SUPPLEMENT TO "ON THE LIMITS OF COMMUNICATION IN MULTIDIMENSIONAL CHEAP TALK: A COMMENT"

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In this appendix we extend the results of our paper to a general family of single-peaked preferences (similar to those of Crawford and Sobel, but adapted to the multidimensional state space). We show that when the conflict between the sender and the receiver is large, these preferences have similar characteristics to those of the lexicographic preferences analyzed in the paper. We prove that communication is restricted in this environment as well.

IN THIS APPENDIX we maintain all the assumptions as in the model considered in Section 2 in the paper, except for the sender's preferences, which we assume to be represented by

$$V(a,\theta)=U(b_x+\theta_x-a_x,\ b_y+\theta_y-a_y),$$

where the function U is strictly decreasing in $|b_x+\theta_x-a_x|$, and in $|b_y+\theta_y-a_y|$, and attains its unique maximum at $(0,0)$. We interpret the vector $\mathbf{b}=(b_x,b_y)\in\mathbb{R}^2$ as the conflict between the sender and the receiver. We assume that at least for some $i\in\{x,y\}$, $b_i\neq 0$. Without loss of generality, we assume that $b_y\neq 0$. Let $\beta\in\mathbb{R}$ be defined by $b_x=\beta b_y$ and let $b=\|(b_x,b_y)\|$. Fixing β, we consider sequences of vectors $\{(b_x^n,b_y^n)\}_{n=1}^{\infty}$ for which $b_x^n=\beta b_y^n$ for any $n\in\{1,2,\dots,\infty\}$ and $b\to\infty$. Finally, assume that U is twice differentiable and that the following limits are well defined[1]:

$$\alpha^*\equiv\lim_{x\to\infty,y\to\infty,x/y\to\beta}\left(-\frac{U_1(x,y)}{U_2(x,y)}\right)$$

and

$$\gamma^*\equiv\lim_{x\to\infty,y\to\infty,x/y\to\beta}\left(-\frac{U_{11}(x,y)+\alpha^*U_{12}(x,y)}{\alpha^*U_{22}(x,y)+U_{12}(x,y)}\right).$$

We say that action a is *induced in* $\Theta\subseteq\mathbb{R}^2$ if there exists $\theta\in\Theta$ that induces it. When a is induced in $\mathbb{R}^2$, we simply say that a is *induced*.

LEMMA S1: *For any compact and convex subset $\Theta\subseteq\mathbb{R}^2$, for any $\varepsilon>0$, $\exists\bar{b}<\infty$ such that for any two actions a and a' in Θ that are induced in Θ, if $b>\bar{b}$, then* (i) $|(a_y-a_y')/(a_x-a_x')-\alpha^*|<\varepsilon$ *and* (ii) *for any θ and θ' in Θ who are indifferent between the two actions,* $|(\theta_y-\theta_y')/(\theta_x-\theta_x')-\gamma^*|<\varepsilon$.

[1]All our assumptions are satisfied, for example, by the family of utility functions $V((\sum_{i\in\{x,y\}}\lambda_i|a_i-(b_i+\theta_i)|^p)^{1/p})$, where $V(\cdot)$ is strictly decreasing and $1<p<\infty$.

PROOF: (i) Because a and a' are induced in Θ, by continuity and the convexity of Θ, there must be a type $\theta \in \Theta$ that is indifferent between the two actions. Consider then the indifference curve of θ that goes through a. The slope of the indifference curve is

$$\frac{da_y}{da_x} = -\frac{U_1(\theta_x + b_x - a_x, \theta_y + b_y - a_y)}{U_2(\theta_x + b_x - a_x, \theta_y + b_y - a_y)}.$$

Consider sequences of vectors $\{(b_x^n, b_y^n)\}_{n=1}^{\infty}$ for which $b_x^n = \beta b_y^n$ for a fixed β and for any $n \in \{1, 2, \dots, \infty\}$, and let $b \to \infty$. By the compactness of Θ, $(\theta_x + b_x - a_x)/(\theta_y + b_y - a_y) \to_{b\to\infty} \beta$ and thus the limit of da_y/da_x does not depend on either θ or a. This convergence is uniform with respect to θ and a; hence (i) follows.

(ii) Consider some type θ who is indifferent between a and a':

$$U(\theta_x + b_x - a_x, \theta_y + b_y - a_y) = U(\theta_x + b_x - a'_x, \theta_y + b_y - a'_y).$$

Total differentiation with respect to θ, along with the mean value theorem, implies (assuming without loss of generality that $a_x \neq a'_x$)

$$\begin{aligned}\frac{d\theta_y}{d\theta_x} = -\Bigg(&U_{11}(\theta_x + b_x - \hat{a}_x, \theta_y + b_y - \hat{a}_y) \\ &+ U_{21}(\theta_x + b_x - \hat{a}_x, \theta_y + b_y - \hat{a}_y)\frac{(a'_y - a_y)}{(a'_x - a_x)}\Bigg) \\ \Bigg/\Bigg(&U_{22}(\theta_x + b_x - \check{a}_x, \theta_y + b_y - \check{a}_y)\frac{(a'_y - a_y)}{(a'_x - a_x)} \\ &+ U_{21}(\theta_x + b_x - \check{a}_x, \theta_y + b_y - \check{a}_y)\Bigg)\end{aligned}$$

for $\hat{a}$ and $\check{a}$ that are between a and a'. As above, $\frac{\theta_x + b_x - \hat{a}_x}{\theta_y + b_y - \hat{a}_y} \to_{b\to\infty} \beta$ and $\frac{\theta_x + b_x - \check{a}_x}{\theta_y + b_y - \check{a}_y} \to_{b\to\infty} \beta$. By (i) above and by the compactness of Θ, we have that $\frac{d\theta_y}{d\theta_x} \to_{b\to\infty} \gamma^*$ uniformly (with respect to θ and a). By continuity, the curve of indifferent types is connected. By compactness and the uniform convergence, (ii) follows. *Q.E.D.*

We now prove that communication is bounded also when the sender's preferences are single-peaked. To this end, we modify some of the definitions introduced in Section 2. First, denote the set of induced actions by A^* and let $\tilde{a}^*$ be the receiver's equilibrium choice of action.

DEFINITION S1: For a finite k, an equilibrium has *k actions up to ε* if

$$k = \min_{\substack{A' \subseteq A^* \\ |A'| < \infty}} \{|A'|;\ \Pr(\tilde{a}^* \in A') > 1 - \varepsilon\}.$$

Second, we can span the state space according to the α-dimension, which parallels lines with slope α^* and with a generic coordinate θ_α, and the γ-dimension, which parallels lines with slope γ^* and with a generic coordinate θ_γ. We can then define the conditional expectations $E(\tilde{\theta}_\gamma|\theta_\alpha)$ in the two-dimensional space spanned by the γ and α dimensions, and extend the definition of the k-crossing property analogously.

PROPOSITION S1: *Suppose that F satisfies the k-crossing property with respect to the γ and α dimensions. Then for any $\varepsilon > 0$, there exists a $\bar{b} < \infty$ such that for all $b > \bar{b}$, any equilibrium has at most k actions up to ε.*

PROOF: We start with some definitions and notation. Recall that equilibrium is defined by a pair of strategies $(a^*(\cdot), m^*(\cdot|\theta))$. Let $M^* = \bigcup_{\theta\in\mathbb{R}^2} \operatorname{Supp}\{m^*(\cdot|\theta)\}$. For the purpose of the proof, we can restrict attention to equilibria in which $m = a^*(m)$ for any $m \in M^*$. This implies that $M^* = A^*$.[2]

The sender's strategy induces a measure on $\mathbb{R}^2 \times \mathbb{R}^2$: $g(\theta, m) = m^*(m|\theta) \times f(\theta)$. For any $A \subset A^*$ in the Borel σ-algebra of $\mathbb{R}^2$, the density of θ conditional on an action in A being induced is

$$g(\theta|A) = \int_A \frac{g(\theta, m)}{\int_{\mathbb{R}^2}\int_A g(\theta', m')\, d\theta'\, dm'}\, dm.$$

For any $A \subset A^*$ in the Borel σ-algebra of $\mathbb{R}^2$, the probability measure of the set of types that induce actions in A is $G(A) = \int_{\mathbb{R}^2}\int_A g(\theta, m)\, dm\, d\theta$. The set of all types who induce some action a is termed the *support set* of action a and is denoted by $S(a)$.

We will use two geometrical constructions in the proof of Proposition S1. First, S^η is the compact subset of $\mathbb{R}^2$, bounded by a square constructed symmetrically around the prior expectation and parallel to the x and y dimensions, that has a measure of $1 - \eta$ according to F. For any η and any two parallel lines l' and l'', let $S^\eta_{l',l''}$ be the subset of S^η that is between l' and l''.

Second, when we span the space according to the α-dimension and the γ-dimension, we denote by μ_γ (μ_α) the γ-coordinate (α-coordinate) of the prior expectation. Finally, let L denote the set of lines with slope γ^*, such that the γ-coordinate of the conditional expectation of the subset of $\mathbb{R}^2$ on each side of the line equals μ_γ. We denote an element of L by l.

The proof of Proposition S1 follows three steps. Step 1 below will allow us to translate results on compact sets to noncompact ones when b is large. Step 2 focuses on compact subsets and invokes Lemma S1 to show how, when b is large, the support sets of equilibrium actions are characterized by lines in L. In Step 3, we show that when F satisfies the k-crossing property with respect

[2]The set M^* is in the Borel σ-algebra of $\mathbb{R}^2$ because $m^*(m|\theta)$ is measurable in m and θ. Thus, also A^* is in the Borel σ-algebra of $\mathbb{R}^2$.

to the y and α dimensions, then the number of lines in L is $k-1$. Finally, we combine the three steps to conclude that for high enough b there will be at most k actions up to ε in equilibrium.

STEP 1: *For any $\delta > 0$, there exists an $\eta' > 0$, such that for any $\eta < \eta'$, in any equilibrium, there is at most a probability measure δ of types in S^η who induce actions in $(S^\eta)^c$.*[3]

PROOF: In what follows, we set "east" to be the upward direction on the x-axis. Consider the subset of $\mathbb{R}^2$ to the east of S^η that is bounded above and below by the extensions of the north and south sides of S^η. Denote this set by E^η. Fix an equilibrium and let $E^{\eta*} = A^* \cap E^\eta$.

Assume, by way of contradiction, that there exists a sequence $\{\eta_n\}_{n=1}^\infty$ that converges to zero, for which the probability measure of the set of types in S^{η_n} who support actions in $E^{\eta_n *}$ is $\delta/4$ for some $\delta > 0$ (in what follows, we suppress the subscript n). Let $g(\theta_x|E^{\eta*})$ be the marginal distribution of $g(\theta|E^{\eta*})$ on the x-axis.

We first focus on the case in which the set of types that induce actions in $E^{\eta*}$ comprises (i) in S^η, the set of measure $\delta/4$ under $F(\cdot)$ that is easternmost in S^η and (ii) in $(S^\eta)^c$, those to the easternmost in $(S^\eta)^c$. Let $\bar{g}(\theta_x|E^{\eta*})$ denote the relevant marginal distribution for this specification.

The x-coordinate of the conditional expectation over the union of the support sets of all actions in $E^{\eta*}$ is $\int_{S^\eta} \theta_x \bar{g}(\theta_x|E^{\eta*})\,d\theta_x + \int_{(S^\eta)^c} \theta_x \bar{g}(\theta_x|E^{\eta*})\,d\theta_x$. Given F, $\bar{g}(\theta, m)$ is a proper density function, and thus $\int_{(S^\eta)^c} \theta_x \bar{g}(\theta_x| E^{\eta*})\,d\theta_x \to_{\eta\to 0} 0$ and $\int_{S^\eta} \theta_x \bar{g}(\theta_x|E^{\eta*})\,d\theta_x \to_{\eta\to 0} \bar{k}_\delta$ for some finite $\bar{k}_\delta$. Where as we focused on the case in which the support sets of actions in $E^{\eta*}$ are to the easternmost of both S^η and $(S^\eta)^c$, then in any equilibrium, $\limsup_{\eta\to 0}\{\int_{S^\eta} \theta_x g(\theta_x|E^{\eta*})\,d\theta_x + \int_{(S^\eta)^c} \theta_x g(\theta_x|E^{\eta*})\,d\theta_x\} \le \bar{k}_\delta$.

Thus, there exists an η' such that for all $\eta < \eta'$, the conditional expectation over the support sets of actions in $E^{\eta*}$ is actually not in $E^{\eta*}$. This is a contradiction, because the conditional expectation over the support sets must equal the expectation over the actions, where the latter are in E^η. Finally, the same exercise can be applied to other subsets analogous to E^η in $(S^\eta)^c$. *Q.E.D.*

STEP 2: *For any $\zeta > 0$ there exists an $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$, there exists a $\bar{b}(\eta) < \infty$, such that for all $b > \bar{b}(\eta)$, for any $a \in S^\eta$ that is induced in S^η, there exist $l, l' \in L$ such that $|G(a) - F(S^\eta_{l,l'})| < \zeta$.*

PROOF: Take some $\hat{a} \in A^* \cap S^\eta$ that is induced in S^η. Denote by $\bar{l}$ a line parallel to the y-axis that intersects the closure of $S(\hat{a})$ in S^η and separates S^η

[3]Because A^*, S^η, and $(S^\eta)^c$ are in the Borel σ-algebra of $\mathbb{R}^2$, their intersections also are in the Borel σ-algebra of $\mathbb{R}^2$. By the measurability of $m^*(\cdot|\theta)$, this implies that the statement in Step 1 is well defined in all equilibria.

so that $S(\hat{a}) \cap S^\eta$ is below it; similarly, let $\underline{l}$ be a line parallel to the γ axis that intersects the closure of $S(\hat{a})$ and separates S^η so that $S(\hat{a}) \cap S^\eta$ is above it. Let $\hat{l}$ be an element of $\{\bar{l}, \underline{l}\}$.

The proof proceeds as follows. In Claim 1, we show that any line $\hat{l} \in \{\bar{l}, \underline{l}\}$ separates S^η in the sense that the measure of types above (below) it who support actions below (above) it is small. We use this result in Claim 2, where we prove that for high b, the support set of $\hat{a}$ nearly equals the set of all types in S^η that are below $\bar{l}$ and above $\underline{l}$. In Claim 3, we show that the lines $\bar{l}$ and $\underline{l}$ must converge to lines in L.

CLAIM 1: *For any $\delta > 0$, $\exists \eta'$ such that for all $\eta < \eta'$, there exists a $b'(\eta) < \infty$ such that for all $b > b'(\eta)$ the probability measure of all types above* (*below*) *$\hat{l}$ and in S^η that support actions below* (*above*) *$\hat{l}$ and in S^η, is bounded by δ.*

PROOF: We focus on proving that the probability measure of all types above $\bar{l}$ and in S^η that support actions below $\bar{l}$ and in S^η, is bounded by δ, because the proof for the other cases is analogous. Let $A' \subset A^*$ be the set of actions below $\bar{l}$ and in S^η that are supported by some types above $\bar{l}$ and in S^η (note that $\hat{a} \notin A'$). Let $\hat{\theta}$ be in the intersection of $\bar{l}$ and the closure of $S(\hat{a}) \cap S^\eta$. A curve of types in S^η who are indifferent between $\hat{a}$ and some $a' \in A'$ must separate $\hat{\theta}$ from the types who induce a', because $\hat{\theta}$ weakly prefers $\hat{a}$. Such a curve cannot therefore be strictly below $\bar{l}$.

In cases in which all types below each such curve prefer a' to $\hat{a}$, Lemma S1 guarantees that the measure of the set of types above $\bar{l}$ in S^η that prefer a' to $\hat{a}$ converges to zero (uniformly) as $b \to \infty$.

In what follows we focus, therefore, on cases in which all types below each such curve prefer $\hat{a}$ to a'. Suppose by way of contradiction that the measure of the set of types above $\bar{l}$ in S^η who induce actions in A' is bounded from below by a strictly positive number. In the cases we now focus on, Lemma S1 guarantees that the measure of the set of types below $\bar{l}$ and in S^η that prefer a' to $\hat{a}$ converges to zero (uniformly) as $b \to \infty$. From this it follows that the set of types in $\mathbb{R}^2$ who induce actions in A' converges, as $\eta \to 0$ and $b \to \infty$, to the set of types above $\bar{l}$ and in S^η who induce actions in A'.

Where we assume that the measure of this set is strictly positive, this implies that the measure of this set of types under $F(\cdot)$ is strictly positive. Because $F(\cdot)$ is continuous, this set of types must have a strictly positive width (the width of a set is defined as the infimum of the distance between any two parallel lines that contain the set and if this is not possible, the width is defined as infinite). Because the set has a strictly positive width and is above $\bar{l}$, its conditional expectation under $g(\theta|A')$ is both above $\bar{l}$ and bounded away from it. This, however, is a contradiction because the conditional expectation over actions in A', which must be below $\bar{l}$, must accord with the conditional expectation over the set of types in $\mathbb{R}^2$ who induce actions in A'. *Q.E.D.*

CLAIM 2: *For any $\zeta > 0$, there exists an $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$ there exists a $\bar{b}(\eta) < \infty$, such that for all $b > \bar{b}(\eta)$, $|G(\hat{a}) - F(S^{\eta}_{\bar{l},\underline{l}})| < \zeta$.*

PROOF: We will show that nearly all types in $S^{\eta}_{\bar{l},\underline{l}}$ support $\hat{a}$ and that nearly all types that support $\hat{a}$ are in $S^{\eta}_{\bar{l},\underline{l}}$. For the first observation, we have to consider types in $S^{\eta}_{\bar{l},\underline{l}}$ who induce actions in $S^{\eta}_{\bar{l},\underline{l}}$ (but not $\hat{a}$), in $S^{\eta}\backslash S^{\eta}_{\bar{l},\underline{l}}$, or in $(S^{\eta})^c$. For the second observation, because by construction $(S(\hat{a}) \cap S^{\eta}) \subseteq S^{\eta}_{\bar{l},\underline{l}}$, we only have to consider types in $(S^{\eta})^c$ who support $\hat{a}$.

Consider a type in $S^{\eta}_{\bar{l},\underline{l}}$ that induces some $a' \in S^{\eta}_{\bar{l},\underline{l}}$ for $a' \neq \hat{a}$. The curve of types who are indifferent between $\hat{a}$ and a' must cross either $\underline{l}$ or $\bar{l}$. By Lemma S1, for any such a', any such curve converges in S^{η} (uniformly with respect to a) to either $\underline{l}$ or $\bar{l}$ when $b \to \infty$. In other words, for any η, there exists a $b(\eta) < \infty$ such that for all $b > b(\eta)$, the measure of those who induce actions in $S^{\eta}_{\bar{l},\underline{l}}$ other than $\hat{a}$ is bounded by $\zeta/4$.

By Claim 1, for any $\zeta > 0$, $\exists\eta'$ such that for all $\eta < \eta'$ there exists $\bar{b}(\eta) = \max\{b'(\eta), b(\eta)\}$ such that for all $b > \bar{b}(\eta)$ the measure of types who are in $S^{\eta}_{\bar{l},\underline{l}}$ and support actions in $S^{\eta}\backslash S^{\eta}_{\bar{l},\underline{l}}$ is less than $\zeta/4$. By Step 1, we can choose $\bar{\eta} < \eta'$ such that there is at most a probability measure $\zeta/4$ of types in $S^{\eta}_{\bar{l},\underline{l}}$ that induce actions in $(S^{\eta})^c$. Finally, we can choose $\bar{\eta} < \zeta/4$ so that there is at most a probability measure $\zeta/4$ of types in $(S^{\eta})^c$. *Q.E.D.*

In the following claim, let the distance between two sets A and B be defined as $d(A, B) = \sup_{a\in A}\{\inf_{b\in B} \|a - b\|\}$.

CLAIM 3: *For any $\zeta > 0$, there exists an $\bar{\eta} > 0$ such that for all $\eta < \bar{\eta}$, there exists a $\bar{b}(\eta) < \infty$ such that for all $b > \bar{b}(\eta)$, $d(\bar{l} \cap S^{\eta}, l \cap S^{\eta}) < \zeta$ and $d(\underline{l} \cap S^{\eta}, l' \cap S^{\eta}) < \zeta$ for some l and l' in L.*

PROOF: Focus on the subset of S^{η} above or below $\bar{l}$ that has the larger measure (say it is the subset below $\bar{l}$). For this subset, consider the set of all actions induced in S^{η} and contained in this subset. By Lemma S1, the γ coordinate of the expectation over these actions must converge to μ_{γ} as $b \to \infty$. However, the expectation over these actions must also equal the conditional expectations over the union of their support sets. According to Claim 1 and Step 1, this union coincides with the set of all types below $\bar{l}$ (which is measurable under F) up to a probability measure of 4ζ (either those in S^{η} and above $\bar{l}$ that support actions in S^{η}, and below $\bar{l}$ and vice versa, or those in S^{η} that support actions in $(S^{\eta})^c$ and vice versa). Thus, by choosing a small enough η and accordingly a large enough b, the γ-coordinate of the conditional expectation over the set of types below $\bar{l}$ converges to μ_γ. By the continuity of F, this implies that $\bar{l}$ converges (uniformly) to some $l \in L$ in S^η. The same argument applies to show that $\underline{l}$ converges to some $l' \in L$. *Q.E.D.*

Claims 2 and 3, and the compactness of S^η, which implies uniform convergence with respect to a and θ, prove the statement in Step 2. *Q.E.D.*

STEP 3: *If F satisfies the k-crossing property with respect to the γ and α dimensions, then $|L| = k - 1$.*

PROOF: The proof follows that of Proposition 1 in the text. *Q.E.D.*

We can now combine the three steps above to show that for a high enough b, any equilibrium has at most k actions up to ε. By Step 1, for any ε, there exists η' such that for all $\eta \leq \eta'$ the probability measure of types who support actions in $(S^\eta)^c$ is at most $\varepsilon/3$. By Step 2, for any ε, there exists an $\bar{\eta}$ and a $\bar{b}(\varepsilon, \bar{\eta})$ such that for all $\eta < \bar{\eta}$ and $b > \bar{b}(\varepsilon, \eta)$, the set of types that is in between any two neighboring lines in L must belong, but for a measure of $\varepsilon/3k$, to a support set of only one action. By Step 3, there are at most k such sets and each has a strictly positive measure. Therefore, for any ε, we can choose $\eta < \min\{\eta', \bar{\eta}, \varepsilon/3\}$ and, hence, there exists a $\bar{b}(\varepsilon, \eta)$ such that for all $b > \bar{b}(\varepsilon, \eta)$, in any equilibrium, the probability measure of those who support the $k' \leq k$ actions in S^η is at least $1 - \varepsilon$. *Q.E.D.*

REMARK 1: In this supplementary appendix we extended our results for the model with one sender and lexicographic preferences to the case of single-peaked preferences. In Section 3 in the text, we considered the case of multiple senders (with lexicographic preferences) and noisy signals. We have assumed that on the dimension of conflict, a sender compares lotteries according to their expectations. Intuitively, this will arise when a sender is risk neutral on this dimension.

One can also extend our analysis in Section 3 to the case of single-peaked preferences. We can then prove a result analogous to Lemma S1 about how a sender compares lotteries when b is large. Again, this will depend on the risk preferences of the sender on the dimension of conflict. In particular, one can show that whenever $\lim_{x\to\infty, y\to\infty, x/y\to\beta} \sqrt{U_x^2 + U_y^2}$ exists, then the single-peaked preferences converge to be lexicographic where the dimension that is orthogonal to α^* becomes the dimension that takes precedence. If $\lim_{x\to\infty, y\to\infty, x/y\to\beta} \sqrt{U_x^2 + U_y^2}$ is finite (and nonzero), then a sender is risk neutral on this dimension, and compares two lotteries $\tilde{a}$ and $\tilde{a}'$ according to the expected action of the receiver on this dimension, as we assumed in Section 3.

If $\lim_{x\to\infty,y\to\infty,x/y\to\beta}\sqrt{U_x^2+U_y^2}=\infty$, then the sender becomes infinitely risk averse on this dimension.

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