Econometrica Supplementary Material

SUPPLEMENT TO “DUAL-SELF REPRESENTATIONS OF AMBIGUITY PREFERENCES”

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THIS SUPPLEMENT IS ORGANIZED as follows. Appendix S.1 povides the proofs for the generalizations of DSEU considered in Section 4.3. Appendix S.2 presents additional content for Section 3.2: a characterization of full dynamic consistency under DSEU, and some supporting examples for Remark 2 on updating under the Amarante and GMM representations. Appendix S.3 considers the representation obtained by inverting the order of moves of Optimism and Pessimism. Appendix S.4 presents an incompatibility result for source dependence under Klibanoff, Marinacci, and Mukerji's (2005) smooth model.

S.1. PROOFS FOR SECTION 4.3

S.1.1. *Proof of Theorem 3*

We will invoke the following result from MMR:

LEMMA S.1.1—Lemma 28 in MMR: *Preference $\succsim$ satisfies Axioms* 1–4 *and Axiom* 10 *if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a normalized niveloid $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Recall that functional $I : U \to \mathbb{R}$ is a ***niveloid*** if $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$ for all $\phi, \psi \in U$. Lemma 25 in MMR shows that I is a niveloid if and only if it is monotonic and constant-additive.

Based on this result, the necessity direction of Theorem 3 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and Axiom 10. Let I, u, and U be as given by Lemma S.1.1. Since I is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset $\hat{U} \subseteq \operatorname{int} U$ with $U \setminus \hat{U}$ of Lebesgue measure 0 such that I is differentiable on $\hat{U}$. Define $\mu_\psi := \nabla I(\psi)$ and $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$ for each $\psi \in \hat{U}$. By Lemma A.4

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and the fact that niveloids are monotonic and constant-additive, $\mu_\psi \in \Delta(S)$ for all $\psi \in \hat{U}$. For each $\psi \in U$, define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let $\mathbb{D} := \{D_\psi : \psi \in U\}$. The following lemma implies that each D_ψ is nonempty; note also that it is closed, convex, and bounded below.

LEMMA S.1.2: *For every $\phi, \psi \in U$, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ with equality if $\phi = \psi$.*

PROOF: First, consider any $\phi, \psi \in \hat{U}$. Let $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$ be as in Lemma A.6. Note that $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$, so that

$$\inf_{\xi \in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as D_ψ is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi), \tag{23}$$

where, by definition of D_ψ, (23) holds with equality if $\psi = \phi$.

Next, consider any $\phi, \psi \in U$. Take sequences $\phi_n \to \phi$, $\psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$ for each n, where we choose $\phi_n = \psi_n$ if $\phi = \psi$. For each n, the previous paragraph yields some $(\mu_n, w_n) \in D_{\psi_n}$ such that $\mu_n \cdot \phi_n + w_n = \min_{(\mu,w)\in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$, with equality if $\phi = \psi$. Thus, for each n, we have $I(\psi_n) - \mu_n \cdot \psi_n \leq w_n \leq I(\phi_n) - \mu_n \cdot \phi_n$. Since $\phi_n \to \phi$, $\psi_n \to \psi$, and I is continuous, this implies that sequence (w_n) is bounded. Thus, up to restricting to a suitable subsequence, we can assume that $(\mu_n, w_n) \to (\mu_\infty, w_\infty)$ for some $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$. Then $(\mu_\infty, w_\infty) \in D_\psi$ and $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$ by continuity of I, with equality if $\phi = \psi$. Thus, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w = \inf_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$, with equality if $\phi = \psi$, where the minimum is attained since D_ψ is closed and bounded below. *Q.E.D.*

Finally, we obtain a dual-self variational representation of $\succsim$ as follows. For each $D \in \mathbb{D}$, define $c_D : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$ for each $\mu \in \Delta(S)$, where, by convention, the infimum of the empty set is ∞. Note that c_D is convex for all D by convexity of D. Moreover, for all $\phi \in U$, $\min_{(\mu,w)\in D} \mu \cdot \phi + w = \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu)$. Thus, Lemma S.1.2 implies

$$I(\phi) = \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu) \tag{24}$$

for all $\phi \in U$. Since I is normalized, applying (24) to any constant vector $\underline{a} \in U$ yields $I(\underline{a}) = a + \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} c_D(\mu) = a$. Hence, $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$ satisfies $\max_{c\in\mathbb{C}^*} \min_{\mu\in\Delta(S)} c(\mu) = 0$ and $(\mathbb{C}^*, u)$ is a dual-self variational representation of $\succsim$ by Lemma S.1.1.

REMARK 3: We note that our characterization of the set of relevant priors under DSEU generalizes to the dual-self variational model. Specifically, let $\text{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$ denote the effective domain of any cost function. Then there exists a unique closed, convex set C such that $C \subseteq \overline{\text{co}}(\bigcup_{c\in\mathbb{C}} \text{dom}(c))$ for all dual-self variational representations of

$\succsim$, with equality for the representation $\mathbb{C}^*$ we constructed in the proof of Theorem 3. Moreover, it can again be shown that C is the Bewley set of the unambiguous preference $\succsim^*$. The argument relies on the observation that $C = \overline{\text{co}}(\bigcup_{\phi \in \text{int}\, U} \partial I(\phi))$, where I is the utility act functional obtained in the proof of Theorem 3 and U its domain. Details are available on request.

S.1.2. *Proof of Theorem 4*

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

LEMMA S.1.3: *Preference $\succsim$ satisfies Axioms 1–4 and 11 if and only if there exists a non-constant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a monotonic, normalized, and continuous functional $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem 4 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and 11. Let I, u, and U be as given by Lemma S.1.3. Define $D_\psi := \{(\mu, I(\psi) - \mu \cdot \psi) \in \mathbb{R}^S_+ \times \mathbb{R} : \mu \in \mathbb{R}^S_+\}$ for each $\psi \in U$. Note that D_ψ is nonempty and convex. Let $I_\psi(\phi) := \inf_{(\mu,w) \in D_\psi} \mu \cdot \phi + w$ for each $\phi, \psi \in U$.

Take any $\phi, \psi \in U$. Observe that

$$I_\psi(\phi) = \inf_{\alpha > 0, s \in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) & \text{if } \phi \geq \psi, \\ -\infty & \text{if } \phi \ngeq \psi. \end{cases}$$

Thus, $I(\phi) \geq I_\psi(\phi)$ by monotonicity of I, with equality if $\phi = \psi$. That is, for each $\phi \in U$,

$$I(\phi) = \max_{\psi \in U} I_\psi(\phi). \tag{25}$$

For each $\psi \in U$, define a function $G_\psi : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by

$$G_\psi(t, \mu) = \sup\{I_\psi(\xi) : \xi \in U, \xi \cdot \mu \leq t\}$$

for each (t, μ). The map is quasi-convex (Lemma 31 in CMMM) and increasing in t.

LEMMA S.1.4: *We have $I_\psi(\phi) = \inf_{\mu \in \Delta(S)} G_\psi(\mu \cdot \phi, \mu)$ for each $\phi, \psi \in U$.*

PROOF: Observe that RHS $= \inf_{\mu \in \Delta(S)} \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$. To see that LHS $\leq$ RHS, observe that $I_\psi(\phi) \leq \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$ holds for any $\mu \in \Delta(S)$. To see that LHS $\geq$ RHS, note first that if $\phi \geq \psi$, then LHS $= I(\psi)$ and RHS $\in \{I(\psi), -\infty\}$, so the inequality clearly holds. If $\phi \ngeq \psi$ then $\phi_s < \psi_s$ for some $s \in S$. Thus, by taking $\mu = \delta_s$, any ξ with $\xi \cdot \mu \leq \phi \cdot \mu$ satisfies $\xi_s \leq \phi_s$, which implies $\xi \ngeq \psi$, whence $I_\psi(\xi) = -\infty$. *Q.E.D.*

Setting $\mathbb{G} = \{G_\phi : \phi \in U\}$, Lemma S.1.4 and (25) ensure that the functional W given by (14) represents $\succsim$ and is continuous. Finally, note that since I is normalized, we have $a = I(\underline{a}) = \max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(a, \mu)$ for any $a \in \mathbb{R}$, as required.

S.2. ADDITIONAL MATERIAL FOR SECTION 3.2

S.2.1. *Characterization of Dynamic Consistency*

Fix any partition Π of S and a family of conditional preferences $\{\succsim_E\}_{E\in\Pi}$. Consider the following strengthening of C-dynamic consistency (Axiom 9):

AXIOM 12—Dynamic Consistency: *For all $f, g \in \mathcal{F}$, $f \succsim_E g \Leftrightarrow fEg \succsim g$.*

Epstein and Schneider (2003) showed that prior-by-prior updating under the maxmin model satisfies Axiom 12[30] for each $E \in \Pi$ if and only if the ex ante set of priors P is *rectangular* with respect to partition Π, meaning that there exist belief-sets $Q^0 \subseteq \Delta(\Pi)$ and $Q^E \subseteq \Delta(E)$ for each $E \in \Pi$ such that[31]

$$P = Q^0 \times \left(Q^E\right)_{E\in\Pi} := \left\{\mu \in \Delta(S) : \mu(\cdot) = \sum_{E\in\Pi} \nu^0(E)\nu^E(\cdot) \text{ for some } \nu^0 \in Q^0, \nu^E \in Q^E\right\}.$$

We show that for prior-by-prior updating under DSEU, Axiom 12 in turn characterizes the following extension of the notion of rectangularity to belief-set collections. Say that $\mathbb{P}$ is a ***rectangular belief-set collection*** (with respect to Π) if there exist belief-set collections $\mathbb{Q}^0 \subseteq \mathcal{K}(\Delta(\Pi))$ and $\mathbb{Q}^E \subseteq \mathcal{K}(\Delta(E))$ for each $E \in \Pi$ such that

$$\mathbb{P} = \mathbb{Q}^0 \times \left(\mathbb{Q}^E\right)_{E\in\Pi} := \left\{Q^0 \times \left(Q^E\right)_{E\in\Pi} : Q^0 \in \mathbb{Q}^0, Q^E \in \mathbb{Q}^E \ \forall E \in \Pi\right\}.$$

Note that this is stronger than requiring each $P \in \mathbb{P}$ to be rectangular. Say that $E \in \Pi$ is ***strongly non-null*** if for all $f \in \mathcal{F}$ and $p, q \in \Delta(Z)$ with $p \succ q$, we have $pEf \succ qEf$.

THEOREM S.2.1: *Suppose that $\succsim$ satisfies Axioms* 1–5, *that each $E \in \Pi$ is strongly non-null, and that each $(\succsim_E)_{E\in\Pi}$ is an Archimedean weak order. Then, the following are equivalent*:

1. *Each pair $(\succsim, \succsim_E)_{E\in\Pi}$ satisfies Axiom* 12.
2. *There exist a rectangular belief-set collection $\mathbb{P}$ and a nonconstant affine utility u such that $(\mathbb{P}, u)$ is a DSEU representation of $\succsim$ and $(\mathbb{P}_E, u)$ is a DSEU representation of $\succsim_E$ for each $E \in \Pi$.*

S.2.1.1. *Proof of Theorem S.2.1*

We will invoke the following lemma:[32]

AXIOM 13—Consequentialism: *If $f(s) = g(s)$ for all $s \in E$, then $f \sim_E g$.*

LEMMA S.2.1: *Suppose $\succsim$ and each $(\succsim_E)_{E\in\Pi}$ are weak orders. The following are equivalent*:

1. *Each pair $(\succsim, \succsim_E)_{E\in\Pi}$ satisfies Axiom* 12.

[30] Epstein and Schneider (2003) used an alternative formulation of dynamic consistency, which is equivalent to Axiom 12 in our setting (cf. Lemma S.2.1).

[31] In the following, we identify $\Delta(E)$ with the subset $\{\mu \in \Delta(S) : \mu(E) = 1\} \subseteq \Delta(S)$.

[32] For the direction (1) $\Rightarrow$ (2), Hubmer and Ostrizek (2015) observed that dynamic consistency implies consequentialism.

2. *Each* $(\succsim_E)_{E\in\Pi}$ *satisfies Axiom* 13 *and, for all* $f, g \in \mathcal{F}$,

$$[f \succsim_E g\ \forall E \in \Pi] \implies f \succsim g; \tag{26}$$

$$[f \succsim_E g\ \forall E \in \Pi \text{ and } f \succ_E g \text{ for some } E \in \Pi] \implies f \succ g. \tag{27}$$

PROOF: **(1.)** $\implies$ **(2.):** Suppose each $(\succsim, \succsim_E)_{E\in\Pi}$ satisfies Axiom 12. To show Axiom 13, consider any $f, g \in \mathcal{F}$ and $E \in \Pi$ with $f(s) = g(s)$ for all $s \in E$. Then $fEg \sim gEg$ since $\succsim$ is reflexive, which implies $f \sim_E g$ by Axiom 12.

Then, for any $f, g, h \in \mathcal{F}$ and $E \in \Pi$, Axioms 12 and 13 imply

$$f \succsim_E g \underbrace{\iff}_{\text{Ax. } 13} fEh \succsim_E gEh \underbrace{\iff}_{\text{Ax. } 12} fEh \succsim gEh. \tag{28}$$

To show (26), suppose $f \succsim_E g \forall E \in \Pi$. Then enumerating $\Pi = \{E_1, \dots, E_n\}$ and applying (28) iteratively, we have

$$f = fE_1f \succsim gE_1f \succsim gE_1(gE_2f) \succsim gE_1\bigl(gE_2(gE_3f)\bigr) \succsim \cdots \succsim g,$$

as required. Moreover, if $f \succ_{E_i} g$ for some i, then the above ensures $f \succ g$, so (27) holds.

(2.) $\implies$ **(1.):** For each $f, g \in \mathcal{F}$ and $E \in \Pi$, since $\succsim_E$ is a weak order and satisfies Axiom 13, we have

$$f \succsim_E g \iff fEg \succsim_E g;$$

moreover, for each $F \in \Pi \setminus \{E\}$,

$$fEg \sim_F g.$$

Thus, if $f \succsim_E g$, then $fEg \succsim g$ by (26). If not $f \succsim_E g$, then $g \succ_E f$ since $\succsim_E$ is a weak order, which implies $g \succ fEg$ by (27). *Q.E.D.*

PROOF OF THEOREM S.2.1: **(2.)** $\implies$ **(1.):** Since each $\succsim_E$ admits the updated DSEU representation $(\mathbb{P}_E, u)$, it satisfies Axiom 13. Thus, to prove that $(\succsim, \succsim_E)_{E\in\Pi}$ satisfies Axiom 12, it suffices by Lemma S.2.1 to verify (26)–(27).

Observe that since $\mathbb{P} = \mathbb{Q}^0 \times (\mathbb{Q}^E)_{E\in\Pi}$ is rectangular, the prior-by-prior updates $\mathbb{P}_E$ satisfy $\mathbb{P}_E = \mathbb{Q}^E$ for each $E \in \Pi$. Thus, each $\succsim_E$ is represented by the functional $W_E(f) = \max_{Q^E\in\mathbb{Q}^E} \min_{\nu^E\in Q^E} \nu^E \cdot u(f)$. Moreover, $\succsim$ is represented by the functional

$$\begin{aligned} W(f) &= \max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot u(f) \\ &= \max_{Q^0\in\mathbb{Q}^0} \min_{\nu^0\in Q^0} \sum_E \nu^0(E) \max_{Q^E\in\mathbb{Q}^E} \min_{\nu^E\in Q^E} \nu^E \cdot u(f) \\ &= \max_{Q^0\in\mathbb{Q}^0} \min_{\nu^0\in Q^0} \sum_E \nu^0(E) W_E(f). \end{aligned}$$

Thus, for any $f, g \in \mathcal{F}$, if $W_E(f) \geq W_E(g)$ for all $E \in \Pi$, then $W(f) \geq W(g)$, verifying (26). To verify (27), suppose $W_E(f) > W_E(g)$ for some $E \in \Pi$ and $W_F(f) \geq W_F(g)$ for all $F \in \Pi \setminus \{E\}$. Pick $p, q \in \Delta(Z)$ such that $u(p) = W_E(f)$ and $u(q) = W_E(g)$. Then

$$W(f) = W(pEf) > W(qEf) \geq W(qEg) = W(g),$$

where the strict inequality holds since each E is strongly non-null.

(1.) $\Longrightarrow$ (2.): Since $\succsim$ satisfies Axioms 1–5, Lemma B.1 yields a nonconstant, affine u and monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ such that $f \succsim g$ iff $I(u(f)) \geq I(u(g))$. Up to applying a positive affine transformation, we can assume that $u(\Delta(Z)) \supseteq [-1, 1]$. Since Axiom 12 implies Axiom 9, each $\succsim_E$ admits some DSEU representation $(\mathbb{Q}^E, u)$ by Theorem 2. Let $I_E : \mathbb{R}^S \to \mathbb{R}$ denote the corresponding monotonic, constant-linear functional given by $I_E(\phi) = \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi$.

For each $\phi^0, \psi^0 \in \mathbb{R}^\Pi$, write $\phi^0 \succsim^* \psi^0$ if there exist $\phi, \psi \in \mathbb{R}^S$ such that $I(\phi) \geq I(\psi)$ and

$$\phi^0(E) = I_E(\phi), \qquad \psi^0(E) = I_E(\psi), \quad \forall E \in \Pi. \tag{29}$$

Note that $\succsim^*$ is a weak order. Indeed, for any $\phi^0 \in \mathbb{R}^\Pi$, define $G(\phi^0) = \phi \in \mathbb{R}^S$ by $\phi(s) = \phi^0(E)$ for each $E \in \Pi$ and $s \in E$. Then, by construction of I_E, we have $\phi^0(E) = I_E(\phi)$ for all E. Moreover, note that for any other $\phi' \in \mathbb{R}^S$ with $\phi^0(E) = I_E(\phi')$, we have $I(\phi) = I(\phi')$: To see this, take $\alpha > 0$ small enough that $\alpha\phi, \alpha\phi' \in (u(\Delta(Z)))^S$. Since $I_E(\alpha\phi) = I_E(\alpha\phi')$ for each E (as I_E is constant-linear), the implication (26) of Axiom 12 in Lemma S.2.1 yields $I(\alpha\phi) = I(\alpha\phi')$. Thus, $I(\phi) = I(\phi')$ (as I is constant-linear). Taken together, this shows that for any $\phi^0, \psi^0 \in \mathbb{R}^\Pi$, $\phi^0 \succsim^* \psi^0$ if and only if $I(G(\phi^0)) \geq I(G(\psi^0))$, that is, $\succsim^*$ is represented by the functional $I_0 := I \circ G : \mathbb{R}^\Pi \to \mathbb{R}$.

Note that I_0 is monotonic, as I is monotonic and $\phi^0 \geq \psi^0$ implies $G(\phi^0) \geq G(\psi^0)$. Moreover, I_0 is constant-linear, as I is constant-linear and for any $\phi^0 \in \mathbb{R}^\Pi$, $\alpha > 0$, and $\beta \in \mathbb{R}$, we have $G(\alpha\phi^0 + \underline{\beta}) = \alpha G(\phi^0) + \underline{\beta}$. Thus, by the proof of Theorem 1, there is a belief-set collection $\mathbb{Q}^0 \subseteq 2^{\Delta(\Pi)}$ such that $I_0(\phi^0) = \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \phi^0$ for each $\phi^0 \in \mathbb{R}^\Pi$.

Set $\mathbb{P} := \{Q^0 \times (Q^E)_{E \in \Pi} : Q^0 \in \mathbb{Q}^0, Q^E \in \mathbb{Q}^E \forall E \in \Pi\}$, which is rectangular. Then for each $\phi \in \mathbb{R}^S$,

$$\begin{aligned}
\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi &= \max_{Q^0 \in \mathbb{Q}^0} \max_{Q^E \in \mathbb{Q}^E, \forall E} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \min_{\nu^E \in Q^E} \nu^E \cdot \phi \\
&= \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi.
\end{aligned}$$

We claim that $(\mathbb{P}, u)$ is a DSEU representation of $\succsim$. Indeed, for any f, g with $\phi = u(f)$, $\psi = u(g)$, define $\phi^0, \psi^0 \in \mathbb{R}^\Pi$ by $\phi^0(E) = I_E(\phi)$, $\psi^0(E) = I_E(\psi)$ for each $E \in \Pi$. Then

$$\begin{aligned}
f \succsim g \quad &\Longleftrightarrow \quad \phi^0 \succsim^* \psi^0 \\
&\Longleftrightarrow \quad \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \phi^0 \geq \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \psi^0 \\
&\Longleftrightarrow \quad \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi \\
&\qquad\quad \geq \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \psi \\
&\Longleftrightarrow \quad \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi \geq \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \psi.
\end{aligned}$$

Finally, by construction, we have $\mathbb{Q}^E = \mathbb{P}_E$ for each $E \in \Pi$, and thus $(\mathbb{P}_E, u)$ is a DSEU representation of $\succsim_E$. *Q.E.D.*

S.2.2. *Details for Remark 2*

We elaborate on some difficulties, outlined in Remark 2, with extending prior-by-prior updating to GMM and Amarante's representations of invariant biseparable preferences.

S.2.2.1. *GMM*

Suppose the ex ante preference $\succsim$ admits a GMM representation (1) with parameters $(\alpha(\cdot), C, u)$. As in Remark 2, consider the following potential extension of prior-by-prior updating: Define the conditional preference $\succsim_E$ by updating the set of relevant priors C prior-by-prior to C_E, while holding the weight function $\alpha(\cdot)$ and utility u fixed; that is, $\succsim_E$ is represented by

$$W_E(f) = \alpha(f) \min_{\mu \in C_E} \mathbb{E}_\mu\big[u(f)\big] + \big(1 - \alpha(f)\big) \max_{\mu \in C_E} \mathbb{E}_\mu\big[u(f)\big].$$

The following example highlights several difficulties that arise for this updating rule: (i) the induced $\succsim_E$ need not be invariant biseparable, as it can violate monotonicity; and (ii) $\succsim_E$ may violate consequentialism. In particular, this implies (by Theorem 2) that this updating rule does not in general satisfy C-dynamic consistency (Axiom 9).

EXAMPLE 3: Take $S = \{1, 2, 3\}$, and a nonconstant affine utility u with range $[0, 1]$. Write $f = (f_1, f_2, f_3)$ for the act f that yields the lottery f_s in state s.

Suppose $\succsim$ is induced by an α-MEU representation (8) with $\alpha = 1/2$, utility u, and belief-set $P = \Delta(S)$. Then $\succsim$ equivalently admits a GMM representation $(\alpha(\cdot), C, u)$, where:[33]

- The set of relevant priors is $C = \text{co}\{(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$.
- The function $\alpha(\cdot)$ is defined, for all f with nonconstant utility profile $(u(f_1), u(f_2), u(f_3))$, by

$$\alpha(f) = \frac{\text{med}\big(u(f)\big) - \min\big(u(f)\big)}{\max\big(u(f)\big) - \min\big(u(f)\big)},$$

  where $\max(u(f)) = \max\{u(f_1), u(f_2), u(f_3)\}$, $\min(u(f)) = \min\{u(f_1), u(f_2), u(f_3)\}$, and $\text{med}(u(f))$ is the median value in $\{u(f_1), u(f_2), u(f_3)\}$. For instance, if f satisfies $u(f_1) > u(f_2) > u(f_3)$, then $\alpha(f) = (u(f_2) - u(f_3))/(u(f_1) - u(f_3))$.

Consider the event $E = \{1, 2\}$. The prior-by-prior update of C is $C_E = \text{co}\{(1, 0, 0), (0, 1, 0)\}$. Thus, the conditional preference $\succsim_E$ induced by the above prior-by-prior updating rule for GMM is represented by the functional

$$W_E(f) = \alpha(f) \min\big\{u(f_1), u(f_2)\big\} + \big(1 - \alpha(f)\big) \max\big\{u(f_1), u(f_2)\big\}.$$

Consider two acts f and g such that $u(f_1) = u(g_1) = 1$, $u(f_2) = 1/2$, and $u(f_3) = u(g_2) = u(g_3) = 0$. Then $\alpha(f) = 1/2$ and $\alpha(g) = 0$. Hence, $W_E(f) = 3/4$ and $W_E(g) = 1$. This shows that $g \succ_E f$ despite the fact that $f(s) \succsim_E g(s)$ for all $s \in S$. Thus, $\succsim_E$ violates monotonicity (Axiom 2) and hence is not an invariant biseparable preference.

Next, consider the same act f as above and some $\tilde{g}$ with $\tilde{g}_1 = f_1$, $\tilde{g}_2 = f_2$, and $u(\tilde{g}_3) = 1/2$. We have $\alpha(\tilde{g}) = 0$, and hence $W_E(\tilde{g}) = 1 > W_E(f)$, which implies $\tilde{g} \succ_E f$. This shows that $\succsim_E$ violates consequentialism (Axiom 13), as $f(s) = \tilde{g}(s)$ for all $s \in E = \{1, 2\}$.

[33]Indeed, note that the corresponding utility act functional $I(v) = \frac{1}{2} \min_{i=1,2,3} v_i + \frac{1}{2} \max_{i=1,2,3} v_i$ is piecewise linear with three slopes given by $\mu \in \{(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$, so C is the convex hull of these three beliefs. Given this, $\alpha(\cdot)$ is determined by setting $\alpha(f) \min_C \mu \cdot u(f) + (1 - \alpha(f)) \max_C \mu \cdot u(f) = I(u(f))$.

An alternative approach to extend prior-by-prior updating to GMM's representation is to impose C-dynamic consistency on $(\succsim, \succsim_E)$. This uniquely pins down a conditional preference $\succsim_E$, which is invariant biseparable (as can be seen from Theorem 2). Thus, the conditional preference $\succsim_E$ induced in this manner must admit some GMM representation $(\alpha^E(\cdot), C^E, u)$. However, we note that obtaining the conditional parameters $\alpha^E(\cdot)$ and C^E directly from the parameters $\alpha(\cdot)$ and C of the ex ante representation can be difficult, as $\alpha^E(\cdot)$ and C^E can each depend jointly on both $\alpha(\cdot)$ and C (in a way that involves solving a fixed-point problem).[34] Notably, the following example illustrates that when $\alpha(\cdot) \not\equiv 0, 1$, the set C^E, that is, the set of relevant priors of the conditional preference $\succsim_E$, need *not* be equal to the prior-by-prior update C_E of the ex ante set of relevant priors C:

EXAMPLE 4: As in Example 3, let $S = \{1, 2, 3\}$ and suppose the ex ante preference $\succsim$ is an α-MEU preference with $\alpha = 1/2$, nonconstant utility u, and belief-set $P = \Delta(S)$. As noted, the set of relevant priors of $\succsim$ is $C = \text{co}\{(\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$.

Again, consider event $E = \{1, 2\}$, but now suppose the conditional preference $\succsim_E$ is pinned down from $\succsim$ by C-dynamic consistency. Note that, for any act f with utility profile $(u(f_1), u(f_2), u(f_3))$, the condition $fEp \sim p$ is equivalent to

$$\frac{1}{2}\min\{u(f_1), u(f_2), u(p)\} + \frac{1}{2}\max\{u(f_1), u(f_2), u(p)\} = u(p),$$

that is, to

$$\frac{1}{2}u(f_1) + \frac{1}{2}u(f_2) = u(p).$$

Thus, by C-dynamic consistency, the conditional preference $\succsim_E$ is the SEU preference with belief $(1/2, 1/2, 0)$. Hence, the set of relevant priors of $\succsim_E$ is $C^E = \{(1/2, 1/2, 0)\}$, which is a strict subset of the prior-by-prior update $C_E = \text{co}\{(1, 0, 0), (0, 1, 0)\}$ of C.

S.2.2.2. *Amarante*

We first restate an example from Frick, Iijima, and Le Yaouanq (2022), which illustrates that, under the α-MEU model, if belief-sets are updated prior-by-prior, then conditional preferences are not uniquely pinned down from the ex ante preference and instead depend on the choice of ex ante representation:

EXAMPLE 5: Suppose $S = \{1, 2, 3\}$. Fix any nonconstant affine utility u, and consider the two α-MEU representations (α, P, u) and (α', P', u), where

$$\alpha = \frac{3}{4}, \qquad P = \text{co}\left\{\left(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}\right), \left(\frac{1}{6}, \frac{5}{12}, \frac{5}{12}\right)\right\},$$
$$\alpha' = 1, \qquad P' = \text{co}\left\{\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}.$$

[34]Specifically, to obtain $(\alpha^E(\cdot), C^E)$ directly from $(\alpha(\cdot), C)$, one must first obtain $\succsim_E$ from $\succsim$ via C-dynamic consistency. For each act f, this involves finding a constant act p_f that solves the fixed-point problem $fEp_f \sim p_f$, and then defining $f \succsim_E g \Leftrightarrow p_f \succsim_E p_g$.

The two representations represent the same ex ante preference $\succsim$, since for all f,

$$\frac{3}{4} \min_{\mu \in \mathrm{co}\{(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}), (\frac{1}{6}, \frac{5}{12}, \frac{5}{12})\}} \mathbb{E}_\mu\big[u(f)\big] + \frac{1}{4} \max_{\mu \in \mathrm{co}\{(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}), (\frac{1}{6}, \frac{5}{12}, \frac{5}{12})\}} \mathbb{E}_\mu\big[u(f)\big]$$
$$= \min_{\mu \in \mathrm{co}\{(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}} \mathbb{E}_\mu\big[u(f)\big].$$

Now, consider the event $E = \{1, 2\}$. The prior-by-prior Bayesian updates of P and P' are

$$P_E = \mathrm{co}\left\{\left(\frac{10}{11}, \frac{1}{11}, 0\right), \left(\frac{2}{7}, \frac{5}{7}, 0\right)\right\}, \qquad P'_E = \mathrm{co}\left\{\left(\frac{4}{5}, \frac{1}{5}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right\}.$$

Consider an act f with utility profile $u(f) = (1, 0, 0)$. The value of this act under the updated model (α, P_E, u) equals

$$\frac{3}{4} \min\left\{\frac{10}{11}, \frac{2}{7}\right\} + \frac{1}{4} \max\left\{\frac{10}{11}, \frac{2}{7}\right\} = \frac{34}{77},$$

and therefore the DM is ex post indifferent between f and the constant act p with utility $34/77$. However, under the updated model (α', P'_E, u), the value of f equals $1/2$, and thus the DM strictly prefers p to f ex post under this model. This shows that (α, P_E, u) and (α', P'_E, u) do not represent the same conditional preference.

Now, consider an Amarante representation (2) with utility u and capacity ν defined on some $P \subseteq \Delta(S)$. Natural updating rules for this representation seem less apparent: The literature has considered several updating rules for the special case of Choquet expected utility (see the survey by Gilboa and Marinacci (2016)), but directly applying these rules to Amarante's model would require one to observe ex post preferences $\succsim_Q$ conditional on subsets $Q \subseteq P$ of beliefs, rather than conditional on subsets E of states.

One potential extension of prior-by-prior updating might be to hold fixed the utility u and consider the updated capacity ν_E, which is defined on the set P_E by $\nu_E(Q) := \nu(\{\mu \in P : \mu_E \in Q\})$ for each $Q \subseteq P_E$; that is, ν_E transfers all weight that ν assigns to any prior belief to its posterior. However, this rule gives rise to the same issue as in Example 5, that is, conditional preferences are not uniquely pinned down from the ex ante preference. To see this, we use the observation from Amarante (2009) that any α-MEU representation (α, P, u) is equal to the Amarante representation with utility u and capacity ν defined on P by $\nu(Q) = \alpha$ for all $\emptyset \neq Q \subsetneq P$, $\nu(\emptyset) = 0$, and $\nu(P) = 1$. This induces an updated capacity ν_E that is defined on P_E and satisfies $\nu_E(Q) = \alpha$ for all $\emptyset \neq Q \subsetneq P_E$, $\nu_E(\emptyset) = 0$, and $\nu_E(P_E) = 1$. Thus, the induced conditional Amarante representation is equal to the α-MEU representation (α, P_E, u). Given this, the multiplicity of conditional preferences in Example 5 also applies to this updating rule for the Amarante model.

S.3. MINMAX DSEU REPRESENTATION

While DSEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for DSEU.

THEOREM S.3.1: *Preference $\succsim$ satisfies Axioms* 1–5 *if and only if $\succsim$ admits a minmax DSEU representation*, *that is*, *there exists a belief-set collection $\mathbb{Q}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu \big[u(f)\big]$$

represents $\succsim$.

Our construction of the maxmin DSEU representation in the proof of Theorem 1 uses the belief-set collection $\mathbb{P}^* = \mathrm{cl}\{P^*_\phi : \phi \in \mathbb{R}^S\}$ with $P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$. Analogously, it can be shown that the belief-set collection $\mathbb{Q}^* := \mathrm{cl}\{Q^*_\phi : \phi \in \mathbb{R}^S\}$ with $Q^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$ yields a minmax DSEU representation. Paralleling Section 2.3, it is straightforward to show that $C := \partial I(\underline{0})$ again corresponds to the smallest set of priors that is contained in $\overline{\mathrm{co}} \bigcup_{Q \in \mathbb{Q}} Q$ for all minmax DSEU representations $\mathbb{Q}$ of $\succsim$, with equality for representation $\mathbb{Q}^*$.

While the different shades of ambiguity aversion in Section 3.1.1 are most conveniently characterized using the maxmin DSEU representation, the minmax DSEU representation is useful for characterizing ambiguity-seeking attitudes. Indeed, one can derive analogs of Propositions 2 and 3 that characterize the ambiguity-seeking counterparts of Axioms 6, 7, and 8 in terms of the intersection of belief-sets in $\mathbb{Q}$.

S.4. SOURCE DEPENDENCE AND THE SMOOTH MODEL

Recall that under Klibanoff, Marinacci, and Mukerji's (2005) (henceforth, KMM's) smooth model, $\succsim$ is represented by the functional

$$W(f) = \int \phi\big(u(f) \cdot \mu\big)\, d\nu(\mu), \tag{30}$$

for some Borel probability measure $\nu \in \Delta(\Delta(S))$ over beliefs, nonconstant affine $u : \Delta(Z) \to \mathbb{R}$, and strictly increasing $\phi : u(Z) \to \mathbb{R}$. For expositional simplicity, we consider $Z = [0, 1]$. Assume that u is strictly increasing and continuous on Z with $u(0) = 0$, and that ϕ is twice continuously differentiable with $\phi'(0), \phi''(0) \neq 0$.

Analogously to Corollary 4 for the α-MEU model, the following claim establishes a sense in which the smooth model is incompatible with source-dependent negative and positive ambiguity attitudes:

CLAIM 1: *Suppose that $\succsim$ admits a representation* (30). *Then there do not exist events $E, F, G \subseteq S$ such that for all $x > 0$*,

$$xE0 \succ xF0 \succ xG0 \quad \textit{and} \quad xE^c0 \succ xF^c0 \succ xG^c0 \tag{31}$$

and such that $\mu(F)$ is constant across all μ in the support of ν.[35]

PROOF: Suppose towards a contradiction that such events E, F, G exist. For each event $A \subseteq S$ and $\Delta \in [0, u(1)]$, let

$$W_A(\Delta) := \int \phi\big(\mu(A)\Delta\big)\, d\nu(\mu).$$

[35] See Theorem 3 in KMM for a behavioral characterization of such unambiguous events F.

Then $W(xA0) = W_A(u(x))$ for all $x > 0$. Thus, (31) implies that, for all $\Delta \in [0, u(1)]$,

$$W_E(\Delta) > W_F(\Delta) > W_G(\Delta) \quad \text{and} \quad W_{E^c}(\Delta) > W_{F^c}(\Delta) > W_{G^c}(\Delta). \tag{32}$$

Observe that, for each A, we have $W_A(0) = \phi(0)$, and

$$\begin{aligned}\frac{\partial}{\partial\Delta} W_A(\Delta) &= \int \phi'\big(\mu(A)\Delta\big)\mu(A)\, d\nu(\mu) \\ &= \phi'(0) \int \mu(A)\, d\nu(\mu) \quad \text{at } \Delta = 0, \\ \frac{\partial^2}{\partial\Delta^2} W_A(\Delta) &= \int \phi''\big(\mu(A)\Delta\big)\mu(A)^2\, d\nu(\mu) \\ &= \phi''(0) \int \mu(A)^2\, d\nu(\mu) \quad \text{at } \Delta = 0.\end{aligned}$$

Let α be the constant such that $\alpha = \mu(F)$ for all μ in the support of ν. Then, performing a first-order Taylor approximation, the first inequalities in (32) imply $\int \mu(E)\, d\nu(\mu) \geq \alpha \geq \int \mu(G)\, d\nu(\mu)$. Likewise, the second inequalities in (32) imply $\int \mu(E^c)\, d\nu(\mu) \geq 1 - \alpha \geq \int \mu(G^c)\, d\nu(\mu)$. Thus,

$$\int \mu(E)\, d\nu(\mu) = \alpha = \int \mu(G)\, d\nu(\mu). \tag{33}$$

Note that it is not the case that $\mu(E) = \alpha$ for ν-almost every μ, as this would imply $W_E(\Delta) = W_F(\Delta)$, contradicting $W_E(\Delta) > W_F(\Delta)$. Likewise, it is not the case that $\mu(G) = \alpha$ for ν-almost every μ, as this would contradict $W_F(\Delta) > W_G(\Delta)$. Thus, by Jensen's inequality,

$$\int \mu(E)^2\, d\nu(\mu), \int \mu(G)^2\, d\nu(\mu) > \alpha^2.$$

Hence, performing a second-order Taylor approximation, $W_E(\Delta) > W_F(\Delta)$ and (33) implies that $\phi''(0) > 0$. Likewise, $W_F(\Delta) > W_G(\Delta)$ and (33) implies that $\phi''(0) < 0$. This is a contradiction. *Q.E.D.*

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