delta(X)$ is IC and first-order measurable, then it is virtually Bayesian implementable.*

PROOF: Fix any $\varepsilon \in (0,1)$. Let (A, g) be the mechanism given in Proposition S.1. Define a social choice function $h(\theta) \equiv (1-\varepsilon)f(\theta) + \varepsilon g(v(\theta), b^{1,-i}(\theta))$. We show that there exists a mechanism (A^*, g^*) such that $\mu(\{\theta \in \Theta : g^* \circ \sigma(\theta) = h(\theta)\}) = 1$ for every BNE σ in (A^*, g^*).

Define $A_i^* = \Theta_i \times (V_i \times \Delta(V_{-i})) \times \mathbb{Z}_+$ and $A^* = \prod_{i\in I} A_i^*$. Denote the elements of A_i^* by $a_i = (\theta_i, v_i, b_i^{1,-i}, z_i)$. For each $a \in A^*$, define a set $\Psi(a)$ and an allocation $x(a)$ as follows:

$$\Psi(a) = \Big\{ i \in I : z_i > 0 \text{ or } \big(v_i(\theta_i), b_i^{1,-i}(\theta_i)\big) \neq \big(v_i, b_i^{1,-i}\big) \Big\},$$

$$x(a) = (q_i = 0, m_i = -z_i \times 1_{(z_i = \max_{j\in I} z_j)})_{i\in I},$$

where $1_{(\cdot)}$ is the indicator function, that is, $\Psi(a)$ is the set of agents who report $z_i > 0$ or her reported type is inconsistent with her reported first-order belief; $x(a)$ is the allocation such that no one gets the object and agent i gets paid z_i if it is the highest among all z_j and 0 otherwise.

Define the outcome function g^* as follows. For any $a = (\theta_i, v_i, b_i^{1,-i}, z_i)_{i\in I} \in A^*$,

$$g^*(a) = \begin{cases} (1-\varepsilon)f(\theta) + \varepsilon g\big(v(\theta), b^{1,-i}(\theta)\big), & \text{if } \big|\Psi(a)\big| \le 1; \\ \delta_{\{x(a)\}}, & \text{otherwise.} \end{cases}$$

That is, the outcome is determined by $(1-\varepsilon)f + \varepsilon g$ if 1 or 0 agent reports a positive integer or an inconsistent profile of type and first-order belief; otherwise, with probability 1 (δ stands for the Dirac measure), no one gets the object, the person who announces the highest integer gets paid the integer she announces, and other players have no payment.

First, $\sigma^* = (\sigma_i^*)_{i\in I}$ with $\sigma_i^*(\theta_i) \equiv (\theta_i, v_i(\theta_i), b_i^{1,-i}(\theta_i), 0)$ is a BNE in (A^*, g^*) on Θ. This follows from IC of f and Proposition S.1.

Second, we prove $\mu(\{\theta \in \Theta : g^* \circ \sigma(\theta) = h(\theta)\}) = 1$ for any BNE,

$$\sigma = \big(\alpha_i : \Theta_i \to \Theta_i, \beta_i : \Theta_i \to V_i \times \Delta(V_{-i}), \gamma_i : \Theta_i \to \mathbb{Z}_+\big)_{i\in I},$$

in (A^*, g^*) in four steps.

Step 1. For every $(i, \theta) \in I \times \Theta$, $b_i(\theta_i)(\Phi_{-i}^{\sigma}) = 1$, where

$$\Phi_{-i}^{\sigma} = \big\{ \theta_{-i} \in \Theta_{-i} : \big(v_j\big(\alpha_j(\theta_j)\big), b_j^1\big(\alpha_j(\theta_j)\big)\big) = \beta_j(\theta_j) \\ \text{and } \gamma_j(\theta_j) = 0, \forall j \neq i \big\}.$$

That is, i believes with probability 1 that all of her opponents report consistent profile of type and first-order belief and the zero integer. If not, i can deviate to report a positive integer z_i, so that $|\Psi((\theta_i, v_i, b_i^{1,-i}, z_i), \sigma_{-i}(\theta_{-i}))| \geq 2$ for any $\theta_{-i} \in \Theta_{-i} \setminus \Phi_{-i}^{\sigma}$. As $z_i \to \infty$, the probability of z_i being the highest among all z_j (with $j \in I$) goes to 1 on $\Theta_{-i} \setminus \Phi_{-i}^{\sigma}$. Agent i thus gets paid $z_i \to \infty$ with a probability converging to $b_i(\theta_i)(\Theta_{-i} \setminus \Phi_{-i}^{\sigma}) > 0$. Therefore, i finds it profitable to deviate to report a sufficiently large z_i.

Step 2. $\mu(\Phi^{\sigma}) = 1$, *where*

$$\Phi^{\sigma} = \{\theta \in \Theta : v_i(\alpha_i(\theta_i)), b_i^1(\alpha_i(\theta_i)) = \beta_i(\theta_i) \text{ and } \gamma_i(\theta_i) = 0, \forall i \in I\}.$$

That is, μ assigns probability 1 to the event that all players report consistent profile of type and first-order belief and the zero integer. This is immediately implied by step 1 and equation (2) in Chen and Xiong (2013).

Step 3. $\beta(\theta) = (v(\theta), b^{1,-i}(\theta)), \forall \theta \in \Theta$.

Suppose to the contrary that there is some $\theta_i' \in \Theta_i$ with $\beta_i(\theta_i') \neq (v_i(\theta_i'), b_i^{1,-i}(\theta_i'))$. Then, by Proposition S.1, β is not a BNE in (A, g). Hence, some $\theta_i \in \Theta_i$ can deviate to play some $\widetilde{\beta}_i(\theta_i)$and get a strictly higher payoff in (A, g). That is,

$$\int_{\Theta_{-i}} \int_X u_i(x|\theta) g(\widetilde{\beta}_i(\theta_i), \beta_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}]$$
$$> \int_{\Theta_{-i}} \int_X u_i(x|\theta) g(\beta_i(\theta_i), \beta_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}].$$

By step 1, $b_i(\theta_i)(\Phi_{-i}^{\sigma}) = 1$. Then, it follows from the definition of g^* that, by playing $(\alpha_i(\theta_i), \widetilde{\beta}_i(\theta_i), \gamma_i(\theta_i))$ in (A^*, g^*), θ_i gets the expected payoff

$$\begin{aligned}
&(1-\varepsilon) \int_{\Theta_{-i}} \int_X u_i(x|\theta) f(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}] \\
&\quad + \varepsilon \int_{\Theta_{-i}} \int_X u_i(x|\theta) g(\widetilde{\beta}_i(\theta_i), \beta_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}] \\
&> (1-\varepsilon) \int_{\Theta_{-i}} \int_X u_i(x|\theta) f(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}] \\
&\quad + \varepsilon \int_{\Theta_{-i}} \int_X u_i(x|\theta) g(\beta_i(\theta_i), \beta_{-i}(\theta_{-i}))[dx] b_i(\theta_i)[d\theta_{-i}],
\end{aligned}$$

where the latter is the expected payoff of θ_i by playing $(\alpha_i(\theta_i), \beta_i(\theta_i), \gamma_i(\theta_i))$. Hence, it is profitable for θ_i to deviate from $(\alpha_i(\theta_i), \beta_i(\theta_i), \gamma_i(\theta_i))$ to play $(\alpha_i(\theta_i), \widetilde{\beta}_i(\theta_i), \gamma_i(\theta_i))$. This contradicts the assumption that σ is a BNE. Hence, step 3 follows.

Step 4. $\mu(\{\theta \in \Theta : f(\theta) = f(\alpha(\theta))\}) = 1.$
Step 2 implies

$$\mu\big(\big\{\theta \in \Theta : \big(v_i\big(\alpha_i(\theta_i)\big), b_i^1\big(\alpha_i(\theta_i)\big)\big) = \beta_i(\theta_i), \forall i \in I\big\}\big) = 1. \tag{S.9}$$

Then, step 3 and (S.9) imply

$$\mu\big(\big\{\theta \in \Theta : \big(v_i\big(\alpha_i(\theta_i)\big), b_i^1\big(\alpha_i(\theta_i)\big)\big) = \big(v_i(\theta_i), b_i^1(\theta_i)\big), \forall i \in I\big\}\big) = 1.$$

That is, the first-order beliefs of θ and $\alpha(\theta)$ match with probability 1. Since f is a first-order mechanism, we have

$$\mu\big(\big\{\theta \in \Theta : f(\theta) = f\big(\alpha(\theta)\big)\big\}\big) = 1.$$

Finally,

$$\begin{aligned}
&\mu\big(\big\{\theta \in \Theta : g^* \circ \sigma(\theta) = h(\theta)\big\}\big) \\
&\quad = \mu\big(\big\{\theta \in \Theta : \\
&\qquad (1-\varepsilon) f\big(\alpha(\theta)\big) + \varepsilon g\big(\beta(\theta)\big) = (1-\varepsilon) f(\theta) + \varepsilon g\big(v(\theta), b^{1,-i}(\theta)\big)\big\}\big) \\
&\quad \geq \mu\big(\big\{\theta \in \Theta : f\big(\alpha(\theta)\big) = f(\theta)\big\} \\
&\qquad \cap \big\{\theta \in \Theta : g\big(\beta(\theta)\big) = g\big(v(\theta), b^{1,-i}(\theta)\big)\big\}\big) \\
&\quad = 1,
\end{aligned}$$

where the first equality follows from step 2 and the definition of g^*, and the last equality follows from steps 3 and 4. *Q.E.D.*

S.2.1. *Proof of Proposition S.1*

First, fix a player i. Note that the collection of bounded rectangles in $V_{-i} = [0,1]^{n-1}$ with rational endpoints

$$\mathcal{R}_{\mathbb{Q}}^{n-1} = \left\{\prod_{j \neq i} (a_j, b_j] : a_j, b_j \in \mathbb{Q}, j \neq i\right\}$$

forms a π-system that generates the Borel σ-algebra on V_{-i} (see Billingsley (1995, pp. 176–177)). Since $\mathcal{R}_{\mathbb{Q}}^{n-1}$ is countable, we enumerate $\mathcal{R}_{\mathbb{Q}}^{n-1}$ as $\{V_{-i,k}\}_{k=1}^{\infty}$ of V_{-i}. By Billingsley (1995, Theorem 10.3), for any measures $b_i^{1,-i}, b_i^{1,-i\prime} \in \Delta(V_{-i})$,

$$b_i^{1,-i} = b_i^{1,-i\prime} \quad \text{iff} \quad b_i^{1,-i}(V_{-i,k}) = b_i^{1,-i\prime}(V_{-i,k}) \quad \text{for every } k \in \mathbb{N}. \tag{S.10}$$

For any $k \in \mathbb{N}$, define $S_{i,k} : \Delta(V_{-i}) \times V_{-i} \to \mathbb{R}$ as follows:

$$S_{i,k}\big(b_i^{1,-i}, v_{-i}\big) \equiv \begin{cases} 2b_i^{1,-i}(V_{-i,k}) - b_i^{1,-i}(V_{-i,k})^2, & \text{if } v_{-i} \in V_{-i,k}, \\ -b_i^{1,-i}(V_{-i,k})^2, & \text{if } v_{-i} \notin V_{-i,k}. \end{cases}$$

Observe that $S_{i,k}$ is the quadratic scoring rule (to elicit $b_i^{1,-i}(V_{-i,k})$). Then, define

$$S_i\big(b_i^{1,-i}, v_{-i}\big) = \sum_{k=1}^{\infty} \frac{1}{2^k} S_{i,k}\big(b_i^{1,-i}, v_{-i}\big),$$

and we show that S_i is the scoring rule to elicit $b_i^{1,-i}$. Formally, we prove that

$$\big\{b_i^{1,-i\prime}\big\} = \arg \max_{b_i^{1,-i} \in \Delta(V_{-i})} \int_{V_{-i}} S_i\big(b_i^{1,-i}, v_{-i}\big) b_i^{1,-i\prime}(dv_{-i}). \tag{S.11}$$

Suppose that agent i gets $S_i(b_i^{1,-i}, v_{-i})$ if she reports $b_i^{1,-i}$ and her opponents truthfully report v_{-i}. Then, (S.11) implies that agent i with $b_i^{1,-i\prime}$ must truthfully report $b_i^{1,-i\prime}$.

For any $b_i^{1,-i}, b_i^{1,-i\prime} \in \Delta(V_{-i})$,

$$\begin{aligned} &\int_{V_{-i}} S_i\big(b_i^{1,-i}, v_{-i}\big) b_i^{1,-i\prime}(dv_{-i}) \\ &\quad = \sum_{l=1}^{\infty} \frac{1}{2^l} \big[2b_i^{1,-i}(V_{-i,l}) b_i^{1,-i\prime}(V_{-i,l}) - b_i^{1,-i}(V_{-i,l})^2\big] \\ &\quad = \sum_{l=1}^{\infty} \frac{1}{2^l} \big[b_i^{1,-i\prime}(V_{-i,l})^2 - \big(b_i^{1,-i\prime}(V_{-i,l}) - b_i^{1,-i}(V_{-i,l})\big)^2\big]. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{V_{-i}} S_i\big(b_i^{1,-i\prime}, v_{-i}\big) b_i^{1,-i\prime}(dv_{-i}) - \int_{V_{-i}} S_i\big(b_i^{1,-i}, v_{-i}\big) b_i^{1,-i\prime}(dv_{-i}) \\ &\quad = \sum_{l=1}^{\infty} \frac{1}{2^l} \big(b_i^{1,-i\prime}(V_{-i,l}) - b_i^{1,-i}(V_{-i,l})\big)^2 \geq 0, \quad \forall b_i^{1,-i} \in \Delta(V_{-i}), \end{aligned}$$

which implies $b_i^{1,-i\prime} \in \arg\max_{b_i^{1,-i} \in \Delta(V_{-i})} \int_{V_{-i}} S_i(b_i^{1,-i}, v_{-i}) b_i^{1,-i\prime}(dv_{-i})$. Furthermore, (S.10) implies $b_i^{1,-i\prime}$ is the unique maximizer, that is, (S.11) holds.

We are ready to define the mechanism $(A, g : A \to \Delta(X))$ in Proposition S.1. Define $A_i = V_i \times \Delta(V_{-i})$ and $A = \prod_{i \in I} A_i$. Let $a = (v_i, b_i^{1,-i})_{i \in I}$ be a message profile. We specify the distribution $g(a)$ (on X) as follows: with probability

$v_i/|I|$, player i gets the object and pays $v_i/2 - S_i(b_i^{1,-i}, v_{-i})$, and every player $j \neq i$ pays $-S_j(b_j^{1,-j}, v_{-j})$; with probability $1 - \sum_{j\in I} v_j/|I|$, no player gets the object, and every player j pays $-S_j(b_j^{1,-j}, v_{-j})$. Formally, for each $i \in I$, define $(\overline{q}_j(i), \overline{m}_j(i))_{j=1}^n$ as follows:

$$\overline{q}_j(i) = \begin{cases} 1, & \text{if } j = i; \\ 0, & \text{if } j \neq i; \end{cases}$$

$$\overline{m}_j(i) = \begin{cases} \dfrac{v_j}{2} - S_j\big(b_j^{1,-j}, v_{-j}\big), & \text{if } j = i; \\ -S_j\big(b_j^{1,-j}, v_{-j}\big), & \text{if } j \neq i. \end{cases}$$

Then, define

$$g(a)\big[\big(\overline{q}_j(i), \overline{m}_j(i)\big)_{j=1}^n\big] = v_i/|I|, \quad \forall i \in I;$$

$$g(a)\big[\big(0, -S_j\big(b_j^{1,-j}, v_{-j}\big)\big)_{j=1}^n\big] = 1 - \sum_{j\in I} v_j/|I|.$$

We now show that $\sigma = (\sigma_i)_{i\in I}$ is a BNE in $(\mathcal{A}, g)$ on Θ iff, for any i, $\sigma_i(\theta_i) = (v_i(\theta_i), b_i^{1,-j}(\theta_i))$ for all θ_i.

For any $(i, \theta) \in I \times \Theta$, let $\sigma_i(\theta_i) = (\sigma_i^v(\theta_i), \sigma_i^b(\theta_i)) \in V_i \times \Delta(V_{-i})$ be the strategy of type θ_i. Given σ, the expected payoff of θ_i is

$$\text{(S.12)} \qquad \frac{\sigma_i^v(\theta_i)}{|I|}\left(v_i(\theta_i) - \frac{\sigma_i^v(\theta_i)}{2}\right) + \int_{\Theta_{-i}} S_i\big(\sigma_i^b(\theta_i), \sigma_{-i}^v(\theta_{-i})\big) b_i(\theta_i)(d\theta_{-i}).$$

FOC implies that $\sigma_i^v(\theta_i) = v_i(\theta_i)$ in any BNE. That is, every type must truthfully reports her true value. Given this, the expected payoff of θ_i is

$$\text{(S.13)} \qquad \frac{[v_i(\theta_i)]^2}{2|I|} + \int_{\Theta_{-i}} S_i\big(\sigma_i^b(\theta_i), v_{-i}(\theta_{-i})\big) b_i(\theta_i)(d\theta_{-i})$$
$$= \frac{[v_i(\theta_i)]^2}{2|I|} + \int_{V_{-i}} S_i\big(\sigma_i^b(\theta_i), v_{-i}\big) b_i^{1,-i}(\theta_i)(dv_{-i}),$$

which implies $\sigma_i^b(\theta_i) = b_i^{1,-i}(\theta_i)$ in any BNE, by (S.11).

Finally, $\sigma_i(\theta_i) = (v_i(\theta_i), b_i^{1,-i}(\theta_i))$ for all θ_i and $i \in I$ is indeed a BNE in $(\mathcal{A}, g)$. Note that if $\sigma_j(\theta_j) = (v_j(\theta_j), b_j^{1,-j}(\theta_j))$ for all θ_j and $j \neq i$, then by (S.12) and (S.13), the type θ_i's expected payoff is

$$\frac{\sigma_i^v(\theta_i)}{|I|}\left(v_i(\theta_i) - \frac{\sigma_i^v(\theta_i)}{2}\right) + \int_{V_{-i}} S_i\big(\sigma_i^b(\theta_i), v_{-i}\big) b_i^{1,-i}(\theta_i)(dv_{-i}).$$

Hence, $\sigma_i^v(\theta_i) = v_i(\theta_i)$ and $\sigma_i^b(\theta_i) = b_i^{1,-i}(\theta_i)$ maximize the payoff of θ_i. This completes the proof of Proposition S.1. *Q.E.D.*

S.3. FINER TOPOLOGIES

We first provide a sufficient condition for the genericity of FSE under a finer topology than the weak* topology. Recall that

$$\mathcal{F}^{\mathrm{cm}} \equiv \{\mu \in \mathcal{P} : \mu \text{ admits FSE in a CM mechanism}\},$$
$$\mathcal{P}_n^f \equiv \{\mu \in \mathcal{P}^f : |\Theta_i^\mu| = n, \forall i \in I\},$$
$$\mathcal{F}^* \equiv \left\{\mu \in \bigcup_{n=1}^{\infty} \mathcal{P}_n^f : \mu \text{ has full rank, full support, and distinct values}\right\}.$$

LEMMA S.1: $\mathcal{F}^* \subset \mathcal{F}^{\mathrm{cm}} \cap \mathcal{M}$.

$\mathcal{F}^*$, $\mathcal{F}^{\mathrm{cm}}$, and $\mathcal{M}$ are defined independent of the topology on priors. The proof of Lemma S.1 can be found in Chen and Xiong (2013).

THEOREM S.2: *Given a topology on $\mathcal{P}$ which is finer than the weak* topology, if $\mathcal{F}^*$ is dense in $\mathcal{P}$, then $\mathcal{F}$ is generic in $\mathcal{P}$.*

The proof of Theorem S.2 is the same as the proofs of Theorem 1 in Chen and Xiong (2013).

S.3.1. *Weak*-Hausdorff Topology*

Recall that the Prohorov metric $d_{\mathcal{P}}$ is defined as

$$d_{\mathcal{P}}(\mu, \mu') = \inf\{\varepsilon > 0 : \mu(E) \leq \mu'(E^\varepsilon) + \varepsilon, \forall \text{Borel set } E \subset \Theta^*\},$$
$$\forall \mu, \mu' \in \Delta(\Theta^*),$$

where $E^\varepsilon \equiv \{\theta' : \inf_{\theta \in E} d(\theta', \theta) < \varepsilon\}$.

Consider the Hausdorff metric $d_{\mathcal{P}}^{\mathrm{H}}$ defined as

$$d_{\mathcal{P}}^{\mathrm{H}}(\mu, \mu') = \max\left\{\sup_{\theta \in \Theta^\mu} \inf_{\theta' \in \Theta^{\mu'}} d(\theta, \theta'), \sup_{\theta' \in \Theta^{\mu'}} \inf_{\theta \in \Theta^\mu} d(\theta, \theta')\right\},$$
$$\forall \mu, \mu' \in \Delta(\Theta^*),$$

where d is the metric on Θ^*, and Θ^μ and $\Theta^{\mu'}$ denote the supports of μ and μ', respectively.

Define a new metric on priors as

$$d_{\mathcal{P}}^{\mathrm{WH}}(\mu, \mu') = \max\{d_{\mathcal{P}}(\mu, \mu'), d_{\mathcal{P}}^{\mathrm{H}}(\mu, \mu')\}, \quad \forall \mu, \mu' \in \Delta(\Theta^*),$$

that is, the topology induced by $d_{\mathcal{P}}^{\mathrm{WH}}$ is the weakest topology which is finer than both the weak* topology and the Hausdorff topology.

LEMMA S.2: *$\mathcal{P}^f$ is dense in $\mathcal{P}$ under the topology induced by $d_{\mathcal{P}}^{\mathrm{WH}}$.*

LEMMA S.3: *$\mathcal{F}^*$ is dense in $\mathcal{P}$ under the topology induced by $d_{\mathcal{P}}^{\mathrm{WH}}$.*

As in Chen and Xiong (2013), Lemma S.3 is an immediate consequence of two facts: (i) $\mathcal{F}^*$ is dense in $\mathcal{P}^f$ (see footnote 14 in Chen and Xiong (2013)); (ii) $\mathcal{P}^f$ is dense in $\mathcal{P}$ (Lemma S.2). We prove Lemma S.2 below.

Before proving Lemma S.2, we review the definition of d on Θ^* (i.e., the product topology) as follows.

Let $d_i^0(\theta_i, \theta_i') = |v_i(\theta_i) - v_i(\theta_i')|$. Recursively, for any integer $k \geq 1$, and $\theta_i, \theta_i' \in \Theta_i^*$, let

$$d_i^k(\theta_i, \theta_i') = \max_{k' \leq k-1} \{ d_i^{k'}(\theta_i, \theta_i'), \rho_i^k(b_i(\theta_i), b_i(\theta_i')) \},$$

where ρ_i^k is the Prohorov distance on $\Delta(\Theta_{-i}^*)$ when Θ_{-i}^* is endowed with the metric d_{-i}^{k-1}, that is,

$$\rho_i^k(b_i(\theta_i), b_i(\theta_i')) \equiv \inf\{ \varepsilon > 0 : b_i(\theta_i)(E_{-i}) \leq b_i(\theta_i')(E_{-i}^{k-1,\varepsilon}) + \varepsilon, \\ \forall \text{Borel set } E_{-i} \subset \Theta_{-i}^* \},$$

where $E_{-i}^{k-1,\varepsilon} \equiv \{\theta_{-i}' : \inf_{\theta_{-i} \in E_{-i}} d_{-i}^{k-1}(\theta_{-i}', \theta_{-i}) < \varepsilon\}$. Note that d_i^k is also a metric on Θ_i^* and

$$d_i^{k'}(\theta_i, \theta_i') \leq d_i^k(\theta_i, \theta_i') \leq 1, \quad \forall k' \leq k. \tag{S.14}$$

Then,

$$d_i(\theta_i, \theta_i') \equiv \sum_{k=1}^{\infty} 2^{-k} d_i^k(\theta_i, \theta_i'); \tag{S.15}$$

$$d(\theta, \theta') \equiv \max_{i \in I} d_i(\theta_i, \theta_i').$$

PROOF OF LEMMA S.2: Fix any $\mu \in \mathcal{P}$, $k \in \mathbb{N}$, and $\varepsilon > 0$. Since Θ^μ is compact, we can partition Θ_i^μ (i.e., the projection of Θ^μ on Θ_i^*) as $\{\Theta_{i,1}, \Theta_{i,2}, \ldots, \Theta_{i,N}\}$ for some $N \in \mathbb{N}$ such that (A) each Θ_{i,n_i} has nonempty interior; (B) the diameter (measured by d_i^k) of Θ_{i,n_i} is strictly less than ε. Note that (A) implies that

$$\mu_i[\Theta_{i,n_i}] > 0; \tag{S.16}$$

(B) implies that

$$\sup_{\theta_i, \theta_i' \in \Theta_{i,n_i}} d_i^k(\theta_i, \theta_i') < \varepsilon. \tag{S.17}$$

By (S.14) and (S.17), it follows that

$$\sup_{\theta_i, \theta_i' \in \Theta_{i,n_i}} d_i^{k'}(\theta_i, \theta_i') < \varepsilon, \quad \forall k' \leq k. \tag{S.18}$$

Now consider an (abstract) finite type space $(\widehat{\Theta}_i, \widehat{v}_i, \widehat{b}_i)_{i \in I}$ defined as follows (see Section 4.4 in Chen and Xiong (2013)):

$$\widehat{\Theta}_i = \{\widehat{\Theta}_{i,1}, \widehat{\Theta}_{i,2}, \dots, \widehat{\Theta}_{i,N}\}.$$

That is, each type in the finite type space corresponds to an element in the partition $\{\Theta_{i,1}, \Theta_{i,2}, \dots, \Theta_{i,N}\}$. To make a distinction, we use Θ_{i,n_i} to denote a partition element and use $\widehat{\Theta}_{i,n_i}$ to denote the corresponding finite type in the finite type space.

First, μ naturally induces a probability measure $\widehat{\mu} \in \Delta(\prod_{i \in I} \widehat{\Theta}_i)$ such that

$$\widehat{\mu}\big[(\widehat{\Theta}_{i,n_i})_{i \in I}\big] \equiv \mu\left[\prod_{i \in I} \Theta_{i,n_i}\right] \quad \text{for any } n_i \in \{1, 2, \dots, N\}. \tag{S.19}$$

Similarly, we can define the marginal distribution $\widehat{\mu}_i$ of $\widehat{\mu}$ accordingly: $\widehat{\mu}_i[\widehat{\Theta}_{i,n_i}] \equiv \mu_i[\Theta_{i,n_i}]$ for any $i \in I$ and any $n_i \in \{1, 2, \dots, N\}$.

Then, we define

$$v_i(\widehat{\Theta}_{i,n_i}) = \frac{\displaystyle\int_{\Theta_{i,n_i}} v_i(\theta_i)\mu_i[d\theta_i]}{\widehat{\mu}_i[\widehat{\Theta}_{i,n_i}]}; \tag{S.20}$$

$$b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] = \frac{\widehat{\mu}[(\widehat{\Theta}_{i,n_i}, \widehat{\Theta}_{-i,n_{-i}})]}{\widehat{\mu}_i[\widehat{\Theta}_{i,n_i}]}. \tag{S.21}$$

Note that (S.20) and (S.21) are well defined because $\widehat{\mu}_i[\widehat{\Theta}_{i,n_i}] \equiv \mu_i[\Theta_{i,n_i}] > 0$ by (S.16). That is, $\widehat{\mu}$ is a prior of the finite type space since $b_i(\widehat{\Theta}_{i,n_i})$ is induced from $\widehat{\mu}$ by Bayes' rule. We identify $\widehat{\Theta}$ with its corresponding belief subspace $\eta(\widehat{\Theta}) \subset \Theta^*$ and $\widehat{\mu}$ with the induced finite prior on $\eta(\widehat{\Theta})$ (see Section 4.4 in Chen and Xiong (2013)). Finally, we show that

$$d_i^{k'}(\theta_i, \widehat{\Theta}_{i,n_i}) \leq (k'+1)\varepsilon, \quad \forall \theta_i \in \Theta_{i,n_i}, \forall k' \leq k, \forall i \in I. \tag{$\bigstar$}$$

Note that ($\bigstar$) and (S.15) imply that

$$d_i(\theta_i, \widehat{\Theta}_{i,n_i}) \leq (k+1)\varepsilon + \left(\frac{1}{2}\right)^k, \quad \forall i \in I, \forall \theta_i \in \Theta_{i,n_i},$$

which further implies that

$$d_{\mathcal{P}}(\mu, \widehat{\mu}) \leq (k+1)\varepsilon + \left(\frac{1}{2}\right)^k$$

and

$$d_{\mathcal{P}}^{\mathrm{H}}(\mu, \widehat{\mu}) \leq (k+1)\varepsilon + \left(\frac{1}{2}\right)^k,$$

that is, $d_{\mathcal{P}}^{\mathrm{WH}}(\mu, \widehat{\mu}) \leq (k+1)\varepsilon + (\frac{1}{2})^k$. Since ε is arbitrary, take $\varepsilon = \frac{1}{k^2}$ and we obtain

$$d_{\mathcal{P}}^{\mathrm{WH}}(\mu, \widehat{\mu}) \leq \frac{(k+1)}{k^2}\varepsilon + \left(\frac{1}{2}\right)^k.$$

Since k is arbitrary, it follows that $d_{\mathcal{P}}^{\mathrm{WH}}(\mu, \widehat{\mu}) \to 0$ as we choose $k \to \infty$. Therefore, $\mathcal{P}^f$ is dense in $\mathcal{P}$ under the topology induced by $d_{\mathcal{P}}^{\mathrm{WH}}$.

We now prove ($\bigstar$) by induction on k' ($\leq k$). The case of $k' = 0$ follows directly from (S.18) and (S.20). For the induction step, we assume that ($\bigstar$) holds for $k' - 1$ and prove that ($\bigstar$) holds for k' in the following three steps.

Step 1. For any $\widehat{\Theta}_{i,n_i}$ and $\widehat{\Theta}_{-i,n_{-i}}$, we have

$$b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] = \frac{1}{\mu_i[\Theta_{i,n_i}]} \int_{\Theta_{i,n_i}} b_i(\theta_i)[\Theta_{-i,n_{-i}}]\mu_i[d\theta_i]. \tag{S.22}$$

It follows from (S.19) and (S.21) that

$$b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] = \frac{\widehat{\mu}[(\widehat{\Theta}_{i,n_i}, \widehat{\Theta}_{-i,n_{-i}})]}{\widehat{\mu}_i[\widehat{\Theta}_{i,n_i}]} = \frac{\mu[(\Theta_{i,n_i}, \Theta_{-i,n_{-i}})]}{\mu_i[\Theta_{i,n_i}]}. \tag{S.23}$$

Moreover, by (2) in Chen and Xiong (2013),

$$\mu[\Theta_{i,n_i} \times \Theta_{-i,n_{-i}}] = \int_{\Theta_{i,n_i}} b_i(\theta_i)[\Theta_{-i,n_{-i}}]\mu_i[d\theta_i]. \tag{S.24}$$

Then, (S.22) follows from (S.23) and (S.24).

Step 2. For any $\widehat{\Theta}_{i,n_i}$ and $\widehat{\Theta}_{-i,n_{-i}}$, we have $b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] \leq b_i(\theta_i)[\Theta_{-i,n_{-i}}^{k'-1,\varepsilon}] + \varepsilon, \forall \theta_i \in \Theta_{i,n_i}$.

By (S.18), $d_i^{k'}(\theta_i, \theta_i') \leq \varepsilon$, which implies that

$$b_i(\theta_i')[\Theta_{-i,n_{-i}}] \leq b_i(\theta_i)\big[\Theta_{-i,n_{-i}}^{k'-1,\varepsilon}\big] + \varepsilon. \tag{S.25}$$

Thus, given any $\theta_i \in \Theta_{i,n}$,

$$b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] = \frac{1}{\mu_i[\Theta_{i,n_i}]}\int_{\Theta_{i,n_i}} b_i(\theta_i')[\Theta_{-i,n_{-i}}]\mu_i[d\theta_i'] \le \frac{1}{\mu_i[\Theta_{i,n_i}]}\int_{\Theta_{i,n_i}} \big(b_i(\theta_i)\big[\Theta^{k'-1,\varepsilon}_{-i,n_{-i}}\big] + \varepsilon\big)\mu_i[d\theta_i'] = b_i(\theta_i)\big[\Theta^{k'-1,\varepsilon}_{-i,n_{-i}}\big] + \varepsilon, \tag{S.26}$$

where the first equality follows from (S.22) and the inequality follows from (S.25).

Step 3. For any $\widehat{\Theta}_{i,n_i}$, we have $d_i^{k'}(\theta_i, \widehat{\Theta}_{i,n_i}) \le (k'+1)\varepsilon$ for any $\theta_i \in \Theta_{i,n_i}$.

By the induction hypothesis, $d_j^{k'-1}(\theta_j, \widehat{\Theta}_{j,n_j}) \le k'\varepsilon$ for any $\theta_j \in \Theta_{j,n_j}$. Thus,

$$\Theta^{k'-1,\varepsilon}_{-i,n_{-i}} \subset \big\{\widehat{\Theta}^{k'-1}_{-i,n_{-i}}\big\}^{(k'+1)\varepsilon}. \tag{S.27}$$

Hence, for any $\theta_i \in \Theta_{i,n}$, (S.26) and (S.27) imply that

$$b_i(\widehat{\Theta}_{i,n_i})[\widehat{\Theta}_{-i,n_{-i}}] \le b_i(\theta_i)\big[\Theta^{k'-1,\varepsilon}_{-i,n_{-i}}\big] + \varepsilon \le b_i(\theta_i)\big[\big\{\widehat{\Theta}^{k'-1}_{-i,n_{-i}}\big\}^{(k'+1)\varepsilon}\big] + (k'+1)\varepsilon.$$

Therefore, $d_i^{k'}(\theta_i, \widehat{\Theta}_{i,n_i}) \le (k'+1)\varepsilon$ for any $\theta_i \in \Theta_{i,n_i}$. This completes the induction step. *Q.E.D.*

S.3.2. *The Topology Induced by the Total Variation Norm*

Consider the total variation norm ρ^T defined as

$$\rho^T(\mu, \mu') = \sup\{|\mu(E) - \mu'(E)| : E \subset \Theta^*\}, \quad \forall \mu, \mu' \in \Delta(\Theta^*).$$

We show below that (i) $\mathcal{F}$ is nongeneric in $\mathcal{P}$ under ρ^T; (ii) ρ^T induces the discrete topology on $\mathcal{M}^f \equiv \mathcal{M} \cap P^f$.

PROPOSITION S.2: *$\mathcal{F}$ is nongeneric in $\mathcal{P}$ under ρ^T.*

PROOF: Consider $\widehat{\mu}$ as follows:

$\widehat{\mu}$	$\widehat{\theta}_2$ with $v_2(\widehat{\theta}_2) = 0$
$\widehat{\theta}_1$ with $v_1(\widehat{\theta}_1) = 1$	$\frac{1}{2}$
$\widehat{\eta}_1$ with $v_1(\widehat{\eta}_1) = 1/2$	$\frac{1}{2}$

.

Consider

$$\widehat{\mathcal{P}} = \{\mu \in \mathcal{P} : \mu(\Theta^{\widehat{\mu}}) > 0\}.$$

Clearly, $\widehat{\mathcal{P}}$ is open in $\mathcal{P}$ under ρ^T. $\widehat{\mathcal{P}}$ is also dense in $\mathcal{P}$ under ρ^T. Take any $\mu \in \mathcal{P}$. Define, for any n, $\mu_n \equiv (1-\frac{1}{n})\mu + \frac{1}{n}\widehat{\mu}$. Then, $\mu_n \in \widehat{\mathcal{P}}$ and $\rho^T(\mu_n, \mu) \to 0$ as $n \to \infty$. Hence, $\widehat{\mathcal{P}}$ is open and dense and hence generic.

We show that

$$\widehat{\mathcal{P}} \cap \mathcal{F} = \emptyset. \tag{S.28}$$

Since $\widehat{\theta}_1$ and $\widehat{\eta}_1$ have the same interim beliefs but different values, for any mechanism (q, m) that is IR and IC and achieves ε-SE on $\Theta^{\widehat{\mu}}$ with $\varepsilon \in (0, 1/8)$, we derive a contradiction by showing that $U_1(\widehat{\theta}_1|q, m) > \frac{1}{4}$. To see this, note that since bidder 1 has a strictly higher value, in any mechanism that achieves ε-SE on $\Theta^{\widehat{\mu}}$ with $\varepsilon \in (0, 1/8)$, $q_1(\widehat{\theta}_1, \widehat{\theta}_2) \geq 1 - 2\varepsilon$ and $q_1(\widehat{\eta}_1, \widehat{\theta}_2) \geq 1 - 4\varepsilon$. Hence, by IC,

$$\begin{aligned}
U_1(\widehat{\theta}_1|q, m) - \frac{1}{4} &\geq U_1(\widehat{\eta}_1|\widehat{\theta}_1, q, m) - \frac{1}{4} \\
&= (1 - 2\varepsilon) - m_1(\widehat{\eta}_1, \widehat{\theta}_2) - \frac{1}{4} \\
&> \frac{1}{2} - m_1(\widehat{\eta}_1, \widehat{\theta}_2) \\
&\geq U_1(\widehat{\theta}_1|q, m) \geq 0,
\end{aligned}$$

where the first inequality follows from IC and the last inequality follows from IR. Hence, FSE is impossible on Θ^{μ}. Thus, (S.28) holds. As a result, $\mathcal{F}$ is non-generic in $\mathcal{P}$. *Q.E.D.*

PROPOSITION S.3: *ρ^T induces the discrete topology on $\mathcal{M}^f$.*

PROOF: We prove that, for any $\mu \in \mathcal{M}^f$, there is some $\varepsilon_\mu > 0$ such that, for any $\mu' \in \mathcal{M}^f$, $\rho^T(\mu', \mu) < \varepsilon_\mu$ only if $\mu' = \mu$, which implies that $\{\mu\}$ is open. Define $\varepsilon_\mu = \frac{\min\{\mu[\{\theta\}]:\theta\in\Theta^\mu\}}{2} > 0$. If $\rho^T(\mu', \mu) < \varepsilon_\mu$, then $\mu'[\{\theta\}] > 0$ for all $\theta \in \Theta^\mu$. That is, $\Theta^{\mu'} \supseteq \Theta^\mu$. Since Θ^μ is a belief subspace, and μ' is a model with $\mu'(\Theta^\mu) > 0$, by M2 in Proposition S.4 below, $\Theta^{\mu'} = \Theta^\mu$. By Lemma 2.9.2 in Mertens, Sorin, and Zamir (1994), we know that $\mu' = \mu$. *Q.E.D.*

S.4. MODEL AND MINIMAL BELIEF SUBSPACE

In this section, we provide two alternative definitions of models (i.e., (M2) and (M3) in Proposition S.4) and prove that they are equivalent to the one we adopted in Chen and Xiong (2013) (recall that a model is a prior μ such that there exist no priors π and v and $\alpha \in (0,1)$ such that $\Theta^\pi \neq \Theta^v$ and $\mu = \alpha\pi + (1-\alpha)v$). We then construct a model μ whose support properly contains a belief subspace and hence is not a minimal belief subspace. We argue that it is reasonable to regard such μ as a model that confronts a mechanism designer following our argument in Chen and Xiong (2013, Section 3.4).

PROPOSITION S.4: *The following three statements for a prior μ are equivalent*:
(M1) *μ is a model.*
(M2) *There exists no belief subspace Θ such that $\Theta \subsetneq \Theta^\mu$ and $\mu(\Theta) > 0$.*
(M3) *For μ-almost all θ, Θ^μ is the minimal belief subspace containing θ_i for all $i \in I$.*

PROOF: (M2) $\Rightarrow$ (M1): Suppose that μ is not a model, that is, there exist two priors π and v and $\alpha \in (0,1)$ such that $\Theta^\pi \neq \Theta^v$ and $\mu = \alpha\pi + (1-\alpha)v$. Hence, $(\Theta^\pi \setminus \Theta^v) \cup (\Theta^v \setminus \Theta^\pi) \neq \emptyset$. Without loss of generality, suppose that $\Theta^v \setminus \Theta^\pi \neq \emptyset$, that is, $\Theta^\pi \subsetneq \Theta^\mu$. Then, Θ^π is a belief subspace, and moreover,

$$\mu(\Theta^\pi) = \alpha\pi(\Theta^\pi) + (1-\alpha)v(\Theta^\pi) \geq \alpha\pi(\Theta^\pi) = \alpha > 0.$$

Since $\Theta^\pi \subsetneq \Theta^\mu$, μ does not satisfy (M2). Hence, (M2) $\Rightarrow$ (M1).

(M3) $\Rightarrow$ (M2): Suppose that (M2) does not hold for μ, that is, there exists a belief subspace Θ such that $\Theta \subsetneq \Theta^\mu$ and $\mu(\Theta) > 0$. Since $\Theta \subsetneq \Theta^\mu$, for each $\theta \in \Theta$, Θ^μ is not the minimal belief subspace containing θ_i for all $i \in I$. Since $\mu(\Theta) > 0$, μ does not satisfy (M3).

(M1) $\Rightarrow$ (M3): Suppose that (M3) does not hold for μ, that is, there is some $\Theta' \subset \Theta^\mu$ with $\mu(\Theta') > 0$ such that, for every $\theta \in \Theta'$, Θ^μ is not the minimal belief subspace containing θ_i for some i. We make the following four claims which will be proved in Sections S.4.1–S.4.4.

CLAIM 1: *There exists a belief subspace $\Theta \subsetneq \Theta^\mu$ such that $\mu(\Theta) \in (0,1)$.*

CLAIM 2: $\mu(\prod_i \Theta_i^c) = 1 - \mu(\Theta)$, *where $\Theta_i^c \equiv \Theta_i^\mu \setminus \Theta_i$ for every $i \in I$.*

CLAIM 3: $\mu(\cdot|\Theta)$ *is a prior.*

CLAIM 4: $\mu(\cdot|\prod_i \Theta_i^c)$ *is a prior.*

By Claims 1–4, we show that μ is not a model. For notational ease, define

$$\pi(\cdot) \equiv \mu(\cdot|\Theta) \quad \text{and} \quad v(\cdot) \equiv \mu\left(\cdot|\prod_i \Theta_i^c\right).$$

First, since $(\prod_i \Theta_i^c) \cap \Theta = \emptyset$, Claim 2 implies that

$$\mu\left(\Theta^\mu \setminus \left(\Theta \cup \left(\prod_i \Theta_i^c\right)\right)\right) = 0. \tag{S.29}$$

Thus, for every $E \subset \Theta^*$,

$$\begin{aligned}
\mu(E) &= \mu(\Theta \cap E) + \mu\left(\left(\prod_i \Theta_i^c\right) \cap E\right) \\
&= \mu(\Theta) \times \frac{\mu(\Theta \cap E)}{\mu(\Theta)} + \mu\left(\prod_i \Theta_i^c\right) \times \frac{\mu\left(\left(\prod_i \Theta_i^c\right) \cap E\right)}{\mu\left(\prod_i \Theta_i^c\right)} \\
&= \mu(\Theta) \times \frac{\mu(\Theta \cap E)}{\mu(\Theta)} + \left[1 - \mu(\Theta)\right] \times \frac{\mu\left(\left(\prod_i \Theta_i^c\right) \cap E\right)}{\mu\left(\prod_i \Theta_i^c\right)} \\
&= \mu(\Theta) \times \pi(E) + \left[1 - \mu(\Theta)\right] \times \nu(E),
\end{aligned} \tag{S.30}$$

where the first equality follows because $(\prod_i \Theta_i^c) \cap \Theta = \emptyset$ and (S.29) holds; the third follows from Claim 2. Then, (S.30), together with Claims 3 and 4, implies that μ is not a model if $\Theta^\pi \neq \Theta^\nu$. Suppose to the contrary that $\Theta^\pi = \Theta^\nu$. Since Θ is a belief subspace, it is closed. Furthermore, $\pi(\Theta) = 1$. Hence,

$$\Theta^\pi \subset \Theta. \tag{S.31}$$

Since $\Theta^\pi = \Theta^\nu$, it follows from (S.30) that

$$\begin{aligned}
\mu\left(\Theta^\pi\right) &= \mu(\Theta) \times \pi\left[\Theta^\pi\right] + \left[1 - \mu(\Theta)\right] \times \nu\left(\Theta^\nu\right) \\
&= \mu(\Theta) + \left[1 - \mu(\Theta)\right] = 1.
\end{aligned}$$

As a result,

$$\Theta^\mu \subset \Theta^\pi. \tag{S.32}$$

Finally, (S.31) and (S.32) imply $\Theta^\mu \subset \Theta$, contradicting $\Theta \subsetneq \Theta^\mu$. Therefore, $\Theta^\pi \neq \Theta^\nu$ and hence μ is not a model. Thus, (M1) $\Rightarrow$ (M3). *Q.E.D.*

S.4.1. *Proof of Claim 1*

Since $\Theta^\mu \subset \Theta^*$ is a compact metric space, pick a countable dense set $\{\theta^n\}_{n=1}^\infty \subset \Theta^\mu$. For every $\theta_i \in \Theta_i^\mu$, let Θ^{θ_i} denote the minimal belief subspace containing θ_i. Since Θ^μ is a belief subspace containing θ for every $\theta \in \Theta^\mu$, we have $\Theta^{\theta_i} \subset \Theta^\mu$ for every $\theta_i \in \Theta_i^\mu$.

Recall that $\mu(\Theta') > 0$ and, for every $\theta \in \Theta'$, Θ^μ is not the minimal belief subspace containing θ_i for some i. Since μ is a prior, it follows from (2) in Chen and Xiong (2013) that $\mu(\{\theta : \theta_{-i} \in \operatorname{supp} b_i(\theta_i)\}) = 1$. Hence, there is $\Theta'' \subset \Theta'$ such that $\mu(\Theta'') > 0$ and $\theta_{-i} \in \operatorname{supp} b_i(\theta_i)$ for all $\theta \in \Theta''$.

We thus have

$$\text{(S.33)} \qquad \Theta'' \subset \bigcup_{i \in I} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Theta_{i,n,m}, \quad \text{where}$$

$$\Theta_{i,n,m} \equiv \bigcup_{\theta_i \in \Theta_i^\mu \text{ and } d(\theta, \theta^n) \geq 1/m, \forall \theta \in \Theta^{\theta_i}} \Theta^{\theta_i},$$

where $\Theta_{i,n,m}$ is the union of all minimal belief subspaces Θ^{θ_i} such that $\theta_i \in \Theta_i^\mu$ and the Hausdorff metric between Θ^{θ_i} and $\{\theta^n\}$ is greater than $1/m$. Equation (S.33) holds because, for every $\theta \in \Theta''$, Θ^μ is not the minimal belief subspace containing θ_i for some i, that is, $\Theta^{\theta_i} \subsetneq \Theta^\mu$ and hence the Hausdorff metric between Θ^{θ_i} and $\{\theta^n\}$ is greater than $1/m$ for some integers m, n due to the denseness of $\{\theta^n\}_{n=1}^\infty \subset \Theta^\mu$; moreover, $\theta \in \Theta^{\theta_i}$ since $\theta_{-i} \in \operatorname{supp} b_i(\theta_i)$ (and thus $\theta \in \{\theta_i\} \times \operatorname{Supp} b_i(\theta) \subset \Theta^{\theta_i}$).

Since $\mu(\Theta'') > 0$, (S.33) implies that $\mu(\Theta_{i,n,m}) > 0$ for some (i, m, n). Clearly, $\overline{\Theta}_{i,n,m}$ (the closure of $\Theta_{i,n,m}$) is a belief subspace and $\mu(\overline{\Theta}_{i,n,m}) > 0$. Moreover, since $d(\theta, \theta^n) \geq 1/m$ for all $\theta \in \Theta_{i,n,m}$, we thus have $\overline{\Theta}_{i,n,m} \subsetneq \Theta^\mu$. This further implies that $\mu(\overline{\Theta}_{i,n,m}) < 1$. Therefore, we find a belief subspace $\overline{\Theta}_{i,n,m}$ such that $\mu(\overline{\Theta}_{i,n,m}) \in (0, 1)$.

S.4.2. *Proof of Claim 2*

First, since μ is a prior,

$$\begin{aligned}
\mu(\Theta) &= \int_{\Theta_i^*} \left(\int_{\Theta_{-i}^*} 1_\Theta(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}] \right) \mu_i[d\theta_i] \\
&= \int_{\Theta_i} \left(\int_{\Theta_{-i}^*} 1_\Theta(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}] \right) \mu_i[d\theta_i] \\
&= \int_{\Theta_i} \left(\int_{\Theta_{-i}^*} b_i(\theta_i) \big[\theta_{-i} \in \Theta_{-i}^* : (\theta_i, \theta_{-i}) \in \Theta\big] \right) \mu_i[d\theta_i] \\
&= \mu_i(\Theta_i),
\end{aligned}$$

where the second equality holds because $1_\Theta(\theta_i, \theta_{-i}) = 0$ for all θ_{-i} if $\theta_i \notin \Theta_i$; the last equality holds because Θ is a belief subspace. It then follows that

$$\mu(\Theta) = \mu_i(\Theta_i) \geq \mu\left(\prod_i \Theta_i\right) \geq \mu(\Theta).$$

Thus,

$$\mu(\Theta) = \mu\left(\prod_i \Theta_i\right). \tag{S.34}$$

Second, observe that the collection $\mathcal{X} = \{\prod_i X_i : X_i \in \{\Theta_i, \Theta_i^c\}\}$ is a partition of Θ^μ. First, for any $X \in \mathcal{X}$, $\mu(X) = 0$ if $X_i = \Theta_i$ and $X_{i'} = \Theta_{i'}^c$ for some $i \neq i'$. Indeed, since μ is a prior,

$$\mu(X) = \int_{\Theta_i} b_i(\theta_i)\left[\prod_{j\neq i} X_j\right]\mu_i[d\theta_i] = 0, \tag{S.35}$$

where the second equality follows because $\Theta_{i'} \cap X_{i'} = \emptyset$ and $b_i(\theta_i)[\Theta_{-i}] = 1$ for every $\theta_i \in \Theta_i$. Hence, $\mu(X) > 0$ only if $X = \prod_i \Theta_i$ or $X = \prod_i \Theta_i^c$. It follows that

$$\mu\left(\prod_i \Theta_i^c\right) = 1 - \mu\left(\prod_i \Theta_i\right). \tag{S.36}$$

By (S.34) and (S.36), we obtain Claim 2.

S.4.3. *Proof of Claim 3*

Observe that for any $E_i \subset \Theta_i$,

$$\pi_i(E_i) = \frac{\mu(E_i \times \Theta_{-i})}{\mu(\Theta)} = \frac{\int_{E_i} b_i(\theta_i)[\Theta_{-i}]\mu_i[d\theta_i]}{\mu(\Theta)} = \frac{\mu_i(E_i)}{\mu(\Theta)}, \tag{S.37}$$

where the first equality follows from (S.34); the second equality follows from the fact that μ is a prior; the last equality follows because $b_i(\theta_i)[\Theta_{-i}] = 1$, for all $\theta_i \in E_i \subset \Theta_i$.

We show that, for any bounded measurable function $\varphi : \Theta^* \to \mathbb{R}$,

$$\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\pi_i[d\theta_i] = \int_{\Theta^*} \varphi(\theta)\pi[d\theta], \quad \forall i. \tag{S.38}$$

Define $\varphi' : \Theta^* \to \mathbb{R}$ such that $\varphi'(\theta) = 1_{\prod_i \Theta_i}(\theta)\varphi(\theta)$.

First, consider the left-hand side of (S.38). We have

$$\begin{aligned}
&\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\pi_i[d\theta_i] \\
&= \int_{\Theta_i}\left(\int_{\Theta_{-i}^*} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\pi_i[d\theta_i] \\
&= \int_{\Theta_i}\left(\int_{\Theta_{-i}} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\pi_i[d\theta_i] \\
&= \frac{1}{\mu(\Theta)}\int_{\Theta_i}\left(\int_{\Theta_{-i}} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i] \\
&= \frac{1}{\mu(\Theta)}\int_{\Theta_i}\left(\int_{\prod_{j\neq i}\Theta_j} \varphi(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i] \\
&= \frac{1}{\mu(\Theta)}\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*} \varphi'(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i],
\end{aligned} \tag{S.39}$$

where the first equality follows because $\pi_i(\Theta_i) = 1$; the second equality holds because $b_i(\theta_i)[\Theta_{-i}] = 1$ for every $\theta_i \in \Theta_i$; the third equality follows from (S.37); the fourth equality follows because $b_i(\theta_i)[\Theta_{-i}] = 1 = b_i(\theta_i)[\prod_{j\neq i}\Theta_j]$ for all $\theta_i \in \Theta_i$.

Second, consider the right-hand side of (S.38). We have

$$\int_{\Theta^*} \varphi(\theta)\pi[d\theta] = \frac{1}{\mu(\Theta)}\int_{\Theta} \varphi(\theta)\mu[d\theta] = \frac{1}{\mu(\Theta)}\int_{\Theta^*} \varphi'(\theta)\mu[d\theta], \tag{S.40}$$

where the second equality follows because $\mu((\prod_i \Theta_i)\setminus\Theta) = 0$ by (S.34). Third, since φ' is a measurable function and μ is a prior,

$$\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*} \varphi'(\theta_i, \theta_{-i}) b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i] = \int_{\Theta^*} \varphi'(\theta)\mu[d\theta]. \tag{S.41}$$

Finally, (S.38) follows from (S.39), (S.40), and (S.41). Consequently, π is a prior.

S.4.4. *Proof of Claim 4*

We divide the proof into three steps:

Step 1. For any $i \in I$, $\mu_i(\Theta_i^{c,*}) = \mu_i(\Theta_i^c)$, where

$$\Theta_i^{c,*} \equiv \left\{\theta_i \in \Theta_i^c : b_i(\theta_i)\left[\prod_{j\neq i}\Theta_j^c\right] = 1\right\}.$$

Suppose otherwise, that is, $\mu_i(\Theta_i^c \setminus \Theta_i^{c,*}) > 0$ for some i. Consider the finite partition $\mathcal{X}_{-i} = \{\prod_{j\neq i} X_j : X_j \in \{\Theta_j, \Theta_j^c\}\}$ of Θ_{-i}^μ. For each $X_{-i} \in \mathcal{X}_{-i}$, let

$$\Theta_i^{c,X_{-i}} \equiv \big\{\theta_i \in \Theta_i^c : b_i(\theta_i)[X_{-i}] > 0\big\}.$$

By the definition of $\Theta_i^{c,*}$, for any $\theta_i \in \Theta_i^c \setminus \Theta_i^{c,*}$, we have $\theta_i \in \Theta_i^{c,X_{-i}}$ for some $X_{-i} \in \mathcal{X}_{-i} \setminus \{\prod_{j\neq i} \Theta_j^c\}$, that is,

$$\Theta_i^c \setminus \Theta_i^{c,*} \subset \bigcup_{X_{-i}\in\mathcal{X}_{-i}\setminus\{\prod_{j\neq i}\Theta_j^c\}} \Theta_i^{c,X_{-i}}.$$

Furthermore, since $\mathcal{X}_{-i}$ is finite and $\mu_i(\Theta_i^c \setminus \Theta_i^{c,*}) > 0$, it follows that $\mu_i(\Theta_i^{c,X_{-i}}) > 0$ for some $X_{-i} \in \mathcal{X}_{-i} \setminus \{\prod_{j\neq i} \Theta_j^c\}$. Hence,

$$\mu\big(\Theta_i^{c,X_{-i}} \times X_{-i}\big) = \int_{\Theta_i^{c,X_{-i}}} b_i(\theta_i)[X_{-i}]\mu_i[d\theta_i] > 0,$$

which implies

$$\mu\big(\Theta_i^c \times X_{-i}\big) \geq \mu\big(\Theta_i^{c,X_{-i}} \times X_{-i}\big) > 0, \quad \text{where} \quad X_{-i} \in \mathcal{X}_{-i} \setminus \Big\{\prod_{j\neq i}\Theta_j^c\Big\}. \tag{S.42}$$

Clearly, $\Theta_i^c \times X_{-i} \in \mathcal{X} \setminus \{\prod_j \Theta_j, \prod_j \Theta_j^c\}$, and we thus have

$$\mu\big(\Theta_i^c \times X_{-i}\big) = 0, \quad \text{by (S.35)}. \tag{S.43}$$

Equation (S.42) contradicts (S.43). Therefore, $\mu_i(\Theta_i^{c,*}) = \mu_i(\Theta_i^c)$ for every i.

Step 2. $\nu_i(E_i) = \frac{\mu_i(E_i)}{\mu(\prod_i \Theta_i^c)}$ *for any* $E_i \subset \Theta_i^{c,*}$.

Observe that

$$\nu_i(E_i) = \frac{\mu\Big(E_i \times \Big(\prod_{j\neq i}\Theta_j^c\Big)\Big)}{\mu\Big(\prod_i \Theta_i^c\Big)} \tag{S.44}$$

$$= \frac{\int_{E_i} b_i(\theta_i)\Big[\prod_{j\neq i}\Theta_j^c\Big]\mu_i[d\theta_i]}{\mu\Big(\prod_i \Theta_i^c\Big)} = \frac{\mu_i(E_i)}{\mu\Big(\prod_i \Theta_i^c\Big)},$$

where the second equality follows from the fact that μ is a prior; the last equality follows because $b_i(\theta_i)[\prod_{j\neq i}\Theta_j^c] = 1$ for all $\theta_i \in E_i \subset \Theta_i^{c,*}$ (see the definition of $\Theta_i^{c,*}$ in step 1).

Step 3. ν is a prior.

We now show that, for any bounded measurable function $\varphi : \Theta^* \to \mathbb{R}$,

$$\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)v_i[d\theta_i] = \int_{\Theta^*}\varphi(\theta)v[d\theta], \quad \forall i. \tag{S.45}$$

Define $\varphi' : \Theta^* \to \mathbb{R}$ such that $\varphi'(\theta) = 1_{\prod_i \Theta_i^c}(\theta)\varphi(\theta)$.

First, consider the left-hand side of (S.45). We have

$$\begin{aligned}
&\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)v_i[d\theta_i] \\
&\quad = \int_{\Theta_i^{c,*}}\left(\int_{\Theta_{-i}^*}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)v_i[d\theta_i] \\
&\quad = \int_{\Theta_i^{c,*}}\left(\int_{\prod_{j\neq i}\Theta_j^c}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)v_i[d\theta_i] \\
&\quad = \frac{1}{\mu\left(\prod_i \Theta_i^c\right)}\int_{\Theta_i^{c,*}}\left(\int_{\prod_{j\neq i}\Theta_j^c}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i] \\
&\quad = \frac{1}{\mu\left(\prod_i \Theta_i^c\right)}\int_{\Theta_i^{c}}\left(\int_{\prod_{j\neq i}\Theta_j^c}\varphi(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i] \\
&\quad = \frac{1}{\mu\left(\prod_i \Theta_i^c\right)}\int_{\Theta_i^{*}}\left(\int_{\Theta_{-i}^*}\varphi'(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i],
\end{aligned} \tag{S.46}$$

where the first equality follows because $v_i(\Theta_i^{c,*}) = v_i(\Theta_i^c) = 1$ by steps 1 and 2; the second equality holds because $b_i(\theta_i)[\prod_{j\neq i}\Theta_j^c] = 1$ for every $\theta_i \in \Theta_i^{c,*}$by step 1; the third equality follows from step 2; the fourth equality follows from step 1.

Second, consider the right-hand side of (S.45). We have

$$\begin{aligned}
\int_{\Theta^*}\varphi(\theta)v[d\theta] &= \frac{1}{\mu\left(\prod_i \Theta_i^c\right)}\int_{\prod_i \Theta_i^c}\varphi(\theta)\mu[d\theta] \\
&= \frac{1}{\mu\left(\prod_i \Theta_i^c\right)}\int_{\Theta^*}\varphi'(\theta)\mu[d\theta].
\end{aligned} \tag{S.47}$$

Third, since φ' is a measure function and μ is a prior,

$$\int_{\Theta_i^*}\left(\int_{\Theta_{-i}^*}\varphi'(\theta_i,\theta_{-i})b_i(\theta_i)[d\theta_{-i}]\right)\mu_i[d\theta_i]=\int_{\Theta^*}\varphi'(\theta)\mu[d\theta]. \tag{S.48}$$

Finally, (S.45) follows from (S.46), (S.47), and (S.48). Consequently, v is a prior.

S.4.5. *A Model Whose Support Is Not Minimal*

We now present an example of a model whose support is not a minimal belief subspace. Consider a two-bidder common-prior type space $\Theta_1 = \{\theta_{1,1},\theta_{1,2},,\ldots\}$ and $\Theta_2=\{\theta_{2,1},\theta_{2,2},\ldots\}$. Define

$$v_1(\theta_{1,n})=v_2(\theta_{2,n})=\frac{1}{n},\quad \forall n.$$

The prior μ on this type space is defined as

$$\mu\big[(\theta_{1,n},\theta_{2,n})\big]=\mu\big[(\theta_{1,n},\theta_{2,n+1})\big]=\frac{1}{2^{n+1}},\quad \forall n.$$

That is,

μ	$\theta_{2,1}$	$\theta_{2,2}$	$\theta_{2,3}$	$\theta_{2,4}$	$\cdots$
$\theta_{1,1}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\cdots$
$\theta_{1,2}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\cdots$
$\theta_{1,3}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\cdots$
$\theta_{1,4}$	0	0	0	$\frac{1}{32}$	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

.

Note that μ is not a convex combination of two other priors, that is, μ is indeed an extreme point in $\mathcal{P}$.[1] However, Θ^μ is not a minimal belief subspace. Observe that $(\theta_{1,n},\theta_{2,n})\to(\theta_{1,0},\theta_{2,0})$, where $(\theta_{1,0},\theta_{2,0})$ is the type profile with common knowledge of the value being 0. Then, $(\theta_{1,0},\theta_{2,0})$ is a belief subspace and $(\theta_{1,0},\theta_{2,0})\subset\Theta^\mu$.

[1]To see this, suppose that $\mu=\alpha\pi+(1-\alpha)\nu$ for some $\pi,\nu\in\mathcal{P}$ and $\alpha\in(0,1)$. Since μ is a model, it follows that $\Theta^\pi=\Theta^\nu=\Theta^\mu$. Since $\Theta^\pi=\Theta^\mu$, $x\equiv\pi(\{(\theta_{1,1},\theta_{2,1})\})>0$. Then, $\pi(\{(\theta_{1,1},\theta_{2,2})\})=x$, $\pi(\{(\theta_{1,2},\theta_{2,2})\})=\pi(\{(\theta_{1,1},\theta_{2,3})\})=x/2$, and so on. Moreover, since $\alpha\in(0,1)$ and $\mu(\{(\theta_{1,0},\theta_{2,0})\})=0$, $\pi(\{(\theta_{1,0},\theta_{2,0})\})=0$. It follows that $2x(1+1/2+1/4+\cdots)=1$. Hence, $x=1/4$ and we conclude that $\pi=\mu$. Similarly, $\nu=\mu$. That is, μ is an extreme point.

Note that while Θ^μ is not a minimal belief subspace, for μ-almost all θ, Θ^μ is the minimal belief subspace that contains θ_i for every i. Hence, μ is a model by Proposition S.4. Thus, if we ask the agents to report the minimal belief subspace containing their actual types, under truthtelling Θ^μ will be reported with μ-probability 1.

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