Econometrica Supplementary Material

SUPPLEMENT TO "THE ANALYTIC THEORY OF A MONETARY SHOCK"

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This supplement contains all the proofs of the paper "The Analytic Theory of a Monetary Shock." The document also contains two applications of the method developed in the paper for multi-product firms and for a general random fixed cost problem.

KEYWORDS: Menu costs, impulse response, dominant eigenvalue, selection.

APPENDIX A: PROOFS OF PROPOSITIONS IN THE BODY OF THE PAPER

PROOF PROPOSITION 1: Using the definitions of $\mathcal{H}$, $\mathcal{G}$, and τ, we have the following recursion:

$$\mathcal{H}(f)(x,t) = \mathcal{G}(f)(x,t) + \mathbb{E}\big[1_{\{t>\tau\}}\mathcal{H}(f)\big(x^*, t-\tau\big)\big|x\big] \quad \text{for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t > 0.$$

Let us begin by defining the following object: $D(x,t) \equiv \mathbb{E}[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)|x]$. We first consider case (i) and show that $D(x,t) = 0$ for all x and all t. This follows since $\mathcal{H}(f)(x^*, s) = 0$ for all s. This in turn follows because f is antisymmetric; thus, we have $\mathbb{E}[f(x(t))|x(\tau) = x^*] = 0$, which follows immediately by the symmetry of the distribution $g(x,t)$ and the antisymmetric property of f. It follows that $\mathbb{E}[1_{\{t\geq\tau\}}f(x(t))|x(0) = x] = 0$. Hence, since $\mathcal{H} = \mathcal{G}$, this implies that $G(t) = H(t)$ for any $p(\cdot, t)$.

Now we turn to case (ii). We note that $D(x,t)$ is symmetric in x around $x^* = (\underline{x} + \bar{x})/2$. This follows since the law of motion of x is symmetric so $g(x,t)$ is symmetric around x^*. This in turn implies that the probability of hitting either barrier at time s, starting with $x(0) = x$, is symmetric in x, which directly implies the symmetry of $D(x,t)$. Now we use that $D(x,t)$ is symmetric and that

$$H(t,f,p) - G(t,f,p) = \int_{\underline{x}}^{\bar{x}} D(x,t)\big(p(x,0) - \bar{p}(x)\big)\,dx.$$

Since $D(x,t)$ is symmetric and $p(x,0) - \bar{p}(x)$ is antisymmetric, we have that the right-hand side is zero so that $H(t) = G(t)$. *Q.E.D.*

PROOF OF PROPOSITION 2: We analyze the eigenvalue-eigenfunction problem defined in equation (12) for $\{\lambda_j, \gamma_j\}$, which can be rewritten as

$$\lambda_j\gamma_j(x) = \frac{\sigma^2}{2}\gamma''(x) - V(x)\gamma_j(x) \quad \text{where } V(x) \equiv \xi(x) + \frac{1}{2}\frac{\mu^2}{\sigma^2}. \tag{A.1}$$

As a matter of notation, we refer to the bounded domain case when $-\infty < \underline{x} < \bar{x} < +\infty$, and to the unbounded domain case when $-\infty = \underline{x} < \bar{x} = +\infty$. We use results from

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Section 3 of Chapter 4 of Zettl (2010) for the bounded domain case, and from Section 2.3 of Chapter 2 of Berezinn and Shubin (1991) for the unbounded domain case. Both references use a different notation from each other, which also differs from the one we use. Relative to the notation in Chapter 4 of Zettl (2010), our boundary condition of the o.d.e. for the eigenvalue-eigenfunction pair in the bounded domain case corresponds to the "Separated self-adjoint BC," our $\sigma^2 > 0$ corresponds to a constant and positive function $p > 0$, our function V corresponds to the function q, and the function w can be taken to be identically 1. The notation in Chapter 2 of Berezinn and Shubin (1991) corresponds to the case where we divide both sides of equation (A.1) by $\sigma^2/2$. Equivalently, we can assume that $\sigma^2/2 = 1$, in which case our potential $V(x)$ corresponds to $v(x)$ in the notation of Chapter 2 of Berezinn and Shubin (1991). Relative to notation convention in both references, our eigenvalues correspond to minus theirs, since the term with the product of the eigenvalue times the eigenfunction is on the other side of the inequality.

- (E_1) The existence of a countably many eigenvalues follows from the spectral theorem for compact self-adjoint operators. where in our case the operator on an arbitrary function f is defined as $L(f)(x) = \frac{\sigma^2}{2} f''(x) - V(x)f(x)$ for $x \in [\underline{x}, \bar{x}]$ and $f(\bar{x}) = f(\underline{x}) = 0$.

  In particular, for the bounded domain case, it follows from Theorem 4.3.1 in Zettl (2010). For the unbounded domain case, it follows from Theorem 3.1 part 1 in Berezinn and Shubin (1991).
- (E_2) That the eigenvalues are all real follows immediately because the operator L defined above is Hermitian or self-adjoint. That L is self-adjoint follows by direct computation, using integration by parts, and the boundary conditions. This is a standard result for Sturm–Liouville equations.

  That the eigenvalues are strictly ordered and that they diverge follows from Theorem 4.3.1 parts 4 and 6 in in Zettl (2010) in the bounded domain case. That the eigenvalues are ordered and that they diverge follows from Theorem 3.1 in Berezinn and Shubin (1991) in the unbounded domain case.

  That the eigenvalues are non-repeated, that is, that each eigenvalue is associated with only one linearly independent eigenfunction, follows from Proposition 3.3 in Berezinn and Shubin (1991) in the unbounded domain case and from part 6 of Theorem 4.3.1 in Zettl (2010).

  The the eigenvalues are negative follows from $\sigma^2 > 0$ and $V \geq 0$. To see why, take $\lambda_j \gamma_j = L(\gamma_j)$, multiply it by γ_j, and integrate it between $\underline{x}$ and $\bar{x}$. Integrating by parts and using the boundary conditions, we obtain $\lambda_j |\gamma_j|^2 = -\frac{\sigma^2}{2} \int_{\underline{x}}^{\bar{x}} \gamma_j'(x)^2 \, dx - \int_{\underline{x}}^{\bar{x}} V(x)\gamma_j(x)^2 \, dx < 0$ since $|\gamma_j| = 1$.
- (E_3) That the eigenfunctions $\{\gamma_j\}_{j=1}^{\infty}$ form a complete orthonormal base in L^2 follows from Theorem 2.27 in Al-Gwaiz (2008) for the bounded domain case, and from Theorem 3.1 in Berezinn and Shubin (1991) in the unbounded domain case. Equivalently, for any $g \in L^2$, we have $\|g - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \gamma_j \|_2 = 0$, where, for any $g, h \in L^2$, we define the standard L^2 norm an inner product as $\langle g, h \rangle_2 \equiv \int_{\underline{x}}^{\bar{x}} g(x)h(x) \, dx$ and where $\|g\|_2^2 \equiv \langle g, g \rangle$.

  Next, we extend the result to show that $\{\varphi_j\}_{j=1}^{\infty}$ form an orthonormal base for L_w^2. Take any $f \in L_w^2$, or equivalently,

$$\langle f, f \rangle = \int_{\underline{x}}^{\bar{x}} \big(f(x)\big)^2 e^{\frac{2\mu}{\sigma^2} x} \, dx = \int_{\underline{x}}^{\bar{x}} \big(f(x) e^{\frac{\mu}{\sigma^2} x}\big)^2 \, dx = \int_{\underline{x}}^{\bar{x}} \big(g(x)\big)^2 \, dx = \langle g, g \rangle_2,$$

where we define $g(x) \equiv f(x)e^{\frac{\mu}{\sigma^2}x}$, and hence $g \in L^2$. By the result above, we have

$$\begin{aligned}
0 &= \int_{\underline{x}}^{\bar{x}} \left(g(x) - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \gamma_j(x) \right)^2 dx \\
&= \int_{\underline{x}}^{\bar{x}} \left(g(x)w(x)^{-\frac{1}{2}} - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \gamma_j(x)w(x)^{-\frac{1}{2}} \right)^2 w(x)\, dx \\
&= \int_{\underline{x}}^{\bar{x}} \left(g(x)e^{-\frac{\mu}{\sigma^2}x} - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \gamma_j(x)e^{-\frac{\mu}{\sigma^2}x} \right)^2 w(x)\, dx \\
&= \int_{\underline{x}}^{\bar{x}} \left(f(x) - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \varphi_j(x) \right)^2 w(x)\, dx = \left\| f - \sum_{j=1}^{\infty} \langle g, \gamma_j \rangle_2 \varphi_j \right\|^2.
\end{aligned}$$

Finally, notice that

$$\begin{aligned}
\langle g, \gamma_j \rangle_2 &= \int_{\underline{x}}^{\bar{x}} g(x)\gamma_j(x)\, dx = \int_{\underline{x}}^{\bar{x}} g(x)w(x)^{-\frac{1}{2}}\gamma_j(x)w(x)^{-\frac{1}{2}}w(x)\, dx \\
&= \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j(x)w(x)\, dx = \langle f, \varphi_j \rangle.
\end{aligned}$$

Thus, we have shown that for any arbitrary $f \in L_w^2$, we have $0 = \| f - \sum_{j=1}^{\infty} \langle f, \varphi_j \rangle \varphi_j \|$.

- (E_4) That the eigenfunctions can be indexed by the number of zeros follows from part 6 of Theorem 4.3.1 in Zettl (2010) for the bounded domain case, and from Theorem 3.5 in Berezinn and Shubin (1991) for the unbounded domain case.
- (E_5) The parity of the eigenfunctions follows immediately from the assumption of symmetry of the problem. Under the symmetry assumption, let us normalize the values of $x^* = 0$ so that $\underline{x} = -\bar{x}$ and $V(-x) = \xi(-x) = \xi(x) = V(x)$. In this case, one can easily check that γ_j will be of the form $\gamma_j(x) = c_j \gamma_j(-x)$ for some nonzero constant c_j solving the o.d.e and the Dirichlet boundary condition. Since there is only one linearly independent eigenfunction for each eigenvalue, this is the form of the eigenfunctions. It will be symmetric or antisymmetric depending on the sign of c_j. Since γ_j has exactly $j - 1$ zeros, then $c_j > 0$ for $j = 1, 3, \ldots$ and $c_j < 0$ for $j = 2, 4, \ldots$.

Q.E.D.

PROOF OF THEOREM 1: The result follows by Proposition 2, the definition of the projection coefficients $\langle \varphi_j, f \rangle$ and $\langle \varphi_j, \hat{P}/w \rangle$, and the definition of the response function in equation (7). In particular, let us start with the definition of IRF G in equation (7) as an integral of $\mathcal{G}$ where $\mathcal{G}$ is the conditional expectation given by (7). Thus, fixing f, the function $\mathcal{G}$ when viewed as a function of (x, t) must satisfy the following Kolmogorov Backward p.d.e. with boundary conditions:

$$\begin{aligned}
\partial_t \mathcal{G}(f)(x, t) &= \mu \partial_x \mathcal{G}(f)(x, t) + \frac{\sigma^2}{2} \partial_{xx} \mathcal{G}(f)(x, t) \\
&\quad - \xi(x)\mathcal{G}(f)(x, t) \quad \text{for all } x \in [\underline{x}, \bar{x}], \text{ and } t > 0, \qquad \text{(A.2)}
\end{aligned}$$

$$0 = \mathcal{G}(f)(\bar{x}, t) = \mathcal{G}(f)(\underline{x}, t) = 0, \quad \text{for all } t > 0, \tag{A.3}$$

$$f(x) = \mathcal{G}(f)(x, 0) \quad \text{for all } x \in [\underline{x}, \bar{x}]. \tag{A.4}$$

We postulate that this equation has as solution:

$$\mathcal{G}(f)(x, t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle f, \varphi_j \rangle \varphi_j(x) \quad \text{for all } x \in [\underline{x}, \bar{x}] \text{ and } t \geq 0. \tag{A.5}$$

To see that equation (A.5) is the solution, first we check that, for each j, the function $e^{\lambda_j t}\varphi_j(x)$ satisfies the p.d.e. in equation (A.2), and the Dirichlet boundary condition in equation (A.3). Substituting this guess, and dividing both sides by $e^{\lambda_j t}$, this function solves the p.d.e. and Dirichlet boundary condition if φ_j satisfies the following o.d.e.:

$$\lambda_j \varphi_j(x) = \mu \partial_x \varphi_j'(x) + \frac{\sigma^2}{2} \varphi_j''(x) - \xi(x)\varphi_j(x) \quad \text{for all } x \in [\underline{x}, \bar{x}] \tag{A.6}$$

and $\varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0$. Using that $\varphi_j(x) = \gamma_j(x)e^{-\frac{\mu}{\sigma^2}x}$ and that the pair $\{\lambda_j, \gamma_j\}$ satisfies the o.d.e. and Dirichlet boundary condition in equation (12), a direct computation of the derivatives of φ_j shows that the pair $\{\lambda_j, \varphi_j\}$ satisfies equation (A.6) and its Dirichlet boundary condition. Since the p.d.e. in equation (A.2) is linear, any linear combination of $e^{\lambda_j t}\varphi_j(x)$ satisfies it, too. The coefficients in the linear combination in equation (A.5) are chosen to solve the space boundary condition equation (A.4) at $t = 0$. This can be done since, as shown in Proposition 2, the set of eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal base of L_w^2. Finally, using the definition of the IRF G, we replace $\mathcal{G}$ by equation (A.5) and integrate each of the terms of the infinite sum with respect to $\hat{P}$, use the definition of $\langle \cdot, \cdot \rangle$ to reinterpret the integrals, and that $\hat{P}$ is a CDF with finitely many mass points, to obtain the desired expression. *Q.E.D.*

PROOF OF COROLLARY 2: Straightforward differentiation of the density function $\bar{p}(x)$ gives

$$\bar{p}'(x) = \begin{cases} -\dfrac{\theta^2[-e^{-\theta x} - e^{2\theta\bar{x}}e^{\theta x}]}{2[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}]} & \text{if } x \in [-\bar{x}, 0], \\ -\dfrac{\theta^2[e^{\theta x} + e^{2\theta\bar{x}}e^{-\theta x}]}{2[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}]} & \text{if } x \in [0, \bar{x}], \end{cases}$$

where $\theta \equiv \bar{x}^2\zeta/\sigma^2$.

The linear projection of $\bar{p}'(x)$ onto φ_j gives the projection coefficients. Let us compute $\int_{-\bar{x}}^{\bar{x}} \bar{p}'(x)\varphi_j(x)\,dx = 2\int_{-\bar{x}}^{0} \bar{p}'(x)\varphi_j(x)\,dx$ for $j = 2, 4, 6, \ldots$. The function $\bar{p}'$ is antisymmetric and φ_j is antisymmetric for j even, with respect to $x = 0$. For $j = 1, 3, 5, \ldots$, this integral is zero, since φ_j is symmetric; see equation (13). For $j = 2, 4, \ldots$, we thus have

$$\langle \varphi_j, \bar{p}' \rangle = 2\int_{-\bar{x}}^{0} \bar{p}'(x)\varphi_j(x)\,dx = \frac{\theta^2}{[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}]} \times \int_{-\bar{x}}^{0} [e^{-\theta x} + e^{2\theta\bar{x}}e^{\theta x}] \frac{1}{\sqrt{\bar{x}}} \sin\left(\frac{(x + \bar{x})}{2\bar{x}} j\pi\right) dx$$

$$= \frac{e^{\bar{x}\theta}4\theta^2\bar{x}}{\sqrt{\bar{x}}\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]}\frac{\left[j\pi\left(1-\cosh(\bar{x}\theta)(-1)^{j/2}\right)\right]}{4\theta^2\bar{x}^2+j^2\pi^2}$$

$$= \frac{8\phi e^{\sqrt{2\phi}}}{\bar{x}^{3/2}\left[1-2e^{\sqrt{2\phi}}+e^{2\sqrt{2\phi}}\right]}\frac{\left[j\pi\left(1-\cosh(\sqrt{2\phi})(-1)^{j/2}\right)\right]}{8\phi+\pi^2 j^2}$$

$$= \frac{j\pi}{4\bar{x}^{3/2}}\frac{(-2)}{\left(1+\dfrac{j^2\pi^2}{8\phi}\right)}\frac{1-\cosh(\sqrt{2\phi})(-1)^{j/2}}{1-\cosh(\sqrt{2\phi})},$$

where we used that $\theta\bar{x}=\sqrt{2\phi}$ and that $\cosh(x)=(1+e^x)/(2e^x)$. Combining it with the expression for $\langle\varphi_j, f\rangle$ in equation (16) gives the desired result. *Q.E.D.*

PROOF OF PROPOSITION 3: Let us define the centered even kth moment for the variable x: $M_k(t,\delta)\equiv\mathbb{E}_\delta(x(t)-\mathbb{E}(x(t)))^k$, where $k=2,4,\dots$ and the subscript δ denotes that probabilities are those of an impulse response following a marginal shock δ to the invariant distribution of gaps at zero inflation.

The objective is to show that $\frac{\partial}{\partial\delta}M_k(t,\delta)|_{\delta=0}=0$ for all t, that is, that a marginal shock δ has no first-order effect on the even centered moments at every t. The proof follows two steps: first, to show that the impulse response of any *even* moment is flat at zero, and second, to show that the impulse response of any *centered* moment is well approximated, up to second-order terms, by the impulse response of the corresponding non-centered moment.

The first step is readily established since a marginal shock triggers an antisymmetric displaced distribution $\hat{p}(x,0)=\bar{p}'(x)\delta$, whose projection coefficients on all even-indexed eigenfunctions $j=2,4,\dots$ are zero (since such eigenfunctions are symmetric). Note next that even (non-centered) moments $k=2,4,\dots$ are symmetric by definition, which immediately implies that their projection coefficients on all odd-indexed eigenfunctions $j=1,3,\dots$ are zero. It follows that none of the eigenfunctions will have a nonzero coefficient. This proves the first step.

To prove the second step write in terms of the non-centered moments

$$\begin{aligned}M_k(t,\delta)&=B_0\mathbb{E}_\delta\big(x(t)^k\big)+B_1\mathbb{E}_\delta\big(x(t)^{k-1}\big)\mathbb{E}_\delta\big(x(t)\big)+\cdots\\&\quad+B_{k-1}\mathbb{E}_\delta\big(x(t)\big)\big(\mathbb{E}_\delta\big(x(t)\big)\big)^{k-1}+B_k\big(\mathbb{E}_\delta\big(x(t)\big)\big)^k,\end{aligned}$$

where the B_j are the binomial coefficients. Next, let us replace each of the moments with its first-order expansion in δ, namely let $\mathbb{E}_\delta(x(t)^k)=a_k\delta+o(\delta)$ where a_k is moment-k first derivative. We get

$$M_k(t,\delta)=B_0\big(a_k\delta+o(\delta)\big)+B_1\big(a_{k-1}\delta+o(\delta)\big)\big(a_1\delta+o(\delta)\big)+\cdots+B_k\big(a_1\delta+o(\delta)\big)^k.$$

It is apparent that the only first-order term in δ is a_k, that is, the coefficient of the non-centered moment. This concludes the proof. *Q.E.D.*

PROOF OF PROPOSITION 4: The equation for eigenvalue-eigenfunction pair $\{\lambda_j,\varphi_j\}$ for the case where ξ is quadratic, that is, when $\xi(x)=\xi_0+\frac{1}{2}\xi_2x^2$ and where $-\underline{x}=\bar{x}=+\infty$, is, after a change in variables, identical to the one-dimensional time-independent

Schrodinger equation for the eigenstate Ψ_j. This equation is typically written as

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi_j(x)+\frac{1}{2}m\omega^2x^2\Psi_j(x)=E_j\Psi_j(x)\quad \text{for } x\in\mathbb{R}, \tag{A.7}$$

where Ψ_j is the jth eigenstate, $\hbar$ is the Planck constant, E_j is the energy of the eigenstate, ω is the natural frequency, and m is the mass of the particle. As can be seen in Chapter 2, Section 3 of Griffiths (2015), the solutions for the energy levels and for the eigenstates are

$$\begin{aligned} E_j &= \left(j+\frac{1}{2}\right)\hbar\omega \quad \text{and} \\ \Psi_j(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^j j!}}e^{-\eta^2\frac{x^2}{2}}H_j(\eta x)\quad \text{for all } x \text{ and } j=0,1,2,\ldots, \end{aligned} \tag{A.8}$$

where $\eta=(\frac{m\omega}{\hbar})^{1/2}$ and where H_j is the physicist jth Hermite's polynomial. Note that in equation (A.8), we are following the convention, common in physics, of labeling the state with the smaller energy as $j=0$. Thus, Φ_j corresponds to our φ_{j+1}, E_j corresponds to our $-\lambda_{j+1}-\xi_0$, $m\omega^2$ corresponds to ξ_2, and $\frac{\hbar^2}{m}$ corresponds to our σ^2. So we can set $m=1$, $\sigma=\hbar$, and $\omega=\sqrt{\xi_2}$. *Q.E.D.*

PROOF OF PROPOSITION 5: We first show that $\frac{\partial}{\partial\mu}Y(t;f,\mu,a)|_{\mu=0,a=0}=0$ as in equation (28) holds. To simplify the notation, we omit a in the expression in this part of the proof. The proof proceeds in several steps. First, we analyze properties of the decision rules (optimal thresholds) as a function of μ. Second, we analyze the direct and indirect (i.e., via the decision rules) implications of μ for the transition probabilities of the state at a given horizon t. Third, we establish a symmetry property of the impulse $\hat{p}(\cdot;\delta,\mu)$ as a function of (μ,δ). Fourth, we use the properties of the transition probabilities and decision rules to derive an antisymmetric property of H, viewed as joint function of (δ,μ) for any fixed t. Fifth, we use this antisymmetric property to obtain a zero cross derivative of H, which implies the desired result.

(1) We write the boundaries of the inaction range and the optimal return point as functions of μ. They satisfy

$$x^*(\mu)=-x^*(-\mu),\ \bar{x}(\mu)=-\underline{x}(-\mu)\quad \text{and}\quad \underline{x}(\mu)=-\bar{x}(-\mu).$$

This can be shown using a guess and verify strategy together with the corresponding guess of the value function $v(x,\mu)=v(-x,-\mu)$.

(2) We define $P_t(y|x;\mu)$ to be the transition function for the state starting at $x(0)=x$ to $x(t)=y$, where the state evolves as follows. For $0<s<t$, then $dx(s)=\mu ds+\sigma dW(s)$ as long as $x(s)\in(\underline{x}(\mu),\bar{x}(\mu))$ and the free adjustment opportunity has not arrived at time s. On the other hand, if $x(s)$ hits either $\bar{x}(\mu)$ or $\underline{x}(\mu)$, or the free adjustment opportunity arrives, then $x_+(s)=x^*(\mu)$, that is, the firm is reinjected at the optimal return point. Using the properties of the decision rules and the symmetry of the innovations in BM, we have

$$P_t(y|x;\mu)=P_t(-y|-x;-\mu).$$

To see why, write $y = x^*(\mu) + \Delta_y$ and $x = x^*(\mu) + \Delta_x$, so that for (y', x') given by $y' = -y$ and likewise $x' = -x$, we have $P_t(y|x;\mu) = P_t(y'|x' - \mu)$. But $y' = x^*(-\mu) - \Delta_y = -x^*(\mu) - \Delta_y = -y$ and likewise $x' = -x'$, establishing the required result.

(3) Recall that $\hat{p}(\cdot;\delta,\mu) = \bar{p}(x+\delta;\mu) - \bar{p}(x;\mu)$. Using the properties of the decision rules and of the Kolmogorov forward equation for the steady-state density $\bar{p}$, we get that $\bar{p}(x,\mu) = \bar{p}(-x,-\mu)$ is symmetric, which can be proved by a guess and verified strategy.

(4) Using P_t, we can write the impulse response as

$$H\big(t; f, \hat{p}(\cdot,\delta,\mu),\mu\big) = \int\int f(y)P_t(y|x;\mu)\hat{p}(x,\mu,\delta)\,dy\,dx.$$

Recall that we define $Y(r;f,\mu) = \frac{\partial}{\partial\delta}H(t;f,\hat{p}(\cdot,\delta,\mu),\mu)|_{\delta=0}$ and thus

$$\frac{\partial}{\partial\mu}Y(r;f,\mu) = \frac{\partial^2}{\partial\delta\partial\mu}H\big(t;f,\hat{p}(\cdot,\delta,\mu),\mu\big)\Big|_{\delta=0,\mu=0}.$$

We will show below that

$$H\big(t;f,\hat{p}(\cdot,\delta,\mu),\mu\big) = -H\big(t;f,\hat{p}(\cdot,-\delta,-\mu),-\mu\big) \tag{A.9}$$

for all μ,δ. Using this, we have

$$\frac{\partial^2}{\partial\mu\partial\delta}H\big(t;f,\hat{p}(\cdot,\delta,\mu),\mu\big) = -\frac{\partial^2}{\partial\mu\partial\delta}H\big(t;f,\hat{p}(\cdot,-\delta,-\mu),-\mu\big),$$

which, evaluated at $(\mu,\delta) = (0,0)$, gives

$$\frac{\partial^2}{\partial\mu\partial\delta}H\big(t;f,\hat{p}(\cdot,0,0),0\big) = -\frac{\partial^2}{\partial\mu\partial\delta}H\big(t;f,\hat{p}(\cdot,0,0),0\big),$$

so the cross derivative has to be zero, establishing the desired results.

To finish the proof, we show equation (A.9) holds. We have

$$\begin{aligned}
&H\big(t;f,\hat{p}(\cdot,\delta,\mu),\mu\big)\\
&\quad= \int\int f(y)P_t(y|x;\mu)\hat{p}(x,\mu,\delta)\,dy\,dx\\
&\quad= -\int\int f(-y)P_t(y|x;\mu)\hat{p}(x,\mu,\delta)\,dy\,dx\\
&\quad= -\int\int f(-y)P_t(-y|-x;-\mu)\hat{p}(x,\mu,\delta)\,dy\,dx\\
&\quad= -\int\int f(-y)P_t(-y|-x;-\mu)\big[\bar{p}(x+\delta,\mu) - \bar{p}(x,\mu)\big]\,dy\,dx\\
&\quad= -\int\int f(-y)P_t(-y|-x;-\mu)\big[\bar{p}(-x-\delta,-\mu) - \bar{p}(-x,-\mu)\big]\,dy\,dx\\
&\quad= -\int\int f(-y)P_t(-y|-x;-\mu)\hat{p}(-x,-\mu,-\delta)\,dy\,dx
\end{aligned}$$

$$= -\int\int f(y')P_t(y'|x'; -\mu)\hat{p}(x', -\mu, -\delta)\,dy'\,dx'$$
$$= -H\big(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu\big),$$

where we use the definition of H, that f is antisymmetric, that P_t is symmetric (as shown above), the definition of $\hat{p}$, the symmetry of $\bar{p}$ (as shown above), the definition of $\hat{p}$ again, a change of variables of integration, and again the definition of H. This finishes the proof.

The proof that $\frac{\partial}{\partial a}Y(t; f, \mu, a)|_{\mu=0,a=0} = 0$ is almost identical to the previous one, step by step replacing μ by a. *Q.E.D.*

APPENDIX B: GENERALIZED RANDOM FIXED-COST MODEL

In this appendix, we write down the problem that the firm solves, which gives rise to the decision rule described by threshold $\bar{x}$, a function $\xi : [-\bar{x}, \bar{x}] \to \mathbb{R}$, and the volatility σ^2. Recall that the Calvo-plus model supplements the traditional Calvo model with the possibility that the firm can change its price by paying a fixed menu cost at any time. The generalization allows the firm to draw a fixed menu cost ψ from a distribution with CDF W at random times—arriving at a Poisson rate $\kappa > 0$. The menu costs drawn by the firm can be zero or strictly positive. If the cost is zero, the firm changes its price to the ideal one (i.e., it "closes its price gap"), just like in Calvo. If the firm draws a strictly positive cost, it will either ignore it or change its price depending on the value of the "price gap" relative to the realization of the fixed cost. In particular, the optimal decision rule will be characterized by a threshold rule that gives the maximum adjustment cost that the firm is willing to pay for adjustment. For all fixed costs smaller than the threshold the firm changes its price, while for larger costs it keeps the price unchanged.

We also allow the firm to have a price change at any time by paying a large fixed cost, which we denote by $\Psi > 0$ and refer to as the "deterministic fixed cost." We let $\bar{x}$ be the threshold so that if $|x| \geq \bar{x}$, the firm will pay the deterministic fixed cost Ψ and adjust its price. If $\Psi = \infty$, then the firm has no such alternative. We can write the value function of the firm, $v(x)$, as

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi} \min\left\{\psi + \min_{x'} v(x') - v(x), 0\right\} dW(\psi),\right.$$
$$\left. r\Big(\Psi + \min_{x'} v(x')\Big)\right\}.$$

If $\Psi = \infty$, then $\bar{x} = \infty$, and thus there is no second branch in the Bellman equation. The term $\min_{x'} v(x')$ is the value right after adjustment, and given the symmetry of the return function, $x^* = 0$ or $v(0) = \min_{x'} v(x')$. Thus, we can simply write that for all x,

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi} \min\{\psi + v(0) - v(x), 0\}\,dW(\psi), r\big(\Psi + v(0)\big)\right\}.$$

It is easy to verify that v is increasing in $|x|$. We also have the following smooth pasting and optimal return point conditions:

$$v'(-\bar{x}) = v'(\bar{x}) = v'(0) = 0. \tag{B.1}$$

We are now ready to define the *generalized hazard function* corresponding to this model, $\xi : (-\bar{x}, \bar{x}) \to \mathbb{R}_+$, which gives the probability (per unit of time) that a firm with

$x \in (-\bar{x}, \bar{x})$ will change its price. Note that, conditional on changing its price, the price change is $-x$, that is, it closes the price gap. The function ξ is defined by the optimal decision rule, by the Poisson arrival rate $\kappa > 0$ and by the distribution of fixed cost W as follows:

$$\xi(x) = \kappa W\big(v(x) - v(0)\big) \quad \text{for all } x \in (-\bar{x}, \bar{x}). \tag{B.2}$$

The function ξ is symmetric around $x = 0$ and weakly increasing in $|x|$, inheriting these properties from $v(x)$. It is continuous at x if W is continuous at $\psi = v(x) - v(0)$, and bounded above by κ. While the function ξ is not defined at $x = \pm\bar{x}$, we abuse notation and let $\xi(\bar{x}) = \lim_{x\to\bar{x}} \xi(x) = \kappa W(\Psi) = \kappa$.

In Alvarez, Lippi, and Oskolkov (2020), we showed that for every function $\xi : (-\bar{x}, \bar{x}) \to \mathbb{R}_+$ that is piecewise continuous, positive, symmetric around $x = 0$, increasing in $|x|$, and bounded above, there is a distribution of cost with CDF W that rationalizes it.

B.1. *Example: Quadratic Hazard Model*

We let $S(t)$ be the survival function for the case of a quadratic function $\xi(x) = \xi_0 + \xi_2 x^2/2$, with $\bar{x} = -\underline{x} = \infty$, and let $h(t)$ be the corresponding hazard rate as function of the duration of the price spell. We have the following:

PROPOSITION 6: *The survival function $S(t)$ and the hazard rate $h(t)$ are*

$$S(t) = \sum_{n=0}^{\infty} (-1)^n \sqrt{2/\pi} \frac{\Gamma\left(\frac{1}{2} + n\right)}{n!} e^{\lambda_{2n} t} \quad \textit{for all } t \geq 0, \tag{B.3}$$

$$h(t) = -\sum_{n=0}^{\infty} \lambda_{2n} \mathcal{H}_n(t) \quad \textit{for all } t > 0 \textit{ where} \tag{B.4}$$

$$\mathcal{H}_n(t) \equiv \frac{(-1)^n \dfrac{\Gamma\left(\frac{1}{2} + n\right)}{n!} e^{\lambda_{2n} t}}{\displaystyle\sum_{m=0}^{\infty} (-1)^m \frac{\Gamma\left(\frac{1}{2} + m\right)}{m!} e^{\lambda_{2m} t}} \quad \textit{for all } n = 0, 1, 2, \ldots. \tag{B.5}$$

Moreover, let $S_0(t)$ be the survival function for $\xi_0 = 0$, so using the identity for competing risk durations, we have $S(t) = e^{-\xi_0 t} S_0(t)$. For the case of $\xi_0 = 0$, we have

$$S_0(t) = \sqrt{\operatorname{sech}\big(t\sqrt{\sigma^2 \xi_2}\big)}, \tag{B.6}$$

$$N = \frac{\sqrt{\sigma^2 \xi_2}}{4\sqrt{2/\pi}\,\Gamma\left(\frac{5}{4}\right)^2} \quad \textit{and} \quad \lambda_j = -N\left(j - \frac{1}{2}\right) 4\sqrt{2/\pi}\,\Gamma\left(\frac{5}{4}\right)^2. \tag{B.7}$$

The expression for $S(t)$ follows directly from the general expression of the survival function in Theorem 1 using $f(x) = 1$ and a degenerate initial condition concentrated at

$x=0$, and the expressions for the eigenvalues and eigenfunction for the quadratic case in Proposition 4. The expression for $h(t)$ follows from differentiating $S(t)$. The expression for S_0 can be verified by comparing the series expansion of $S(t)$ when $\xi_0=0$. Alternatively, one can use the Laplace transform of the square of the integral of a standard Brownian W—see Example 1 on page 11 in Kac (1949)—and any constant u gives

$$\mathbb{E}\big[e^{-u\int_0^t (W(s))^2\,ds}\big]=\sqrt{\operatorname{sech}(t\sqrt{2u})}, \tag{B.8}$$

and setting $u=\sigma^2\xi_2/2$, we obtain equation (B.6). The expression for N in the case of $\xi_0=0$ uses that the expected duration of price spells, or its reciprocal, the expected number of price changes for the case satisfy

$$\frac{1}{N}=\int_0^\infty S(t)\,dt=\int_0^\infty \sqrt{\operatorname{sech}\big(\sqrt{\sigma^2\xi_2}t\big)}\,dt=4\sqrt{\frac{2}{\pi\sigma^2\xi_2}}\Gamma\left(\frac{5}{4}\right)^2.$$

Using this expression in the general expression of the eigenvalues, we eliminate *eta* to obtain the desired expression.

B.2. *Example: Absolute Value Generalized Hazard Function*

In this appendix, we characterize the odd (antisymmetric) eigenvalues and eigenfunctions for the absolute value $\xi(x)=A|x|$. The eigenfunctions are given by displaced Airy functions and the eigenvalues are the zeros of the Airy functions $Ai(\cdot)$. We give formulas and numerical implementations for the eigenvalues λ_k in equation (B.10), the antisymmetric eigenfunctions $\varphi_j(\cdot)$ in equation (B.11), the invariant distribution $\bar{p}(\cdot)$ in equation (B.12), the expected number of price changes N in equation (B.13), and the projections $\langle\varphi_j(\cdot),-x\rangle$ in equation (B.14) and $\langle\varphi_j(\cdot),\bar{p}'(\cdot)\rangle$ in equation (B.15).

Let us start with the equation we wish to solve:

$$[Ax+\lambda_k]\varphi_k(x)=\frac{\sigma^2}{2}\varphi_k''(x)\quad\text{for } x\ge 0,\ \varphi_k(x)=-\varphi_k(-x)\text{ and } k=2,4,6,\ldots.$$

First, let $z=bx$ for some $b>$ or $x=z/b$ and define $\tilde{\varphi}_k(z)=\varphi_k(z/b)$ so that $\tilde{\varphi}_k''(z)=\varphi_k''(z)/b^2$ or $\varphi_k''(z)=b^2\tilde{\varphi}_k''(z)$ and thus

$$\tilde{\varphi}_k(z)\left[z\frac{A}{b}+\lambda_k\right]=\frac{\sigma^2}{2}b^2\tilde{\varphi}_k''(z)\quad\text{or}\quad\tilde{\varphi}_k(z)\left[z\frac{A}{b^3\sigma^2/2}+\frac{\lambda_k}{b^2\sigma^2/2}\right]=\tilde{\varphi}_k''(z).$$

Set b:

$$b\equiv\left(\frac{2A}{\sigma^2}\right)^{1/3}\quad\text{thus }\tilde{\varphi}_k(z)[z+\tilde{\lambda}_k]=\tilde{\varphi}_k''(z)\text{ where }\tilde{\lambda}_k\equiv\lambda_k\frac{\left(\frac{\sigma^2/2}{A}\right)^{2/3}}{\sigma^2/2}.$$

The Airy function $Ai(z)$ solves $Ai(z)z=Ai''(z)$ and has $Ai(z)\to 0$ as $z\to+\infty$. Moreover, it has infinitely many negative zeros, denoted by $0>a_1>a_2>\cdots$. Thus, the solution for $\tilde{\varphi}_{2k+1}$ is

$$\tilde{\varphi}_{2k+2}(z)=Ai(z+a_k)\quad\text{for all } z>0\text{ and }\tilde{\lambda}_{2k+2}=a_{k+1}\text{ for }k=0,1,\ldots.$$

While there are no closed expressions for the zeros of the Airy functions, there are excellent approximations, which can be used to find numerically exact values of them. For instance,

$$a_k = -\frac{1}{4}\left(m^2+20\right)^{1/3} + \bar{E}_k \frac{457}{\left(m^3\left(m^2+40\right)^{1/6}\right.}, \quad \text{where } m = (12k-3)\pi \text{ for } k = 1, 2, \ldots,$$

where $\bar{E}_k$ is an approximation error that is less than 1 in absolute value. See Theorem 10 in Krasikov (2014). Thus, our analytical approximation to the odd eigenvalues is

$$\lambda_{2k+2} = a_{k+1} \frac{\sigma^2/2}{\left(\frac{\sigma^2/2}{A}\right)^{2/3}} \quad \text{for } k = 0, 1, 2, \ldots \tag{B.9}$$

$$\approx -\frac{1}{4}\left(\left[12(k+1) - 3\pi\right]^2 + 20\right)^{1/3} \frac{\sigma^2/2}{\left(\frac{\sigma^2/2}{A}\right)^{2/3}} \tag{B.10}$$

and the antisymmetric eigenfunctions are

$$\varphi_{2k+2}(x) = \beta Ai\left(a_{k+1} + \left(\frac{2A}{\sigma^2}\right)^{1/3} x\right) \quad \text{for } x \geq 0 \text{ for } k = 0, 1, 2, \ldots,$$

where β is a normalizing constant. We will need to normalize the eigenfunctions by $2\int_0^\infty \varphi_k(x)^2\, dx = 1$. For this, we note that for any c,

$$\int_c^\infty \left[Ai(z)\right]^2 dz = -c\left[Ai(c)\right]^2 + \left[Ai'(c)\right]^2.$$

Thus,

$$\begin{aligned}
1 &= 2\int_0^\infty \varphi_{2k+2}(x)^2\, dx = 2\beta^2 \int_0^\infty Ai(a_{k+1} + bx)^2\, dx \\
&= \frac{2\beta^2}{b} \int_0^\infty Ai(a_{k+1} + bx)^2\, dbx \\
&= 2\beta^2 \frac{1}{b} \int_0^\infty Ai(a_{k+1} + z)^2\, dz = \frac{2\beta^2}{b} \int_{a_{k+1}}^\infty Ai(s)^2\, ds \\
&= \frac{2\beta^2}{b}\left[-a_{k+1}\left[Ai(a_{k+1})\right]^2 + \left[Ai'(a_{k+1})\right]^2\right] \\
&= \frac{2\beta^2}{b}\left[Ai'(a_{k+1})\right]^2.
\end{aligned}$$

Thus, the normalized eigenfunctions are given by

$$\varphi_{2k+1}(x) = \frac{\sqrt{\left(\frac{2A}{\sigma^2}\right)^{1/3}}}{\sqrt{2}\left|Ai'(a_{k+1})\right|} Ai\left(a_{k+1} + \left(\frac{2A}{\sigma^2}\right)^{1/3} x\right)$$

$$\text{for } x \geq 0 \text{ for } k = 0, 1, 2, \dots. \tag{B.11}$$

Likewise, for the invariant distribution $\bar{p}$, satisfying $\bar{p}(x)Ax = \sigma^2/2\bar{p}(x)$ for $x > 0$. We define again $z = bx$ and $\tilde{p}(z) = \bar{p}(z/b)$ so that $\tilde{p}(z)Az/b = \sigma^2/2\tilde{p}(z)b^2$, and setting again $b = (2A/\sigma^2)^{1/3}$, we get $\tilde{p}(z)z = \tilde{p}''(z)$, which is solved by the Airy function. Thus,

$$\bar{p}(x) = Ai\big(\big(2A/\sigma^2\big)^{1/3}x\big)/\alpha \quad \text{with } \alpha = 2\int_0^\infty Ai\big(\big(2A/\sigma^2\big)^{1/3}x\big)\,dx.$$

We can use that

$$\int_0^\infty Ai(z)\,dz = 1/3 \quad \text{and thus} \quad \alpha = \frac{2}{b}\int_0^\infty Ai(bx)\,dbx = \frac{2}{3b}$$

to obtain

$$\bar{p}(x) = \frac{3\big(2A/\sigma^2\big)^{1/3}}{2} Ai\big(\big(2A/\sigma^2\big)^{1/3}x\big) \quad \text{for } x \geq 0. \tag{B.12}$$

Note that $N = -\sigma^2\bar{p}'(0)$ and thus

$$N = -\sigma^2\bar{p}'(x)|_{x=0} = -\sigma^2\frac{3\big(2A/\sigma^2\big)^{2/3}}{2} Ai'\big(\big(2A/\sigma^2\big)^{1/3}x\big)\Big|_{x=0};$$

using that $Ai'(0) = -1/(3^{1/3}\Gamma(1/3))$, we have $N = -\sigma^2\frac{3(2A/\sigma^2)^{2/3}}{2}Ai'(0)$; using that $Ai'(0) = 1/(3^{1/3}\Gamma(1/3))$, we have

$$N = \frac{\sigma^2}{2}\left(\frac{2A}{\sigma^2}\right)^{2/3}\frac{3}{3^{1/3}\Gamma(1/3)}. \tag{B.13}$$

Let φ_{2k+2} be an antisymmetric eigenfunction. Then the projections for the IRF are

$$\langle \varphi_{2k+2}, -x\rangle = -\frac{\sqrt{b}}{b^2\sqrt{2}\big|Ai'(a_{k+1})\big|}2\int_0^\infty zAi(a_{k+1}+z)\,dz, \tag{B.14}$$

$$\big\langle \varphi_{2k+2}, \bar{p}'\big\rangle = \frac{\sqrt{b}}{\sqrt{2}\big|Ai'(a_{k+1})\big|}2b\frac{3}{2}\int_0^\infty Ai'(z)Ai(a_{k+1}+z)\,dz. \tag{B.15}$$

Note that the product $\langle \varphi_{2k+2}, -x\rangle\langle \varphi_{2k+2}, \bar{p}'\rangle$ is independent of b.

APPENDIX C: MONETARY PROPAGATION WITH VOLATILITY SHOCKS

This section discusses the effect that changes to the volatility of shocks exert on the propagation of monetary shocks. The issue matters to, for example, the effectiveness of monetary policy in recessions versus boom, when the state of the economy is assumed to feature, respectively, high versus low volatility of shocks as in Vavra (2014). Our method provides a sharp analytic answer to this question.

For concreteness, we illustrate the problem by using the pure menu cost model (without Calvo adjustment, i.e., $\zeta = 0$ so that $\phi = \ell = 0$), whose output response to a small monetary shock was given in equation (18). We conduct a comparative static exercise to analyze how the propagation is affected by an innovation of the "volatility shocks," namely a permanent change in the common value of the idiosyncratic volatility σ.[1]

We start with a steady state for the model with idiosyncratic volatility σ. We characterize the effect of a small monetary shock, $\delta > 0$, which occurs $s \geq 0$ periods after a change in idiosyncratic volatility from σ to $\tilde{\sigma}$, so that $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma})\sigma$. In particular, we let $Y(t; s, d\sigma/\sigma)\delta$ denote the output's IRF $t \geq 0$ periods after an unexpected monetary shock of size δ starting with a cross-sectional distribution that has evolved s periods since the change in σ.[2]

While we characterize Y for all $t > 0$ and $s \geq 0$, two interesting cases are worthwhile to mention separately: the short-run and the long-run effect of volatility. The short-run effect, defined as $Y(t; 0, d\sigma/\sigma)$ or $s = 0$, consists of considering a simultaneous permanent change of both σ (to $\tilde{\sigma}$) and $\delta > 0$. After the shock, the forward looking firm's decision rule adjusts immediately to the new volatility $\tilde{\sigma}$, while the initial distribution of price gaps corresponds to the stationary distribution obtained under the old decision rule. The long-run effect, denoted by $Y(t; \infty, d\sigma/\sigma)$ or $s \to \infty$, is equivalent to computing the effect of a monetary shock δ for a new steady state with volatility $\tilde{\sigma}$. We refer to this as the *long-run* effect since it is the effect of an unanticipated monetary shock once the distribution of price gaps has achieved its new invariant distribution. In this case, the firm's decision rule corresponds to the new volatility $\tilde{\sigma}$ and the economy is described by the new invariant distribution of price gaps.

The general case characterizes an IRF whose coefficients are indexed by the parameter $0 < s < \infty$. The key feature of this case is that the monetary shock δ occurs s periods after the volatility shock, thus displacing a cross-section distribution of price gaps that is in a transition towards the new invariant distribution. Our analytic method allows us to exactly compute the evolution of this distribution and hence the effect of a monetary shock.

The next proposition uses the notation introduced above, where $Y(t; 0, 0)$ denotes the impulse before any change in volatility occurs, which we use as a benchmark. Also, the difference $Y(t; s, \frac{d\sigma}{\sigma}) - Y(t; \infty, \frac{d\sigma}{\sigma})$ is the correction to the long-run effect of a volatility shock $d\sigma/\sigma$ due to a finite duration s.

PROPOSITION 7: *Let $Y(t; s, \frac{d\sigma}{\sigma})$ denote the time-t value of the output marginal impulse response that occurs s periods after a volatility increase from σ to $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma})\sigma$. The* long-run *effect* ($s \to \infty$) *of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is*

$$Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right) \quad \textit{for all } t \geq 0. \tag{C.1}$$

The short-run *effect* ($s \to 0$) *of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is*

$$Y\left(t; 0, \frac{d\sigma}{\sigma}\right) = \left(1 + \frac{d\sigma}{\sigma}\right) Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right) \quad \textit{for all } t \geq 0. \tag{C.2}$$

[1]For simplicity and clarity of the results, we consider here once and for all shocks to volatility. It is simple to modify the setup to consider a two-state Markov switching volatility process and to solve the associated firm's decision rules.

[2]We will keep using the notation of Y as the output's IRF per unit of monetary shock, and then omit the δ in the expressions below.

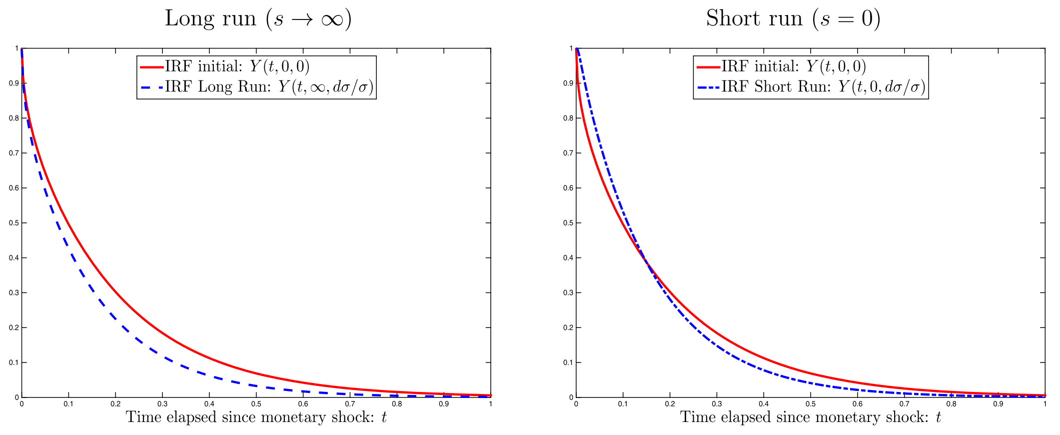

FIGURE S1.—Short-run and long-run IRF versus IRF before volatility increases. *Note*: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

The deviation from the long-run response as a function of s is given by

$$Y\left(t; s, \frac{d\sigma}{\sigma}\right) - Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k} t} b_{2k}[f] b_{2k}\left[\hat{p}'(\cdot, s)\right] \quad \textit{for all } t, s \geq 0, \tag{C.3}$$

where $\hat{p}'(\cdot, s)$ *is the initial condition* (*i.e.*, *a displaced cross section*) *at the time of the monetary shock*, *s periods after the change in volatility*, *whose projection coefficients are given by*

$$b_{2k}\left[\hat{p}'(\cdot, s)\right] = \frac{d\sigma}{\sigma} \frac{1}{\bar{x}^{\frac{3}{2}}} \sum_{j=1,3,5,\ldots}^{\infty} e^{\lambda_j s} \left(2 \frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\pi)^2}\right) \left(\frac{4kj}{(4k^2 - j^2))}\right), \quad k = 1, 2, 3, \ldots, \tag{C.4}$$

and where $b_{2k}[f] = 2\bar{x}^{3/2}/(k\pi)$ *as in equation* (16).

A few comments are in order.

(i) Figure S1 illustrates the difference between the short-run and long-run effect of an increase in volatility on the output's response to a monetary shock. The left panel compares the IRF with no change in volatility, $Y(t; 0, 0)$, to the one where the volatility increase has occurred $s \to \infty$ periods ago, that is, $Y(t; \infty, d\sigma/\sigma)$ the long-run effect. The right panel compares the IRF with no change in volatility, $Y(t; 0, 0)$, to the one where the volatility increase has occurred at the same time as the monetary shock $s = 0$ periods ago, that is, $Y(t; 0, d\sigma/\sigma)$ the short-run effect.

(ii) For this proposition, we use the form of the decision rules for the threshold $\bar{x}$, which, as the discount rate goes to zero, is $\bar{x} = (6\frac{\psi}{B}\sigma^2)^{\frac{1}{4}}$ where ψ is the fixed cost—as fraction of the frictionless profit and B is the curvature of the profit function around the frictionless profit. This implies that the elasticity of $\bar{x}$ to σ is $1/2$. This elasticity is the so-called "option value" effect on the optimal decision rules.

(iii) The rescaling of time in $Y(t(1+\frac{d\sigma}{\sigma}); 0, 0)$ in the expressions for the long- and short-run effect of volatility reflects the change in the eigenvalues, which depend on the value

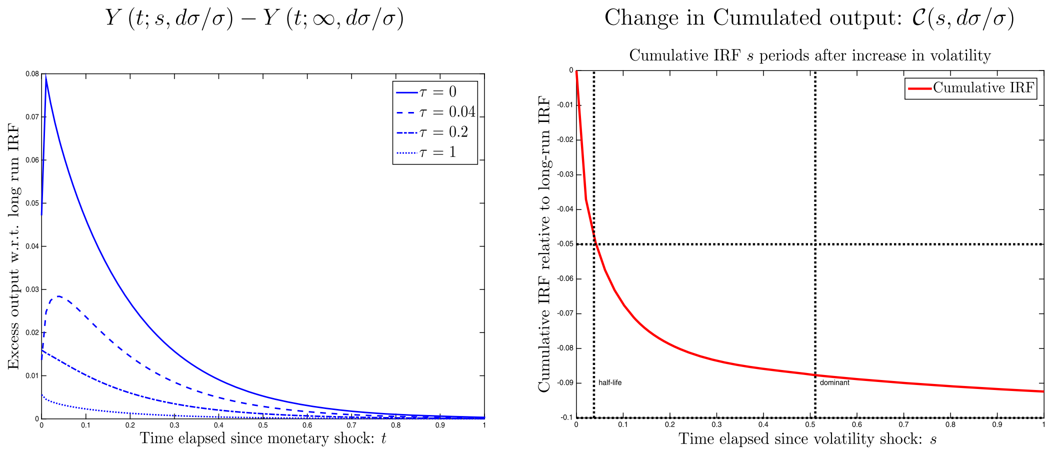

FIGURE S2.—The propagation of monetary shocks as s grows. *Note*: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

of N, the implied average number of price changes per unit of time, as $\lambda_j = -N(\pi j)^2/8$ (see (21) for $\zeta = 0$). Recall that $N = (\sigma/\bar{x})^2$, and hence all the eigenvalues change proportionally with σ.

(iv) For the case of the impact effect and in which $\tilde{\sigma} > \sigma$, the invariant distribution just before the monetary shock is narrower than the range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level $(1 + \frac{d\sigma}{\sigma})$ in the impact effect in equation (C.2): since firms have price gaps that are discretely away from the inaction bands, then prices react more slowly, generating the extra effect on output. The logic for the case where $\tilde{\sigma} < \sigma$ is similar.

(v) In equation (C.3), we use only the even terms for the projections, that is, the index for the projection $b_{2k}[\cdot]$ runs on $2k$ because f is antisymmetric. For these coefficients, as was the case without volatility shocks, the eigenvalues that control the effect of the horizon t in the IRF are the even ones, that is, $\lambda_2, \lambda_4, \ldots$, starting with the leading one λ_2.

(vi) The expressions in equation (C.3) and equation (C.4) show that what governs the difference between the long-run and the short-run volatility effects are the odd eigenvalues, that is, $\lambda_1, \lambda_3, \ldots$, since these are the only elements where s affects the expressions. In particular, λ_1 is the dominant eigenvalue.

(vii) We note that the expression for the correction term in equation (C.3) involves no parameter for the model with the exception of N, which enters only in the eigenvalues $\lambda_j = -N(j\pi)^2/8$. This gives a meaning to the units of t and s, which are measured relative to the (new) steady-state duration of price changes $1/N$. This remark is needed to interpret the time units in the horizontal axes of both panels of Figure S2.

(vii) To illustrate the general case of $0 < s < \infty$ in Figure S2, we display two plots. First, the left panel of Figure S2 plots equation (C.3), evaluated at four values of s. It is apparent that as s becomes bigger, monetary policy becomes less effective and gradually converges to the long-run value. This can be seen by comparing the correction for any given t across the four values of s. Second, the right panel plots the cumulated IRF of a monetary shock s periods after the volatility shock, relative to the cumulative IRF of a monetary shock when there is no volatility shock. In particular, we plot

$$\mathcal{C}(s, d\sigma/\sigma) \equiv \frac{\displaystyle\int_0^\infty Y(t, s, d\sigma/\sigma)\, dt}{\displaystyle\int_0^\infty Y(t, 0, 0)\, dt} - 1.$$

We use the cumulated IRF to obtain a simple one-dimensional summary of this effect across all times t. Notice the following properties of $\mathcal{C}$: for all s, we have $\mathcal{C}(s, d\sigma/\sigma) = (d\sigma/\sigma)\mathcal{C}(s, 1)$, since it is based on a derivative, and for extreme values of s, we have $\mathcal{C}(\infty, d\sigma/\sigma) = -d\sigma/\sigma$, and $\mathcal{C}(0, d\sigma/\sigma) = 0$. From Figure S2, it is clear that the transition to the higher volatility occurs very fast; a cumulative effect of $\mathcal{C}$ half as large as half of the one in $s \to \infty$ will occur when $s_{1/2} \approx 0.05/N$, a half-life indicated by a vertical bar in the right panel. More precisely, $s_{1/2}$ is defined as $\mathcal{C}(s_{1/2}, d\sigma/\sigma) = -(1/2)(d\sigma/\sigma)$. This effect is much faster than the half-life corresponding to the dominant eigenvalue $\lambda_1 = -N\pi^2/8$, which is given by $t_{1/2} \equiv -8\log(0.5)/(N\pi^2) \approx 0.56/N$, and it is indicated by another vertical bar in the right panel. The ratio of the two times is very large: $t_{1/2}/s_{1/2} \approx 12$, and it is independent of any parameter of the model.[3] From this comparison, we conclude that for this particular model, using exclusively the dominant eigenvalue λ_1 to approximate the time it takes for the distribution to converge after the change in volatility will be misleading. Summarizing, in the Golosov–Lucas model, the short-run effect of the volatility change is only relevant when the monetary shock occurs almost immediately after the volatility change.

C.1. *Proofs for Uncertainty Shocks*

PROOF OF PROPOSITION 7: First, we consider case (i), that is, the *long-run* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma})\sigma$ on the impulse response of output to a monetary shock. We note that the expression for $Y(t)$ for the Golosov–Lucas model does not feature $\bar{x}$, which is a function of σ (see equation (18)). Indeed, the only place where σ enters in the expression for $Y(t)$ is in the eigenvalues (the parameter $N(j\pi)^2/8$ in equation (18)). Since $N = \sigma^2/\bar{x}^2$ and $\bar{x} = (6\frac{\psi}{B}\sigma^2)^{\frac{1}{4}}$, then $d\log\bar{x} = 1/2 d\log\sigma$ and $d\log N = 2(d\log\bar{x} - d\log\sigma)$, hence $d\log N = d\log\sigma$. Substituting this into the eigenvalue, $\lambda_j = -\tilde{N}(j\pi)^2/8 = -(1 + \frac{d\sigma}{\sigma})N(j\pi)^2/8$ where N is the average number of price changes before the volatility shock. Using the expression for the impulse response in terms of the post-shock objects, we have

$$Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) = \sum_{j=1}^{\infty} \langle\varphi_j, f\rangle\langle\varphi_j, \bar{p}'\rangle e^{-(1+\frac{d\sigma}{\sigma})N\frac{(j\pi)^2}{8}t} = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right),$$

and we obtain the desired result.

Now we consider the short-run effect, that is, the *impact* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = (1 + \frac{d\sigma}{\sigma})\sigma$ on the impulse response of output to a monetary shock. As in the previous case, the eigenvalues can be written as functions of the shock and the old value of the expected number of price changes. Also as the previous case, we have

[3]The vertical distance on the correction between $s = 0$ and $s \to \infty$ plotted on the right panel is $d\sigma/\sigma$, which is 0.1 for this example. For other values, the vertical axis scales proportionally.

$f(x) = -x$. The difference is on the initial distribution $p(x, 0)$. The initial condition is given by $p(x, 0) = \bar{p}(x + \delta; \bar{x}(\sigma))$ where we write $\bar{x}(\sigma)$ to indicate that the distribution depends on σ. Indeed, since we are using the expression for $Y(t, 0, \frac{d\sigma}{\sigma})$ in terms of the value of $\bar{x}$ that corresponds to the post-shock value of σ, we need to consider the effect on $\bar{x}$ of a decrease of σ in the proportion $d\sigma/\sigma$. To do this, we take a second-order expansion of $p(x, 0) = \bar{p}(x + \delta; \bar{x}(\sigma))$ with respect to δ and σ evaluated at $\delta = 0$ and $d\sigma = 0$:

$$
\begin{aligned}
p(x; 0) &\equiv \bar{p}\big(x + \delta; \bar{x}(\sigma)\big) \\
&= \bar{p}(x) + \frac{\partial}{\partial \delta} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \delta - \frac{\partial}{\partial \bar{x}} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \frac{\partial \bar{x}(\sigma)}{\partial \sigma} d\sigma \\
&\quad + \frac{1}{2} \frac{\partial^2}{\partial \delta^2} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \delta^2 \\
&\quad + \frac{1}{2} \frac{\partial^2}{\partial \bar{x}^2} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \left(\frac{\partial \bar{x}(\sigma)}{\partial \sigma}\right)^2 d\sigma^2 \\
&\quad + \frac{1}{2} \frac{\partial}{\partial \bar{x}} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \frac{\partial^2 \bar{x}(\sigma)}{\partial \sigma^2} d\sigma^2 \\
&\quad - \frac{\partial^2}{\partial \bar{x} \partial \delta} \bar{p}\big(x + \delta; \bar{x}(\sigma)\big)\Big|_{\delta=0} \frac{\partial \bar{x}(\sigma)}{\partial \sigma} d\sigma \delta + o\big(\|(\delta, d\sigma)\|^2\big)
\end{aligned}
$$

for $x \in [\underline{x}, \bar{x}]$ and $x \neq 0$. Recall that the invariant distribution for this model is the triangular density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x}, \bar{x})$. Using this functional form, we have

$$
\frac{\partial}{\partial \delta} \bar{p}(\delta + x; \bar{x}) = \begin{cases} +\dfrac{1}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0), \\ -\dfrac{1}{\bar{x}^2} & \text{if } x \in (0, \bar{x}], \end{cases}
$$

$$
\frac{\partial}{\partial \bar{x}} \bar{p}(\delta + x; \bar{x})\Big|_{\delta=0} = \begin{cases} -\dfrac{x}{\bar{x}^2} \dfrac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0), \\ +\dfrac{x}{\bar{x}^2} \dfrac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}], \end{cases}
$$

$$
\frac{\partial^2}{\partial \delta \partial \bar{x}} \bar{p}(x + \delta; \bar{x}) = \begin{cases} -\dfrac{1}{\bar{x}^2} \dfrac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0), \\ +\dfrac{1}{\bar{x}^2} \dfrac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}], \end{cases}
$$

$$
\frac{\partial^2}{\partial \bar{x}^2} \bar{p}(\delta + x; \bar{x}) = \begin{cases} +\dfrac{x}{\bar{x}^2} \dfrac{6}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0), \\ -\dfrac{x}{\bar{x}^2} \dfrac{6}{\bar{x}^2} & \text{if } x \in (0, \bar{x}]. \end{cases}
$$

Notice that the first-order derivatives with respect to δ as well as the cross-partial derivative are antisymmetric functions of x around $x = 0$, while the derivatives with respect to $\bar{x}$ are symmetric functions of x. Finally, we have $\frac{\partial^2}{\partial \delta^2} \bar{p}(x + \delta; \bar{x}) = 0$.

Now we use the expansion and compute the impulse response coefficients $\beta_j \equiv \langle \varphi_j, f \rangle \langle \varphi_j, p(\cdot, 0) \rangle$. The first-order term for $d\sigma$ is zero because f is antisymmetric (so that

$\langle \varphi_j, f\rangle = 0$ for $j = 2, 4, 6, \ldots$) and the first derivative with respect to $\bar{x}$ is symmetric (so that $\langle \varphi_j, p(\cdot, 0)\rangle = 0$ for $j = 1, 3, 5, \ldots$), hence the $\beta_j = 0$ for $j = 1, 2, 3, 4, \ldots$. Likewise, the second-order terms for $d\sigma^2$ are zero since f is antisymmetric and the first and second derivative with respect to $\bar{x}$ are symmetric. The second-order term δ^2 is zero because the second derivative with respect to δ is zero. This leaves us with two nonzero terms: the first-order term on δ, which is the term for the IRF with respect to a monetary shock, and the second-order term corresponding to the cross-derivative. For the cross-partial term, we note that, using that $\bar{x}$ has elasticity 1/2 with respect to σ, we can write

$$
\begin{aligned}
&-\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\,d\sigma\delta \\
&\quad= -\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\bar{x}(\sigma)\left[\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\frac{\sigma}{\bar{x}(\sigma)}\right]\frac{d\sigma}{\sigma}\delta \\
&\quad= -\frac{2}{\bar{x}(\sigma)}\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\bar{x}(\sigma)\frac{1}{2}\frac{d\sigma}{\sigma}\delta = -\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\frac{d\sigma}{\sigma}\delta.
\end{aligned}
$$

Thus, we have that each β_j term is given by the sum of the (nonzero) terms corresponding to the first-order term on δ and the second-order term corresponding to the cross-derivative:

$$
\langle\varphi_j, f\rangle\big\langle\varphi_j, \bar{p}'(\cdot)\big\rangle\delta + \langle\varphi_j, f\rangle\big\langle\varphi_j, \bar{p}'(\cdot)\big\rangle\delta\frac{d\sigma}{\sigma} = \langle\varphi_j, f\rangle\big\langle\varphi_j, \bar{p}'(\cdot)\big\rangle\delta\left(1+\frac{d\sigma}{\sigma}\right).
$$

This gives the projection coefficients for the short-run impact that appear in equation (C.2) in the proposition. In particular, it shows that the coefficients for the short run are equal to the ones for the long run multiplied by the factor $(1 + d\sigma/\sigma)$.

Finally, we consider the case of a monetary shock that occurs s periods after the volatility shock. We proceed in three steps.

Step 1: Find Initial Signed Measure $\hat{p}(x, p)$. For a small σ shock, the signed measure $\hat{p}(x, 0)$ right after the uncertainty shock is given by

$$
\hat{p}(x,0) \equiv \bar{p}\big(x;\bar{x}(\sigma)\big) - \bar{p}\big(x;\bar{x}(\tilde{\sigma})\big) = \bar{p}\big(x;\bar{x}(\sigma)\big) - \bar{p}\big(x;\bar{x}(\sigma+d\sigma)\big),
$$

which is by the difference between the original invariant distribution and the new long-run distribution. We now take an expansion around the original invariant distribution and write

$$
\hat{p}(x,0) = \bar{p}\big(x;\bar{x}(\sigma)\big) - \bar{p}\big(x;\bar{x}(\sigma)\big) - \bar{p}_{\bar{x}}\big(x;\bar{x}(\sigma)\big)\bar{x}'\,d\sigma + o(d\sigma),
$$

where $\bar{p}_{\bar{x}}$ is the derivative of the density function with respect to $\bar{x}$. For the pure menu cost model, we have $\bar{p}(x,\bar{x}) = 1/\bar{x} - (1/\bar{x}^2)|x|$ for $x \in (\bar{x}, \bar{x})$, so we have

$$
\bar{p}_{\bar{x}}(x,\bar{x}) = \frac{1}{\bar{x}^2}\left(-1+\frac{2|x|}{\bar{x}}\right) \quad\text{and}\quad \frac{\partial}{\partial\sigma}\bar{x}(\sigma) = \frac{1}{2}\frac{\bar{x}}{\sigma},
$$

where we use that $\bar{x}(\sigma) = (6\psi/B\sigma^2)^{1/4}$. Replacing into the expression for $\hat{p}$, we have

$$
\hat{p}(x,0) = -\bar{p}_{\bar{x}}\big(x;\bar{x}(\sigma)\big)\bar{x}'\,d\sigma + o(d\sigma) = \frac{1}{\bar{x}}\left(\frac{|x|}{\bar{x}}-\frac{1}{2}\right)\frac{d\sigma}{\sigma} + o(d\sigma) \quad \text{for } x \in (-\bar{x}, \bar{x}).
$$

Step 2: Find Signed Measure After s Periods $\hat{p}(x,s)$. The function $\hat{p}(x,s)$ describes the evolution of this signed measure s periods after the uncertainty shock. We use our characterization of the density of transition function $\sum_j \exp(\lambda_j s)\phi_j(x_s)\phi(x_0)$ between time $t=0$ and $t=s$ with eigenfunctions φ_j and eigenvalues λ_j with $\bar{x}(\tilde{\sigma})$ and $N=\tilde{\sigma}^2/\bar{x}(\tilde{\sigma})^2$ to construct the evolution of $\hat{p}(x,s)$. We represent the signed measure (deviation from the invariant distribution) as follows:

$$\hat{p}(x,s)=\frac{d\sigma}{\sigma}\sum_{j=1,3,5,\dots} e^{\lambda_j s}\varphi_j(x)\langle\varphi_j,\hat{p}(\cdot,0)\rangle, \tag{C.5}$$

where the projection coefficients $\langle\varphi_j,\hat{p}\rangle=0$ for $j=2,4,6,\dots$ since the function $\hat{p}(x,0)$ is a symmetric function while the even-indexed φ_j functions are antisymmetric. The nonzero coefficients are

$$\langle\varphi_j,\hat{p}(\cdot,0)\rangle=\int_{-\bar{x}}^{\bar{x}}\varphi_j(x)\hat{p}(x,0)\,dx=2\int_0^{\bar{x}}\frac{1}{\bar{x}}\left(\frac{x}{\bar{x}}-\frac{1}{2}\right)\varphi_j(x)\,dx \quad \text{for } j=1,3,5,\dots.$$

Direct calculation gives

$$\langle\varphi_j,\hat{p}(\cdot,0)\rangle=\frac{2}{\bar{x}^{1/2}}\frac{4(-1)^{\frac{j+3}{2}}-j\pi}{(j\pi)^2} \quad \text{for } j=1,3,5,\dots. \tag{C.6}$$

Step 3: Find Excess Impulse Response $Y(t;s,\frac{d\sigma}{\sigma})-Y(t;\infty,\frac{d\sigma}{\sigma})$. The cross-section distribution right after the monetary shock is

$$p(x+\delta,s;\tilde{\sigma})=\bar{p}(x+\delta;\tilde{\sigma})+\hat{p}(x+\delta,\tau).$$

The first term is the invariant distribution (under the new variance $\tilde{\sigma}$) which will settle in the long run; the second term is the deviation between the current cross-section distribution and the invariant, discussed above. Then

$$p(x+\delta,s;\tilde{\sigma})-\bar{p}(x;\tilde{\sigma})\approx\delta\big(\bar{p}'(x;\tilde{\sigma})+\hat{p}'(x,s)\big).$$

We let $Y(t;\infty,\frac{d\sigma}{\sigma})=\sum_{k=1}^{\infty}e^{\lambda_{2k}t}b_{2k}[f]b_{2k}[\bar{p}'(\cdot;\tilde{\sigma})]$ be the long-run output response to a monetary shock, after the initial uncertainty shock has settled down (i.e., for $s\to\infty$). Our main proposition implies that impulse response to a monetary shock s periods after the uncertainty shock is

$$Y\left(t;s,\frac{d\sigma}{\sigma}\right)=\sum_{k=1}^{\infty}e^{\lambda_{2k}t}b_{2k}[f]b_{2k}\big[\hat{p}'(\cdot,s)\big]+Y\left(t;\infty,\frac{d\sigma}{\sigma}\right). \tag{C.7}$$

Note that the above summation only uses even-indexed eigenfunctions since the function of interest for the output $f(x)=-x$ is antisymmetric; we know that all $b_{2k+1}[f]=0$ for $k=1,2,3,\dots$..

Now we turn to the computation of $b_{2k}[\hat{p}'(\cdot,\tau)]$, given by $b_{2k}[\hat{p}'(\cdot,\tau)]\equiv\int_{-\bar{x}}^{\bar{x}}\varphi_{2k}(x)\times\hat{p}'(x,s)\,dx$. Note that from equation (2) we can write $\hat{p}'(\cdot,\tau)$ as

$$\hat{p}'(x,s)=\frac{d\sigma}{\sigma}\sum_{j=1,3,5,\dots}e^{\lambda_j s}\varphi_j'(x)\langle\varphi_j,\hat{p}(\cdot,0)\rangle.$$

Using equation (C.5) and the form of the eigenfunctions:

$$b_{2k}\big[\hat{p}'(\cdot,s)\big] = \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x)\hat{p}'(x,s)\,dx$$
$$= \frac{d\sigma}{\sigma} \sum_{j=1,3,5,\dots} e^{\lambda_j s}\langle \varphi_j, \hat{p}(\cdot,0)\rangle b_{2k}\big[\varphi_j'\big], \quad k = 1,2,3,\dots.$$

Direct computation for $k = 1, 2, 3, \dots$ and $j = 1, 3, 5, \dots$ gives

$$b_{2k}\big[\varphi_j'\big] \equiv \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x)\varphi_j'(x)\,dx$$
$$= \frac{j\pi}{2\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(k\pi\left(\frac{x+\bar{x}}{\bar{x}}\right)\right)\cos\left(j\pi\left(\frac{x+\bar{x}}{2\bar{x}}\right)\right)dx = \frac{4kj}{\bar{x}\big(4k^2-j^2)\big)}$$

and hence

$$b_{2k}\big[\hat{p}'(\cdot,s)\big] = \frac{d\sigma}{\sigma}\frac{1}{\bar{x}^{3/2}} \sum_{j=1,3,5,\dots}^{\infty} e^{\lambda_j s}\left(2\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\pi)^2}\right)\left(\frac{4kj}{\big(4k^2-j^2)\big)}\right),$$
$$k = 1,2,3,\dots. \qquad \textit{Q.E.D.}$$

APPENDIX D: ADDITIONAL RESULTS ON THE CALVO-PLUS MODEL

Next, we discuss whether it is possible to approximate the impulse response function in a parsimonious way, a question that is naturally related to the shape of the impulse response. A natural candidate would be to analyze the impulse response associated to the leading eigenvalue as defined in Section 4.2, namely the largest eigenvalue associated with nonzero projection coefficient b_j in equation (14), for a case in which the IRF is close to exponential. We analyze this question by focusing on a small monetary shock that causes a marginal displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo-plus model as well as for a model with price-plans.

The next proposition gives a characterization of the ratio between the true area under the output impulse response and the approximate one, computed using only the leading eigenvalue:

PROPOSITION 8: *Consider the marginal impulse response for output, so that $f(x) = -x$ and $\hat{p} = \delta\bar{p}'$. Define the ratio of the approximate cumulative impulse response based on the leading eigenvalue relative to the area under the impulse response as*

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty}\beta_j(\phi)/\lambda_j(\phi)} = 2\frac{\big[1+\cosh(\sqrt{2\phi})\big]}{\big[\cosh(\sqrt{2\phi})-1-\phi\big]\left[1+\frac{\pi^2}{2\phi}\right]^2}.$$

We note that $m_2(0) = \frac{16}{\pi^4}6 \approx 0.98$, $m_2'(\phi) > 0$, and $m_2(\phi) \to 2$ as $\phi \to \infty$.

Proposition 8 shows that the leading eigenvalue provides an accurate approximation of the total cumulative IRF for most variants of the Calvo-plus model. At values of $\ell \approx 0.7$,

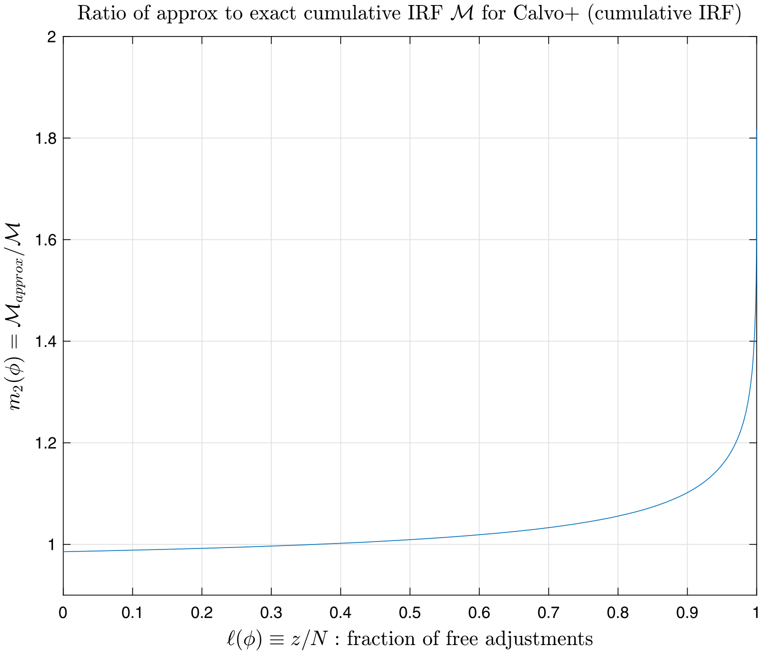

FIGURE S3.—Calvo-plus model.

the approximate function is close to 95% of the true effect. Figure S3 shows that accuracy degenerates as the model converges towards a pure Calvo model $\ell \to 1$.

We can also use the expression for the coefficients of the impulse response to show that the slope of Y at $t = 0$ is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

PROPOSITION 9: *The derivative of the IRF with respect to t at $t = 0$ is given by*

$$\frac{\partial}{\partial t} Y(t)\bigg|_{t=0} = -\infty \quad \text{for } 0 \leq \phi < \infty.$$

Note that when $\phi \to \infty$, so we get the pure Calvo model, then the impulse response becomes $Y(t) = \exp(-Nt)$, and thus $Y'(0)$ is finite.

D.1. *Proofs*

PROOF OF PROPOSITION 8: Rewriting the expression for m_2:

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} \beta_j(\phi)/\lambda_j(\phi)} = \frac{\beta_2(\phi)/\lambda_2(\phi)}{Kurt(\phi)/(6N)}$$

$$= \frac{\left[\frac{1+\cosh(\sqrt{2\phi})}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{8(2\phi)}{4(2\phi)+4\pi^2}\right]}{N\ell(\sqrt{2\phi})\left[1+\frac{\pi^2}{2\phi}\right]}$$

$$\times \frac{N\big(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2\big)^2}{\big(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})\big)\big(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2-2\phi\big)}$$

$$= 2\frac{\left[1+\cosh(\sqrt{2\phi})\right]}{\left[\cosh(\sqrt{2\phi})-1-\phi\right]\left[1+\frac{\pi^2}{2\phi}\right]^2},$$

where the first line follows from the definition, and the first equality from the sufficient statistic result in Alvarez, Le Bihan, and Lippi (2016). The second line uses the expression for β_2, λ_2 derived above, as well as the expression for the kurtosis derived in Alvarez, Le Bihan, and Lippi (2016). The third line uses the expression for ℓ. The remaining lines are simplifications. *Q.E.D.*

PROOF OF PROPOSITION 9: First use Proposition 2 to write

$$\frac{\partial}{\partial t}Y(t)\bigg|_{t=0} = \lim_{M\to\infty}\sum_{j=1}^{M}\beta_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}[\beta_{2+4i}\lambda_{2+4i}+\beta_{4+4i}\lambda_{4+4i}].$$

Using the coefficients for β_j in Proposition 2 and the expression for the eigenvalues in equation (21), we write

$$\frac{\partial}{\partial t}Y(t)\bigg|_{t=0} = -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh(\sqrt{2\phi})}{\cosh(\sqrt{2\phi})-1}\right]-1\right)$$

$$= -2N\ell(\phi)\lim_{M\to\infty}M\left(\left[\frac{1+\cosh(\sqrt{2\phi})}{\cosh(\sqrt{2\phi})-1}\right]-1\right),$$

which diverges towards minus infinity for any $0 \le \phi < \infty$. *Q.E.D.*

APPENDIX E: THE IMPULSE RESPONSE IN TIME-DEPENDENT MODELS

Suppose that the probability of a price change is time-dependent, described by a survival function $S(t)$ giving the probability that a new price survives for at least t periods after reset. A famous example is $S(t) = e^{-\zeta t}$ giving rise to the constant hazard rate of the Calvo model, but many others are of course embedded in the frame. The key assumption is that $S(t)$ does not depend on the state of the firm problem, for example, its price gap x.

Suppose a model where the firm has a price gap x (log points deviation from the static profit maximizing profit) and is minimizing the following quadratic cost function:

$$v(x) = \mathbb{E}\int_0^\infty e^{-\rho t}\big(S(t)\big(x(t)\big)^2 + \big(1-S(t)\big)v\big(x^*\big)\big)\,dt \quad \text{where } x(0) = x,$$

where ρ is the time discount and $v(x^*)$ is the value of the minimized value function. Let the law of motion of the price gap be given by the diffusion $dx = -\mu dt + \sigma dW$, where μ is the inflation rate and W is a standard BM.

It is immediate to use the law of motion to solve for expectations and rewrite the value function as

$$v(x) = \int_0^\infty e^{-\rho t}\big(S(t)\big((x-\mu t)^2 + \sigma^2 t\big) + \big(1 - S(t)\big)v\big(x^*\big)\big)\, dt.$$

The first-order condition for the optimal reset state x^* conditional on the firm receiving an adjustment opportunity is $v'(x^*) = 0 = \int_0^\infty e^{-\rho t} S(t)(x^* - \mu t)\, dt$, which gives the optimality condition for the reset price

$$x^* = \mu \frac{\displaystyle\int_0^\infty e^{-\rho t} S(t) t\, dt}{\displaystyle\int_0^\infty e^{-\rho t} S(t)\, dt} = \mu\hat{\alpha}, \tag{E.1}$$

where $\hat{\alpha}$ is a constant summarizing the optimal degree of inflation front-loading. For instance, in the case of Calvo pricing where $S(t) = e^{-\zeta t}$, the optimality condition gives $\hat{\alpha} = \frac{1}{\zeta + \rho}$ so that, for example, the firm is front-loading into the price the inflation that will be recorded over the expected duration of the price spell (equal to $1/\zeta$). The important property of equation (E.1) is that the optimal reset point x^* is proportional to the inflation rate, namely $x^* = \mu\hat{\alpha}$.

E.1. *Impulse Response*

The fact that the adjustment probability does not depend on the state of the firm makes the characterization of the impulse response extremely simple. Let us consider an aggregate shock δ reducing all price gaps by the same amount. Let $f(x, t, \delta)$ be the cross-sectional distribution of the price gaps, t periods after an aggregate shock of size δ.

The first and second (non-centered) moments t periods after the shock can be written as

$$\tilde{M}_1(t, \mu, \delta) \equiv \int_{-\infty}^{\infty} x f(x, t, \delta)\, dx, \qquad \tilde{M}_2(t, \mu, \delta) \equiv \int_{-\infty}^{\infty} x^2 f(x, t, \delta)\, dx.$$

Let $N \equiv \frac{1}{\int_0^\infty S(t)\, dt}$ be the average number of price changes (the reciprocal of the expected duration) and define the density of prices with duration larger than t as $A(\tau) \equiv \frac{S(\tau)}{\int_0^\infty S(t)\, dt} = NS(\tau)$. Let $n(x, m, u)$ denote the density of a normal distribution with mean m and variance u. It is easy to see that

$$\begin{aligned} f(x, t, \delta) = {} & \int_0^t A(\tau) n\big(x, \mu(\hat{\alpha} - \tau), \tau\sigma^2\big)\, d\tau \\ & + \int_t^\infty A(\tau) n\big(x, \mu(\hat{\alpha} - \tau) - \delta\big), \tau\sigma^2\big)\, d\tau, \end{aligned} \tag{E.2}$$

where the first integral takes into account all firms that have adjusted the price after t periods (using the survival function), while the second integral takes into account all other firms that have not yet adjusted their price. This explains the different mean and variance of the two groups: all firms reset their price gap at $\hat{\alpha}\mu$, which then changes at rate $-\mu dt$ due to inflation. The group of firms that have not adjusted since the shock has a lower mean by $-\delta$ due to the shock. The variance for both groups increases linearly in time, as implied by the random walk nature of the idiosyncratic shocks.

Let us now use equation (E.2) to compute the non-centered moments $\tilde{M}_1$ and $\tilde{M}_2$, and then compute the centered moment $M_2 \equiv \tilde{M}_2 - \tilde{M}_1^2$ to analyze the impulse response of various moments following a monetary shock δ.

Using the above definitions, we have

$$\begin{aligned}\tilde{M}_2(t,\mu,\delta) = \int_{-\infty}^{\infty}\bigg(&\int_0^t A(\tau)x^2 n\big(x,\mu(\hat{\alpha}-\tau),\tau\sigma^2\big)\,d\tau \\ &+ \int_t^{\infty} A(\tau)x^2 n\big(x,\mu(\hat{\alpha}-\tau)-\delta,\tau\sigma^2\big)\,d\tau\bigg)\,dx;\end{aligned}$$

exchanging the time and state integrals, we have

$$\begin{aligned}\tilde{M}_2(t,\mu,\delta) = &\int_0^t A(\tau)\bigg(\int_{-\infty}^{\infty} x^2 n\big(x,\mu(\hat{\alpha}-\tau),\tau\sigma^2\big)\,dx\bigg)\,d\tau \\ &+ \int_t^{\infty} A(\tau)\bigg(\int_{-\infty}^{\infty} x^2 n\big(x,\mu(\hat{\alpha}-\tau)-\delta,\tau\sigma^2\big)\,dx\bigg)\,d\tau.\end{aligned}$$

The state integrals are solved using that n is normal density. Collecting terms gives

$$\begin{aligned}\tilde{M}_2(t,\mu,\delta) = &\big(\sigma^2 - 2\mu^2\hat{\alpha}\big)\int_0^{\infty} A(\tau)\tau\,d\tau + \mu^2\hat{\alpha}^2 + \mu^2\int_0^{\infty} A(\tau)\tau^2\,d\tau \\ &+ \big(\delta^2 - 2\delta\mu\hat{\alpha}\big)\int_t^{\infty} A(\tau)\,d\tau + 2\delta\mu\int_t^{\infty} A(\tau)\tau\,d\tau, \end{aligned} \tag{E.3}$$

where we use $\int_0^{\infty} A(\tau)\,d\tau = 1$ and the expressions in the first line give the steady-state value of the second (non-centered) moment.

This equation can be used to compute the steady-state moments of this economy (as $t \to \infty$), or a parameterization without inflation ($\mu = 0$). Following a similar logic, we get the first moment

$$\begin{aligned}\tilde{M}_1(t,\mu,\delta) = &\int_0^t A(\tau)\bigg(\int_{\infty}^{\infty} x n\big(x,\mu(\hat{\alpha}-\tau),\tau\sigma^2\big)\,dx\bigg)\,d\tau \\ &+ \int_t^{\infty} A(\tau)\bigg(\int_{\infty}^{\infty} x n\big(x,\mu(\hat{\alpha}-\tau)-\delta,\tau\sigma^2\big)\,dx\bigg)\,d\tau,\end{aligned}$$

which gives

$$\tilde{M}_1(t,\mu,\delta) = -\mu\int_0^{\infty} A(\tau)\tau\,d\tau + \mu\hat{\alpha} - \delta\int_t^{\infty} A(\tau)\,d\tau. \tag{E.4}$$

From this expression, we recover the steady-state mean

$$\tilde{M}_1(\infty,\mu,0) = \mu\bigg(\hat{\alpha} - \int_0^{\infty} A(\tau)\tau\,d\tau\bigg). \tag{E.5}$$

Define the impulse response (in deviation from the steady state)

$$H_1(t,\mu,\delta) \equiv \tilde{M}_1(t,\mu,\delta) - \tilde{M}_1(\infty,\mu,0), \tag{E.6}$$

so we have $H_1(t,\mu,\delta) = -\delta \int_t^\infty A(\tau)\,d\tau$, which shows that the impulse response is linear in the shock size (as common for time-dependent models) and that inflation only affects the steady-state average level of x, through equation (E.5), but not the shape of the impulse response (as is intuitive since the survival function is unchanged).

Using the expressions above, it is immediate to compute the expression for the second centered moment of x as

$$M_2(t,\mu,\delta) = \tilde{M}_2(t,\mu,\delta) - \big(\tilde{M}_1(t,\mu,\delta)\big)^2;$$

we have

$$\begin{aligned} M_2(t,\mu,\delta) &= \big(\sigma^2 - 2\mu^2\hat{\alpha}\big)\int_0^\infty A(\tau)\tau\,d\tau + \mu^2\int_0^\infty A(\tau)\tau^2\,d\tau \\ &\quad + \delta^2\int_t^\infty A(\tau)\,d\tau + 2\delta\mu\int_t^\infty A(\tau)\tau\,d\tau\bigg(1 - \int_0^\infty A(\tau)\tau\,d\tau\bigg) \\ &\quad - \mu^2\bigg(\int_0^\infty A(\tau)\tau\,d\tau\bigg)^2 - \delta^2\bigg(\int_t^\infty A(\tau)\,d\tau\bigg)^2. \end{aligned} \tag{E.7}$$

The steady-state value of the centered second moment then is

$$\begin{aligned} M_2(\infty,\mu,0) &= \big(\sigma^2 - 2\mu^2\hat{\alpha}\big)\int_0^\infty A(\tau)\tau\,d\tau \\ &\quad + \mu^2\int_0^\infty A(\tau)\tau^2\,d\tau - \mu^2\bigg(\int_0^\infty A(\tau)\tau\,d\tau\bigg)^2. \end{aligned} \tag{E.8}$$

It appears that inflation does have an effect on the steady-state dispersion of price gaps that is proportional to μ^2. This means that the derivative of dispersion around zero inflation is zero, so that a small inflation only has second-order effects on the steady-state dispersion of price gaps, namely

$$\begin{aligned} &\frac{\partial}{\partial\mu}M_2(\infty,\mu,0) \\ &\quad = 2\mu\bigg(-2\hat{\alpha}\int_0^\infty A(\tau)\tau\,d\tau + \int_0^\infty A(\tau)\tau^2\,d\tau - \bigg(\int_0^\infty A(\tau)\tau\,d\tau\bigg)^2\bigg), \end{aligned} \tag{E.9}$$

which is zero when evaluated at $\mu = 0$.

Let us conclude by inspecting the response of the second moment after a shock δ:

$$H_2(t,\mu,\delta) \equiv M_2(t,\mu,\delta) - M_2(\infty,\mu,0); \tag{E.10}$$

we have

$$H_2(t,\mu,\delta) = \delta^2\int_t^\infty A(\tau)\,d\tau\bigg(1 - \int_t^\infty A(\tau)\,d\tau\bigg)$$

$$+2\delta\mu \int_t^\infty A(\tau)\tau\, d\tau \left(1 - \int_0^\infty A(\tau)\tau\, d\tau\right), \tag{E.11}$$

which shows that the impulse response is made of the second-order terms δ^2 and $\delta\mu$. This shows that for a small inflation and for a small shock, the impulse response does not feature a first-order term in δ.

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