SUPPLEMENT TO "BELIEF-FREE EQUILIBRIA IN GAMES WITH INCOMPLETE INFORMATION"

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THIS SUPPLEMENT CONTAINS some omitted details on the existence of belief-free equilibria for two families of games that are studied in the literature on reputation. Namely, for the specific cases considered in footnotes 15 and 19, we claim that it is possible to find games arbitrarily close to the respective original reputation games such that V^* has nonempty interior. In this supplementary material, we explain this in greater detail.

Consider first a one-sided incomplete information game Γ with known-own payoffs, where player 2's payoff matrix is u_2, while player 1's payoff is u_1 in state $j = 1$ and $-u_2$ in state $j = 2$. In footnote 15 of the paper, we claim that there exists a game $\hat{\Gamma}$ arbitrarily close to Γ for which the set of belief-free equilibria is nonempty.

Let us start with a two-player full information game where u_1 and u_2 are players' payoff matrixes, and assume that the set of individually rational payoffs of this game has nonempty interior (otherwise the question of reputation is trivial). Consider a complete information zero-sum two-player game Γ^0, where player 2's payoff matrix is u_2 with value $v_1 = v$, $v_2 = -v$. Let (s_1^*, s_2^*) denote a saddle point of this game. Let M_i denote the highest feasible payoff for player i and let a^i denote the action profile attaining this payoff. First, we shall show that there always exists a perturbation of payoffs in Γ^0 that generates a full information game Γ' arbitrarily close to Γ^0 and whose set of individually rational payoffs has nonempty interior. To this purpose we perturb payoffs in such a way that (s_1^*, s_2^*) remains an equilibrium of Γ', but there exists a feasible payoff Pareto dominating (s_1^*, s_2^*).

1. s_i^* is not completely mixed, for some $i = 1, 2$:

(a) $M_i > v_i$: Let s_i' denote some action assigned zero probability by s_i^*, and increase $u_{-i}(s_i', s_{-i})$ by $\varepsilon > 0$ for all s_{-i}. Call u' the new payoff matrix. Since player i is not using s_i', s_{-i}^* remains a best reply to s_i^*, and since i's payoff matrix has not changed, s_i^* also remains a best reply to s_{-i}^*. So s^* remains an equilibrium. Because player i does not use s_i', it means that $u_i(s_i', s_{-i}^*) \leq v_i$ and so $u_{-i}(s_i', s_{-i}^*) + \varepsilon > v_{-i}$, while also $u_i(s_i', s_{-i}^*) + u_{-i}(s_i', s_{-i}^*) + \varepsilon > 0$ (since the game is zero sum), that is, $u_i'(s_i', s_{-i}^*) + u_{-i}'(s_i', s_{-i}^*) > 0$. As a^i denotes an action profile such that $u_i(a^i) = M_i$, there exists a mixture $\lambda a^i + (1 - \lambda)(s_i', s_{-i}^*)$ that strictly improves upon the Nash equilibrium (s_1^*, s_2^*).

(b) $M_i = v_i$: This means that player i is getting his maximal payoff from playing s_i^* independently of player $-i$'s action, so that *any* strategy profile (s_i^*, s_{-i}), $s_{-i} \in A_{-i}$, is a saddle point. So pick one action s_{-i} and consider the game in which $u_i'(s_i, s_{-i}) = u_i(s_i, s_{-i}) + \varepsilon$ for all $s_i \in A_i$ and some $\varepsilon > 0$, and

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all other payoffs remain unchanged. Clearly, (s_i^*, s_{-i}) is an equilibrium point of the game u', and in this equilibrium point, player i receives $v_i + \varepsilon$. We then proceed as in the previous case: There exists a mixture $\lambda a^{-i} + (1-\lambda)(s_i^*, s_{-i})$ that strictly improves upon the Nash equilibrium (s_1^*, s_2^*).

2. Both s_1^* and s_2^* are completely mixed: $u(a^1) \in \mathbf{R}^2$ and $u(a^2) \in \mathbf{R}^2$ are the two extremes of the set of feasible payoffs that is a segment with slope $-45°$, while $u(s^*)$ is somewhere on the interior of this segment. Let $a^1 = (a_1, a_2)$ and $a^2 = (a_1', a_2')$. Both a_i and a_i' are in the support of s_i^* for $i = 1, 2$, and let α_i, α_i' denote the probabilities of those actions given s_i^*. Consider the payoffs u' such that $u'(a_1, a_2) = u(a_1, a_2) + (\varepsilon/\alpha_2, \varepsilon/\alpha_1)$, $u'(a_1, a_2') = u(a_1, a_2') + (-\varepsilon/\alpha_2', -\varepsilon/\alpha_1)$, $u'(a_1', a_2) = u(a_1', a_2) + (-\varepsilon/\alpha_2, -\varepsilon/\alpha_1')$, and $u'(a_1', a_2') = u(a_1', a_2') + (\varepsilon/\alpha_2', \varepsilon/\alpha_1')$ (all other entries are left unchanged). By construction, s^* remains an equilibrium in game u', leading to payoffs $v_1 = v$ and $v_2 = -v$. Now the action profiles (a_1, a_2) and (a_1', a_2') provide two points that are above the $-45°$ line, namely $(M_1 + \varepsilon, -M_1 + \varepsilon)$ and $(-M_2 + \varepsilon, M_2 + \varepsilon)$, respectively. Hence there exists a convex combination of a^1 and a^2 that is a Pareto improvement with respect to the Nash equilibrium (s_1^*, s_2^*).

Let $\hat{\Gamma}$ be the one-sided incomplete information game with known-own payoffs where player 2's payoff matrix is u_2', while player 1's payoff is u_1 in state $j = 1$ and u_1' in state $j = 2$. Here u_1' and u_2' are obtained as described above and are such that u_1' and u_2' are arbitrarily close to $-u_2$ and u_2, respectively. The purpose is to show that the set of belief-free equilibria in $\hat{\Gamma}$ is nonempty. Consider the following construction.

Let α^* be the occupation measure generated by strategy profile (s_1^*, s_2^*). Let A^{IR_2} be the set of occupation measures leading to payoffs that are individually rational for player 2. This set has nonempty interior and includes α^*.[1] Let $\alpha^{1,j}$ be the $\alpha \in A^{\mathrm{IR}_2}$ preferred by player 1 in state $j = 1, 2$. The payoffs originated by $\alpha^{1,1}$ and $\alpha^{1,2}$ are incentive compatible for player 1 (and generically strictly incentive compatible provided $|A| > 3$). We shall show that $\alpha^{1,1}$ and $\alpha^{1,2}$ generate strictly individually rational payoffs for player 1, that is to say, player 2 has a strategy $\hat{s}_2$ that punishes player 1 in the two states. Let B^j be player 1's best reply correspondence in state j. Note first that payoffs that strictly Pareto dominate $(u_1'(s_1^*, s_2^*), u_2'(s_1^*, s_2^*))$ exist by construction of Γ' and are reachable with occupation measures that are in A^{IR_2}. Thus, $u_1'(\alpha^{1,2}) > u_1'(s_1^*, s_2^*)$. Also, since s^* minmaxes player 2's payoff, it results in $u_2'(B^1(s_2^*), s_2^*) \geq u_2'(s_1^*, s_2^*)$ and hence $u_1(\alpha^{1,1}) \geq u_1(B^1(s_2^*), s_2^*)$, as $(B^1(s_2^*), s_2^*)$ is in A^{IR_2}. Let $\varepsilon := (u_1'(\alpha^{1,2}) - u_1'(s_1^*, s_2^*))/2 > 0$. Then there are strategies s_2 close to s_2^* such that $u_1'(B^2(s_2), s_2) < u_1'(s_1^*, s_2^*) + \varepsilon < u_1'(\alpha^{1,2})$. Thus, we can define player 2's punishment strategy, $\hat{s}_2$, as the s_2 that solves

$$\inf_{s_2} u_1(B^1(s_2), s_2)$$

[1]Recall that the set of individually rational payoffs in the initial game has nonempty interior.

s.t.

$$u_1'(B^2(s_2), s_2) < u_1'(s_1^*, s_2^*) + \varepsilon.$$

Noting that $u_1(B^1(\hat{s}_2), \hat{s}_2) \leq u_1(B^1(s_2^*), s_2^*) \leq u_1(\alpha^{1,1})$ and considering that the set of individually rational payoff has nonempty interior in the full information game corresponding to state $j = 1$, it follows that generically $u_1(B^2(\hat{s}_2), \hat{s}_2) < u_1(\alpha^{1,1})$. Finally note that $\alpha^{1,1}$ and $\alpha^{1,2}$ are individually rational for player 2 as they are in A^{IR_2}. Strict individual rationality can be obtained by slightly perturbing $\alpha^{1,1}$ and $\alpha^{1,2}$ if necessary. This can be done without violating player 1 individual rationality and incentive compatible constraint since these constraints are strictly satisfied at $\alpha^{1,1}$ and $\alpha^{1,2}$.

A similar, but simpler, construction works for the case of dominant action games (footnote 19 in the paper): Pick, for instance, the commitment type's payoff (for whom the payoff from the dominant action is only "nearly" independent of his opponent's action) to be such that the ranking over player 2's pure actions (given his own dominant action) is the same for both types. Then, since the minmax action of player 2 is independent of player 1's type, we can find a distribution over action profiles that is both weakly incentive compatible and strictly individual rational for both players (take a "pooling" distribution in which player 1 plays his dominant action and player 2 does not play for sure his minmax action)[2]; since there are two states and four action profiles, there will also be strictly individually rational, strictly incentive compatible distributions.

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[2]More precisely, this works if, as in the example in the paper, the dominant action for player 1 is not the action that minmaxes player 2; otherwise, player 1 must also play another action with small probability.