Econometrica Supplementary Material

SUPPLEMENT TO "REPEATED GAMES WHERE THE PAYOFFS AND MONITORING STRUCTURE ARE UNKNOWN"

(*Econometrica*, Vol. 78, No. 5, September 2010, 1673–1710)

BY DREW FUDENBERG AND YUICHI YAMAMOTO

S.1. PROOF OF THEOREM 1

THEOREM 1: *If a subset W of $\mathbf{R}^{I\times|\Omega|}$ is bounded and ex post self-generating with respect to δ, then $W\subseteq E(\delta)$.*

PROOF: Let $v\in W$. We will construct a PPXE that yields v. Since $v\in B(\delta,W)$, there exist a profile α and a function $w:Y\to W$ such that (α,v) is ex post enforced by w. Set the action profile in period one to be $s|_{h^0}=\alpha$ and for each $h^1=y^1\in Y$, set $v|_{h^1}=w(h^1)\in W$. The play in later periods is determined recursively, using $v|_{h^t}$ as a state variable. Specifically, for each $t\geq 2$ and for each $h^{t-1}=(y^\tau)_{\tau=1}^{t-1}\in H^{t-1}$, given a $v|_{h^{t-1}}\in W$, let $\alpha|_{h^{t-1}}$ and $w|_{h^{t-1}}:Y\to W$ be such that $(\alpha|_{h^{t-1}},v|_{h^{t-1}})$ is ex post enforced by $w|_{h^{t-1}}$. Then set the action profile after history h^{t-1} to be $s|_{h^{t-1}}=\alpha|_{h^{t-1}}$ and for each $y^t\in Y$, set $v|_{h^t=(h^{t-1},y^t)}=w|_{h^{t-1}}(y^t)\in W$.

Because W is bounded and $\delta\in(0,1)$, payoffs are continuous at infinity, so finite approximations show that the specified strategy profile $s\in S$ generates v as an average payoff, and its continuation strategy $s|_{h^t}$ yields $v|_{h^t}$ for each $h^t\in H^t$. Also, by construction, nobody wants to deviate at any moment of time, given any state $\omega\in\Omega$. Because payoffs are continuous at infinity, the one-shot deviation principle applies, and we conclude that s is a PPXE, as desired. *Q.E.D.*

S.2. PROOF OF THEOREM 2

THEOREM 2: *If a subset W of $\mathbf{R}^{I\times|\Omega|}$ is compact, convex, and locally ex post generating, then there is $\overline{\delta}\in(0,1)$ such that $W\subseteq E(\delta)$ for all $\delta\in(\overline{\delta},1)$.*

PROOF: Suppose that W is locally ex post generating. Since $\{U_v\}_{v\in W}$ is an open cover of the compact set W, there is a subcover $\{U_{v^m}\}_m$ of W. Let $\overline{\delta}=\max_m\delta_{v^m}$. Choose $u\in W$ arbitrarily and let U_{v^m} be such that $u\in U_{v^m}$. Since $W\cap U_{v^m}\subseteq B(\delta_{v^m},W)$, there exist α_u and $w_u:Y\to W$ such that (α_u,u) is ex post enforced by w_u for δ_{v^m}. Given a $\delta\in(\overline{\delta},1)$, let

$$w(y)=\frac{\delta-\delta_u}{\delta(1-\delta_u)}u+\frac{\delta_u(1-\delta)}{\delta(1-\delta_u)}w_u(y)$$

for all $y\in Y$. Then it is straightforward that (α_u,u) is enforced by $(w(y))_{y\in Y}$ for δ. Also, $w(y)\in W$ for all $y\in Y$, since u and $w(y)$ are in W and W is convex. Therefore, $u\in B(\delta,W)$, meaning that $W\subseteq B(\delta,W)$ for all $\delta\in(\overline{\delta},1)$. (Recall

 DOI: 10.3982/ECTA8565

that u and δ are arbitrarily chosen from W and $(\overline{\delta}, 1)$.) Then, from Theorem 1, $W \subseteq E(\delta)$ for $\delta \in (\overline{\delta}, 1)$, as desired. *Q.E.D.*

S.3. PROOF OF LEMMA 2

LEMMA 2: *For every $\delta \in (0,1)$, $E(\delta) \subseteq E^*(\delta) \subseteq Q$, where $E^*(\delta)$ is the convex hull of $E(\delta)$.*

PROOF: It is obvious that $E(\delta) \subseteq E^*(\delta)$. Suppose $E^*(\delta) \not\subseteq Q$. Then, since the score is a linear function, there is $v \in E(\delta)$ and λ such that $\lambda \cdot v > k^*(\lambda)$. In particular, since $E(\delta)$ is compact, there exist $v^* \in E(\delta)$ and λ such that $\lambda \cdot v^* > k^*(\lambda)$ and $\lambda \cdot v^* \geq \lambda \cdot \tilde{v}$ for all $\tilde{v} \in E^*(\delta)$. By definition, v^* is enforced by $(w(y))_{y \in Y}$ such that $w(y) \in E(\delta) \subseteq E^*(\delta) \subseteq H(\lambda, \lambda \cdot v^*)$ for all $y \in Y$. But this implies that $k^*(\lambda)$ is not the maximum score for direction λ, a contradiction. *Q.E.D.*

S.4. PROOF OF LEMMA 3

LEMMA 3: *For any smooth set W in the interior of Q, there is $\overline{\delta} \in (0,1)$ such that $W \subseteq E(\delta)$ for $\delta \in (\overline{\delta}, 1)$.*

PROOF: Since W is bounded, it suffices to show that it is also locally ex post generating, that is, for each $v \in W$, there exist $\delta_v \in (0,1)$ and an open neighborhood U_v of v such that $W \cap U_v \subseteq B(\delta_v, W)$.

First, consider $v \in \operatorname{bd} W$. Let λ be normal to W at v and let $k = \lambda \cdot v$. Since $W \subset Q \subseteq H^*(\lambda)$, there exist α, $\tilde{v}$, and $(\tilde{w}(y))_{y \in Y}$ such that $\lambda \cdot \tilde{v} > \lambda \cdot v = k$, $(\alpha, \tilde{v})$ is enforced using continuation payoffs $(\tilde{w}(y))_{y \in Y}$ for some $\tilde{\delta} \in (0,1)$, and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$. For each $\delta \in (\tilde{\delta}, 1)$ and $y \in Y$, let

$$w(y, \delta) = \frac{\delta - \tilde{\delta}}{\delta(1-\tilde{\delta})} v + \frac{\tilde{\delta}(1-\delta)}{\delta(1-\tilde{\delta})}\left(\tilde{w}(y) + \frac{v - \tilde{v}}{\tilde{\delta}}\right).$$

By construction, (α, v) is enforced by $(w(y,\delta))_{y \in Y}$ for δ, and there is $\kappa > 0$ such that $|w(y,\delta) - v| < \kappa(1-\delta)$. Also, since $\lambda \cdot \tilde{v} > \lambda \cdot v = k$ and $\tilde{w}(y) \in H(\lambda, \lambda \cdot \tilde{v})$ for all $y \in Y$, there is $\varepsilon > 0$ such that $\tilde{w}(y) - \frac{v - \tilde{v}}{\tilde{\delta}}$ is in $H(\lambda, k - \varepsilon)$ for all $y \in Y$, thereby

$$w(y, \delta) \in H\left(\lambda, k - \frac{\tilde{\delta}(1-\delta)}{\delta(1-\tilde{\delta})}\varepsilon\right)$$

for all $y \in Y$. Then, as in the proof of FL's Theorem 3.1, it follows from the smoothness of W that $w(y, \delta) \in \operatorname{int} W$ for sufficiently large δ, that is, (α, v) is enforced with respect to $\operatorname{int} W$. To enforce u in the neighborhood of v, use α and a translate of $(w(y,\delta))_{y \in Y}$.

Next, consider $v \in \operatorname{int} W$. Choose λ arbitrarily, and let α and $(w(y,\delta))_{y\in Y}$ be as in the above argument. By construction, (α, v) is enforced by $(w(y,\delta))_{y\in Y}$. Also, $w(y,\delta) \in \operatorname{int} W$ for sufficiently large δ, since $|w(y,\delta) - v| < \kappa(1-\delta)$ for some $\kappa > 0$ and $v \in \operatorname{int} W$. Thus, (α, v) is enforced with respect to $\operatorname{int} W$ when δ is close to 1. To enforce u in the neighborhood of v, use α and a translate of $(w(y,\delta))_{y\in Y}$, as before. *Q.E.D.*

S.5. ALTERNATE PROOF OF LEMMA 6

LEMMA 6: *Suppose that a profile α has statewise full rank for (i,ω) and $(j,\tilde{\omega})$ satisfying $\omega \neq \tilde{\omega}$, and that α has individual full rank for all players and states. Then $k^*(\alpha,\lambda) = \infty$ for direction λ such that $\lambda_i^\omega \neq 0$ and $\lambda_j^{\tilde{\omega}} \neq 0$.*

PROOF: Let (i,ω) and $(j,\tilde{\omega})$ be such that $\lambda_i^\omega \neq 0$, $\lambda_j^{\tilde{\omega}} \neq 0$, and $\tilde{\omega} \neq \omega$. Let α be a profile that has statewise full rank for all (i,ω) and $(j,\tilde{\omega})$ satisfying $\omega \neq \tilde{\omega}$.

First, we claim that for every $K > 0$, there exist $z_i^\omega = (z_i^\omega(y))_{y\in Y}$ and $z_j^{\tilde{\omega}} = (z_j^{\tilde{\omega}}(y))_{y\in Y}$ such that

$$\text{(S1)} \qquad \pi^\omega(a_i, \alpha_{-i}) \cdot z_i^\omega = \frac{K}{\delta \lambda_i^\omega}$$

for all $a_i \in A_i$,

$$\text{(S2)} \qquad \pi^{\tilde{\omega}}(a_j, \alpha_{-j}) \cdot z_j^{\tilde{\omega}} = 0$$

for all $a_j \in A_j$, and

$$\text{(S3)} \qquad \lambda_i^\omega z_i^\omega(y) + \lambda_j^{\tilde{\omega}} z_j^{\tilde{\omega}}(y) = 0$$

for all $y \in Y$. To prove that this system of equations indeed has a solution, eliminate (S3) by solving for $z_j^{\tilde{\omega}}(y)$. Then there remain $|A_i| + |A_j|$ linear equations, and its coefficient matrix is $\Pi_{(i,\omega)(j,\tilde{\omega})}(\alpha)$. Since statewise full rank implies that this coefficient matrix has rank $|A_i| + |A_j|$, we can solve the system.

Next, for each $(l, \overline{\omega}) \in \mathbf{I} \times \Omega$, we choose $(\tilde{w}_l^{\overline{\omega}}(y))_{y\in Y}$ so that

$$\text{(S4)} \qquad (1-\delta) g_l^{\overline{\omega}}(a_l, \alpha_{-l}) + \delta \pi^{\overline{\omega}}(a_l, \alpha_{-l}) \cdot \tilde{w}_l^{\overline{\omega}} = 0$$

for all $a_l \in A_l$. Note that this system has a solution, since α has individual full rank. Intuitively, continuation payoffs $\tilde{w}^{\overline{\omega}}$ are chosen so that players are indifferent over all actions and their payoffs are zero.

Let $K > \max_{y\in Y} \lambda \cdot \tilde{w}(y)$, and choose $(z_i^\omega(y))_{y\in Y}$ and $(z_j^{\tilde{\omega}}(y))_{y\in Y}$ to satisfy (S1)–(S3). Then let

$$w_l^{\overline{\omega}}(y) = \begin{cases} \tilde{w}_i^\omega(y) + z_i^\omega(y), & \text{if } (l,\overline{\omega}) = (i,\omega), \\ \tilde{w}_j^{\tilde{\omega}}(y) + z_j^{\tilde{\omega}}(y), & \text{if } (l,\overline{\omega}) = (j,\tilde{\omega}), \\ \tilde{w}_l^{\overline{\omega}}(y), & \text{otherwise} \end{cases}$$

for each $y \in Y$. Also, let

$$v_l^{\overline{\omega}} = \begin{cases} \dfrac{K}{\lambda_i^\omega}, & \text{if } (l, \overline{\omega}) = (i, \omega), \\ 0, & \text{otherwise.} \end{cases}$$

We claim that this (v, w) satisfies constraints (i) through (iii) in LP Average. It follows from (S4) that constraints (i) and (ii) are satisfied for all $(l, \overline{\omega}) \in (\mathbf{I} \times \Omega) \setminus \{(i, \omega), (j, \tilde{\omega})\}$. Also, using (S1) and (S4), we obtain

$$\begin{aligned} &(1-\delta) g_i^\omega(a_i, \alpha_{-i}) + \delta \pi^\omega(a_i, \alpha_{-i}) \cdot w_i^\omega \\ &\quad = (1-\delta) g_i^\omega(a_i, \alpha_{-i}) + \delta \pi^\omega(a_i, \alpha_{-i}) \cdot (\tilde{w}_i^\omega + z_i^\omega) \\ &\quad = \frac{K}{\lambda_i^\omega} \end{aligned}$$

for all $a_i \in A_i$. This shows that (v, w) satisfies constraints (i) and (ii) for (i, ω). Likewise, from (S2) and (S4), (v, w) satisfies constraints (i) and (ii) for $(j, \tilde{\omega})$. Furthermore, using (S3) and $K > \max_{y \in Y} \lambda \cdot \tilde{w}(y)$,

$$\begin{aligned} \lambda \cdot w(y) &= \lambda \cdot \tilde{w}(y) + \lambda_i^\omega z_i^\omega(y) + \lambda_j^{\tilde{\omega}} z_j^{\tilde{\omega}}(y) \\ &= \lambda \cdot \tilde{w}(y) < K = \lambda \cdot v \end{aligned}$$

for all $y \in Y$, and hence constraint (iii) holds.

Therefore, $k^*(\alpha, \lambda) \geq \lambda \cdot v = K$. Since K can be arbitrarily large, we conclude $k^*(\alpha, \lambda) = \infty$. *Q.E.D.*

Dept. of Economics, Harvard University, Cambridge, MA 02138, U.S.A.; dfudenberg@harvard.edu

and

Dept. of Economics, Harvard University, Cambridge, MA 02138, U.S.A.; yamamot@fas.harvard.edu.

Manuscript received May, 2009; final revision received May, 2010.