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SUPPLEMENT TO "BARGAINING AND EXCLUSION WITH MULTIPLE BUYERS"

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EXAMPLE FROM SECTION 3 WITH RANDOM MATCHING

WE REVISIT THE EXAMPLE FROM SECTION 3 in which $n = 3$, $q = 2$ and $a_1 = 4$, $a_2 = 3$, $a_3 = 1$ in the context of the model with random matching introduced in Section 5. In the version of the game in which the seller is matched to bargain with equal probability with one of the remaining buyers in every state, there exist three classes of MPEs that are asymptotically equivalent to those derived for the benchmark model.

For a given MPE of the game with random matching and discount factor δ, let $\alpha_i(S)$ denote the probability that the seller reaches an agreement with buyer i conditional on being matched with buyer i in state S. We use the same notation for payoffs (and the convention to write α_i and u_i for MPE variables corresponding to the initial state N) as in the benchmark model. Note that $\delta u_0(S) + \delta u_i(S) > a_i + \delta u_0(S \setminus \{i\})$ implies that $\alpha_i(S) = 0$, and the opposite strict inequality implies that $\alpha_i(S) = 1$. If $\delta u_0(S) + \delta u_i(S) = a_i + \delta u_0(S \setminus \{i\})$, then the seller and buyer i are indifferent between trading and not trading when matched in the state S, and receive payoffs $\delta u_0(S)$ and $\delta u_i(S)$, respectively, if they trade.

Payoffs in two-buyer subgames following the first trade have the same asymptotic values for $\delta \to 1$ as in the benchmark model. Suppose buyers i and j with $i < j$ are competing in a subgame for the remaining unit. In this subgame there is a unique MPE, in which the seller and buyer i trade with probability 1 conditional on being matched ($\alpha_i(\{i, j\}) = 1$). If $a_j < a_i/2$, then for $\delta \in [4a_j/(a_i + 2a_j), 1)$, the outside option of trading with buyer j is not binding in the MPE ($\alpha_j(\{i, j\}) = 0$): the seller trades exclusively with buyer i, and $u_0(\{i, j\}) = u_i(\{i, j\}) = a_i/(2(2 - \delta))$, $u_j(\{i, j\}) = 0$. If $a_j \geq a_i/2$, then for any $\delta \in (0, 1)$, the outside option is binding: the seller trades with buyer j with positive but vanishing probability as $\delta \to 1$ ($\alpha_j(\{i, j\}) > 0$ but $\lim_{\delta \to 1} \alpha_j(\{i, j\}) = 0$). In this case, for sufficiently high δ, we have that $\alpha_j(\{i, j\}) \in (0, 1)$, implying that $\delta u_0(\{i, j\}) + \delta u_j(\{i, j\}) = a_j$. Then, $u_j(\{i, j\}) = \alpha_j(\{i, j\}) \delta u_j(\{i, j\})/2$ leads to $u_j(\{i, j\}) = 0$. Hence, $u_0(\{i, j\}) = a_j/\delta$, and $u_i(\{i, j\})$ solves

$$u_0(\{i, j\}) = \frac{1}{4}\big(a_i - \delta u_i(\{i, j\})\big) + \frac{3}{4}\delta u_0(\{i, j\})\big),$$

so $u_i(\{i, j\}) = a_i/\delta - a_j(4 - 3\delta)/\delta^2$.

As in the analysis of this example in the benchmark model, it is convenient to define $A_k := a_k + \delta u_0(N \setminus \{k\})$ for $k \in N$, with $u_0(N \setminus \{k\})$ derived above for δ close to 1.

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We now prove the existence of the counterpart of the first class of MPEs discussed in Section 3. In this class, we have $\alpha_1 = 1$ and $\alpha_2 \in (0,1)$ and $\alpha_3 \in (0,1)$ for high δ. Then, the MPE variables must satisfy

$$u_0 = \frac{1}{6}(A_1 - \delta u_1) + \frac{5}{6}\delta u_0, \tag{S1}$$

$$A_2 = \delta u_0 + \delta u_2, \tag{S2}$$

$$A_3 = \delta u_0 + \delta u_3. \tag{S3}$$

The payoff equations for buyers reduce to

$$u_1 = \frac{1}{6}(A_1 - \delta u_0) + \frac{\alpha_2}{3}\delta u_1(\{1,3\}) + \frac{\alpha_3}{3}\delta u_1(\{1,2\}) + \frac{5 - 2\alpha_2 - 2\alpha_3}{6}\delta u_1, \tag{S4}$$

$$u_2 = \frac{1}{3}\delta u_2(\{2,3\}) + \frac{\alpha_3}{3}\delta u_2(\{1,2\}) + \frac{2-\alpha_3}{3}\delta u_2, \tag{S5}$$

$$u_3 = \frac{1}{3}\delta u_3(\{2,3\}) + \frac{\alpha_2}{3}\delta u_3(\{1,3\}) + \frac{2-\alpha_2}{3}\delta u_3. \tag{S6}$$

Conversely, any solution $(u_0, u_1, u_2, u_3, \alpha_2, \alpha_3)$ to the above system of equations with $\alpha_2, \alpha_3 \in [0,1]$ for which $A_1 - \delta u_1 \geq \delta u_0$ (or equivalently, $u_0 \geq 0$) satisfies $u_i \geq 0$ for all players i, and characterizes payoffs and behavior in the initial state of an MPE for the game with discount factor δ.

Fix a discount factor δ close to 1. Since $u_3(\{2,3\}) = u_3(\{1,3\}) = 0$, equation (S6) requires that $u_3 = 0$. Solving for u_0 in (S3), we get $u_0 = A_3/\delta$. We then obtain u_1 and u_2 from (S1) and (S2), respectively: $u_1 = A_1/\delta - A_3(6-5\delta)/\delta^2$ and $u_2 = (A_2 - A_3)/\delta$. Noting that $u_2(\{1,2\}) = 0$ and plugging in the computed value of u_2 in (S5), we obtain $\alpha_3 = 2 - 3/\delta + \delta u_2(\{2,3\})/(A_2 - A_3)$. Finally, given the values of u_0, u_1, and α_3, we can solve for α_2 in (S4) as long as its coefficient $\delta(u_1(\{1,3\}) - u_1)/3$ is not zero. As δ goes to 1, this coefficient converges to $1/6$, so the computation is feasible for δ close to 1. Both α_2 and α_3 converge to $1/2$ for $\delta \to 1$. It follows that for δ close to 1, the computed values solve the system of equations (S1)–(S6) and belong to the range required to define an MPE.

Payoffs in this class of MPEs have the same asymptotic values as in the first class of MPEs discussed in Section 3. Since both α_2 and α_3 converge to $1/2$ as δ goes to 1, the probabilities with which buyers 1, 2, and 3 trade first in the game with random matching converge to $(1/2, 1/4, 1/4)$ like in the benchmark model (e.g., the limit probability of trading with buyer 1 first is given by Bayes' rule: $(1 \times 1/3)/(1 \times 1/3 + 1/2 \times 1/3 + 1/2 \times 1/3)$). These limit values can also be characterized by high-level arguments using analogues of equations (3) and (4) from Section 3.

We next describe the equilibria corresponding to the second class of MPEs from Section 3. In this class, we have $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 \in (0,1)$. Like in the first class, it must be that $u_3 = 0$ and $u_0 = A_3/\delta$. Then,

$$u_0 = \frac{1}{6}(A_2 - \delta u_2) + \frac{5}{6}\delta u_0$$

leads to $u_2 = A_2/\delta - A_3(6-5\delta)/\delta^2$. We substitute this value of u_2 to solve for α_3 in

$$u_2 = \frac{1}{6}(A_2 - \delta u_0) + \left(\frac{5}{6} - \frac{\alpha_3}{3}\right)\delta u_2 + \frac{\alpha_3}{3}\delta u_2(\{1,2\}),$$

where we know that $u_2(\{1,2\}) = 0$. We obtain

$$\alpha_3 = \frac{3(1-\delta)\big((6-4\delta)A_3 - \delta A_2\big)}{\delta\big(\delta A_2 - (6-5\delta)A_3\big)},$$

which converges to 0 as δ goes to 1, and is positive for δ near 1 because $\lim_{\delta\to 1} A_2 = 5$ and $\lim_{\delta\to 1} A_3 = 4$. It follows that $\alpha_3 \in (0,1)$ for δ near 1. It can be immediately checked that for δ sufficiently close to 1, the computed variables characterize an MPE for the game with random matching in which payoffs have the same limits for $\delta \to 1$ as in the second class of MPEs for the benchmark model and $\alpha_1 = 0$ is consistent with the MPE constraints.

The existence proof for the third class of MPEs is analogous to that for the second class.

APPLYING PROPOSITION 1 TO THE EXAMPLE FROM SECTION 3

We illustrate how Proposition 1 (along with Theorem 1) can be used to quickly derive buyers' limit payoffs and trading probabilities in the example from Section 3. In that example, there are three classes of MPEs for high δ. In the first class, the seller mixes with full support over the three buyers in the initial state, and Proposition 1 pins down the limit payoffs for every buyer i: $\bar{u}_i = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_0(N)$, where $\bar{u}_0(N \setminus \{i\})$ and $\bar{u}_0(N)$ are given by Theorem 1. As explained in Section 3, one can substitute these values in the limit payoff equations for buyers 1 and 2 to solve for the seller's limit mixing probabilities in the initial state. In the second class of MPEs, the support of the seller's mixing in the initial state is formed by buyers 2 and 3, and Proposition 1 immediately determines $\bar{u}_2$ and $\bar{u}_3$. This information can be plugged in the limit payoff equation of buyer 2 to infer that $\bar{\pi}_2 = 1$. Hence, $\bar{\theta}_1(\{1,3\}) = 1$, and Proposition 1 leads to $\bar{u}_1 = a_1 + \bar{u}_0(\{3\}) - \bar{u}_0(\{1,3\})$, where $\bar{u}_0(\{1,3\}) = a_1/2$ and, by definition, $\bar{u}_0(\{3\}) = 0$. Similarly, limit buyer payoffs and trading probabilities in the third class of MPEs can be directly derived via Proposition 1 and the limit buyer payoff equations.

ANALYSIS OF THE EXAMPLE IN FOOTNOTE 13

Consider the benchmark game with supply $q = 2$ in which the seller bargains with $n = 3$ buyers who have values $a_1 = 5$, $a_2 = 4$, $a_3 = 3$. We derive MPEs for high δ that are differentiated by the subsets of buyers over which the seller randomizes in the initial state. In this example, it turns out that the seller and buyer 3 trade with positive limit probability in all MPEs for $\delta \to 1$. Hence, all MPEs are asymptotically inefficient.

In the first class of MPEs, we have $\pi_1 = 0$ and $\pi_2, \pi_3 > 0$. The corresponding MPE variables must solve the following system of equations

$$u_0 = \frac{1}{2}\big(a_2 + \delta u_0(\{1,3\}) - \delta u_2\big) + \frac{1}{2}\delta u_0,$$

$$u_i = \pi_i\left(\frac{1}{2}\big(a_i + \delta u_0(N \setminus \{i\}) - \delta u_0\big) + \frac{1}{2}\delta u_i\right) + \sum_{k \in N \setminus \{i\}} \pi_k \delta u_i\big(N \setminus \{k\}\big) \quad \text{for } i = 1, 2, 3,$$

$$a_2 + \delta u_0\big(\{1,3\}\big) - \delta u_2 = a_3 + \delta u_0\big(\{1,2\}\big) - \delta u_3,$$

$$\pi_2 + \pi_3 = 1,$$

and additionally obey the constraint $a_2 + \delta u_0(\{1,3\}) - \delta u_2 \geq a_1 + \delta u_0(\{2,3\}) - \delta u_1$. Any solution satisfies equation (7) in the paper, which for $i = 2, 3$ yields

$$u_i = \frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i}\frac{a_i + \delta u_0(\{1,j\})}{2} + \frac{\pi_j(2-\delta)}{2-\delta-\delta\pi_i}\delta u_i(\{1,i\}),$$

where $j = 5 - i$. Plugging the expressions for u_2 and u_3 above in the constraint $a_2 + \delta u_0(\{1,3\}) - \delta u_2 = a_3 + \delta u_0(\{1,2\}) - \delta u_3$, multiplying out the two denominators $2 - \delta - \delta\pi_2$ and $2-\delta-\delta\pi_3$, and substituting $\pi_3 = 1-\pi_2$ leads to a quadratic equation in π_2. Coefficients in this quadratic involve functions of δ, including the cumbersome payoff formulae for two-buyer subgames from the Appendix (in this example, the outside option is binding in all such subgames). With the help of Mathematica, we solved the quadratic analytically and obtained the two roots as functions of δ. We then computed the limits of the roots for $\delta \to 1$: $-3 - 5/\sqrt{2}$ and $-3 + 5/\sqrt{2}$; only the latter limit belongs to $[0, 1]$. Hence, an MPE with this structure exists for sufficiently high δ, and limit mixing probabilities for $\delta \to 1$ in this family of MPEs are given by $\bar{\pi}_2 = -3 + 5/\sqrt{2} \approx 0.5355$ and $\bar{\pi}_3 = 1 - \bar{\pi}_2$. Limit buyer payoffs can then be derived via Proposition 1 taking into account that $\bar{\theta}_1(\{1,2\}) = \bar{\pi}_3$, $\bar{\theta}_1(\{1,3\}) = \bar{\pi}_2$ and $\bar{\theta}_2(N) = \bar{\theta}_3(N) = 1$. We obtain $\bar{u}_1 = \bar{\pi}_2 \times 2 + (1 - \bar{\pi}_2) \times 1 = 1 + \bar{\pi}_2$, $\bar{u}_2 = a_2 + \bar{u}_0(\{1,3\}) - \bar{u}_0 = 0$ and $\bar{u}_3 = a_3 + \bar{u}_0(\{1,2\}) - \bar{u}_0 = 0$. The condition necessary for the optimality of the seller's choice of $\pi_1 = 0$ for δ close to 1 is immediately verified.

A similar exercise characterizes the second class of MPEs, in which $\pi_2 = 0$. For this class, we find that $\bar{\pi}_1 = (-11+\sqrt{177})/4 \approx 0.5760$, leading to $\bar{u}_1 = 1$, $\bar{u}_2 = \bar{\pi}_1 \times 1 + \bar{\pi}_3 \times 0 = \bar{\pi}_1$, and $\bar{u}_3 = 0$.

In a possible third class of MPEs, the seller mixes with positive probability between all three buyers. Then, Proposition 1 pins down all limit buyer payoffs: $\bar{u}_1 = 1$, $\bar{u}_2 = \bar{u}_3 = 0$. The seller's limit mixing probabilities in the initial state can then be computed like in the Section 3 example. We obtain $\bar{\pi}_1 = \bar{\pi}_2 = 0$, $\bar{\pi}_3 = 1$.

In the first and second classes of MPEs—unlike in the third class—the limit buyer payoff equations do not pin down the corresponding limit mixing probabilities in the initial state. For instance, for the family of MPEs with $\pi_1 = 0$ and $\pi_2, \pi_3 > 0$, equation (5) in the paper leads to $\bar{u}_2 = \bar{\pi}_2\bar{u}_2 + \bar{\pi}_3\bar{u}_2(\{1,2\})$, which does not impose constraints on $\bar{\pi}_2$ and $\bar{\pi}_3$ because $\bar{u}_2 = \bar{u}_2(\{1,2\}) = 0$. The exact values of π_2 and π_3 for high δ are determined by small differences between u_2 and $u_2(\{1,2\})$.

As in the example from Section 3, we can rule out other possibilities for the support of the seller's mixing in the initial state of any MPE for sufficiently high δ. We conclude that under any convergent family of MPEs for $\delta \to 1$, the seller inefficiently trades with the set of buyers $\{1,3\}$ with positive limit probability.

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