SUPPLEMENT TO "ENDOGENOUS COMPLETENESS OF DIFFUSION DRIVEN EQUILIBRIUM MARKETS"

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In this document, we give the proofs that were omitted from the main text and provide details for some of the arguments used in the proofs of our main results. We also discuss some additional results concerning the assumption of real analyticity in the space variables and economies with terminal dividends, heterogenous discount rates, and time-dependent aggregate consumption.

CONTENTS

A. RESULTS WITHOUT REAL ANALYTICITY IN THE SPACE VARIABLES

IN THIS SECTION, we show that analyticity in the space variables is not necessary to obtain the existence of an equilibrium with complete markets and we provide the counterpart of our results in the absence of this property. To work in this relaxed setting, we replace Assumptions A and B with the following assumption:

ASSUMPTION A.1:
(a) $n = d$ and $\operatorname{rank}(\sigma_X(t, x)) = d$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.
(b) *The functions* μ_X *and* σ_X *are jointly* C^5 *in* $(t, x) \in [0, T] \times \mathbb{R}^n$ *and are time-independent if the economy has an infinite horizon. Furthermore, all their space derivatives are real analytic in* $t \in [0, T]$.
(c) *The vector of state variables takes values in* $\mathcal{X} \subseteq \mathbb{R}^d$ *and admits a transition density* $p(t, x, \tau, y)$ *that is jointly* C^7 *in* (t, x, τ, y) *for all* $(x, y) \in \mathcal{X}^2$ *and* $t < \tau \in [0, T]$ *for* $t < \tau$.
(d) *There are locally bounded functions* (K, L), *a metric* d *that is locally equivalent to the Euclidean metric, and constants* $\varepsilon, \alpha, \phi > 0$ *such that* $p(t, x, \tau, y)$ *is analytic with respect to* $t \neq \tau$ *in the set*

$$\mathcal{P}_\varepsilon^2 \equiv \left\{ (t, \tau) \in \mathbb{C}^2 : \Re \tau \leq T \text{ and } |\Im(\tau - t)| \leq \varepsilon \Re(\tau - t) \right\}$$

and satisfies

$$|p(t, x, \tau, y)| \leq K(x)L(y)|\tau - t|^{-\alpha} e^{\phi|\tau-t|-d(x,y)^2/|\tau-t|} \equiv \overline{p}(t, x, \tau, y)$$

for all $(t, \tau, x, y) \in \mathcal{P}_\varepsilon^2 \times \mathcal{X}^2$.

ASSUMPTION A.2: *The function* $u_a \in C^5(0, \infty)$ *is increasing, is strictly concave, and satisfies the Inada conditions* $u_a'(0) = \infty$, $u_a'(\infty) = 0$. *The dividend rates* g_i *and individual endowments* ℓ_a *belong to* $C^4(\mathcal{X})$.

PROPOSITION A.1: *The candidate price function* $S(t, x)$ *and its gradient* $\frac{\partial}{\partial x} S(t, x)$ *are real analytic in* $t \in (0, T)$ *and* $S(t, x) \in C^3((0, T] \times \mathcal{X})$.

PROOF: The proof is entirely analogous to that of Proposition 2. The only new claim is the real analyticity of $\frac{\partial}{\partial x} S(t, x)$. Using the fact that the candidate price function solves

$$\frac{\partial S(t, x)}{\partial t} = -\mathcal{A}(S(t, x)) - g(x)$$

in conjunction with the Cauchy formula, we obtain that $\mathcal{A}(S(t, x))$ is real analytic in $t \in (0, T)$ and the real analyticity in $t \in (0, T)$ of the derivative $\frac{\partial}{\partial x} S(t, x)$ now follows by standard ellipticity arguments; see Shimakura (1992). *Q.E.D.*

The following theorems constitute the direct counterpart to Theorem 1 under the weaker assumptions of this section.

THEOREM A.1: *If* $\det(\sigma_g(T, x)) \neq 0$ *for almost every* $x \in \mathcal{X}$, *then there exists an equilibrium with dynamically complete markets*.

PROOF: It follows from Proposition A.1 and the same approximation argument as in the proof of Theorem 1 that $\det(\sigma_S(t, x))$ is real analytic in t and almost everywhere nonzero for t sufficiently close to T. The claim follows now from Lemma A.1 below. *Q.E.D.*

Similarly, Theorems 2 and 3 have direct analogs whose proofs follow, by the same argument as Theorem A.1, from Lemma A.1 below.

THEOREM A.2: *Assume that the utility functions are real analytic and that the relative risk aversion of all agents is bounded between* γ_1 *and* γ_2 *for some* $0 < \gamma_1 \leq \gamma_2$. *If* $\det(B_{a,1}(x)) + \cdots + \det(B_{a,d}(x)) \neq 0$ *for some* a *and almost every* $x \in \mathcal{X}$, *then an equilibrium with dynamically complete markets exists for all matrix* η *of initial endowments outside of a closed set of measure zero*.

The proof is completely analogous to that of Theorem 2. We omit the details.

THEOREM A.3: *In the infinite horizon case, if the relative risk aversion of all agents is bounded between γ_1 and γ_2 for some $0 < \gamma_1 \leq \gamma_2$, and either*

(a) $\det(\sigma_g(x)) \neq 0$ *for almost every $x \in \mathcal{X}$ or*

(b) *the utility functions are real analytic and* $\det(B_{a,1}(x)) + \cdots + \det(B_{a,d}(x)) \neq 0$ *for some a and almost every $x \in \mathcal{X}$,*

then an equilibrium with dynamically complete markets exists for all η and $\rho > R$ outside of a closed set of measure zero.

Note that the only difference between these results and Theorems 1–3 is that the condition is required here to hold for almost every x, whereas in Theorems 1–3, we require it to hold in a single point. The reason is that in Theorems 1–3, all functions are real analytic in x and therefore are automatically almost everywhere nonzero if they are nonzero at a single point.

REMARK A.1: The reason we have to assume that the utility functions are real analytic in Theorems A.2 and assertion (b) of Theorem A.3 is that we need the consumption price function $m(t, x, \lambda)$ to be real analytic in λ so as to apply the result of Lemma A.1 below.

LEMMA A.1: *Assume that $F: \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}$ is continuous and real analytic in $t \in \mathbb{R}_+$ and that $F(\cdot, x) \not\equiv 0$ for Lebesque almost every $x \in \mathcal{X}$. Then there exists a countable set $\mathcal{O} \subset \mathbb{R}_+$ such that*

$$F(t, x) \neq 0$$

for each fixed $t \in \mathbb{R}_+ \setminus \mathcal{O}$ and almost every $x \in \mathcal{X}$.

PROOF: Suppose to the contrary that there exists a bounded uncountable[1] set $\mathcal{O}$ and a family $\{A_t : t \in \mathcal{O}\}$ of Lebesque-measurable subsets of $\mathcal{X}$ of positive measure such that

$$\{(t, x) : x \in A_t\} \subseteq \mathcal{Z}_F \equiv \{(t, x) \in \mathbb{R}_+ \times \mathcal{X} : F(t, x) = 0\}$$

for each fixed $t \in \mathcal{O}$. By application of Lemma A.2 below, this implies that there exists a countably infinite sequence $\{t_k\}_{k=1}^\infty \subseteq \mathcal{O}$ such that the set $A_\infty = \bigcap_{k=1}^\infty A_{t_k}$ has strictly positive Lebesgue measure. By construction, $F(t_k, x) = 0$ for all k and all $x \in A_\infty$. By the uniqueness theorem for analytic functions of one variable, $F(t, x) \equiv 0$ for any $x \in A_\infty$, which is a contradiction.   Q.E.D.

LEMMA A.2: *Let $\Sigma \subseteq \mathbb{R}$ be an uncountable set and let $\{A_\sigma : \sigma \in \Sigma\}$ denote a family of Lebesque-measurable subsets of $\mathcal{X}$ such that A_σ has strictly positive Lebesgue measure for every $\sigma \in \Sigma$. Then there exists a countably infinite set $\Phi \subseteq \Sigma$ such that $A^* \equiv \bigcap_{\sigma \in \Phi} A_\sigma$ has strictly positive Lebesgue measure.*

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[1]If $\mathcal{O}$ is unbounded and uncountable, then there exists an $n \in \mathbb{N}$ such that $\mathcal{O} \cap [0, n]$ is uncountable (since otherwise $\mathcal{O}$ itself would be countable) and we may simply replace $\mathcal{O}$ with $\mathcal{O} \cap [0, n]$.

PROOF: Let ν denote Lebesgue measure on $\mathbb{R}^n$. Since $\{1_{A_\sigma} : \sigma \in \Sigma\}$ is a non-empty subset of $L^1(\mathcal{X}, \nu)$, it follows from well known results in functional analysis that there exists a nontrivial subset A^c of $\mathcal{X}$ and a sequence $\{A_{\sigma_k}\}_{k=1}^\infty$ of elements of $\{A_\sigma : \sigma \in \Sigma\}$ such that

$$\lim_{k \to \infty} \int_{\mathcal{X}} \left| 1_{A_{\sigma_k}}(x) - 1_{A^c}(x) \right| dx = \lim_{k \to \infty} \nu\left(A_{\sigma_k} \triangle A^c\right) = 0,$$

where $\triangle$ denotes the set theoretic symmetric difference. Let now $\Phi = \{\phi_n\}_{n=1}^\infty$ be a subsequence of $\{\sigma_k\}_{k=1}^\infty$ such that

$$\nu\left(A_{\phi_n} \triangle A^c\right) \leq 2^{-(n+1)} \nu(A^c)$$

for all $n \geq 1$ and set $A^* \equiv \bigcap_{\sigma \in \Phi} A_\sigma = \bigcap_{n=1}^\infty A_{\phi_n}$. With this notation

$$\nu(A^*) \geq \nu(A^* \cap A^c) \geq \nu(A^c) - \nu(A^* \triangle A^c)$$

$$\geq \nu(A^c) - \sum_{n=1}^\infty \nu(A_{\phi_n} \triangle A^c)$$

$$\geq \nu(A^c) - \sum_{n=1}^\infty 2^{-(n+1)} \nu(A^c) = \frac{\nu(A^c)}{2}$$

and the desired result now follows from the fact that the set A^c has strictly positive measure by construction.   *Q.E.D.*

B. A GENERAL MODEL WITH TERMINAL DIVIDENDS

To establish a direct connection with Anderson and Raimondo (2008), we show in this section how our results can be modified to include terminal dividends, heterogenous discount rates, and time-dependent aggregate consumption.

Consider a finite horizon economy as in the paper and assume that instead of paying only intermediate dividends, stock i is represents a claim to a cumulative dividend process of the form[2]

$$D_{it} \equiv \int_0^{t \wedge T} g_i(\tau, X_\tau) \, d\tau + 1_{\{t \geq T\}} G_i(X_T),$$

where $g_i \geq 0$ is a real analytic function that represents a flow rate of dividends and the function $G_i \geq 0$ represents a terminal lump dividend. As in Anderson and Raimondo (2008), we assume that the function G_i is continuous and that

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[2]If the economy has an infinite horizon, then we naturally assume that stocks do not pay terminal dividends and we set $G_i \equiv 0$ for all i.

there exists a open set $\mathcal{V} \subseteq \mathcal{X}$ such that $G_i \in C^1(\mathcal{V})$ for every i. Furthermore, we assume that the state variables satisfy Assumptions A and D.

Since the stocks now pay lump-sum dividends at the terminal time, we need to assume that agents derive utility from terminal wealth as well as from intermediate consumption. Specifically, we assume that the preferences of agent a over lifetime consumption plans are represented by an expected utility index of the form[3]

$$U_a(c, C) \equiv E_0\left[\int_0^T e^{-\rho_a \tau} u_a(c_\tau)\, d\tau + e^{-\rho_a T} v_a(C)\right].$$

In the above equation, the constant $\rho_a \geq 0$ is the agent's subjective rate of time preferences and (u_a, v_a) is a pair of utility functions that satisfy Assumption B. As in the paper, agent a is endowed with $\eta_{ai} \in [0, 1]$ units of stock i and receives income at rate $\ell_a(t, X_t) \geq 0$ for some real analytic function ℓ_a.

The following result provides a characterization of the equilibrium consumption price process as the marginal utility of a representative agent and constitutes the counterpart of Proposition 1 for the present model.

PROPOSITION B.1: *Assume that*

$$(\text{B.1}) \qquad E\Bigg[\int_0^T e^{-\rho_a t} u_a'(\overline{g}(t, X_t)/A)\overline{g}(t, X_t)\, dt$$

$$+ e^{-\rho_a T} v_a'(\overline{G}(X_T)/A)\overline{G}(X_T)\Bigg] < \infty$$

for each $a \leq A$, where $\overline{G} = G^\top \mathbf{1}_A$ denotes the aggregate terminal dividend. Then the set of Arrow–Debreu equilibria is nonempty and in any such equilibrium,

$$m_t = m(t, X_t) \equiv 1_{\{t<T\}} \frac{\partial u}{\partial c}(t, \lambda, \overline{g}(t, X_t)) + 1_{\{t=T\}} \frac{\partial v}{\partial C}(\lambda, \overline{G}(X_T))$$

for some vector $\lambda \in \mathcal{S}_+$ of Pareto weights, where

$$v(\lambda, C) \equiv \max_{\alpha \in \mathcal{S}_+} \sum_{a=1}^A \lambda_a e^{-\rho_a T} v_a(\alpha_a C),$$

$$u(t, \lambda, c) \equiv \max_{\alpha \in \mathcal{S}_+} \sum_{a=1}^A \lambda_a e^{-\rho_a t} u_a(\alpha_a c),$$

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[3]If the economy has an infinite horizon or if there are no terminal dividends, then we naturally assume that agents have no utility for terminal consumption and set $v_a \equiv 0$ for all a.

and $\mathcal{S}$ denotes the unit simplex. In particular, the equilibrium consumption price function m is real analytic in $(t, x, \lambda) \in (0, T) \times \mathcal{X} \times \mathcal{S}_+$.[4]

PROOF: The fact that condition (B.1) is sufficient to guarantee the existence of an Arrow–Debreu equilibrium follows from Proposition C.1 below. The second part of the statement follows from arguments similar to those used in the proof of the second part of Proposition 1.   *Q.E.D.*

As shown by the above result, the equilibrium consumption price is continuous on $[0, T)$ but has a predictable jump

$$\Delta(X_T) \equiv \frac{\partial v}{\partial c}(\lambda, \overline{G}(X_T)) - \frac{\partial u}{\partial c}(T, \lambda, \overline{g}(T, X_T))$$

at the terminal time. In the present context, this somehow unnatural jump reflects the potential misalignment between the agents' preferences for intermediate and terminal consumption on the one hand, and between intermediate and terminal dividends on the other.[5] It can be avoided by assuming that either dividends and utilities are aligned in the sense that $\overline{g}(T, x) = \overline{G}(x)$ and $u_a(c) = v_a(c)$ for each a or that the stocks do not pay lump-sum dividends at the terminal date.

Consider now a fixed Arrow–Debreu equilibrium and denote by $m(t, X_t)$ the corresponding consumption price. Using arguments similar to those of the main text, we have that the candidate prices are given by $B_t = \exp(A_t)$ and

$$S_t = 1_{\{t < T\}} P(t, X_t) + 1_{\{t = T\}} G(X_T),$$

where

$$A_t = -\int_0^{t \wedge T} \frac{\mathcal{D}(m(\tau, X_\tau))}{m(\tau, X_\tau)} d\tau + 1_{\{t \geq T\}} \log\left(1 - \frac{\Delta(X_T)}{m(T, X_T)}\right)$$

and P is the function defined by

$$(B.2) \qquad P(t, X_t) \equiv E_t\left[\int_t^T \frac{m(\tau, X_\tau)}{m(t-, X_t)} g(\tau, X_\tau) d\tau + \frac{m(T, X_T)}{m(t-, X_t)} G(X_T)\right].$$

As can be seen from these definition, the candidate price processes are continuous on $[0, T)$ and have a jump at the terminal time which they inherit from

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[4]If the traded security do not pay terminal dividends, the statement can be strengthened to show that m is real analytic in $(t, x, \lambda) \in (0, T] \times \mathcal{X} \times \mathcal{S}_+$.

[5]Other examples of diffusion driven equilibrium models where prices have singular components include the case of finite marginal utility at zero studied by Karatzas, Lehoczky, and Shreve (1991), the liquidity constraints model of Detemple and Serrat (2003), and the portfolio insurance model of Basak (1995), among others.

the discontinuity in the equilibrium consumption price. However, and as required to guarantee that prices are arbitrage-free, the relative jumps on all traded securities are equal:

$$\lim_{t \to T} \frac{B_T}{B_t} = \lim_{t \to T} \frac{S_{iT}}{S_{it}} = Q(X_T) \equiv 1 - \frac{\Delta(X_T)}{m(T, X_T)}.$$

If this were not the case, then trivial arbitrages could be implemented by buying a security with a larger relative jump and short-selling the same amount of another with a smaller relative jump just prior to the terminal time.

The following result shows that market completeness obtains provided that the volatility $\sigma_P(t, x) = \frac{\partial P}{\partial x}(t, x)\sigma_X(t, x)$ of the candidate pre-horizon price function is invertible at one point of the state space and constitutes the counterpart of Propositions 2 and 3 for the model of this section.

PROPOSITION B.2: *Assume that the conditions of Proposition* B.1 *are satisfied and that either Assumptions* A *and* C *or Assumptions* A *and* D *hold true. Then the function P belongs to* $C^\infty((0, T] \times X)$ *and is real analytic in* $(t, x) \in (0, T) \times \mathcal{X}$. *As a result, if* $\det(\sigma_S(t, x)) \neq 0$ *for some point* $(t, x) \in (0, T) \times \mathcal{X}$, *then there exists an equilibrium with dynamically complete markets.*

PROOF: The first part of the statement follows from arguments similar to those of the proof of Theorem 4 below. As in the main text, the second part follows from the real analyticity of $\det(\sigma_S(t, x))$ in $(t, x) \in (0, T) \times \mathcal{X}$ and the fact that a real analytic function is either identically zero or almost everywhere different from zero.   *Q.E.D.*

As we now show, the above result can be used to derive a simple sufficient condition for market completeness in finite horizon economies with terminal dividends. Using the definition of the pre-horizon price function in equation (B.2), we obtain

$$\lim_{t \to T} P(t, x) = \frac{G(x)}{Q(x)} \equiv \mathcal{G}(x),$$

and appealing to Proposition B.2 for the required smoothness gives

$$\frac{\partial P}{\partial x}(t, x) = \frac{\partial \mathcal{G}}{\partial x}(x) + o(1)$$

for all $x \in \mathcal{V}$. This equation shows that close to the terminal date, the volatility of the candidate prices can be approximated by that of the vector $\mathcal{G}(X_t)$ of effective terminal dividends. In particular,

$$\det(\sigma_P(t, x)) = \det(\sigma_{\mathcal{G}}(T, x)) + o(1)$$

for all $x \in \mathcal{V}$, where $\sigma_{\mathcal{G}}(t, x) \equiv \mathcal{G}'(x)\sigma_X(t, x)$ denotes the volatility of the effective terminal dividends, and combining this with the second part of Proposition B.2 delivers the following theorem.

THEOREM B.1: *If* $\det(\sigma_{\mathcal{G}}(T, x)) \neq 0$ *for some* $x \in \mathcal{V}$, *then there exists an equilibrium with dynamically complete markets*.

Theorem B.1 shows that an equilibrium with dynamically complete markets exists as soon as the volatility of effective terminal dividends is nondegenerate at one point of the state space. If one assumes, in addition, that there is no jump in the equilibrium consumption price at the terminal time, then this condition simplifies further and only requires that the volatility $\sigma_G(t, x) \equiv G'(x)\sigma_X(t, x)$ of the terminal dividends be nondegenerate. In that case, our condition is similar to that of Anderson and Raimondo (2008), albeit with a slightly different market structure.[6]

C. EXISTENCE OF ARROW–DEBREU EQUILIBRIA

In this section, we establish an existence result that covers both finite and infinite horizon economies in the setting of Section B above, that is, with terminal dividends, heterogenous discount rates, and time-dependent aggregate consumption.

PROPOSITION C.1: *Assume that*

$$E\left[\int_0^T e^{-\rho_a t} u_a'(\overline{g}(t, X_t)/A)\overline{g}(t, X_t)\,dt + e^{-\rho_a T} v_a'(\overline{G}(X_T)/A)\overline{G}(X_T)\right]$$
$$< \infty$$

for every $a \leq A$. *Then an Arrow–Debreu equilibrium exists*.

PROOF: To facilitate the exposition, we assume throughout that $v_a = G_i \equiv 0$ for all a, i, but our arguments can be extended in a straightforward way to include terminal dividends. Consider the excess utility map $e : \mathbb{R}_{++}^A \to \mathbb{R}^A$ defined by

$$e_a(\lambda) = E\int_0^T \frac{m(t, X_t, \lambda)}{\lambda_a}$$
$$\times \left[I_a\left(\frac{e^{\rho_a t} m(t, X_t, \lambda)}{\lambda_a}\right) - \ell_a(t, X_t) - \eta_a^\top g(t, X_t)\right]dt,$$

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[6]Specifically, Anderson and Raimondo (2008) assumed that in place of instantaneously risk-free bonds, the menu of traded securities includes a zero net supply security with terminal payoff $G_0(x) > 0$ for all $x \in \mathcal{V}$ and price P_0. In that case, a straightforward modification of our arguments shows that Theorem B.1 remains valid provided that we replace P with P/P_0 and $\mathcal{G}$ with G/G_0.

where I_a denotes the inverse marginal utility of agent a. By Lemma C.1 below, we have that e has all the properties of a finite-dimensional excess demand function. Consequently, there exists some $\lambda^* \in \mathcal{S}_+$ such that $e(\lambda^*) = 0$, and it now follows from standard arguments that the consumption price process $m(t, X_t, \lambda^*)$ and the allocation associated with λ^* constitute an Arrow–Debreu equilibrium.   Q.E.D.

LEMMA C.1: *The excess utility map satisfies the following statements*:
  (a) *e is homogeneous of degree zero*.
  (b) $\sum_{a=1}^{A} \lambda_a e_a = 0$.
  (c) *e is continuous in $\mathbb{R}_{++}^A$*.
  (d) *e_a is bounded from above for all a on $\mathcal{S}_+$ and $e_a \to -\infty$ as $\lambda_a \to 0$*.

PROOF: The first two properties are straightforward. To establish the remaining two, observe that since the function $m(t, x, \lambda)$ solves the goods market clearing condition

$$(C.1) \qquad \overline{g}(t, x) = \sum_{a=1}^{A} I_a(e^{\rho_a t} m(t, X_t, \lambda)/\lambda_a),$$

we have that it is continuous in λ and satisfies

$$(C.2) \qquad e^{-\rho_j t} \lambda_j u_j'(\overline{g}(t, x)) < m(t, x, \lambda) < \sum_{a=1}^{A} e^{-\rho_a t} \lambda_a u_a'(\overline{g}(t, x)/A)$$

for all j. Indeed, using equation (C.1) and the fact that the functions I_j are nonnegative gives

$$\overline{g}(t, x) \geq I_j(e^{\rho_j t} m(t, X_t, \lambda)/\lambda_j),$$

and the lower bound of equation (C.2) now follows from the decrease of I_j. Similarly, the definition of the functions I_a implies that

$$\overline{g}(t, x) = \sum_{a=1}^{A} I_a(u_a'(\overline{g}(t, x)/A))$$

$$> \sum_{a=1}^{A} I_a\left(\frac{e^{\rho_a t}}{\lambda_a} \sum_{j=1}^{A} e^{-\rho_j t} \lambda_j u_j'(\overline{g}(t, x)/A) \right)$$

and the upper bound of equation (C.2) follows from the decrease of I_j. Using this upper bound in conjunction with the definition of the aggregate consump-

tion and the fact that $\eta_{ai} \in [0, 1]$, we deduce that

$$
(C.3) \qquad m(t, X_t, \lambda)(\ell_a(t, X_t) + \eta_a^\top g(t, X_t))
$$
$$
\leq m(t, X_t, \lambda)\overline{g}(t, X_t)
$$
$$
< \sum_{a=1}^{A} e^{-\rho_a t} \lambda_a u_a'(\overline{g}(t, x)/A)\overline{g}(t, X_t),
$$

and since

$$
(C.4) \qquad \sum_{a=1}^{A} E\left[\int_0^T e^{-\rho_a t} u_a'(\overline{g}(t, X_t)/A)\overline{g}(t, X_t)\,dt \right] < \infty
$$

by assumption, it follows from the dominated convergence theorem that

$$
f_a(\lambda) \equiv E \int_0^T m(t, X_t, \lambda)(\ell_a(t, X_t) + \eta_a^\top g(t, X_t))\,dt
$$

is continuous in λ. Similarly, since

$$
m(t, X_t, \lambda)I_j(e^{\rho_j t}m(t, X_t, \lambda)/\lambda_j) \leq m(t, X_t, \lambda)\overline{g}(t, X_t)
$$

by application of equation (C.2) it follows from (C.3), (C.4), and the dominated convergence theorem that

$$
h_a(\lambda) \equiv E \int_0^T m(t, X_t, \lambda)I_a(e^{\rho_a t}m(t, X_t, \lambda)/\lambda_a)\,dt
$$

is continuous in λ, and (c) follows by noting that $e_a(\lambda) = (1/\lambda_a)(f_a(\lambda) + h_a(\lambda))$. To show that $e_a \to -\infty$ as $\lambda_a \to 0$, observe that since m is finite and the utility functions satisfy the Inada conditions, we have

$$
(C.5) \qquad \lim_{\lambda_a \to 0} I_a(e^{\rho_a t}m(t, X_t, \lambda_a)) = 0.
$$

Combining this with (C.3), (C.4), and the dominated convergence theorem shows that $\lim_{\lambda_a \to 0} f_a(\lambda) = 0$. As a result, it now suffices to show that h_a is bounded from below on $\mathcal{S}_+$, but this follows from (C.4), the lower bound of equation (C.2), and the fact that since $\lambda \in \mathcal{S}_+$, some of its coordinates must be nonzero. The proof of the fact that e_a is bounded from above on $\mathcal{S}_+$ follows from similar arguments and therefore is omitted.   *Q.E.D.*

D. COUNTEREXAMPLES

In this section, we present counterexamples that show that the requirement that the candidate prices be real analytic in time cannot be relaxed if one wants

to deduce market completeness from the primitives of the economy. To highlight the intuition behind our construction, we start with an explicit example of a single stock economy before we move on to economies with multiple stocks.

D.1. *A Single Stock Economy*

Assume that the uncertainty is generated by a one-dimensional Brownian motion and consider a finite horizon economy populated by a representative agent with initial portfolio $\eta = 1$, time preference rate $\rho \geq 0$, and logarithmic utility for both intermediate and terminal consumption.

Since there is a representative agent, we know that this economy admits a unique equilibrium independently of whether markets are complete or not. Furthermore, exploiting the assumption of logarithmic utility, we obtain that in this equilibrium the stock price is

$$S_t = 1_{\{t<T\}}P(t, X_t) + 1_{\{t=T\}}G(X_T),$$

where the pre-horizon price function is defined by

$$P(t, X_t) = g(t, X_t)\big((1/\rho) + e^{-\rho(T-t)}(1 - 1/\rho)\big).$$

This expression shows that in this example, dynamic market completeness is entirely determined by the intermediate dividends. In particular, if we assume that there exists a set of strictly positive measure $\mathcal{R} \subseteq (0, T)$ such that

(D.1) $$\frac{\partial g}{\partial x}(t, x) = 0$$

for almost every $(t, x) \in \mathcal{R} \times \mathcal{X}$, then the stock volatility vanishes on $\mathcal{R}$ and it follows that markets are dynamically incomplete in equilibrium even though the effective terminal dividend

$$\mathcal{G}(x) = P(T, x) = g(T, x)$$

may be chosen[7] in such a way as to satisfy the nondegeneracy condition of Theorem B.1. The reason the result of Theorem B.1 does not apply here is that if (D.1) holds on a set of strictly positive measure, then the equilibrium price function fails to be real analytic in time over $(0, T)$ and so we cannot propagate the nondegeneracy of the stock volatility from a neighborhood of T to the whole interval.

---

[7]A simple example of a smooth function that satisfies the condition of Theorem B.1 as well as equation (D.1) on a set of strictly positive measure is $\Gamma(t, x) = \alpha + \beta \exp(-x^2/(t - \gamma T)^+)$ for some strictly positive constants α, β, and $\gamma < 1$. In that case, the set over which the price function fails to be real analytic in time is simply given by $\mathcal{R} = (0, \gamma T]$.

D.2. *Multiple Stocks Economies*

Consider now a finite horizon economy where the uncertainty is generated by a Brownian motion of dimension $d > 1$, and assume that the economy is populated by a representative agent with time preference rate $\rho \geq 0$ and utility function u for both intermediate and terminal consumption.

Additionally, let, (g, G) denote the vectors of intermediate and terminal dividends and assume that the dividends of the first stock are uniformly bounded from above and away from zero. The following proposition allows us to extend the construction of the previous example to economies with multiple risky securities.

PROPOSITION D.1: *Consider an economy as above and let $F_1 \in C^{1,2}((0, T) \times \mathcal{X})$ be a nonnegative function. Then there exist nonnegative intermediate dividend functions $g_2, \ldots, g_d$ and a nonnegative constant K such that the equilibrium price of the first stock satisfies $S_{1t} = F_1(t, X_t) + K$ for all $t \in [0, T)$.*

In the equilibrium just described, market completeness depends to a large extent on the choice of the exogenous function F_1. In particular, let $\mathcal{R} \subseteq [0, T]$ be an open set of strictly positive measure and assume that

$$\frac{\partial F_1}{\partial x}(t, x) = \mathbf{0}_d$$

for almost every $(t, x) \in \mathcal{R} \times \mathcal{X}$. In that case, the volatility matrix of the equilibrium prices is automatically degenerate on $\mathcal{R}$ and it follows that markets are incomplete even though the effective terminal dividends of the risky securities,

$$\mathcal{G}(x) = \lim_{t \to T} P(t, x) = (F_1(T, x) + K)\frac{G(x)}{G_1(x)},$$

can be chosen in such a way as to satisfy the nondegeneracy condition of Theorem B.1. As in the previous example, the reason the result of Theorem B.1 does not apply here is that, given our choice for F_1, the equilibrium price function fails to be real analytic over the whole time interval. This clearly shows that real analyticity cannot be dispensed with if one is to deduce dynamic completeness from the properties of the dividends of traded securities.

REMARK D.1: Proposition D.1 bears some close connection with the literature on viable diffusion price processes; see Bick (1993), He and Leland (1993), and Wang (1993), among others. In particular, it complements this literature by showing that in an economy with multiple stocks, fixing the dividends of one stock does not impose any constraint on its equilibrium price except for a lower bound.

PROOF OF PROPOSITION D.1: Assume without loss of generality that $d = 2$ and consider the nonnegative process defined by

$$H_t = H(t, X_t) \equiv E_t\left[e^{-\rho T + \int_t^T \Phi_s \, ds} v'(\overline{G}(X_T))G_1(X_T)\right]$$

with

$$\Phi_t = \Phi(t, X_t) \equiv \frac{g_1(t, x)}{K + F_1(t, x)}.$$

Since all the terms are bounded, we know that H is bounded, and by choosing K large enough, it can be guaranteed that the function

$$g_2(t, x) \equiv (u')^{-1}\left(\frac{e^{\rho t}H(t, x)}{K + F_1(t, x)}\right) - g_1(t, x)$$

is nonnegative for all (t, x). Taking g_2 as the intermediate dividend on the second stock, we obtain that the equilibrium consumption price process is

$$e^{\rho t}m(t, X_t) = 1_{\{t < T\}}u'(\overline{g}(t, X_t)) + 1_{\{t = T\}}v'(\overline{G}(X_t))$$

and it follows that the (pre-horizon) equilibrium price of the first stock is

$$\begin{aligned}
&m(t, X_t)P_1(t, X_t) \\
&= \frac{H_t P_1(t, X_t)}{F_1(t, X_t) + K} \\
&= E_t\left[\int_t^T m(\tau, X_\tau)g_1(\tau, X_\tau) \, d\tau + e^{-\rho T}m(T, X_T)G_1(X_T)\right].
\end{aligned}$$

Using this expression in conjunction with the definition of the functions g_2 and m, we easily get that

$$\text{(D.2)} \qquad d\left(\frac{H_t P_1(t, X_t)}{K + F_1(t, X_t)}\right) = -\Phi_t H_t \, dt + dM_t^P$$

for some uniformly integrable martingale M^P. On the other hand, we have that the dynamics of the process H are given by

$$dH_t = -\Phi_t H_t \, dt + e^{-\int_0^t \Phi(s, X_s) \, ds} \, dM_t = -\Phi_t H_t \, dt + dM_t^H,$$

where M is the bounded martingale defined by

$$\begin{aligned}
M_t &\equiv e^{\int_0^t \Phi_s \, ds}H_t - H_0 \\
&= E_t\left[e^{-\rho T + \int_0^T \Phi_s \, ds}v'(\overline{G}(X_T))G_1(X_T)\right] - H_0.
\end{aligned}$$

Using the boundedness of M in conjunction with the nonnegativity of Φ and the Burkholder–Davis–Gundy inequalities (see Karatzas and Shreve (1998, Theorem 3.28)), we get that

$$E_0\left[\sup_{t\in[0,T]}|M_t^H|\right] \le C \cdot E_0\left[\sup_{t\in[0,T]}|M_t|\right] < \infty$$

for some strictly positive constant C and it follows that M^H is a uniformly integrable martingale. Using this property in conjunction with equation (D.2), we deduce that

$$N_t = H_t\left(1 - \frac{P_1(t,X_t)}{K + F_1(t,X_t)}\right) = N_0 + M_t^H - M_t^P$$

is also a uniformly integrable martingale, and since $\lim_{t\to T} N_t = 0$ by definition of g_2 and H, we finally conclude that $P_1 = K + F_1$.   *Q.E.D.*

E. DETAILS OF SOME ARGUMENTS

DETAILS OF THE PROOF OF PROPOSITION 1: The result follows directly from Assumption A(d), Assumption C, the fact that for any $\nu > 1$ we have $p(0,x,\tau,y) \le C_\nu \bar{p}(0,x,\nu\tau,y)$ for some constant $C_\nu > 0$ and Proposition C.1.   *Q.E.D.*

DETAILS OF THE PROOF OF LEMMA 1: Recall that by the first part of the proof, we have that the transition density is a classical solution to

$$(\text{E.1}) \qquad \mathcal{D}(p(t,x,\tau,y)) = \frac{\partial p(t,x,\tau,y)}{\partial t} + \mathcal{A}(p(t,x,\tau,y))$$

for all $(x,y) \in \mathcal{X}$ and $t \ne \tau$. The proof of the required bound is carried out by induction on k. Since the transition density satisfies the required bound by assumption, the fact that $\partial^{k_0} p/\partial t^{k_0}$ also satisfies it can be established by standard arguments based on Cauchy's theorem (see, e.g., Davies (1997, Proof of Theorems 3 and 4)). Indeed, by application of Cauchy's theorem, we have that

$$\frac{\partial^{k_0} p(t,x,\tau,y)}{\partial t^{k_0}} = \frac{k_0!}{2\pi i}\int_\Gamma \frac{p(z,x,\tau,y)}{(t-z)^{1+k_0}}\,dz,$$

where Γ is a small complex circle centered at t. Since by Assumption A the domain of analyticity of the transition density in (t,τ) contains $\mathcal{P}_\varepsilon^2$, we can choose Γ in such a way that $|z - t| \ge \varepsilon_1|\tau - t|$ for some $\varepsilon_1 > 0$. By analyticity, we know that equation (E.1) holds for $t \ne \tau$ in a small complex neighborhood.

Shrinking Γ if necessary, we can assume that $|z - t| < (1 + \varepsilon)|\tau - t|$ for all $z \in \Gamma$. Therefore, it follows from Assumption A that

$$
\left| \frac{\partial^{k_0} p(t, x, \tau, y)}{\partial t_0^k} \right| \leq \frac{k_0!}{2\pi} \varepsilon_1^{-(1+k_0)} |\tau - t|^{-(1+k_0)} \int_\Gamma |p(z, x, \tau, y)||dz|
$$

$$
\leq \frac{k_0!}{2\pi} \varepsilon_1^{-(1+k_0)} |\tau - t|^{-(\alpha+k_0)} K(x) L(y)
$$

$$
\times e^{\phi(1+\varepsilon)|\tau-t| - d(x,y)^2/((1+\varepsilon)|\tau-t)}
$$

and the arbitrariness of k_0 shows that $\partial^{k_0} p / \partial t^{k_0}$ satisfies the required bound for all $k_0 \in \mathbb{N}$. To proceed further, let us first fix some notation. For a function $u(x)$ and an arbitrary multi-index $k \in \mathbb{N}^d$, we let

$$
D^k u(x) = \frac{\partial^{k_1 + \cdots + k_d}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} u(x)
$$

and define a norm by setting

$$
\|u\|_{k,p,\Omega} \equiv \left(\int_\Omega \sum_{|j| \leq |k|} |D^j u(x)|^p \, dx \right)^{1/p}.
$$

Using the fact that the transition density is a solution[8] to equation (E.1) together with the Calderon–Zygmund estimates (see, e.g., Gilbarg and Trudinger (1983, Theorem 9.13)) shows that

$$
\|u\|_{2,p;B(x,r/2)} \leq C_1 \big(\|u\|_{p,B(x,r)} + \|\mathcal{A}u\|_{p,B(x,r)} \big)
$$

$$
\leq C_2 \sup_{w \in B(x,r)} \big(|u(w)| + |\mathcal{A}u(w)| \big)
$$

for some constants $C_1, C_2 > 0$, where $B(x, r)$ denotes the ball of radius r centered at x in the metric d and we have set $\|u\|_{p,\Omega} = \|u\|_{0,p,\Omega}$. Now the Morrey estimates (see, e.g., Gilbarg and Trudinger (1983, Section 7.9)) imply that

$$
\sup_{w \in B(x,r/2)} \big(|u(w)| + |\nabla u(w)| \big) \leq C_3 \|u\|_{2,p;B(x,r/2)}
$$

for any $p > d$ and some constant $C_3 > 0$ that depends on the size of the derivatives of the coefficients of $\mathcal{A}$ but not on the function u itself. Combining the above estimates shows that

$$
\text{(E.2)} \qquad \sup_{w \in B(x,r/2)} |\nabla u(w)| \leq C_4 \sup_{w \in B(x,r)} \big(|u(w)| + |\mathcal{A}u(w)| \big)
$$

---

[8]In the time-inhomogeneous case, the generator $\mathcal{A}$ has complex coefficients for complex values of t. However, when the complex neighborhood is sufficiently small, this operator can be viewed as a small perturbation of its real part and the results of Auscher (1996) apply.

for some constant $C_4 > 0$. Fix $(t, \tau, y) \in [0, T]^2 \times \mathcal{X}$ with $t \neq \tau$ and let $w = x$, $u(w) = p(t, w, \tau, y)$. Using the first part of the proof in conjunction with the fact that the transition density is a solution to (E.1) then shows that

$$\text{(E.3)} \qquad |\mathcal{A}u(w)| = \left| \frac{\partial p(t, x, \tau, y)}{\partial t} \right|$$

$$\leq A(w)L(y)|\tau - t|^{-(1+\alpha)} e^{\phi(1+\varepsilon)|\tau - t| - d(w,y)^2/((1+\varepsilon)|\tau - t|)}$$

for some locally bounded function A. Furthermore, picking r sufficiently small, we may assume that

$$d(w, y)^2 \geq \frac{d(x, y)^2}{1 + \delta}$$

for all $w \in B(x, r/2)$ and $x \in \mathcal{X}$ such that $d(x, y) > \varepsilon$. Combining this with (E.2), (E.3), and Assumption A(d) then shows that the required bound holds for $k_0 = 0$, $|k| = 1$. Repeating the same argument as in the first part shows that the required bound also holds for $k_0 = |k| = 1$, and since

$$\frac{\partial^2 p(t, x, \tau, y)}{\partial t \, \partial x_i} = \frac{\partial \mathcal{A}(p(t, x, \tau, y))}{\partial x_i} \equiv \nabla_i(\mathcal{A}p(t, x, \tau, y))$$

due to equation (E.1), we conclude that the function $\nabla_i(\mathcal{A}p)$ also satisfies the required bound. Finally, using the Schauder estimates (see, e.g., Gilbarg and Trudinger (1983, Theorem 6.2)), we obtain that

$$\sup_{w \in B(x,r)} \left(\sum_{|j| \leq 2} |D^j u(w)| \right) \leq C_5 \left(\sup_{w \in B(x,r)} |u(w)| + \sup_{w \in B(x,r), i \leq d} \left| \nabla_i(\mathcal{A}u(w)) \right| \right)$$

for some constant $C_5 > 0$. It follows that the required bound holds for $k_0 = 0$ and $|k| = 2$, and repeating the argument of the first part of the proof shows that it also holds for $k_0 = 1$ and $|k| = 2$. To complete the proof, suppose that we have established the required bound for $k_0 = 1$, $|k| \leq \ell$, and let

$$q_i(t, x, \tau, y) \equiv \frac{\partial^{k_1 + \cdots + k_d + 1} p(t, x, \tau, y)}{\partial x_1^{k_1} \cdots \partial x_i^{k_i + 1} \cdots \partial x_d^{k_d}}, \quad i = 1, \ldots, d.$$

Differentiating equation (E.1), we obtain that

$$-\frac{\partial q_i(t, x, \tau, y)}{\partial t} = \mathcal{A}(q_i(t, x, \tau, y)) + \hat{q}_i(t, x, \tau, y),$$

where $\hat{q}_i$ contains derivatives of p of order less than $|\ell|$ as well as partial derivatives of the coefficients σ_X and μ_X of the infinitesimal generator. Since $\hat{q}_i$ satisfies the required bound due to the induction hypothesis, the same argument

as above applies and we get that q_i also satisfies the required bounds. This establishes the required bound for $k_0 = 1$, $k \in \mathbb{N}^d$, and the result for $k_0 > 1$ now follows from the same Cauchy theorem-based argument as in the first part of the proof. *Q.E.D.*

DETAILS OF THE PROOF OF THEOREMS 1 AND 2: Here we provide a proof of the second order expansion of the price volatility matrix given in equation (5) and establish the expansion of the determinant that was used as a basis for the proof of Theorems 1 and 2.

The assumptions of the statement and the definition of the candidate prices imply that $Q \equiv mS$ is a solution to

$$-\frac{\partial Q(t, x)}{\partial t} = m(t, x)g(x) + \mathcal{A}(Q(t, x))$$

for all $(t, x) \in (0, T) \times \mathcal{X}$ and has terminal value zero. Since $m > 0$ is smooth by Proposition 1, this implies that $S(t, x)$ is a solution to

(E.4) $$-\frac{\partial S(t, x)}{\partial t} = g(x) + \frac{\mathcal{A}(mS)(t, x)}{m(t, x)} + S(t, x)\frac{\partial \log m}{\partial t}(t, x)$$

for all $(t, x) \in (0, T) \times \mathcal{X}$ and has terminal value zero. Using the smoothness of the coefficients in conjunction with Propositions 1 and 2, we obtain

$$\lim_{t \to T} S(t, x)\frac{\partial \log m}{\partial t}(t, x) = \lim_{t \to T} \frac{\mathcal{A}(mS)(t, x)}{m(t, x)} = \mathbf{0}_d$$

and it now follows from equation (E.4) that

(E.5) $$\lim_{t \to T} \frac{\partial S(t, x)}{\partial t} = -g(x).$$

Since $S \in C^2((0, T] \times \mathcal{X})$ by Proposition 1, this further implies

$$\lim_{t \to T} \frac{\partial \sigma_S(t, x)}{\partial t} = \lim_{t \to T} \left(\frac{\partial^2 S}{\partial t\, \partial x}\sigma_X + \frac{\partial S}{\partial x}\frac{\partial \sigma_X}{\partial t} \right)(t, x)$$

$$= \lim_{t \to T} \frac{\partial^2 S(t, x)}{\partial t\, \partial x}\sigma_X(t, x)$$

$$= -\sigma_g(T, x).$$

Differentiating with respect to time on both sides of equation (E.4), we obtain

(E.6) $$-\frac{\partial^2 S(t, x)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\mathcal{A}(mS)(t, x)}{m(t, x)} \right)$$

$$+ \frac{\partial S(t, x)}{\partial t}\frac{\partial \log m}{\partial t}(t, x) + S(t, x)\frac{\partial^2 \log m}{\partial t^2}(t, x).$$

Using the assumed regularity of the coefficients in conjunction with Proposition 1, Proposition 2, and equation (E.5), we obtain

$$\text{(E.7)} \qquad \lim_{t \to T} \frac{\partial}{\partial t} \left(\frac{\mathcal{A}(mS)(t, x)}{m(t, x)} \right) = -\frac{\mathcal{A}(m(T, x)g(x))}{m(T, x)}.$$

On the other hand, it follows from equation (E.5) and the continuity of the candidate price function that we have

$$\lim_{t \to T} \left(\frac{\partial S}{\partial t} \frac{\partial \log m}{\partial t} + S \frac{\partial^2 \log m}{\partial t^2} \right)(t, x) = -g(x) \frac{\partial \log m}{\partial t}(T, x),$$

and combining this with equations (E.6) and (E.7), we conclude that

$$\lim_{t \to T} \frac{\partial^2 S(t, x)}{\partial t^2} = \frac{\mathcal{A}(m(T, x)g(x))}{m(T, x)} + g(x) \frac{\partial \log m}{\partial t}(T, x)$$

$$= \frac{\mathcal{D}(m(T, x)g(x))}{m(T, x)}.$$

Since $S \in C^3((0, T] \times \mathcal{X})$ by Proposition 2, this further implies

$$\lim_{t \to T} \frac{\partial^2 \sigma_S(t, x)}{\partial t^2}$$

$$= \lim_{t \to T} \left(\frac{\partial^3 S}{\partial t^2 \partial x} \sigma_X + 2 \frac{\partial^2 S}{\partial t \partial x} \frac{\partial \sigma_X}{\partial t} + \frac{\partial S}{\partial x} \frac{\partial \sigma_X}{\partial t} \right)(t, x)$$

$$= \frac{\partial}{\partial x} \left(\frac{\mathcal{D}(m(T, x)g(x))}{m(T, x)} \sigma_X(T, x) \right) - 2g'(x) \frac{\partial \sigma_X}{\partial t}(T, x)$$

$$= H(x)$$

and, therefore,

$$\Phi(t, x) \equiv \sigma_S(t, x) - (T - t)\sigma_g(T, x) - \frac{1}{2}(T - t)^2 H(x) = o(T - t)^2,$$

where we have used the fact that the candidate price function converges to zero at the terminal time. To obtain an expansion of $\det(\sigma_S(t, x))$, consider the matrix-valued functions defined by

$$A(x) = \sigma_g(T, x), \quad K(t, x) = \frac{\Phi(t, x)}{(T - t)^2},$$

$$B(t, x) = K(t, x) + \frac{1}{2}H(x).$$

With these notations, we have that

$$\sigma_S(t, x) = (T - t)A(x) + (T - t)^2 B(t, x)$$

and, therefore,

(E.8) $I(t, x) \equiv \det(\sigma_S(t, x)) = (T - t)^d \det(A(x) + (T - t)B(t, x)).$

Well known results from linear algebra show that for any two matrices M_1 and M_2, the determinant of the sum $M_1 + M_2$ is given by

$$\det(M_1 + M_2) = \sum_{k=0}^{d} \sum_{1 \le i_1 < \cdots < i_k \le d} \det\big(C_{i_1, \ldots, i_k}\big),$$

where $C_{i_1, \ldots, i_k}$ is obtained from M_1 by replacing its rows numbered $i_1, \ldots, i_k$ with the corresponding rows of the matrix M_2. Applying this to (E.8), we obtain

$$I(t, x) = \sum_{k=0}^{d} (T - t)^{k+d} \sum_{1 \le i_1 < \cdots < i_k \le d} \det\big(C_{i_1, \ldots, i_k}(t, x)\big)$$

$$= (T - t)^d \det(A(x))$$

$$+ \sum_{k=1}^{d} (T - t)^{k+d} \sum_{1 \le i_1 < \cdots < i_k \le d} \det\big(C_{i_1, \ldots, i_k}(t, x)\big),$$

where $C_{i_1, \ldots, i_k}(t, x)$ is obtained from $A(x)$ by replacing its rows numbered $i_1, \ldots, i_k$ with the corresponding rows of the matrix $B(t, x)$. Expanding the second term on the right hand side and using the fact that

$$\det(C_i(t, x)) = \frac{1}{2} \det(B_i(x))$$

for all $1 \le i \le d$, where

$$B_i(x) = \sigma_g(T, x) + \mathbf{e}_i \mathbf{e}_i^\top (H(x) - \sigma_g(T, x))$$

and $\mathbf{e}_i$ is the ith vector in the orthonormal basis of $\mathbb{R}^d$, we thus obtain that the determinant of the price gradient satisfies

$$I(t, x) = (T - t)^d \left(\det(\sigma_g(T, x)) + \frac{1}{2}(T - t) \sum_{i=1}^{d} \det(B_i(x)) \right)$$

$$+ o((T - t)^{1+d}),$$

which is the expansion used as basis for the proof of Theorems 1 and 2.
Q.E.D.

DETAILS OF THE PROOF OF THEOREM 3: To justify the differentiability of G which was used in the proof of Theorem 3, it suffices to prove that the functions

$$j_a(\lambda) \equiv E \int_0^\infty m(t, X_t, \lambda) I_a(e^{\rho t} m(t, X_t, \lambda)/\lambda_a) \, dt,$$

$$k_a(\lambda) \equiv E \int_0^\infty m(t, X_t, \lambda)(\ell_a(X_t) + g_i(X_t)) \, dt$$

are both continuously differentiable with respect to λ. Differentiating the market clearing condition (C.1) with respect to λ_j shows that

$$(E.9) \qquad \frac{1}{m(t, x, \lambda)} \frac{\partial m(t, x, \lambda)}{\partial \lambda_j} = \frac{\lambda_j^{-2} e^{\rho t} m(t, x, \lambda) I_j'(e^{\rho t} m(t, x, \lambda)/\lambda_j)}{\displaystyle\sum_{k=1}^{A} \lambda_k^{-1} e^{\rho t} m(t, x, \lambda) I_k'(e^{\rho t} m(t, x, \lambda)/\lambda_k)}.$$

Since I_k is the inverse of u_k', we have that $I_k'(z) = 1/u_k''(I_k(z))$ and, therefore,

$$\frac{1}{\gamma_2} \le -\frac{z I_k'(z)}{I_k(z)} = -\frac{u_k'(I_k(z))}{I_k(z) u_k''(I_k(z))} \le \frac{1}{\gamma_1}$$

since the agents' relative risk aversions are assumed to be bounded between γ_1 and γ_2. Combining this with (E.9) gives

$$0 \le \frac{1}{m(t, x, \lambda)} \frac{\partial m(t, x, \lambda)}{\partial \lambda_j} < \frac{\gamma_2}{\lambda_j \gamma_1}$$

and, therefore,

$$\left| \frac{\partial m(t, X_t, \lambda)}{\partial \lambda_j}(\ell_a(X_t) + g_i(X_t)) \right| \le \frac{\gamma_2}{\lambda_j \gamma_1} m(t, X_t, \lambda)\overline{g}(X_t)$$

so that the differentiability of k_a now follows from the upper bound of equation (C.2), Lemma 2, Assumption C, and the dominated convergence theorem. Using similar arguments, it can be shown that

$$\left| \frac{\partial m(t, X_t, \lambda)}{\partial \lambda_j} I_a\left(\frac{e^{\rho t} m(t, X_t, \lambda)}{\lambda_a} \right) \right| \le C m(t, X_t, \lambda)\overline{g}(x)$$

for some $C > 0$ and the differentiability of j_a now follows from equation (C.2), Lemma 2, Assumption C, and the dominated convergence theorem. *Q.E.D.*

Verification of the Bound for Square Root Processes: Here we verify the claim that the transition density of the square root process satisfies the bound of Assumption A(d) for real values of $t \neq \tau$. Assume that

$$(E.10) \quad dX_{it} = (a_i - b_i X_{it})\,dt + \sigma_i \sqrt{|X_{it}|}\,dZ_{it}$$

for some constants $a_i > 0$, b_i, and σ_i such that $\nu_i \equiv (2a_i/\sigma_i^2) - 1 > 0$.[9] Using well known results on square root diffusions (see, e.g., Feller (1951) and Cox, Ingersoll, and Ross (1985)), we have that the transition density is given by

$$q_i(t, x, y) = \frac{1}{\tau_i(t)}\left(\frac{y}{e^{-b_i t}x}\right)^{\nu_i/2} e^{-(y+e^{-b_i t}x)/\tau_i(t)} I_{\nu_i}\left(\frac{2}{\tau_i(t)}\sqrt{e^{-b_i t}xy}\right)$$

for all $(t, x, y) \in [0, T]^2 \times (0, \infty)^2$, where the function I_{ν_i} is the modified Bessel function of the first kind with index ν_i and we have set

$$\tau_i(t) \equiv \frac{\sigma_i^2}{2b_i}(1 - e^{-b_i t}).$$

Since the function τ_i is uniformly bounded above and away from zero on $[0, T]$, we have that there are constants $C_i = C_i(T) > 0$ such that

$$C_1 t \leq \tau_i(t) \leq C_2 t.$$

Combining this property with the inequality

$$e^{-(y-e^{-t}x)^2/(1-e^{-2t})} \leq e^{-(1/2t)(y-x)^2+t/2+(x^2-y^2)/2} \quad \forall(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$$

and the fact that $I_{\nu_i}(x) \leq K_0(x/2)^{\nu_i} e^x$ for some strictly positive constant K_0 (see Joshi and Bissu (1991)), we obtain that the transition density satisfies

$$|q_i(t, x, y)| \leq K_0 \tau_i(t)^{-(1+\nu_i)} y^{\nu_i} e^{-(1/\tau_i(t))(\sqrt{y}-\sqrt{e^{-b_i t}x})^2}$$

$$\leq K_1 t^{-(1+\nu_i)} y^{\nu_i} e^{-(1/\tau_i(t))(\sqrt{y}-\sqrt{e^{-b_i t}x})^2}$$

$$\leq K_2 t^{-(1+\nu_i)} y^{\nu_i} e^{K_3(x-y)-(K_4/t)(\sqrt{y}-\sqrt{x})^2}$$

for some strictly positive constants $(K_i)_{i=1}^4$. The desired result now follows by noting that the formula

$$d(x, y)^2 = K_4 \sum_{i=1}^{d}(\sqrt{x_i} - \sqrt{y_i})^2$$

[9]This parametric restriction is meant to guarantee that the solution to (E.10) never reaches zero and can be relaxed at the cost of increased notational burden.

defines a metric that is locally equivalent to the Euclidean distance.

Verification of the Bound for Constant Elasticity of Variance Processes: Here we verify the claim that the transition density of the constant elasticity of variance process satisfies the bound of Assumption A(d) for real values of $t \neq \tau$. Assume that

(E.11) $dX_{it} = \mu_i X_{it} \, dt + \xi_i |X_{it}|^{1+\beta_i} \, dZ_{it}$

for some constants μ_i, ξ_i, and $\beta_i > 0$. To establish the required bound, consider the nonnegative process defined by

$$Y_{it} \equiv (\xi_i \beta_i X_{it}^{\beta_i})^{-2}.$$

Applying Itô's lemma to the right hand side and using equation (E.11) shows that this process evolves according to equation (E.10) with the constants $a_i = 2 + 1/\beta_i$, $b_i = 2\mu_i\beta_i$, and $\sigma_i = -2$. Combining this simple observation with the arguments used in the square root case immediately gives the desired result.

F. PROOFS OMITTED FROM THE MAIN TEXT

PROOF OF PROPOSITION 3: Let $m_t = m(t, X_t, \lambda)$ and $c_{at} = (u_a')^{-1}(e^{\rho t} m_t / \lambda_a)$ be an Arrow–Debreu equilibrium, denote by

$$S_t = S(t, X_t) = \frac{1}{m_t} E_t \int_t^T m_\tau g(X_\tau) \, d\tau$$

the corresponding vector of candidate prices, and assume that the corresponding volatility matrix $\sigma_{St} = \sigma_S(t, X_t)$ is almost surely nondegenerate.

To prove that this Arrow–Debreu equilibrium gives rise to an equilibrium, we start by showing that for each a, the consumption plan c_a belongs to $\mathcal{C}_a(m, B, S)$ and is optimal for agent a. Consider the wealth process defined by

$$W_{at} \equiv \frac{1}{m_t} E_t \int_t^T m_\tau (c_{a\tau} - \ell_a(X_\tau)) \, d\tau.$$

Since the process

$$m_t W_{at} + \int_0^t m_\tau (c_{a\tau} - \ell_a(X_\tau)) \, d\tau$$

is a martingale, it follows from the martingale representation theorem, the definition of B, and the assumed invertibility of the stock volatility matrix that

(F.1) $W_{at} = W_{a0} + \int_0^t \alpha_{a\tau} \, dB_\tau + \int_0^t \pi_{a\tau}^\top d(S + D)_\tau - \int_0^t (c_{a\tau} - \ell_a(X_\tau)) \, d\tau$

for some trading strategy (α_a, π_a). On the other hand, we have

$$\left| \int_t^T m_\tau (c_{a\tau} - \ell_a(X_\tau)) \, d\tau \right| \leq \int_0^T \left| m_\tau (c_{a\tau} - \ell_a(X_\tau)) \right| d\tau$$

$$\leq \int_0^T m_\tau \overline{g}(X_\tau) \, d\tau$$

and, since the right hand side is integrable as a result of (C.3) and Assumption C, it follows from the dominated convergence theorem that

$$\lim_{t \to T} E[m_t W_{at}] = \lim_{t \to T} E \int_t^T m_\tau (c_{a\tau} - \ell_a(X_\tau)) \, d\tau = 0,$$

which, together with equation (F.1) and the definition of W_{at}, implies that the consumption plan c_a is feasible for agent a. To show that it is in fact optimal, let $c' \in \mathcal{C}_a(m, B, S)$ denote another feasible plan and denote by W' the corresponding wealth process. Using the martingale property of the process

$$m_t W_t' + \int_0^t m_\tau (c_\tau' - \ell_a(X_\tau)) \, d\tau$$

together with the definition of the set of feasible strategies and the same argument as above, we deduce that

$$(\text{F.2}) \qquad m_0 W_{a0} = m_0 W_0 = \lim_{\theta \to T} \left(E[m_\theta W_\theta'] + E \int_0^\theta m_\tau (c_\tau' - \ell_a(X_\tau)) \, d\tau \right)$$

$$\geq \lim_{\theta \to T} E \int_0^\theta m_\tau (c_\tau' - \ell_a(X_\tau)) \, d\tau$$

$$= E \int_0^T m_\tau (c_\tau' - \ell_a(X_\tau)) \, d\tau.$$

On the other hand, by concavity of the utility function, we have

$$u_a(c_t') - u_a(c_{at}) \leq u_a'(c_{at})(c_t' - c_{at}),$$

and combining this inequality with (F.2) and the definition of the wealth process associated with c_a shows that

$$E \int_0^T e^{-\rho\tau} (u_a(c_\tau') - u_a(c_{a\tau})) \, d\tau$$

$$\leq E \int_0^T e^{-\rho\tau} u_a'(c_{a\tau})(c_\tau' - c_{a\tau}) \, d\tau$$

$$= E \int_0^T m_\tau (c'_\tau - c_{a\tau}) \, d\tau$$

$$= E \int_0^T m_\tau (c'_\tau - \ell_a(X_\tau)) \, d\tau - m_0 W_{a0} \leq 0$$

and establishes the optimality of c_a. To complete the proof, it remains to show that the financial markets clear. Since

$$\sum_{a=1}^A W_{at} = \sum_{i=1}^n S_{it}$$

by definition of the consumption and wealth processes $(c_a, W_a)_{a=1}^A$, we have that the market for the risky asset clears. On the other hand, applying Itô's lemma to both sides of this equality and matching diffusion terms gives

$$\sigma_{St}^\top \left(\sum_{a=1}^A \pi_{at} - S_t \right) = \mathbf{0}$$

and the clearing of the stock market now follows from the assumed invertibility of the stock volatility matrix.   *Q.E.D.*

PROOF OF LEMMA 2: The fact that $\mathbf{m} > 0$ follows from the fact that $\overline{g}$ is finite. On the other hand, using an argument similar to that which led to (C.2), we obtain

$$\mathbf{m}(x) \leq \sum_{a=1}^A \lambda_a u'_a(\overline{g}(x)/A),$$

which is the desired result.   *Q.E.D.*

PROOF OF LEMMA 5: We first note that

$$\int_0^\infty e^{-\rho t} f(t) \, dt - \int_0^{t_0} e^{-\rho t} f(t) \, dt = O(e^{-\rho t_0})$$

for any $t_0 > 0$ because

$$\int_{t_0}^{+\infty} e^{-\rho t} f(t) \, dt = e^{-\rho t_0} \int_{t_0}^{+\infty} e^{-\rho(t-t_0)} f(t) \, dt.$$

Integrating by parts, we get

$$\int_0^{t_0} e^{-\rho t} f(t) \, dt = \rho^{-1}(f(0) - e^{-\rho t_0} f(t_0)) + \rho^{-1} \int_0^t e^{-\rho t} f'(t) \, dt.$$

Repeating this calculation, we get by induction that

$$\int_0^{t_0} e^{-\rho t} f(t)\, dt = \sum_{i=0}^{k} \rho^{-i-1} f^{(i)}(0)$$

$$+ \rho^{-k-1} \int_0^{t_0} e^{-\rho t} f^{(k+1)}(t)\, dt + O(e^{-\rho t_0})$$

and the desired follows because

$$\int_0^{t_0} e^{-\rho t} f^{(k+1)}(t)\, dt \to 0$$

by application of the dominated convergence theorem. *Q.E.D.*

PROOF OF PROPOSITION 4: The result of Proposition 4 follows directly from that of the following lemma (Lemma F.1). *Q.E.D.*

LEMMA F.1: *Let $T < \infty$ and assume that the state variables evolve according to*

(F.3) $\qquad dX_t = (b(t) - A(t)X_t)\, dt + \sigma_X(t)\, dZ_t$

for some real analytic functions such that $\mathrm{rank}(\sigma_X(t)) = d$ *for all* $t \in [0, T]$. *Then the following assertions hold*:

(a) *The state variables admit a transition density* $p(t, x, \tau, y)$ *that is real analytic in* $(t, \tau) \in [0, T]^2 \setminus \{\tau = t\}$.

(b) *There are constants* $C, \varepsilon > 0$ *as well as a complex neighborhood* $\mathcal{P} \supseteq [0, T]$ *such that* $p(t, x, \tau, y)$ *is analytic for*

$$(t, \tau) \in \mathcal{P}_\varepsilon^2 \equiv \left\{ (t, \tau) \in \mathcal{P}^2 : s = \tau - t \neq 0, |\Im(s)| \leq \varepsilon \Re(s) \right\}$$

and satisfies

$$|p(t, x, \tau, y)| \leq K(x)|\tau - t|^{-d/2} e^{-d(x,y)^2/(C|\tau - t|)}$$

for all quadruples $(x, y, t, \tau) \in \mathcal{X}^2 \times \mathcal{P}_\varepsilon^2$, *where* $d(x, y) = \|x - y\|$ *is the standard Euclidean distance and* K *is a locally bounded function*.

PROOF: By the analytic implicit function theorem (see Lunardi (1995, Theorem 8.3.9)), we have that the unique solution to

$$\frac{d\Phi(t)}{dt} = -A(t)\Phi(t)$$

with initial condition $\mathbb{I}_d$ is analytic in a neighborhood of $[0, T]$. Furthermore, the inverse of this unique solution solves

$$\frac{d(\Phi^{-1}(t))}{dt} = \Phi^{-1}(t)A(t)$$

with initial condition $\mathbb{I}_d$ and hence is analytic as well.

Using the above notation in conjunction with well known results on linear stochastic differential equations (see Karatzas and Shreve (1998, Chapter 5.6)), we obtain that the unique solution to equation (F.3) is a Gaussian process with mean

$$\mu(t, x, \tau) \equiv E[X_\tau | X_t = x] = \Phi(\tau)\Phi^{-1}(t)x + \int_t^\tau \Phi(\tau)\Phi^{-1}(s)b(s)\,ds,$$

and variance–covariance matrix

$$\Omega(t, \tau) \equiv E[X_\tau X_\tau^\top | X_t] - E[X_\tau | X_t]E[X_\tau | X_t]^\top$$
$$= \int_t^\tau (\Phi(\tau)\Phi^{-1}(s)\sigma_X(s))(\Phi(\tau)\Phi^{-1}(s)\sigma_X(s))^\top\,ds.$$

In particular, the transition density of the state variables is given by

(F.4) $$p(t, x, \tau, y) = \phi(y, \mu(t, x, \tau), \Omega(t, \tau)),$$

where

$$\phi(y, m, B) \equiv (2\pi)^{-d/2} |\det(B)|^{-1/2} e^{-(y-m)^\top B^{-1}(y-m)}$$

denotes the d-dimensional Gaussian probability distribution function. Since the functions b, σ_X, Φ, and Φ^{-1} are analytic, we have that the functions μ and Ω are also analytic, and it now follows from equation (F.4) that the transition density exists and is real analytic in $(t, \tau) \in [0, T]^2 \setminus \{t = \tau\}$.

Since the horizon is finite and the matrix $\sigma_X(t)$ is by assumption nondegenerate for all $t \in [0, T]$, we know that there are strictly positive constants ε and δ such that

$$\delta\mathbb{I}_d \geq \Sigma_X(t) \equiv \sigma_X(t)\sigma_X(t)^\top \geq \varepsilon\mathbb{I}_d$$

for all $t \in [0, T]$, where, for two symmetric matrices, $A \geq B$ means that $A - B$ is nonnegative definite. Now consider the function defined by

$$\hat{\Omega}(t, \tau) \equiv \frac{\Omega(t, \tau)}{\tau - t}.$$

Since the function Ω is analytic in a neighborhood of $[0, T]^2$, we know that the function $\hat{\Omega}$ is jointly real analytic for $t \neq \tau$. On the other hand, since

$$\lim_{t \to \tau} \hat{\Omega}(t, \tau) = \Sigma_X(\tau),$$

we have that the singularity at $\tau = t$ is removable and it thus follows from Shabat (1992, Theorem 3, p. 92 and Hartog's Theorem, p. 28) that (i) the function $\hat{\Omega}$ is real analytic in $(t, \tau) \in [0, T]^2$ and satisfies

$$\delta \mathbb{I}_d \geq \hat{\Omega}(t, \tau) \geq \varepsilon \mathbb{I}_d$$

for all $(t, \tau) \in [0, T]^2$, and (ii) there exists a neighborhood $\mathbb{C} \supset \mathcal{P} \supseteq [0, T]$ such that

$$\|\hat{\Omega}(t, \tau) - \hat{\Omega}(\Re t, \Re \tau)\| < \frac{1}{2}\varepsilon$$

for all $(t, \tau) \in \mathcal{P}^2$, where the notation $\|M\|$ denotes the Euclidean norm of the matrix M. Using these properties we readily obtain that

$$\|\hat{\Omega}(t, \tau)\| \geq \|\hat{\Omega}(\Re t, \Re \tau)\| - \|\hat{\Omega}(t, \tau) - \hat{\Omega}(\Re t, \Re \tau)\|$$
$$\geq \frac{h^\top}{\|h\|} \hat{\Omega}(\Re t, \Re \tau) \frac{h}{\|h\|} - \frac{1}{2}\varepsilon \geq \frac{1}{2}\varepsilon$$

for any $h \in \mathbb{R}^d$ and it follows that

(F.5) $$\|\hat{\Omega}^{-1}(t, \tau)\| \leq \varepsilon_1 \equiv \frac{2}{\varepsilon}$$

for all $(t, \tau) \in \mathcal{P}^2$. Using similar arguments and setting $\xi(t, \tau) \equiv \mu(t, x, \tau) - \Phi(\tau)\Phi^{-1}(t)x$, it can be shown that the vector-valued function

$$\eta(t, x, \tau) \equiv \frac{x - \mu(t, x, \tau)}{\tau - t}$$
$$= \frac{(\mathbb{I}_d - \Phi(\tau)\Phi^{-1}(t))}{\tau - t}x - \frac{\xi(t, \tau)}{\tau - t}$$

is real analytic in $(t, \tau) \in [0, T]^2$ and, since real analytic functions are locally bounded, it follows that there exists a constant $C_0 > 0$ such that

(F.6) $$\|\eta(t, x, \tau)\| \leq C_0(1 + \|x\|)$$

for all $(t, \tau, x) \in \mathcal{P}^2 \times \mathcal{X}$. After these lengthy preparations, we now turn to assertion (b). Using the expression of the transition density in conjunction with

equation (F.5) and the fact that

$$\frac{1}{|\det(\Omega(\tau,t))|^{1/2}} = \left|\det(\Omega^{-1}(\tau,t))\right|^{1/2} \le \|\Omega^{-1}(t,\tau)\|^{d/2}$$

$$= \|(\tau-t)^{-1}\hat{\Omega}^{-1}(t,\tau)\|^{d/2} \le \varepsilon_1^{d/2}|\tau-t|^{-d/2},$$

we deduce that

$$|p(t,x,\tau,y)| \le C_p|\tau-t|^{-d/2}e^{-A_1+2A_2+A_3}$$

for all $(t,\tau,x,y) \in \mathcal{P}^2 \times \mathcal{X}^2$ and some constant $C_p > 0$, where

$$A_1 = (y-x)^\top \Re(\Omega^{-1}(t,\tau))(y-x),$$

$$A_2 = \left|(y-x)^\top \Re(\hat{\Omega}^{-1}(t,\tau)\eta(t,x,\tau))\right|,$$

$$A_3 = \left|\Re((\tau-t)\eta(t,x,\tau)^\top \hat{\Omega}^{-1}(t,\tau)\eta(t,x,\tau))\right|.$$

Using equations (F.5) and (F.6), we obtain that

$$A_2 \le \|y-x\| \|\hat{\Omega}^{-1}(t,\tau)\eta(t,x,\tau)\|$$

$$\le \|y-x\| \|\hat{\Omega}^{-1}(t,\tau)\| \|\eta(t,x,\tau)\| \le C_2(1+\|x\|)\|y-x\|,$$

$$A_3 \le |\tau-t| |\eta(t,x,\tau)^\top \hat{\Omega}^{-1}(t,\tau)\eta(t,x,\tau)|$$

$$\le |\tau-t| \|\hat{\Omega}^{-1}(t,\tau)\| \|\eta(t,x,\tau)\|^2 \le C_3(1+\|x\|)^2$$

for all $(t,\tau,x,y) \in \mathcal{P}^2 \times \mathcal{X}^2$ and some nonnegative constants C_i, where the last inequality follows from the boundedness of $\mathcal{P}$. Now let $\phi > 0$ be a fixed constant. Since $\hat{\Omega}^{-1}(t,\tau) \ge \varepsilon_2$ and $\Im(\hat{\Omega}^{-1}(t,\tau)) = 0$ for some constant $\varepsilon_2 > 0$ and all real (t,τ), we can assume by shrinking the neighborhood if necessary that

$$\Re(\hat{\Omega}^{-1}(\tau,t)) \ge \frac{1}{2}\varepsilon_2 \mathbb{I}_d, \quad \|\Im(\hat{\Omega}^{-1}(t,\tau))\| \le \phi\varepsilon_2$$

for all $(t,\tau) \in \mathcal{P}^2$. Using these estimates in conjunction with the fact that $|\Im(\tau-t)| \le \varepsilon\Re(\tau-t)$ in the set $\mathcal{P}_\varepsilon^2$, we deduce that there exists a $C_1 \equiv C_1(\phi) > 0$ such that

$$A_1 = \frac{(y-x)^\top(\Re(\tau-t)\Re(\hat{\Omega}^{-1}(t,\tau)) + \Im(\tau-t)\Im(\hat{\Omega}^{-1}(t,\tau)))(y-x)}{|\tau-t|^2}$$

$$\ge \frac{\Re(\tau-t)}{|\tau-t|} \frac{(1/2-\phi\varepsilon)\varepsilon_2\|y-x\|^2}{|\tau-t|} \ge \frac{(1/2-\phi\varepsilon)\varepsilon_2}{1+\varepsilon} \frac{\|y-x\|^2}{|\tau-t|}$$

for all $(x,y,t,\tau) \in \mathcal{X}^2 \times \mathcal{P}_\varepsilon^2$ and assertion (b) follows. $\quad$ *Q.E.D.*

REMARK F.1: Assume that the economy has an infinite time horizon and that the coefficients σ_X, b, and A are time-independent. In this case, it can be shown that the result of Lemma F.1 remains valid provided that the variance–covariance matrix of the state variables

$$\Omega(\tau) = E\big[(X_\tau - E[X_\tau])(X_\tau - E[X_\tau])^\top\big]$$

is positive definite for all $\tau > 0$ and that either all the eigenvalues of A have strictly positive real parts or the matrix A is diagonalizable and all its eigenvalues have nonnegative real parts.

PROOF OF PROPOSITION 5: Assume that each of the coordinates of the vector of state variables follows an autonomous process as in equation (8), let $\Sigma(x) = \mathrm{diag}(\sigma_1(x_1), \ldots, \sigma_d(x_d))$, and let

$$h(x) = \prod_{i=1}^{d} \frac{1}{2\sigma_i(x_i)} \exp\left(\int_{y_i}^{x_i} \frac{2\mu_i(z)}{\sigma_i(z)^2}\, dz\right),$$

where y in an arbitrary point in the interior of the state space. With these notations, we have that the drift of the state variables can be written as

$$\text{(F.7)} \qquad \mu_i(x) = \frac{1}{2} \sum_{j=1}^{d} \frac{\sqrt{\det(\Sigma(x))}}{h(x)^2} \frac{\partial}{\partial x_j}\left(\frac{h(x)^2}{\sqrt{\det(\Sigma(x))}} \Sigma_{ji}(x)\right).$$

This shows that the infinitesimal generator $\mathcal{A}$ of the state variables is a weighted Laplace–Beltrami operator with weight function h^2 (see Grigor'yan (2006)) and the desired conclusion now follows from the results of Davies (1997). *Q.E.D.*

REMARK F.2: The result of Proposition 4 holds not only for autonomous diffusions, but also for any multidimensional diffusion process whose drift can be expressed as in equation (F.7) for some function $h: \mathcal{X} \to \mathbb{R}$. No particular conditions need to be imposed on the volatility matrix except for the fact that it be of full rank.

PROOF OF THEOREM 4: To establish Theorem 4, it suffices to show that under Assumption D the candidate prices are real analytic in $t \in (0, T)$. As can be seen from the proof of Lemma 3, this follows once we show that there exists a complex neighborhood $\mathcal{P} \supset [0, T]$ such that $m(t, x)g_i(t, x)$ is analytic in $t \in \mathcal{P}$ *for all $x \in \mathcal{X}$* and

$$\text{(F.8)} \qquad \int_0^T \int_{\mathcal{X}} \sup_{\{\theta \in \mathcal{P}: \Re\theta = \tau\}} |m(\theta, x)g_i(\theta, x)|\, \bar{p}(0, x, \nu\tau, y)\, dy\, d\tau < \infty.$$

Using the result of Lemma F.2 below together with Assumption D, we have that there exists a constant K and a complex neighborhood $\mathcal{P} \supset [0, T]$ such that

$$\sup_{\{\theta \in \mathcal{P}:\Re\theta=\tau\}} |m(\theta, x)g_i(\theta, x)| \leq \sum_{a=1}^{A} K_1 \big|u'_a(\overline{g}(\tau, x))\overline{g}(\tau, x)\big|$$

$$\leq \sum_{a=1}^{A} K_2 e^{-\rho_a\tau} \big|u'_a(\overline{g}(\tau, x)/A)\overline{g}(\tau, x)\big|$$

for all $\tau \in [0, T]$ and some $K_1, K_2 > 0$, where the second equality follows from the concavity of the utility function and the finiteness of T. Combining this estimate with Assumption D then shows that (F.8) holds and completes the proof.   *Q.E.D.*

LEMMA F.2: *Under Assumption* D, *there exists a complex neighborhood* $\mathcal{P} \supset [0, T]$ *such that* $m(\theta, x)$ *is analytic in* $\theta \in \mathcal{P}$ *for all* $x \in \mathcal{X}$ *and satisfies*

$$|m(\theta, x)| \leq \sum_{a=1}^{A} K u'_a(\overline{g}(\Re\theta, x))$$

for all $(\theta, x) \in \mathcal{P} \times \mathcal{X}$ *and some constant* $K > 0$.

PROOF: Assumption D implies that for z close to 0 and ∞, all the utility functions behave asymptotically as Constant relative risk aversion (CRRA) functions and, for simplicity of exposition, we assume that all utilities are, in fact, CRRA. The general case follows by a modification of the arguments. Assume that

$$u_a(c) = \frac{c^{1-\gamma_a} - 1}{1 - \gamma_a}$$

for some $\gamma_a > 0$, define $b_a \equiv 1/\gamma_a$, and consider the function

$$\phi(x, z, u) \equiv \sum_{a=1}^{A} e^{-\rho_a b_a z} u^{-b_a} c_{a0} - \overline{g}(z, x).$$

With these notations we have that the consumption price solves $\phi(x, z, m(z, x)) = 0$ and we want to apply the analytic implicit function theorem to obtain analyticity of $m(z, x)$ with respect to $z \in \mathcal{P}$ for some complex neighborhood $\mathcal{P} \supset [0, T]$ that is independent of the choice of $x \in \mathcal{X}$. To this end, we argue as follows. For real values of $t \in [0, T]$, we have that

$$\max_{1 \leq a \leq A} (e^{-\rho_a b_a t} u^{-b_a} c_{a0}) \geq \sum_{a=1}^{A} e^{-\rho_a b_a t} (c_{a0}/A) u^{-b_a} \geq e^{-\rho_i b_i t} u^{-b_i} (c_{i0}/A)$$

for any index $1 \leq i \leq A$. Since the horizon T is finite, it follows that there exists a strictly positive constant C_1 such that

(F.9) $\qquad (1/C_1) \sum_{a=1}^{A} \overline{g}(t, x)^{-\gamma_a} \leq m(t, x) \leq C_1 \sum_{a=1}^{A} \overline{g}(t, x)^{-\gamma_a}.$

By Chang, He, and Pradhu (2003, Theorem 1.1), we know that the radius of time analyticity of the function m depends only on two quantities: a lower bound on $|\partial \phi / \partial u|$ at the point $(x, t, m(t, x))$ and an upper bound on $|\phi(x, z, u)|$ for (x, z, u) in a complex neighborhood of the point $(x, t, m(t, x))$.[10]

By compactness, when $\overline{g}(t, x)$ varies in a fixed bounded interval, the claim follows from the analytic implicit function theorem. Now we have to consider two regimes: that where $\overline{g}(t, x)$ is large and that where it is small. Assume first that $\overline{g}(t, x) > K_4$ for some sufficiently large constant K_4, let $\gamma_{\min} \equiv \min_a \gamma_a$, and consider the function $\ell(z, x) \equiv m(z, x) \overline{g}(z, x)^{\gamma_{\min}}$. By equation (F.9), we have that

$$C_2 + C_2 \sum_{\gamma_a > \gamma_{\min}}^{A} \overline{g}(t, x)^{\gamma_{\min} - \gamma_a} \leq m(t, x) \leq C_3 + C_3 \sum_{\gamma_a > \gamma_{\min}}^{A} \overline{g}(t, x)^{\gamma_{\min} - \gamma_a}$$

for some constants $C_2, C_3 > 0$, and it follows that $\ell(z, x)$ varies in a bounded interval as long as the aggregate dividend is bounded away from zero. On the other hand, a direct calculation shows that $\ell(z, x)$ solves $\psi(x, z, \ell(z, x)) = 0$, where

$$\psi(z, x, u) \equiv \sum_{a=1}^{A} e^{-\rho_a b_a z} \overline{g}(z, x)^{\gamma_{\min}/\gamma_a - 1} u^{-b_a} c_{a0} - 1.$$

In particular, we have that

$$\frac{\partial \psi(z, x, u)}{\partial u} \geq -b_i e^{-\rho_i b_i z} u^{-b_i} c_{i0},$$

where $i = \arg\min_a \gamma_a$ denotes the agent with the smallest risk aversion and it follows that the derivative is uniformly bounded away from zero as $u = \ell(z, x)$ is bounded away from zero. Since

$$|\overline{g}(z, x)| \geq |\Re \overline{g}(z, x)| \geq K_5 |\overline{g}(\Re z, x)| \geq K_4 K_5$$

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[10]Even though the conditions of Chang, He, and Pradhu (2003) only require a bound on $|\phi(x, t, u)|$ for real values of t, a close inspection of their proof shows that one in fact needs a bound on $|\phi(x, z, u)|$ for complex values of z.

for some constant $K_5 > 0$, we can make $\overline{g}(z, x)$ arbitrarily large by increasing the constant K_4. Sending $\overline{g}(z, x)$ to infinity and using the identity $\psi(z, x, \ell(z, x)) = 0$ in conjunction with the fact that $\ell(z, x)$ stays uniformly bounded, we obtain

$$\lim_{\overline{g}(z,x)\to\infty} \sum_{\gamma_a > \gamma_{\min}}^{A} e^{-\rho_a b_a z} \overline{g}(z, x)^{\gamma_{\min}/\gamma_a - 1} u^{-b_a} c_{a0} = 0$$

and, therefore,

$$\lim_{\overline{g}(z,x)\to\infty} \ell(z, x) = e^{-\rho_i z} c_{i0}^{\gamma_{\min}},$$

where the convergence is uniform in $x \in \mathcal{X}$ such that $\overline{g}(t, x) \geq K_4$ for all $t \in [0, T]$ and some sufficiently large K_4. Hence, $|\psi(z, x, u)|$ can be made arbitrarily small when $|u - \ell(t, x)| + |z - t| < \varepsilon$ uniformly in x for some sufficiently small ε and it follows that the radius of time analyticity of $\ell(z, x)$, which is equal to that of $m(z, x)$, is uniformly bounded from below when the aggregate dividend is large.

In the regime where the aggregate dividend is small, the desired result follows from a similar argument using $s(z, y) \equiv m(z, x)\overline{g}(z, x)^{\gamma_{\max}}$ with $\gamma_{\max} \equiv \max_a \gamma_a$ instead of the function $\ell(z, x)$. We omit the details.

To complete the proof, let $\mathcal{P} \supseteq [0, T]$ denote the domain of complex analyticity of $m(z, x)$. Using the definition of the consumption price function in conjunction with the triangle inequality, we obtain that

$$|\overline{g}(z, x)| \leq K_6 \sum_{a=1}^{A} |m(z, x)|^{-b_a}$$

for some constant $K_6 > 0$. As a result, we have that there exists an index $a \equiv a(x)$ as well as a strictly positive constant K_7 such that

$$|m(z, x)|^{-b_{a(x)}} \geq \frac{|\overline{g}(z, x)|}{AK_6} \geq \frac{|\Re\overline{g}(z, x)|}{AK_6} \geq K_7 \overline{g}(\Re z, x)$$

and the desired conclusion follows. *Q.E.D.*

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