Econometrica Supplementary Material

SUPPLEMENT TO "ON MONOTONE RECURSIVE PREFERENCES"

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THIS SUPPLEMENTAL MATERIAL contains two parts. Section S.1 complements Appendix B of the paper by providing an exhaustive proof of Proposition 1 with all technical details included. Section S.2 provides the proof of Proposition 4.

To avoid confusion in the numbering of equations and sections between the main text and this supplement, all numbers in the Supplemental Material will be prefixed by "S." Conversely, numbers without prefix refer to an equation or a section of the main text.

S.1. PROOF OF PROPOSITION 1

S.1.1. *Deriving a System of Distributivity Equations*

Necessity of the axioms is obvious, so we focus on sufficiency. We adopt the same notational conventions as in the main text. From Lemma 1, the preference relation $\succeq$ has a recursive representation (U, W, I). It is w.l.o.g. to assume that $U(D) = [0, 1]$. Fix some integer $m > 2$. Let $\mathcal{W}_0 := [0, 1]^m$ and

$$\mathcal{W}_1 := \{(W(c, x_1), \dots, W(c, x_m)) : c \in C, (x_1, \dots, x_m) \in \mathcal{W}_0\},$$
$$\mathcal{W}_2 := \{(W(c, x_1), \dots, W(c, x_m)) : c \in C, (x_1, \dots, x_m) \in \mathcal{W}_1\}.$$

Note that $\mathcal{W}_0 \supset \mathcal{W}_1 \supset \mathcal{W}_2$.

Now fix a vector $(\pi_1, \dots, \pi_m) \in (0, 1)^m$ such that $\sum_i \pi_i = 1$. For every vector $(x_1, \dots, x_m) \in [0, 1]^m$, let $(\pi_1, x_1; \dots; \pi_m, x_m)$ be the lottery in $M([0, 1])$ that gives x_k with probability π_k. Define a function $G_0 : \mathcal{W}_0 \to [0, 1]$ by

$$G_0(x_1, \dots, x_m) := I((\pi_1, x_1; \dots; \pi_m, x_m)), \quad \forall (x_1, \dots, x_m) \in [0, 1]^m, \tag{S.1}$$

which is the certainty equivalent of the lottery $(\pi_1, x_1; \dots; \pi_m, x_m)$. For $k \in \{1, 2\}$, define a function $G_k : \mathcal{W}_k \to [0, 1]$ inductively by letting

$$G_{k+1}(W(c, x_1), \dots, W(c, x_m)) := W(c, G_k(x_1, \dots, x_m)). \tag{S.2}$$

The functions G_k, $k \in \{1, 2\}$, are well-defined by Monotonicity. For every $c \in C$, let F_c denote the function $x \mapsto W(c, x)$ from $[0, 1]$ into $[0, 1]$. Each function F_c is continuous

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and strictly increasing. With this notation, equation (S.2) becomes, for $k = 1, 2$,

$$\begin{cases} G_1\big(F_c(x_1), \dots, F_c(x_m)\big) = F_c G_0(x_1, \dots, x_m), & c \in C, (x_1, \dots, x_m) \in \mathcal{W}_1, \\ G_2\big(F_c(x_1), \dots, F_c(x_m)\big) = F_c G_1(x_1, \dots, x_m), & c \in C, (x_1, \dots, x_m) \in \mathcal{W}_2, \end{cases} \tag{S.3}$$

which is a system of generalized distributivity equations. The two equations in (S.3) are related through the function G_1, which appears in both.

We now derive standard distributivity equations from the generalized distributivity equations in (S.3). To simplify our notation, let $\beta := W(\underline{c}, 1)$. If $F_{\overline{c}}(0) > \beta$, let c^* be such that $F_{c^*}(0) = \beta$. Alternatively, if $F_{\overline{c}}(0) \leq \beta$, let $c^* := \overline{c}$. In each case, we have $F_c^{-1}[0, \beta] = [0, F_c^{-1}(\beta)] \neq \emptyset$ for every $c < c^*$. Take $c < c^*$, $k \in \{0, 1\}$, and $(x_1, \dots, x_m) \in [0, F_c^{-1}(\beta)]^m \cap \mathcal{W}_k$. Applying $F_{\underline{c}}^{-1}$ to both sides of equation (S.3) implies $F_{\underline{c}}^{-1} G_{k+1}(F_c(x_1), \dots, F_c(x_m)) = F_{\underline{c}}^{-1} F_c G_k(x_1, \dots, x_m)$. Combining the last equation for an arbitrary c with the same equation for $c = \underline{c}$ yields

$$G_k\big(F_{\underline{c}}^{-1} F_c(x_1), \dots, F_{\underline{c}}^{-1} F_c(x_m)\big) = F_{\underline{c}}^{-1} F_c G_k(x_1, \dots, x_m). \tag{S.4}$$

Defining $f_c := F_{\underline{c}}^{-1} F_c$, equation (S.4) becomes, for $c < c^*$, $(x_1, \dots, x_m) \in [0, F_c^{-1}(\beta)]^m \cap \mathcal{W}_k$,

$$G_k\big(f_c(x_1), \dots, f_c(x_m)\big) = f_c G_k(x_1, \dots, x_m), \quad k = 0, 1, \tag{S.5}$$

which are distributivity equations similar to (29).

S.1.2. *Constructing an Iteration Group*

The proof requires some mathematical machinery from Lundberg (1982). First, given a proper interval $A \subset \mathbb{R}$, let $\mathcal{D}(A)$ be the set of all continuous, strictly increasing functions f whose domain and range are intervals contained in A and whose graphs disconnect A^2. Given $\lambda \in \mathbb{R} \cup \{+\infty\}$, a collection $\{f^\alpha\}_{\alpha \in (-\lambda, \lambda)} \subset \mathcal{D}(A)$ is *an iteration group on* A if $f^{\alpha+\alpha'} = f^\alpha f^{\alpha'}$ for all $\alpha, \alpha', \alpha + \alpha' \in (-\lambda, \lambda)$.[1] When no confusion arises, we suppress λ and the interval A and write $\{f^\alpha\}$ for an iteration group. A few remarks about the definition of an iteration group are in order. First, f^0 is necessarily the identity function on A. Moreover, if $1 \in (-\lambda, \lambda)$ and α is any other integer in $(-\lambda, \lambda)$, then f^α is the α-iterate of the function f^1. In fact, let $f := f^1$. We know how to define the α-iterate of f for any integer α. One can think of an iteration group as a way to define an α-iterate of the function f for any real number α, while ensuring (i) that the definition is consistent with the usual definition of an iterate for integer α, and (ii) that the different "iterates," f^α, $f^{\alpha'}$, and $f^{\alpha+\alpha'}$, do in fact "iterate." We should also point out that the index α has no meaning beyond encoding this second property. Formally, let $\gamma \neq 1$ be any real number and, for every $\alpha \in (-\lambda, \lambda)$, define $g^{\alpha\gamma} := f^\alpha$. Then, $\{g^{\tilde{\alpha}}\}_{\tilde{\alpha} \in (-\gamma\lambda, \gamma\lambda)}$ is an iteration group on A and $\{g^{\tilde{\alpha}}\} = \{f^\alpha\}$. Thus, $\{g^{\tilde{\alpha}}\}$ is just a relabeling of $\{f^\alpha\}$. When we specify an iteration group $\{f^\alpha\}$, we assume that the group is *non-trivial*, that is, that $f^\alpha \neq f^0$ for at least one $\alpha \neq 0$. If the group is non-trivial, then $f^\alpha \neq f^0$ for all $\alpha \neq 0$. It should also be observed that $\lambda < +\infty$ whenever A is a bounded interval. For example, if $f^1(x) > x$ for all $x \in A$, then the graph of f^n lies outside of $A \times A$ for all n large enough, so that $f^n \notin \mathcal{D}(A)$. Finally, when we

[1]When A is a proper subset of $\mathbb{R}$, Lundberg (1982) called the iteration group *truncated*. We have no occasion to distinguish between truncated and untruncated groups and use the term iteration group to denote both.

specify an iteration group $\{f^\alpha\}_{\alpha\in(-\lambda,\lambda)}$ on a bounded interval A, we assume that the group is *maximal*, that is, there is no other iteration group $\{g^\alpha\}_{\alpha\in(-\lambda',\lambda')}\subset\mathcal{D}(A)$ such that $\lambda'>\lambda$ and $g^\alpha=f^\alpha$ for all $\alpha\in(-\lambda,\lambda)$.

Let $(f_n)_n$ be a sequence of functions $f_n\in\mathcal{D}(A)$. A function $f\in\mathcal{D}(A)$ is the *closed limit of* $(f_n)_n$, which we denote as $f_n\to_L f$, if the graph of f is the closed limit of the graphs of the functions f_n.[2] If A is a closed interval and the graphs of f_n and f are closed, then $f_n\to_L f$ if and only if the graphs of f_n converge to the graph of f in the Hausdorff metric. We write $f_n\to_H f$ to denote the latter type of convergence. The sequence $(f_n)_n$, $f_n\in\mathcal{D}(A)$, *generates the iteration group* $\{f^\alpha\}$ on A if for every $\alpha\in(-\lambda,\lambda)$, there exists a sequence $(p_n)_n$ of integers such that $f_n^{p_n}\to_L f^\alpha$.

We come back to the proof of the theorem. Let j be the identity function on $[0,1]$. Fix a sequence $(c_n)_n$ such that $c_n\in(\underline{c},c^*)$ for every n and the sequence decreases monotonically to $\underline{c}$. Let $(f_{c_n})_n$ be the associated sequence of functions where $f_{c_n}=F_{\underline{c}}^{-1}F_{c_n}$ for every n. We note several properties of the sequence $(f_{c_n})_n$. First, $f_{c_n}>f_{c_{n+1}}>j$ for every n. Second, each function f_{c_n} has domain $\mathrm{Dom}_n:=[0,F_{c_n}^{-1}(\beta)]$ and range $[f_{c_n}(0),1]$. It follows that the graph of each function f_{c_n} disconnects $[0,1]^2$ so that $f_{c_n}\in\mathcal{D}([0,1])$. Another immediate implication is that $\mathrm{Dom}_n\to_H[0,1]$. The latter implies that for every $x\in(0,1)$, there is $k>0$ such that $f_{c_n}(x)$ is defined for all $n\geq k$. The sequence $(f_{c_k}(x),f_{c_{k+1}}(x),\dots)$ converges to x and the next lemma shows that the convergence is in fact uniform.

LEMMA S.1—Uniform Convergence: $f_{c_n}\to_H j$.

PROOF: Let Gr_n denote the graph of f_{c_n}. Let E' be a limit point of the sequence $(\mathrm{Gr}_n)_n$ in the Hausdorff metric. Let $E:=\{(x,x):x\in[0,1]\}$, that is, E is the diagonal of the unit square $[0,1]^2$. It is also the graph of the identity function j. For every $a\in(0,1)$ and every n large enough, the functions f_{c_n} are defined on the interval $[0,a]$. Since the functions f_{c_n} converge monotonically to the identity function, we can apply Dini's theorem to conclude that the convergence is uniform when the functions are restricted to the interval $[0,a]$. But the uniform convergence of functions is equivalent to the Hausdorff convergence of their graphs. We conclude that $E\cap([0,a]\times[0,1])=E'\cap([0,a]\times[0,1])$. Since this is true for every $a<1$, the intersections of E and E' with $[0,1)\times[0,1]$ coincide. Since the set E' is closed, we know that $(1,1)\in E'$. Moreover, since $f_{c_n}>j$ for all n, the set E' "lie above" E, that is, there is no pair $(1,x)\in[0,1]^2$ such that $x<1$ and $(1,x)\in E'$. We conclude that $E'=E$. Since the limit point E' of $(\mathrm{Gr}_n)_n$ was arbitrary, this concludes the proof. *Q.E.D.*

The next two lemmas are key for solving the distributivity equation.

LEMMA S.2—Constructing an Iteration Group: *There is an iteration group* $\{f^\alpha\}_{\alpha\in(-\lambda,\lambda)}$ *on* $(0,1)$ *such that* $\lambda>1$, $f^\alpha>j$ *for all* $\alpha>0$, *and*

$$f^\alpha G_0(x_1,\dots,x_m)=G_0\big(f^\alpha(x_1),\dots,f^\alpha(x_m)\big),\tag{S.6}$$

for all $(x_1,\dots,x_m)\in[0,1]^m$ *and* $\alpha\in(-\lambda,\lambda)$ *for which the equation is well-defined.*

PROOF: We know that $f_{c_n}\to_L j$, $f_{c_n}\neq j$ for every n, and $\mathrm{Dom}_n\to_L(0,1)$. Theorem 4.16 in Lundberg (1982) shows that $(f_{c_n})_n$ has a subsequence that generates the desired it-

[2] See Aliprantis and Border (1999, p. 109) for the definition of a closed limit.

eration group.[3] Abusing notation, from now on we write $(f_{c_n})_n$ for the latter subsequence. *Q.E.D.*

LEMMA S.3—Constructing an Abel Function: *There is a continuous, strictly increasing function $L : (0,1) \to \mathbb{R}$ such that $f^\alpha(x) = L^{-1}(L(x)+\alpha)$ for all x in the domain of f^α and all $\alpha \in (-\lambda, \lambda)$.*

PROOF: We know that $f^\alpha > j$ for all $\alpha \in (0, \lambda)$. Since f^α is the inverse of $f^{-\alpha}$, the latter implies that $f^\alpha < j$ for all $\alpha \in (-\lambda, 0)$. In particular, none of the functions f^α, $\alpha \neq 0$, has a fixed point. As explained in Section B.1, the iteration group then has an *Abel function*, that is, a continuous function $L : (0,1) \to \mathbb{R}$ such that $f^\alpha(x) = L^{-1}(\alpha + L(x))$ for every $\alpha \in (-\lambda, \lambda)$ and every x in the domain of f^α. Since $f^\alpha > j$ for all $\alpha > 0$, the function L is strictly increasing. *Q.E.D.*

Recall that each function f_{c_n} is defined in a right neighborhood of 0, while its range contains a left neighborhood of 1. It follows that for each $\alpha > 0$, $f^\alpha(0) := \lim_{x\searrow 0} f^\alpha(x)$ and for each $\alpha < 0$, $f^\alpha(1) := \lim_{x\nearrow 1} f^\alpha(x)$ are well-defined. From now on, we assume that $\{f^\alpha\}$ is such that $f^1(0) > 0$ and $f^{-1}(1) < 1$. Section S.1.9 below shows how to modify the proof if either $f^1(0) = 0$ or $f^{-1}(1) = 1$. Under the assumption just made, we have $f^\alpha(0) > 0$ for all $\alpha > 0$ and $f^\alpha(1) < 1$ for all $\alpha < 0$ as well as $L(0) := \lim_{x\searrow 0} L(x) > -\infty$ and $L(1) := \lim_{x\nearrow 1} L(x) < +\infty$. Using the latter, we now argue that the Abel function L can be chosen so that $L(0) = 0$ and $L(1) = 1$. First, observe that if L is an Abel function for the iteration group $\{f^\alpha\}$, then so is the function $L + l$ where $l \in \mathbb{R}$ is a constant. Thus, we can choose L so that $L(0) = 0$. To see that L can be chosen so that $L(1) = 1$, observe that $\lambda = \lim_{\alpha\nearrow\lambda} f^\alpha(0) = L(1)$. Relabeling the iteration group $\{f^\alpha\}_{\alpha\in(-\lambda,\lambda)}$ so that $\lambda = 1$ implies that $L(1) = 1$.

S.1.3. *A Monotone Transformation of Utility*

Since $L : [0,1] \to [0,1]$ is strictly increasing, the function $\widetilde{U} := LU : D \to [0,1]$ represents $\succeq$ on D. Moreover, the function $\widetilde{U}$ is part of a recursive representation $(\widetilde{U}, \widetilde{W}, \widetilde{I})$ where

$$\widetilde{W}(c,x) := LW\big(c, L^{-1}(x)\big) \quad \forall x \in [0,1], c \in C,$$
$$\widetilde{I}(\mu) := LI\big(\mu \circ L^{-1}\big) \quad \forall \mu \in M\big([0,1]\big).$$

For every $c \in C$, let $\widetilde{F}_c := LF_cL^{-1}$. For $k \in \{0,1,2\}$, we define

$$\widetilde{G}_k(x_1, \dots, x_m) := LG_k\big(L^{-1}(x_1), \dots, L^{-1}(x_m)\big). \tag{S.7}$$

As before, define $\widetilde{\mathcal{W}}_0 := [0,1]^m$ and inductively for $k \in \{1,2\}$,

$$\widetilde{\mathcal{W}}_k := \big\{\big(\widetilde{F}_c(x_1), \dots, \widetilde{F}_c(x_m)\big) : c \in C, (x_1, \dots, x_m) \in \widetilde{\mathcal{W}}_{k-1}\big\}. \tag{S.8}$$

By definition, the function $\widetilde{G}_k$, $k \in \{0,1,2\}$, has domain $\widetilde{\mathcal{W}}_k$. Also, $\widetilde{\mathcal{W}}_0 \supset \widetilde{\mathcal{W}}_1 \supset \widetilde{\mathcal{W}}_2$. As in Section B.1, we use the Abel function to prove that $\widetilde{G}_0$ is translation-invariant.

[3]The statement of Theorem 4.16 in in Lundberg (1982) does not say that $\lambda > 1$ and $f^\alpha > j$ for all $\alpha > 0$, but these properties of the iteration group follow from the proof of the theorem and the fact that $f_{c_n} > j$ for every n.

LEMMA S.4—Translation Invariance $\widetilde{G}_0$: *For every* $(x_1, \dots, x_m) \in \widetilde{\mathcal{W}}_0$, $\alpha \in (-1, 1)$ *such that* $(\alpha + x_1, \dots, \alpha + x_m) \in \widetilde{\mathcal{W}}_0$, *we have* $\widetilde{G}_0(\alpha + x_1, \dots, \alpha + x_m) = \alpha + \widetilde{G}_0(x_1, \dots, x_m)$.

PROOF: Let $(x_1, \dots, x_m)$ and α be as in the statement of the lemma. Let $y_i = L^{-1}(x_i)$ for $i = 1, \dots, m$. Then,

$$\begin{aligned}\widetilde{G}_0(\alpha + x_1, \dots, \alpha + x_m) &= LG_0\big(L^{-1}\big(\alpha + L(y_1)\big), \dots, L^{-1}\big(\alpha + L(y_m)\big)\big) \\ &= LG_0\big(f^{\alpha}(y_1), \dots, f^{\alpha}(y_m)\big) = Lf^{\alpha}\big(G_0(y_1, \dots, y_m)\big) \\ &= L\big(G_0(y_1, \dots, y_m)\big) + \alpha = \widetilde{G}_0(x_1, \dots, x_m) + \alpha. \qquad \textit{Q.E.D.}\end{aligned}$$

We now introduce a function which, together with the last lemma, will later on (see Section S.1.6) allow us to derive more tractable, "linearized" versions of the equations in (S.3). Namely, for every $x_1, x_2 \in [0, 1]$, define

$$\phi_0(x_2 - x_1) := \widetilde{G}_0(x_1, x_2, x_2, \dots, x_2) - x_1. \tag{S.9}$$

To see that ϕ_0 is well-defined, take $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_2 - x_1 = y_2 - y_1$ and let $\mathbf{z} := (x_1, x_2, \dots, x_2), \mathbf{z}' := (y_1, y_2, \dots, y_2) \in \widetilde{\mathcal{W}}_0$. Let $\alpha := y_1 - x_1 > 0$. By construction, $\mathbf{z}' = \mathbf{z} + \alpha$ and $\alpha \in (-1, 1)$. But then $\widetilde{G}_0(\mathbf{z} + \alpha) = \widetilde{G}_0(\mathbf{z}) + \alpha$, which is equivalent to $\widetilde{G}_0(x_1, x_2, x_2, \dots, x_2) - x_1 = \widetilde{G}_0(y_1, y_2, y_2, \dots, y_2) - y_1$, showing that ϕ_0 is well-defined. Finally, observe that ϕ_0 has domain $[-1, 1]$. The next lemma characterizes the monotonicity of the functions ϕ_0 and $j - \phi_0$.

LEMMA S.5: *The function* ϕ_0 *is continuous and strictly increasing. The function* $j - \phi_0$ *is strictly decreasing.*

PROOF: The definition (S.9) and the properties of $\widetilde{G}_0$ directly imply that ϕ_0 is continuous and strictly increasing. Regarding the second statement, fix some x in $(-1, 1)$, that is, in the interior of ϕ_0's domain. Then, there is some $x' \in [0, 1]$ such that for all $\varepsilon > 0$ small enough, the vectors $\mathbf{z} := (x', x' + x, \dots, x' + x)$, $\mathbf{z} + \varepsilon$, and $\mathbf{z}' := \mathbf{z} + (0, \varepsilon, \varepsilon, \dots, \varepsilon)$ belong to $\widetilde{\mathcal{W}}_0$. Observe that $\mathbf{z}' < \mathbf{z} + \varepsilon$. It is enough to show that $\phi_0(x + \varepsilon) - \phi_0(x) - \varepsilon < 0$. From the definition of ϕ_0 and Lemma S.4, we can deduce that $\phi_0(x + \varepsilon) - \phi_0(x) - \varepsilon = \widetilde{G}_0(\mathbf{z}') - \widetilde{G}_0(\mathbf{z} + \varepsilon)$, which is less than 0 since $\widetilde{G}_0$ is strictly increasing. *Q.E.D.*

S.1.4. *Solving the Distributivity Equation for G_1*

In this section, we show that the iteration group $\{f^{\alpha}\}$ constructed in Section S.1.2 also solves the following distributivity equation (37):

$$G_1\big(f_c(x_1), \dots, f_c(x_m)\big) = f_c G_1(x_1, \dots, x_m). \tag{S.10}$$

For every $c \in C$, let $A_c := [F_c(0), F_c(1)]$ and let A_c^m be the Cartesian product of m-copies of the set A_c. Observe that $\mathcal{W}_1 = \bigcup_c A_c^m$, that is, the domain of G_1 is the union of product sets situated along the diagonal in $[0, 1]^m$. Lemma S.6 below shows that equation (S.6) continues to hold when the function G_0 is replaced with G_1. One important caveat is that the equation is only guaranteed to hold "locally," that is, within each product set A_c^m rather than across the entire domain $\mathcal{W}_1$ of G_1. The proof of Lemma S.6 clarifies why we can only obtain a local analogue of Lemma S.2.

LEMMA S.6: *For all* $c \in C$, $(x_1, \dots, x_m) \in A_c^m$, *and* $\alpha \in (-1, 1)$ *with* $(f^\alpha(x_1), \dots, f^\alpha(x_m)) \in A_c^m$, *we have* $G_1(f^\alpha(x_1), \dots, f^\alpha(x_m)) = f^\alpha G_1(x_1, \dots, x_m)$.

PROOF: We need to establish a preliminary property of the distributivity equation in (S.10). Fix $c \in C$ and remember that $f = f^1$ is a function such that $f > j$ and $fG_1(x_1, \dots, x_m) = G_1(f(x_1), \dots, f(x_m))$ for all $(x_1, \dots, x_m) \in A_c^m$ such that $(f(x_1), \dots, f(x_m)) \in A_c^m$. Let $p > 1$ be an integer and let $(x_1, \dots, x_m) \in A_c^m$ be such that $(f^p(x_1), \dots, f^p(x_m)) \in A_c^m$. We want to show that $f^p G_1(x_1, \dots, x_m) = G_1(f^p(x_1), \dots, f^p(x_m))$. Suppose first that $p = 2$. Then,

$$f^2 G_1(x_1, \dots, x_m) = ffG_1(x_1, \dots, x_m) = fG_1\big(f(x_1), \dots, f(x_m)\big) \tag{S.11}$$

$$= G_1\big(f^2(x_1), \dots, f^2(x_m)\big), \tag{S.12}$$

as desired. Next, fix an integer $p > 2$. For every integer p' such that $0 < p' < p$, we have

$$(x_1, \dots, x_m) \le \big(f^{p'}(x_1), \dots, f^{p'}(x_m)\big) \le \big(f^p(x_1), \dots, f^p(x_m)\big),$$

where $\le$ is the pointwise order on $\mathbb{R}^m$. Since A_c^m is a product set and the vectors $(x_1, \dots, x_m)$, $(f^p(x_1), \dots, f^p(x_m))$ belong to A_c^m, it follows that the vector $(f^{p'}(x_1), \dots, f^{p'}(x_m))$ belongs to A_c^m. But then a chain of equalities analogous to those in (S.11) and (S.12) shows that f^3 solves the distributivity equation. By induction, so do the functions $f^4, f^5, \dots$, and f^p.

We can now complete the proof of the lemma. Take some $\alpha > 0$; symmetric arguments apply when $\alpha < 0$. Fix any $c \in C$. Take some $(x_1, \dots, x_m)$ in the interior of A_c^m and $\alpha \in (0, \lambda)$ such that $(f^\alpha(x_1), \dots, f^\alpha(x_m)) \in A_c^m$. We know that there is a sequence $(p_n)_n$ of integers such that $f_{c_n}^{p_n} \to_L f^\alpha$. For n large enough, we know that $(f_{c_n}^{p_n}(x_1), \dots, f_{c_n}^{p_n}(x_m)) \in A_c^m$. Since (S.10) holds for each f_{c_n}, we know that $G_1(f_{c_n}^{p_n}(x_1), \dots, f_{c_n}^{p_n}(x_m)) = f_{c_n}^{p_n} G_1(x_1, \dots, x_m)$ for all n large enough. Since $f_{c_n}^{p_n} \to_L f^\alpha$ and G_1 is continuous, we have $G_1(f^\alpha(x_1), \dots, f^\alpha(x_m)) = f^\alpha G_1(x_1, \dots, x_m)$, as desired. *Q.E.D.*

S.1.5. *Translation Invariance for $\widetilde{G}_1$ and $\widetilde{G}_2$*

In this section, we show that $\widetilde{G}_1$ and $\widetilde{G}_2$—defined in equation (S.7) above—are translation-invariant in a "local" sense which we make precise below.

Translation Invariance for $\widetilde{G}_1$.

For every $c \in C$, let $\widetilde{A}_c := [\widetilde{F}_c(0), \widetilde{F}_c(1)]$ and let $\widetilde{A}_c^m$ be the Cartesian product of m-copies of the set $\widetilde{A}_c$. Note that $\widetilde{G}_1$ is defined on $\bigcup_c \widetilde{A}_c^m (= \widetilde{\mathcal{W}}_1)$. The proof of the next lemma parallels that of Lemma S.4 and is omitted. As was the case with Lemmas S.2 and S.6, Lemma S.7 is only a partial analogue of Lemma S.4 in that its conclusion holds only within each separate rectangle $\widetilde{A}_c^m$, rather than within the entire domain of $\widetilde{G}_1$.

LEMMA S.7—Translation Invariance for $\widetilde{G}_1$: *For every* $c \in C$, $(x_1, \dots, x_m) \in \widetilde{A}_c^m$, *and* $\alpha \in (-1, 1)$ *such that* $(\alpha + x_1, \dots, \alpha + x_m) \in \widetilde{A}_c^m$, *we have* $\widetilde{G}_1(\alpha + x_1, \dots, \alpha + x_m) = \alpha + \widetilde{G}_1(x_1, \dots, x_m)$.

Using the above lemma, for every $c \in C$, we can define ϕ_1^c by setting

$$\phi_1^c(x_2 - x_1) := \widetilde{G}_1(x_1, x_2, \dots, x_2) - x_1$$

for all $(x_1, x_2, \dots, x_2) \in \widetilde{A}_c^m$. An analogue of Lemma S.5 shows that ϕ_1^c is a continuous, strictly increasing function and that $j - \phi_1^c$ is a strictly decreasing function. We omit the details. We should observe, however, that if $(x_1, x_2, \dots, x_2) \in \widetilde{A}_c^m$, then $(x_2, x_1, \dots, x_1) \in \widetilde{A}_c^m$. Thus, the domain of ϕ_1^c is an interval of the form $[-a_1^c, a_1^c]$. Since $\widetilde{A}_c \subset [0,1]$, we also know that $[-a_1^c, a_1^c] \subset [-1, 1]$.

Translation Invariance for $\widetilde{G}_2$.

We need an analogous construction for $\widetilde{G}_2$ as well. Fix $c \in C$ and let

$$\widetilde{A}_{cc}^m := \{(\widetilde{F}_c(x_1), \dots, \widetilde{F}_c(x_m) : (x_1, \dots, x_m) \in \widetilde{A}_c^m\}.$$

Again, $\widetilde{G}_2$ is defined on $\bigcup_c \widetilde{A}_{cc}^m (= \widetilde{\mathcal{W}}_2)$. An analogue of Lemma S.7 shows that $\widetilde{G}_2$ is translation-invariant within each rectangle $\widetilde{A}_{cc}^m$. Hence, we can define a function ϕ_2^c such that $\phi_2^c(x_2 - x_1) = \widetilde{G}_2(x_1, x_2, \dots, x_2) - x_1$ for all $(x_1, x_2, \dots, x_2) \in \widetilde{A}_{cc}^m$, exactly as we did for ϕ_1^c. Finally, note that $\widetilde{A}_{cc}^m \subset \widetilde{A}_c^m$ and so ϕ_2^c is defined on an interval $[-a_2^c, a_2^c] \subset [-a_1^c, a_1^c]$.

S.1.6. *Two Linear Distributivity Equations*

The functions $\widetilde{G}_0$, $\widetilde{G}_1$, $\widetilde{G}_2$, and $\widetilde{F}_c$ satisfy analogues of the equations in (S.3). From these equations and from the definitions of $\widetilde{F}_c$, ϕ_0, ϕ_1^c, and ϕ_2^c, we obtain

$$\widetilde{F}_c\big(x_1 + \phi_0(x_2 - x_1)\big) = \widetilde{F}_c(x_1) + \phi_1^c\big(\widetilde{F}_c(x_2) - \widetilde{F}_c(x_1)\big), \tag{S.13}$$

$$\widetilde{F}_c\big(x_1 + \phi_1^c(x_2 - x_1)\big) = \widetilde{F}_c(x_1) + \phi_2^c\big(\widetilde{F}_c(x_2) - \widetilde{F}_c(x_1)\big), \tag{S.14}$$

where the first equation holds for all $c \in C$ and $x_1, x_2 \in [0, 1]$, while the second holds for all c and x_1, x_2 such that $(x_1, x_2, x_2, \dots, x_2) \in \widetilde{A}_c^m$. Equations such as (S.13) and (S.14) were studied in Lundberg (1985). His results, Theorem 11.1 in particular, are applicable since all functions are continuous, $\widetilde{F}_c$, ϕ_0, ϕ_1^c, ϕ_2^c are strictly increasing, and $j - \phi_0$, $j - \phi_1^c$, $j - \phi_2^c$ are strictly decreasing, as shown in Section S.1.5. For any given $c \in C$, Theorem 11.1 in Lundberg (1985) shows that there are four cases for the functions $\widetilde{F}_c$, ϕ_1^c that solve (S.13). As in Lundberg (1985), we enumerate those cases: (a), (b), (c), (d). In addition, we let $\Omega_{(a)}$ be the set of all $c \in C$ such that the functions $\widetilde{F}_c$, ϕ_1^c belong to case (a). The sets $\Omega_{(b)}$, $\Omega_{(c)}$, $\Omega_{(d)}$ are defined analogously. The next lemma shows that all but one of those sets are empty, meaning that the system of equations in (S.13)–(S.14) is solved by functions that belong to the same set.

LEMMA S.8: *$C = \Omega_k$ for some $k \in \{(a), (b), (c), (d)\}$.*

PROOF: The four sets $\Omega_{(a)}$, $\Omega_{(b)}$, $\Omega_{(c)}$, and $\Omega_{(d)}$ form a partition of C. Since C is connected, it is enough to show each of these sets is open in C. For every $c \in C$, write (a_c, b_c) for the interval $(\widetilde{F}_c(0), \widetilde{F}_c(1))$. If $\Omega_{(a)}$ is empty, it is necessarily open. So suppose $\Omega_{(a)}$ is non-empty and fix some $c' \in \Omega_{(a)}$. Since the functions $c \mapsto a_c$ and $c \mapsto b_c$ are continuous, we can find $\varepsilon > 0$ such that for all $c'' \in (c', c' + \varepsilon) \cap C$, we have $a_{c''} \in (a_{c'}, b_{c'})$ and for all $c'' \in (c' - \varepsilon, c') \cap C$, we have $b_{c''} \in (a_{c'}, b_{c'})$. In other words, for all c'' sufficiently close to c', the intervals $(a_{c'}, b_{c'})$ and $(a_{c''}, b_{c''})$ have a non-empty intersection. To show that $\Omega_{(a)}$ is open in C, it is enough to show that the neighborhood $(c' - \varepsilon, c' + \varepsilon) \cap C$ of c' is a subset of $\Omega_{(a)}$. First, take some $c'' \in (c', c' + \varepsilon) \cap C$. For every $x_1, x_2 \in (a_{c''}, b_{c'}) = (a_{c'}, b_{c'}) \cap (a_{c''}, b_{c''})$, we know from the definitions of $\phi_1^{c'}, \phi_1^{c''}$ that

$$\phi_1^{c'}(x_2 - x_1) = \widetilde{G}_1(x_1, x_2, x_2, \dots, x_2) - x_1, \tag{S.15}$$

$$\phi_1^{c''}(x_2 - x_1) = \widetilde{G}_1(x_1, x_2, x_2, \dots, x_2) - x_1. \tag{S.16}$$

Note that if $x_1, x_2 \in (a_{c''}, b_{c'})$, then $x_2 - x_1 \in (a_{c''} - b_{c'}, b_{c'} - a_{c''})$. From (S.15), conclude that $\phi_1^{c'}, \phi_1^{c''}$ coincide on the interval $(a_{c''} - b_{c'}, b_{c'} - a_{c''})$, which is a symmetric, non-trivial neighborhood of 0. From Theorem 11.1 in Lundberg, if $\phi_{c'}^1, \phi_{c''}^1$ belong to different cases, they cannot coincide on any non-trivial interval. We conclude that $c'' \in \Omega_{(a)}$. Analogous arguments show that $(c' - \varepsilon, c') \cap C \subset \Omega_{(a)}$ and, hence, that $\Omega_{(a)}$ is open in C. Similarly, the sets $\Omega_{(b)}, \Omega_{(c)}, \Omega_{(d)}$ are open in C, completing the proof of the lemma. *Q.E.D.*

Cases (b) and (c) can be ruled out. Indeed, for some $c \in C$, equations (S.13) and (S.14) are linked by the functions $\widetilde{F}_c$ and ϕ_1^c, which appear in both equations but in a "different position." However, it is known from Lundberg (1985) that functions that appear in "different positions" have different functional forms, which rules out (b) and (c).

We are thus left with cases (a) and (b), which we refer to as the affine and the non-affine case and which we study in detail below.

S.1.7. *Detailed Analysis of the Affine and Non-Affine Cases*

S.1.7.1. *The Affine Case*

If all functions $\widetilde{F}_c$ solving equations (S.13) and (S.14) belong to case a), then $\widetilde{F}_c(x) = u(c) + b(c)x$ for every $c \in C$ and $x \in [0, 1]$. Moreover, the functions $u, b : C \to \mathbb{R}$ are continuous and $b(C) \subseteq (0, 1)$. If b is a constant function, there is little left to prove since we already know that $\widetilde{G}_0$ is translation-invariant. See Section S.1.8 for the remaining details. Here, suppose that the function b is not constant. We begin with a general lemma concerning the scale invariance of a real-valued function G' defined on a convex set in a Euclidean space.

LEMMA S.9: *Let $\mathcal{W}'$ be a convex set in $\mathbb{R}^m$ containing the origin and G' be a continuous function from $\mathcal{W}'$ into $\mathbb{R}$. Suppose that for every $\mathbf{x} \in \mathcal{W}'$, there is $\varepsilon \in (0, 1)$ such that $G'(\alpha\mathbf{x}) = \alpha G'(\mathbf{x})$ for all $\alpha \in (1 - \varepsilon, 1]$. Then, $G'(\alpha\mathbf{x}) = \alpha G'(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{W}'$ and all $\alpha > 0$ such that $\alpha\mathbf{x} \in \mathcal{W}'$.*

PROOF: Pick $\mathbf{x} \in \mathcal{W}'$. It is enough to show that $G'(\gamma\mathbf{x}) = \gamma G'(\mathbf{x})$ for all $\gamma \in (0, 1]$. We proceed by way of contradiction. Let us assume that there is $\gamma' \in (0, 1)$ such that $G'(\gamma\mathbf{x}) = \gamma G'(\mathbf{x})$ for all $\gamma \in [\gamma', 1]$ and $G'(\gamma\mathbf{x}) \neq \gamma G'(\mathbf{x})$ for all γ in a left neighborhood of γ'. But we know that there is $\varepsilon_{\gamma'\mathbf{x}} > 0$ such that $G'(\alpha\gamma'\mathbf{x}) = \alpha G'(\gamma'\mathbf{x})$ for all $\alpha \in (1 - \varepsilon_{\gamma'\mathbf{x}}, 1]$. Also, by the definition of γ', $\alpha G'(\gamma'\mathbf{x}) = \alpha\gamma' G'(\mathbf{x})$ and, hence, $G'(\alpha\gamma'\mathbf{x}) = \alpha\gamma' G'(\mathbf{x})$ for all $\alpha \in (1 - \varepsilon_{\gamma'\mathbf{x}}, 1]$, contradicting the fact that $G'(\gamma\mathbf{x}) \neq \gamma G'(\mathbf{x})$ for all γ in some left neighborhood of γ'. *Q.E.D.*

When a function $G' : \mathcal{W}' \to \mathbb{R}$ has the property deduced in Lemma S.9, we say that G' is *scale-invariant on* $\mathcal{W}'$.

LEMMA S.10: *If* $b : C \to (0, 1)$ *is non-constant, then* $\widetilde{G}_0$ *is scale-invariant on* $\widetilde{\mathcal{W}}_0$.

PROOF: For every $c, c' \in C$ and $\mathbf{x}$ in the interior of $\widetilde{\mathcal{W}}_0$, let

$$\mathbf{y} := \frac{u(c') - u(c)}{b(c)} + \frac{b(c')}{b(c)}\mathbf{x}. \tag{S.17}$$

Observe that if c' is sufficiently close to c, then $\mathbf{y}$ is close to $\mathbf{x}$ and hence $\mathbf{y} \in \widetilde{\mathcal{W}}_0$. Similarly, we can ensure that $\frac{b(c')}{b(c)}\mathbf{x} \in \widetilde{\mathcal{W}}_0$. From now on, assume that c, c' are chosen so that both inclusions hold. From the definition of $\mathbf{y}$, conclude that $u(c) + b(c)\mathbf{y} = u(c') + b(c')\mathbf{x}$. Hence, $\widetilde{G}_1(u(c) + b(c)\mathbf{y}) = \widetilde{G}_1(u(c') + b(c')\mathbf{x})$. Since $\widetilde{G}_0$ and $\widetilde{G}_1$ satisfy an analogue of equation (S.3), conclude that $u(c) + b(c)\widetilde{G}_0(\mathbf{y}) = u(c') + b(c')\widetilde{G}_0(\mathbf{y})$. Substituting the expression for $\mathbf{y}$ from (S.17), we get

$$\widetilde{G}_0(\mathbf{x}) = \frac{u(c) - u(c')}{b(c')} + \frac{b(c)}{b(c')}\widetilde{G}_0\left(\frac{u(c') - u(c)}{b(c)} + \frac{b(c')}{b(c)}\mathbf{x}\right). \tag{S.18}$$

Since $\mathbf{y} \in \widetilde{\mathcal{W}}_1$ and $\frac{b(c')}{b(c)}\mathbf{x} \in \widetilde{\mathcal{W}}_1$, we can apply Lemma S.4 and deduce that

$$\widetilde{G}_0(\mathbf{x}) = \frac{b(c)}{b(c')}\widetilde{G}_0\left(\frac{b(c')}{b(c)}\mathbf{x}\right).$$

Since b is non-constant, we can also choose c, c' so that $b(c) > b(c')$. Since b is continuous and C a connected set, we can also vary c, c' so that $\frac{b(c')}{b(c)}$ spans an open interval of the form $(1 - \varepsilon, 1]$. It follows from Lemma S.9 that $\widetilde{G}_0$ is scale-invariant on $\widetilde{\mathcal{W}}_0$. *Q.E.D.*

S.1.7.2. *The Non-Affine Case*

If all functions $\widetilde{F}_c$ solving equations (S.13) and (S.14) belong to case (d), then

$$\widetilde{F}_c(x) = \frac{1}{a}\log\big(u(c) + b(c)e^{ax}\big) \quad \forall c \in C, x \in [0, 1], \tag{S.19}$$

where $u, b : C \to \mathbb{R}$ are continuous functions, $a \in (0, +\infty)$, and $b(C) \subset (0, 1)$. Let $H(x) := e^{ax}$ and observe that $H^{-1}(y) = \frac{1}{a}\log y$. For every $c \in C$, let $\widehat{F}_c := H\widetilde{F}H^{-1}$. Each function $\widehat{F}_c$ has domain $[1, e^a]$ and, by construction, $\widehat{F}_c(x) = u(c) + b(c)x$ for every $c \in C$ and $x \in [1, e^a]$. Also, let $\widehat{\mathcal{W}}_0 := [1, e^a]^m$ and

$$\widehat{\mathcal{W}}_k := \big\{\big(\widehat{F}_c(x_1), \dots, \widehat{F}_c(x_m)\big), c \in C, (x_1, \dots, x_m) \in \widehat{\mathcal{W}}_{k-1}\big\}$$

for $k \in \{1, 2\}$. For $k \in \{0, 1, 2\}$ and every $(x_1, \dots, x_m) \in \widehat{\mathcal{W}}_k$, let

$$\widehat{G}_k(x_1, \dots, x_m) := H\widetilde{G}_k\big(H^{-1}(x_1), \dots, H^{-1}(x_1)\big).$$

The next two lemmas focus on the function $\widehat{G}_0$.

LEMMA S.11: *The function $\widehat{G}_0 : \widehat{\mathcal{W}}_0 \to \mathbb{R}$ is scale-invariant on $\widehat{\mathcal{W}}_0$.*

PROOF: Fix $\mathbf{x} = (x_1, \dots, x_m) \in \widehat{\mathcal{W}}_0$ and $\alpha > 0$ such that $\alpha\mathbf{x} \in \widehat{\mathcal{W}}_0$. Since $x_i, \alpha x_i \in [1, e^a]$ for every i, we know that $H^{-1}(x_i) \in [0, 1]$ and $H^{-1}(\alpha x_i) = H^{-1}(x_i) + H^{-1}(\alpha) \in [0, 1]$ for every $i = 1, \dots, m$. Using Lemma S.4 and the definition of $\widehat{G}_0$, deduce that

$$\begin{aligned}\widehat{G}_0(\alpha\mathbf{x}) &= H\widetilde{G}_0\big(H^{-1}(\alpha x_1), \dots, H^{-1}(\alpha x_m)\big)\\ &= H\widetilde{G}_0\big(H^{-1}(x_1) + H^{-1}(\alpha), \dots, H^{-1}(x_m) + H^{-1}(\alpha)\big)\\ &= H\big[\widetilde{G}_0\big(H^{-1}(x_1), \dots, H^{-1}(x_m)\big) + H^{-1}(\alpha)\big] = \alpha H\widetilde{G}_0\big(H^{-1}(x_1), \dots, H^{-1}(x_m)\big)\\ &= \alpha\widehat{G}_0(\mathbf{x}).\end{aligned}$$

Q.E.D.

In order to show that the function $\widehat{G}_0$ is translation-invariant, we first establish the following local result.

LEMMA S.12: *For every $\mathbf{x}$ in the interior of $\widehat{\mathcal{W}}_0$, there is some $\delta_{\mathbf{x}} > 0$ such that $\widehat{G}_0(\mathbf{x} + \delta) = \widehat{G}_0(\mathbf{x}) + \delta$ for all $\delta \in [0, \delta_{\mathbf{x}}]$.*

PROOF: Suppose first that u is a non-constant function. By construction, the functions $\widehat{F}_c$, $\widehat{G}_0$, and $\widehat{G}_1$ satisfy an analogue of equations in (S.3), that is,

$$u(c) + b(c)\widehat{G}_0(\mathbf{x}) = \widehat{G}_1\big(u(c) + b(c)\mathbf{x}\big) \tag{S.20}$$

for every $c \in C$ and $\mathbf{x} \in \widehat{\mathcal{W}}_0$. For every $c, c' \in C$ and every $\mathbf{x}$ in the interior of $\widehat{\mathcal{W}}_0$, let

$$\mathbf{y} := \frac{u(c') - u(c)}{b(c)} + \frac{b(c')}{b(c)}\mathbf{x}.$$

Using (S.20), deduce that

$$\widehat{G}_0(\mathbf{x}) = \frac{u(c) - u(c')}{b(c')} + \frac{b(c)}{b(c')}\widehat{G}_0\left(\frac{u(c') - u(c)}{b(c)} + \frac{b(c')}{b(c)}\mathbf{x}\right). \tag{S.21}$$

If c, c' are close to one another, then $\frac{b(c)}{b(c')}\mathbf{y}, \mathbf{y} \in \widehat{\mathcal{W}}_0$. From Lemma S.11, we can conclude that $\frac{b(c)}{b(c')}\widehat{G}_0(\mathbf{y}) = \widehat{G}_0(\frac{b(c)}{b(c')}\mathbf{y})$. Then, (S.21) becomes

$$\widehat{G}_0(\mathbf{x}) = \frac{u(c) - u(c')}{b(c')} + \widehat{G}_0\left(\frac{u(c') - u(c)}{b(c')} + \mathbf{x}\right). \tag{S.22}$$

Summarizing the arguments so far, we can ensure that (S.22) holds for all $\mathbf{x}$ in the interior of $\widehat{\mathcal{W}}_0$, all $c \in C$, and all c' in some neighborhood $O_{\mathbf{x},c}$ of c. Since $u : C \to \mathbb{R}$ is non-constant, we can choose c such that u is non-constant in some right neighborhood of c. But then (S.22) implies that $\widehat{G}_0(\mathbf{x}) = -\delta + \widehat{G}_0(\delta + \mathbf{x})$ for all $\mathbf{x}$ in the interior of $\widehat{\mathcal{W}}_0$ and all $\delta > 0$ less than some $\delta_{\mathbf{x}} > 0$, as we wanted to prove.

Consider now the case when the function u is constant. The functions $\widehat{F}_c$, $\widehat{G}_0$, $\widehat{G}_1$, and $\widehat{G}_2$ satisfy equations analogous to the equations in (S.3). Deduce that

$$u(c)+b(c)u(c)+b(c)b(c)\widehat{G}_0(\mathbf{x})=\widehat{G}_2\big(u(c)+b(c)u(c)+b(c)b(c)\mathbf{x}\big) \tag{S.23}$$

for every $c \in C$ and $\mathbf{x} \in \widehat{\mathcal{W}_0}$. For every $c \in C$, let $v(c) := u(c)(1+b(c))$ and $\gamma(c) := b(c)b(c)$. Observe that if u is a constant function, then b is necessarily non-constant. Otherwise, $\succeq$ fails to be strictly increasing in the pointwise order on C^∞. Conclude that v is necessarily a non-constant function. Then, (S.23) becomes

$$v(c)+\gamma(c)\widehat{G}_0(\mathbf{x})=\widehat{G}_2\big(v(c)+\gamma(c)\mathbf{x}\big), \tag{S.24}$$

which holds for every $c \in C, \mathbf{x} \in \widehat{\mathcal{W}_0}$. But this equation is an exact analogue of equation (S.20), with the function v non-constant. Hence, the proof can be completed in an identical manner. *Q.E.D.*

The next lemma shows that the local property obtained in Lemma (S.12) "integrates" into a global property. The proof is analogous to that of Lemma S.9 and is omitted.

LEMMA S.13: *For every $\mathbf{x} \in \widehat{\mathcal{W}_0}$ and every $\delta \in \mathbb{R}$ such that $\mathbf{x}+\delta \in \widehat{\mathcal{W}_0}$, we have $\widehat{G}_0(\mathbf{x}+\delta) = \widehat{G}_0(\mathbf{x})+\delta$.*

S.1.8. *Concluding the Proof*

The preceding arguments show that it is always possible to renormalize the utility representation, so as to obtain an affine time aggregator, $W(c,x)=u(c)+b(c)x$, and a renormalized certainty equivalent ($\widetilde{G}_0$ in the affine case and $\widehat{G}_0$ in the non-affine case) which is translation-invariant (Lemmas S.4 and S.13), and furthermore scale-invariant when the function b is not constant (Lemmas S.10 and S.11). Recall from (S.1) that G_0 was defined by fixing $m>1$ and a probability vector $(\pi_1,\dots,\pi_m)$ and projecting I onto $[0,1]^m$. Since m and $(\pi_1,\dots,\pi_m)$ were arbitrary, we obtain that the recursive representation (U,W,I) of $\succeq$ can be renormalized so that:

– case 1: $W(c,x)=u(c)+\beta x$ and I is translation-invariant on $M^f(\mathcal{U})$,

– case 2: $W(c,x)=u(c)+b(c)x$ and I is translation- and scale-invariant on $M^f(\mathcal{U})$,

where $\mathcal{U} := U(D)$ and $M^f(\mathcal{U})$ is the set of simple lotteries with prizes drawn from the interval $\mathcal{U}$. In the first case, $u: C \to \mathbb{R}$ is continuous and $\beta \in (0,1)$. In the second, $u, b: C \to \mathbb{R}$ are continuous and $b(C) \subset (0,1)$. Since $M^f(\mathcal{U})$ is dense in $M(\mathcal{U})$ and the certainty equivalent $I: M(\mathcal{U}) \to \mathcal{U}$ is continuous, we know that if I is translation-invariant on $M^f(\mathcal{U})$, then I is also translation-invariant on $M(\mathcal{U})$. An identical argument holds for scale invariance.

To conclude the proof, it remains to take full account of the implications of Axiom 6 (Deterministic Monotonicity). First, note that the main features of the representations (U,W,I) we have derived so far—that $W(c,x)$ is affine in x and that I is translation- and, in the appropriate case, also scale-invariant—are preserved under positive affine transformations of utility. It is therefore w.l.o.g. to assume that the representations are chosen so that $U(D)=[0,1]$. This normalization, which we maintain in the statement of Proposition 1, makes it possible to express the implications of Axiom 6 in terms of the representation. When discounting is exogenous, everything is standard in that Axiom 6 is equivalent to the strict monotonicity of the function $u : C \to \mathbb{R}$. When discounting is endogenous, Axiom 6 is equivalent to the strict monotonicity of the functions $u, u + b : C \to \mathbb{R}$, provided that $U(D) = [0, 1]$. To see this, note that $U(D) = [0, 1]$ is equivalent to $U(\underline{c}, \underline{c}, \ldots) = 0$ and $U(\overline{c}, \overline{c}, \ldots) = 1$. The latter imply that for all $c \in C$, we have $U(c, \underline{c}, \underline{c}, \ldots) = u(c)$ and $U(c, \overline{c}, \overline{c}, \ldots) = u(c) + b(c)$, from where the strict monotonicity of $u, u + b$ follows.

S.1.9. *Dealing With the Case Where the Abel Function Is not Bounded*

We assume in Section S.1.2 that the iteration group $\{f^\alpha\}$ obtained in Lemma S.2 is such that $f^1(0) > 0$ and $f^{-1}(1) < 1$. This means that the Abel function $L : (0, 1) \to \mathbb{R}$ is bounded, which allows us to extend L continuously from $(0, 1)$ to $[0, 1]$. We show here how to handle the case when either $f^1(0) = 0$ or $f^{-1}(1) = 1$ (or both). To see how this affects the preceding proof, note that we started with a utility function $U : D \to [0, 1]$ and then obtained the desired representations by looking at the functions LU or HLU, depending on whether we were in the affine or non-affine case. If L is unbounded on $(0, 1)$, however, the functions LU or HLU are not well-defined on the entire domain D: we have to exclude the best and worst temporal lotteries in D, namely, the deterministic consumption streams $(\overline{c}, \overline{c}, \ldots)$ and $(\underline{c}, \underline{c}, \ldots)$. In particular, let $D^\circ \subset D$ be the subset of all temporal lotteries whose consumption levels are drawn from the open interval $C^\circ := (\underline{c}, \overline{c})$. Following the preceding arguments, we can then obtain the desired representations on D°. It remains to show that these representations can be extended from D° to the entire domain D. The only non-trivial part in this argument is to show that an Uzawa–Epstein representation on $(C^\circ)^\infty$ can be extended to an Uzawa–Epstein representation on C^∞. The next lemma provides the details, thus completing the proof of Proposition 1. To state the lemma, consider some set $X \subset C$ and say that a preference relation $\succeq$ on X^∞ has an *Uzawa–Epstein representation* (u, b, U) if it is represented by the utility function

$$\begin{aligned} U(c_0, c_1, \ldots) &= u(c_0) + b(c_0)u(c_1) + b(c_0)b(c_1)u(c_2) + \cdots \\ &= u(c_0) + b(c_0)U(c_1, c_2, \ldots), \end{aligned}$$

where $u : X \to \mathbb{R}$ and $b : X \to (0, 1)$ are continuous functions.

LEMMA S.14: *If $\succeq$ is continuous on C^∞ and has an Uzawa–Epstein representation on $(C^\circ)^\infty$, then $\succeq$ has an Uzawa–Epstein representation on the entire domain C^∞.*

PROOF: Let (u, b, U) be the Uzawa–Epstein representation on $(C^\circ)^\infty$. In particular, note that u, b are functions on C° and U is a function on $(C^\circ)^\infty$. First, we are going to show that $\lim_{c \nearrow \overline{c}} U(c, c, \ldots) < +\infty$. Fix some $c', c'' \in C^\circ$ such that $U(c', c', \ldots) < U(c'', c'', \ldots) < U(c', c', \ldots) + \frac{1}{2}$. If $\lim_{c \nearrow \overline{c}} U(c, c, \ldots) = +\infty$, then we can find a sequence $(c_n)_n$ such that $c_n \in C^\circ$ and $b(c')^n U(c_n, c_n, \ldots) \geq 1$ for every n. Consider the consumption streams $d_1 := (c', c_1, c_1, \ldots)$, $d_2 := (c', c', c_2, c_2, \ldots)$, and so on. Since the sequence $(d_n)_n$ converges pointwise to $(c', c', \ldots)$ and $\succeq$ is continuous in the product topology on C^∞, we know that $(c'', c'', \ldots) \succ d_n$ for all n large enough. But for every n,

$$U(d_n) = U(c', c', \ldots)\left(1 - b(c')^n\right) + b(c')^n U(c_n, c_n, \ldots) \geq U(c', c', \ldots)\left(1 - b(c')^n\right) + 1.$$

Hence, $U(d_n) > U(c', c', \ldots) + \frac{1}{2} > U(c'', c'', \ldots)$ for all n large enough, a contradiction.

Next, we are going to show that $\lim_{c\nearrow\overline{c}} b(c) < 1$. The proof is once again by contradiction. Let $(c_n)_n$ be a sequence such that $c_n \nearrow \overline{c}, b(c_n) \nearrow 1$, and $c_n \in C^\circ$ for every n. Fix some $c, c' \in C^\circ$ such that $(\overline{c}, c, c, \dots) \succ (c', c, c, \dots) \succ (c, c, \dots)$. Since $\succeq$ is continuous, we know that $(c_n, c, c, \dots) \succ (c', c, c, \dots)$ for all n large enough. Also,

$$U(c_n, c, c, \dots) = \bigl(1 - b(c_n)\bigr)U(c_n, c_n, \dots) + b(c_n)U(c, c, \dots) \quad \forall n.$$

Since $\lim_n U(c_n, c_n, \dots) < \infty$ and $b(c_n) \nearrow 1$, it follows that $\lim_n U(c_n, c, c, \dots) = U(c, c, \dots)$. But then $U(c_n, c, c, \dots) < U(c', c, c, \dots)$ for all n large enough, contradicting the fact that U represents $\succeq$ on $(C^\circ)^\infty$. Analogous arguments show that $\lim_{c\searrow\underline{c}} U(c, c, \dots) > -\infty$ and $\lim_{c\searrow\underline{c}} b(c) > 0$. Since $u(c) = (1 - b(c))^{-1}U(c, c, \dots)$ for every $c \in C^\circ$ and the function U is bounded, we can conclude that $u : C^\circ \to \mathbb{R}$ is bounded. By taking limits, we can extend the functions $u, b : C^\circ \to \mathbb{R}$ from C° to C. Let (u', b', U') be the ensuing Uzawa–Epstein representation on C^∞. By construction, U' agrees with U on $(C^\circ)^\infty$ and hence represents $\succeq$ on $(C^\circ)^\infty$. Since U' is the continuous extension of U from $(C^\circ)^\infty$ to C^∞, the function U' represents $\succeq$ on C^∞ as well. *Q.E.D.*

S.2. PROOF OF PROPOSITION 4

Before starting the proof, let us recall that the notions of conditional, continuation, and concatenated acts are related to each other. Formally, we have

$$h = (h_0, h_1, h_2, \dots) \in \mathcal{H} \quad \text{and} \quad s \in S \Rightarrow h^s = \bigl(h_0, h^{s,1}\bigr), \tag{S.25}$$

$$\forall c \in C, h \in \mathcal{H} \quad \text{and} \quad s \in S, (c, h)^{s,1} = h. \tag{S.26}$$

We first establish a result similar to Lemma 1, after which the proof of Proposition 1 can be almost readily applied.

LEMMA S.15: *A binary relation $\succeq$ on $\mathcal{H}$ admits a recursive representation* (U, W, I) (*as defined in equation* (26)) *if and only if it fulfills Axioms* 1, 2, A.3, A.4, *and* A.5.

PROOF: Let us start with the necessity of the axioms. It is obvious that representation (26) implies that Axioms 1, 2, and A.4 hold. Remark that (S.26) and (26) imply that for any $c \in C$ and $h \in \mathcal{H}$, we have

$$U(c, h) = W\bigl(c, U(h)\bigr),$$

which proves that Axiom A.5 holds. Last, for Axiom A.3, let us consider two acts $h = (h_0, h_1, \dots)$ and $\hat{h} = (h_0, \hat{h}_1, \dots)$ such that $U(h^s) \geq U(\hat{h}^s)$ for all s. Using (S.25), we get $U(h^s) = W(h_0, U(h^{s,1}))$ and $U(\hat{h}^s) = W(h_0, U(\hat{h}^{s,1}))$, so that the inequality $U(h^s) \geq U(\hat{h}^s)$ for all s provides $U(h^{s,1}) \geq U(\hat{h}^{s,1})$ for all s. Thus, we deduce that $U(h) \geq U(\hat{h})$, which implies that Axiom A.3 holds and concludes the necessity part.

We now demonstrate that the axioms are sufficient. Let us denote by $\mathcal{H}_e$ the set of consumption plans $h = (h_0, h_1, \dots)$ whose consumption h_0 at date 0 is not constrained to be deterministic, that is, the set of C-valued and $\mathcal{G}$-adapted processes. States will be denoted $(s_0, s_1, \dots)$ to emphasize the difference with our setting, where h_0 was supposed to be constant (while here it may depend on s_0). Let $c_0 \in C$ be a given constant consumption level. We define a binary relation $\succeq_e$ on the set $\mathcal{H}_e$ as follows:

$$\forall (h, \hat{h}) \in \mathcal{H}_e^2, h \succeq_e \hat{h} \quad \Leftrightarrow \quad (c_0, h) \succeq (c_0, \hat{h}),$$

where (c, h) is defined similarly to equation (24). Because of Axiom A.4, the binary relation $\succeq_e$ is independent of c_0 and defines a preference relation on $\mathcal{H}_e$. Moreover, for any $c \in C$ and $h \in \mathcal{H}$, Axioms A.4 and A.5 imply that $(c, h) \succeq_e (c, \hat{h}) \Leftrightarrow h \succeq_e \hat{h}$. The preference relation $\succeq_e$ fulfills, therefore, a property similar to the one defined in Axiom A.5. By continuity of $\succeq$ and thus of $\succeq_e$, there exists a continuous utility representation, whose corresponding utility function is denoted U.

For any $c \in C$, the functions $h \in \mathcal{H}_e \mapsto U(h)$ and $h \in \mathcal{H}_e \mapsto U((c, h))$ both represent the preference relation $\succeq_e$. Therefore, there exists a continuous function W, which is increasing in its second argument, such that

$$\forall h \in \mathcal{H}_e, U\big((c, h)\big) = W\big(c, U(h)\big).$$

For any h in $\mathcal{H}_e$ and any $s \in S$, one can define a conditional act $h^s \in \mathcal{H}$ similarly to equation (22). Consider now two acts h and $\hat{h}$ in $\mathcal{H}_e$ such that $h^s \succeq_e \hat{h}^s$ for all s. By definition of $\succeq_e$, we have $(c_0, h^s) \succeq (c_0, \hat{h}^s)$ for all s. Axiom A.3 implies then that $h \succeq_e \hat{h}$. The set $\mathcal{H}_e$ being isomorphic to $\mathcal{H}^S$, we obtain that for any $h \in \mathcal{H}_e$, $U(h) = I(U \circ h)$ where $U \circ h : S \to Im(U) = [0, 1]$ is defined by $(U \circ h)(s) = U(h^s)$ and $I : B_0(\Sigma) \to \mathbb{R}_+$ is a continuous, strictly increasing function. Since any $h = (h_0, h_1, \dots) \in \mathcal{H}$ can be viewed as $(h_0, (h_1, \dots))$ where $(h_1, \dots) \in \mathcal{H}_e$, we obtain

$$U(h_0, \dots) = W\big(h_0, I\big(U \circ h^1\big)\big).$$

It remains to show that I fulfills $I(x) = x$ for all $x \in Im(U)$. For this, consider any act such that $U(h^{s,1}) = x$ for all s. We have $U(h^s) = W(h_0, x)$ which is independent of s. With Axiom A.3, this implies that $U(h^s) = U(h)$ and therefore $I(x) = x$. *Q.E.D.*

To end up proving Proposition 4, it remains to show that Axioms 6 and A.7 hold if and only if one can use a time aggregator W and a certainty equivalent that fulfill the same kind of restrictions as those derived in the risk setting. The sufficiency part of the proof is exactly the same as the one provided for the risk setting. Indeed, in Appendix B, the number $m \in \mathbb{N}_+$ and the probabilities $(\pi_1, \dots, \pi_m) \in (0, 1)^m$ were considered to be fixed. Thus, the whole reasoning that was done considering the simple lottery $(\pi_1, x_1; \dots; \pi_m, x_m) \in M([0, 1])$ can be reproduced here without any change by imposing that $m = \operatorname{card}(S)$ and viewing the $x_1, \dots, x_m$ as state-contingent realizations.

Necessity of Axiom 6 is obvious, as in the risk setting. For the necessity of Axiom A.7, which is a stronger monotonicity requirement than the one imposed in the risk setting, let us define the notion of time-t conditional act as follows. For any $t \geq 1$, any $(\sigma_1, \dots, \sigma_t) \in S^t$, and any act $h \in \mathcal{H}$, we set

$$h^{\sigma_1, \dots, \sigma_t} \in \mathcal{H} : (s_1, s_2, \dots) \in \Omega \to h^{\sigma_1, \dots, \sigma_t}(s_1, s_2, \dots) = h(\sigma_1, \dots, \sigma_t, s_{t+1}, s_{t+2}, \dots).$$

This is a direct generalization of the notion of conditional act defined in equation (22).

Now consider $T > 0$ and an act h such that any time-T conditional act is a constant act (i.e., the act h only depends on the information revealed during the first T periods). We can show that $U(h)$ is given by the terminal point of the backward recursion:

$$\begin{cases} V_t(\sigma_1, \dots, \sigma_t, h) = V\big(h^{\sigma_1, \dots, \sigma_T}\big) & \text{for } t \geq T, \\ V_t(\sigma_1, \dots, \sigma_t, h) = I_{t+1}\big(s \mapsto V_{t+1}(\sigma_1, \dots, \sigma_t, s, h)\big) & \text{for all } 0 \leq t < T, \end{cases} \tag{S.27}$$

where V is the ex post lifetime utility function (i.e., the restriction of U to C^∞) and the I_t are given by $I_t(\mu) = \beta^t I(\frac{1}{\beta^t}\mu)$. The property stated in Axiom A.7 is then found to hold for any pair of acts h and $\hat{h}$ that only depends on information revealed in a finite number of periods. The extension to all pairs of acts is obtained by continuity.

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