

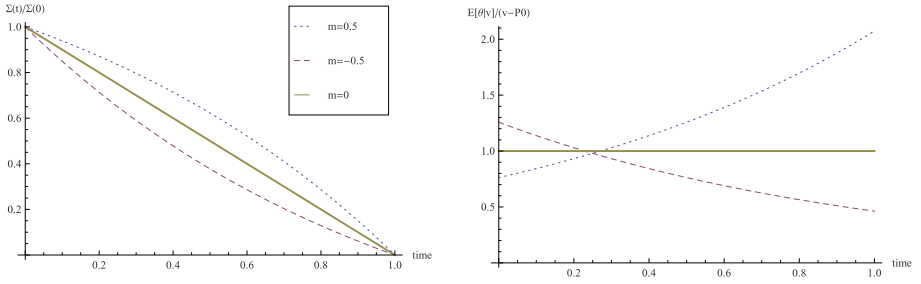
SUPPLEMENT TO “INSIDER TRADING, STOCHASTIC LIQUIDITY,  
AND EQUILIBRIUM PRICES”: ONLINE APPENDIX  
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S1. DETERMINISTIC GROWTH RATE OF NOISE TRADER VOLATILITY

IN GENERAL, WHEN NOISE TRADING VOLATILITY IS STOCHASTIC ( $\nu \neq 0$ ) and there is predictability ( $m_t \neq 0$ ), then price impact is stochastic and negatively correlated (in changes) with noise trading volatility. However, price volatility and the posterior variance of the fundamental value ( $\Sigma_t$ ) are both deterministic and only depend on the unconditional expected path of noise trading volatility. For illustration, Figure S1(a) plots the paths of the posterior variance ( $\Sigma_t$ ) for three cases of constant growth rate  $m = 0.5$ ,  $m = 0$ , and  $m = -0.5$ . It is remarkable that irrespective of  $\nu_t$  (and thus of realized shocks to noise trading, i.e., of realized volume), private information is revealed following a deterministic path, which only depends on the expected rate of change in noise trading volatility, despite the fact that the strategy of the insider is stochastic. This is, of course, the result of the offsetting effect noise trading volatility has on price impact. If the level of noise trading variance increases (decreases) unexpectedly, then the insider trades more (less) aggressively, but price impact decreases (increases) one for one, making price dynamics and information arrival rate independent of the volatility level. Instead, if noise trading variance is expected to increase on average because  $m > 0$ , for example, then the insider is expected to scale back his trading initially and to trade more aggressively later on. As Figure S1(a) shows, this leads to private information getting into prices more slowly initially, and then faster later on. So posterior variance follows a deterministic concave path if noise trading volatility is expected to increase, but a convex path if it is expected to decrease. As a result, the equilibrium price process exhibits deterministic time-varying volatility. Price volatility increases (decreases) exponentially if noise trading volatility is expected to increase (decrease).

In Figure S1(b), we plot the expected optimal trading rate of the insider normalized by the initial undervaluation ( $E[\theta_t|v, \mathcal{F}_0]/(v - P_0)$ ) from equation (27) in the paper) for different levels of constant  $m$ . As we can see, when noise trading volatility is unpredictable ( $m = 0$ ), then we expect the insider to trade at a constant rate as in the Kyle model. Instead, if noise trading volatility is expected to increase ( $m > 0$ ), then, unconditionally, we expect the insider to trade more aggressively on average in the future when more noise trading will occur. In fact, when  $m > 0$ , the insider initially scales back his expected trading relative to the  $m = 0$  case despite the fact that he starts out with the same level of noise trader volatility ( $\sigma_0$ ) in both cases and indeed, despite the fact that the



(a) Path of posterior variance of the insider's private information scaled by the prior variance ( $\Sigma_t/\Sigma_0$ ) for various values of the expected change in noise trading volatility  $m$ .

(b) Trading strategy of the insider normalized by his expected profit ( $E[\theta_t | v, \mathcal{F}_0]/(v - P_0)$ ) plotted against time and for different levels of expected growth rate of noise trading volatility.

FIGURE S1.—Constant growth rate ( $m$ ) of noise trading volatility ( $\sigma_0 = 0.4$ ,  $\Sigma_0 = 0.04$ ,  $T = 1$ ).

variance of noise trading is always expected to be larger in the  $m > 0$  case than in the  $m = 0$  case.<sup>1</sup>

Notice that price volatility is not affected by the realized shocks in noise trading volatility when  $m_t$  is deterministic. Only the unconditional ex ante expected volume matters for the rate at which information will flow into prices ex post and consequently for future price volatility. Even though price impact is stochastic, price volatility is not stochastic, and the model cannot generate any contemporaneous relation between changes in volume and price volatility or between price impact and price volatility. To generate such relations, we need a stochastic growth rate of noise trading volatility.

## S2. MEAN-REVERTING NOISE TRADING VOLATILITY

Here we consider an example where noise trading volatility follows a diffusion process with mean-reversion. Specifically, we consider the case where  $x_t = \log \sigma_t$  follows a mean-reverting Ornstein–Uhlenbeck process:

$$(S1) \quad dx_t = \left( -\frac{\nu^2}{2} - \kappa x_t \right) dt + \nu dW_t.$$

<sup>1</sup>This effect would be even starker, that is, the insider would scale back even more initially in the  $m > 0$  case, if we lowered the initial  $\sigma_0$  in the  $m > 0$  case so as to keep the expected noise trader variance ( $G_0$ ) and thus the insider's expected profit equal across both cases.

We parameterize the drift of  $x_t$  so that, when  $\kappa = 0$ , volatility is a martingale:

$$(S2) \quad \frac{d\sigma_t}{\sigma_t} = -\kappa x_t dt + \nu dW_t.$$

As a result, we can focus on the impact of mean-reversion alone, and use a series expansion in  $\kappa$  around the known solution when  $\kappa = 0$  (derived in Theorem 2 in the paper). The following result characterizes the solution.

**THEOREM S1:** *If the log of noise trading volatility follows a mean-reverting process as given in equation (S2), then the process  $G(t)$  is of the form*

$$(S3) \quad G(t) = \sigma_t^2 A(T - t, x_t, \kappa)^2,$$

where the function  $A(\tau, x, \kappa)$  can be approximated by a series expansion:

$$(S4) \quad A(\tau, x, \kappa) = \sqrt{T - t} \left( 1 + \sum_{i=1}^n (-k\tau)^i \left( \sum_{j=0}^i x^j \sum_{k=0}^{i-j} c_{ijk} t^k \right) + O(\kappa^{n+1}) \right),$$

where the  $c_{ijk}$  are positive constants that depend only on  $\nu^2$  and can be solved explicitly.<sup>2</sup> In that case, private information enters prices at a stochastic rate that depends on the level of noise trading volatility:

$$(S5) \quad \frac{d\Sigma_t}{\Sigma_t} = -\frac{1}{A(T - t, x_t, \kappa)^2} dt.$$

Market depth is stochastic and given by

$$(S6) \quad \lambda_t = \frac{\sqrt{\Sigma_t}}{\sigma_t A(T - t, x_t, \kappa)}.$$

The trading strategy of the insider is

$$(S7) \quad \theta_t = \frac{\sigma_t}{\sqrt{\Sigma_t} A(T - t, x_t, \kappa)} (v - P_t).$$

Stock price dynamics follow a three-factor  $(P, x, \Sigma)$  Markov process with stochastic volatility given by

$$(S8) \quad dP_t = \frac{(v - P_t)}{A(T - t, x_t, \kappa)^2} dt + \frac{\sqrt{\Sigma_t}}{A(T - t, x_t, \kappa)} dZ_t.$$

<sup>2</sup>We provide in the section below the fifth-order solution. Higher-order expansions can be obtained easily using Mathematica (program available upon request).

*In particular, stock price volatility is stochastic and tends to be higher when noise trading volatility is higher. The unconditional expected profit at time zero of the insider is  $T\sigma_v\sigma_0\frac{A_0}{\sqrt{T}}$ .*

PROOF: To prove this result, we observe that  $m_t = -\kappa x_t$  and that  $x_t$  has the following dynamics under the  $\tilde{P}$  measure:

$$(S9) \quad dx_t = \left( \frac{\nu^2}{2} - \kappa x_t \right) dt + \nu d\tilde{W}_t,$$

where, by Girsanov's theorem, we have defined  $\tilde{W}_t = W_t - \nu^2 t$  a standard  $\tilde{P}$ -measure Brownian motion. Thus,  $x_t$  is a one-factor Markov process under  $\tilde{P}$ . Using the Markov property for conditional expectations, we guess that the solution is a function  $A(t, x_t)$ . Substituting the guess from equation (S3) above into the recursive equation satisfied by  $G(t)$ , we see that  $A(t, x)$  satisfies

$$(S10) \quad \tilde{E}_t \left[ dA(t, x_t)/dt + m_t A(t, x_t) + \frac{1}{2A(t, x_t)} \right] = 0.$$

Using Itô's lemma, we obtain the following nonlinear PDE for  $A(T-t, x)$  (where we change variables to  $\tau = T-t$  and drop the argument of the function for simplicity):

$$(S11) \quad \frac{\nu^2}{2}(A_{xx} + A_x) - \kappa x(A_x + A) - A_\tau + \frac{1}{2A} = 0$$

subject to boundary conditions  $A(0, x) = 0$ . When  $\kappa = 0$ , the solution is simply  $A(\tau, x; \kappa = 0) = \sqrt{\tau}$ . Assuming the solution is analytic in its arguments, we seek a series expansion solution of the form given in equation (S4) above. Plugging this guess into the left-hand side of the PDE and Taylor expanding in  $\kappa$ , we find that each term in the series expansion can be set to zero by an appropriate choice of the constants  $c_{ijk}$ . We can thus recursively solve for these constants and obtain an approximate solution to the PDE. In Figure S2, we plot the zeroth-, first-, second-, and fifth-order expansion solution for  $\nu = 0.7$ ,  $T = 1$ ,  $\kappa = 0.25$ , and for three values of  $x_0 = \{-0.3; 0; +0.3\}$ . *Q.E.D.*

The first term in the series expansion of the  $A(\tau, x, \kappa)$  function is instructive. Indeed, we find

$$(S12) \quad A(\tau, x, \kappa) = \sqrt{\tau} \left( 1 - \frac{\kappa}{2} \tau \left( \frac{\nu^2 \tau}{6} + x \right) \right) + O(\kappa^2).$$

This confirms that we need  $\kappa$  to be different from zero for uncertainty about future noise trading volatility to affect the trading strategy of the insider, and equilibrium prices. We see that, for a given expected path of noise trading volatility (e.g., setting  $x = 0$  where it is expected to stay constant), the higher the mean-reversion strength  $\kappa$ , the lower the  $A$  function. This implies that mean-reversion tends to lower the profit of the insider for a given expected path of noise trading volatility (compare his profits to the case where  $\kappa = 0$ ).

Further, we see that the function is decreasing in (log) noise trading volatility if  $\kappa > 0$  (we confirm this for higher-order expansions). This implies that stock price volatility is stochastic and positively correlated with noise trading volatility. Equilibrium prices follow a Bridge process with stochastic volatility that is Markovian in three state variables. Private information gets incorporated into prices faster the higher the level of noise trading volatility, as the insider trades more aggressively in these states. Note that, since the  $A(\tau, \log \sigma, \kappa)$  function is decreasing and convex in volatility, the insider trades more aggressively than in the case where  $\kappa = 0$  (where  $A(t, \log \sigma)$  is independent of volatility). In these high-volatility states, market depth also improves, but less than proportionally to volatility to account for the more aggressive insider trading.

The net effect is that the insider's strategy changes as a function of uncertainty about future noise trading volatility, as the insider can benefit from timing market (liquidity) conditions in this context. In fact, the higher  $\nu^2$  is, the more aggressively does the insider choose to respond to a change in noise trading volatility (as  $A$  is decreasing in  $\nu^2$ ).

### S3. EXPANSION SOLUTION

The fifth-order expansion of the  $A$  function (with  $v = \nu^2$ ) is

$$\begin{aligned}
 A(\tau, x, \kappa) = & \sqrt{t} \left( 1 - \kappa t \left( \frac{vt}{12} + \frac{x}{2} \right) \right. \\
 & + \kappa^2 t^2 \left( \frac{13v^2 t^2}{1440} + x \left( \frac{vt}{12} + \frac{1}{6} \right) + \frac{7vt}{96} + \frac{5x^2}{24} \right) \\
 & - \kappa^3 t^3 \left( \frac{89v^3 t^3}{120,960} + x \left( \frac{3v^2 t^2}{320} + \frac{323vt}{2880} + \frac{1}{24} \right) \right. \\
 & + \frac{11v^2 t^2}{640} + x^2 \left( \frac{59vt}{1440} + \frac{1}{6} \right) + \frac{3vt}{80} + \frac{x^3}{16} \left. \right) \\
 & + \kappa^4 t^4 \left( \frac{1237v^4 t^4}{29,030,400} + \frac{337v^3 t^3}{161,280} \right.
 \end{aligned}$$

$$\begin{aligned}
& + x^2 \left( \frac{71v^2t^2}{16,128} + \frac{2593vt}{34,560} + \frac{59}{720} \right) + \frac{6827v^2t^2}{387,072} \\
& + x \left( \frac{17v^3t^3}{24,192} + \frac{2657v^2t^2}{120,960} + \frac{737vt}{8640} + \frac{1}{120} \right) \\
& + x^3 \left( \frac{vt}{80} + \frac{59}{720} \right) + \frac{31vt}{2160} + \frac{79x^4}{5760} \\
& - \kappa^5 (t^5) \left( \frac{6299v^5t^5}{3,832,012,800} + \frac{193v^4t^4}{1,244,160} + \frac{51,709v^3t^3}{16,588,800} \right. \\
& + x^3 \left( \frac{601v^2t^2}{483,840} + \frac{4673vt}{161,280} + \frac{59}{960} \right) + \frac{18,703v^2t^2}{1,451,520} \\
& + x^2 \left( \frac{4241v^3t^3}{14,515,200} + \frac{7129v^2t^2}{580,608} + \frac{9127vt}{120,960} + \frac{11}{360} \right) \\
& + x \left( \frac{287v^4t^4}{8,294,400} + \frac{49,439v^3t^3}{21,772,800} \right. \\
& + \left. \frac{319,777v^2t^2}{11,612,160} + \frac{2293vt}{48,384} + \frac{1}{720} \right) \\
& + x^4 \left( \frac{431vt}{161,280} + \frac{1}{40} \right) + \left. \frac{vt}{224} + \frac{3x^5}{1280} \right) \\
& + O(\kappa^6).
\end{aligned}$$

We illustrate the convergence of the expansion in Figures S2–S4.

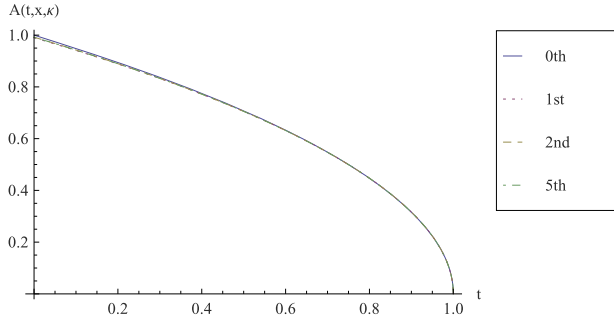


FIGURE S2.— $A$  function expansion solution given in equation (S4) for different order (0, 1, 2, 5) of the expansion for  $x_0 = 0$ . Other parameter values are  $\kappa = 0.25$ ,  $\nu = 0.7$ ,  $T = 1$ .

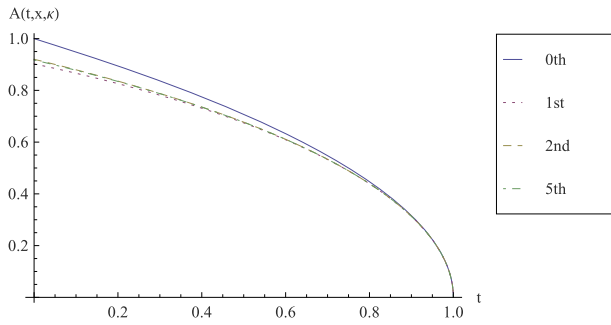


FIGURE S3.— $A$  function expansion solution given in equation (S4) for different order (0, 1, 2, 5) of the expansion for  $x_0 = +0.7$ . Other parameter values are  $\kappa = 0.25$ ,  $\nu = 0.7$ ,  $T = 1$ .

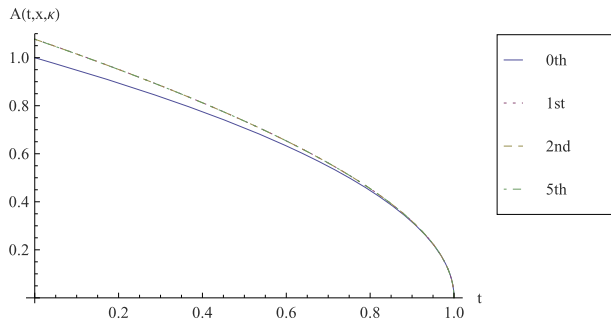


FIGURE S4.— $A$  function expansion solution given in equation (S4) for different order (0, 1, 2, 5) of the expansion for  $x_0 = -0.7$ . Other parameter values are  $\kappa = 0.25$ ,  $\nu = 0.7$ ,  $T = 1$ .

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