Econometrica Supplementary Material

SUPPLEMENT TO "SPURIOUS INFERENCE IN REDUCED-RANK ASSET-PRICING MODELS"

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THIS SUPPLEMENTAL MATERIAL IS STRUCTURED AS FOLLOWS. Section S.1 establishes the equivalence of the CU-GMM estimators with centered and uncentered optimal weighting matrices and serially correlated moment conditions. Section S.2 shows that the result in Theorem 2 in the paper continues to hold when we replace the assumption that $\sqrt{T}\,\mathrm{vec}(\hat{B} - B) \xrightarrow{d} \mathcal{N}(0_{NK}, E[x_t x_t']^{-1} \otimes \Sigma)$ with the assumption that the returns and the factors are jointly elliptically distributed. We refer the readers to the paper for the notation used here.

S.1. EQUIVALENCE BETWEEN CENTERED AND UNCENTERED CU-GMM ESTIMATORS

As pointed out in footnote 4 of the paper, Newey and Smith (2004) and Antoine, Bonnal, and Renault (2007) establish the equivalence of the CU-GMM estimators based on the centered and uncentered optimal weighting matrix in the case when $e_t(\lambda)$ is serially uncorrelated. When $e_t(\lambda)$ is serially correlated, the centered autocorrelation consistent estimator of $V_e(\lambda)$ is given by

$$\hat{V}_e(\lambda) = \sum_{j=-T+1}^{T-1} k(j/m)\hat{\Gamma}_j(\lambda), \tag{S.1}$$

where $k(j/m)$ is a kernel (weight) function and $m < T - 1$ is a lag truncation parameter such that $k(j/m) = 0$ if $j > m$,[1] and

$$\hat{\Gamma}_j(\lambda) = \begin{cases} \dfrac{1}{T}\displaystyle\sum_{t=j+1}^{T} \big(e_t(\lambda) - \bar{e}(\lambda)\big)\big(e_{t-j}(\lambda) - \bar{e}(\lambda)\big)' & \text{for } j \geq 0, \\ \dfrac{1}{T}\displaystyle\sum_{t=-j+1}^{T} \big(e_{t+j}(\lambda) - \bar{e}(\lambda)\big)\big(e_t(\lambda) - \bar{e}(\lambda)\big)' & \text{for } j < 0. \end{cases} \tag{S.2}$$

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[1]More specifically, the kernel function $k(x)$ is defined to be in the class of kernels that satisfy (i) $|k(x)| \leq 1$ and $k(x) = k(-x)$ for all $x \in \mathbb{R}$, (ii) $k(0) = 1$, (iii) $\int_{-\infty}^{\infty} |k(x)|\,\mathrm{d}x < \infty$, and (iv) $k(x)$ is continuous at zero and almost all $x \in \mathbb{R}$ (Andrews (1991)).

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The corresponding uncentered autocorrelation consistent estimator is

$$\tilde{V}_e(\lambda) = \sum_{j=-T+1}^{T-1} k(j/m)\tilde{\Gamma}_j(\lambda), \tag{S.3}$$

where

$$\tilde{\Gamma}_j(\lambda) = \begin{cases} \dfrac{1}{T}\displaystyle\sum_{t=j+1}^{T} e_t(\lambda)e_{t-j}(\lambda)' & \text{for } j \geq 0, \\ \dfrac{1}{T}\displaystyle\sum_{t=-j+1}^{T} e_{t+j}(\lambda)e_t(\lambda)' & \text{for } j < 0. \end{cases} \tag{S.4}$$

Note that

$$\begin{aligned} \hat{\Gamma}_j(\lambda) &= \tilde{\Gamma}_j(\lambda) - \frac{1}{T}\sum_{t=j+1}^{T} e_t(\lambda)\bar{e}(\lambda)' - \bar{e}(\lambda)\frac{1}{T}\sum_{t=j+1}^{T} e_{t-j}(\lambda)' + \bar{e}(\lambda)\bar{e}(\lambda)' \\ &= \tilde{\Gamma}_j(\lambda) - \bar{e}(\lambda)\bar{e}(\lambda)'. \end{aligned} \tag{S.5}$$

Then we have

$$\tilde{V}_e(\lambda) = \hat{V}_e(\lambda) + a\bar{e}(\lambda)\bar{e}(\lambda)', \tag{S.6}$$

where $a = \sum_{j=-T+1}^{T-1} k(j/m)$. Using the identity

$$x'\left(A + axx'\right)^{-1}x = x'\left(A^{-1} - \frac{aA^{-1}xx'A^{-1}}{1 + ax'A^{-1}x}\right)x = \frac{x'A^{-1}x}{1 + ax'A^{-1}x}, \tag{S.7}$$

we obtain

$$\bar{e}(\lambda)'\tilde{V}_e(\lambda)^{-1}\bar{e}(\lambda) = \frac{\bar{e}(\lambda)'\hat{V}_e(\lambda)^{-1}\bar{e}(\lambda)}{1 + a\bar{e}(\lambda)'\hat{V}_e(\lambda)^{-1}\bar{e}(\lambda)}. \tag{S.8}$$

This implies that the CU-GMM estimator of λ is the same regardless of whether we use $\hat{V}_e(\lambda)^{-1}$ or $\tilde{V}_e(\lambda)^{-1}$ as a weighting matrix.

S.2. ASYMPTOTIC DISTRIBUTION OF THE $\mathcal{J}$ TEST UNDER MULTIVARIATE ELLIPTICITY

Suppose R_t and f_t are i.i.d. multivariate elliptically distributed with finite fourth moments and kurtosis parameter $\kappa = \mu_4/(3\sigma^4) - 1$, where σ^2 and μ_4 are the second and fourth central moments of the elliptical distribution. The class of elliptical distributions includes normal, student t, Cauchy, Laplace, symmetric stable, and logistic distributions, among others, as special cases. Note also that under multivariate ellipticity, except for the multivariate normal case, the returns R_t exhibit conditional heteroskedasticity. The generating process for the test asset payoffs is assumed to be

$$R_t = Bx_t + \varepsilon_t. \tag{S.9}$$

For any nonzero vector c, the asymptotic covariance matrix of $\sqrt{T}(P_1'\hat{D}c - P_1'Dc)$ under the multivariate ellipticity assumption is given by

$$\begin{aligned}\mathcal{V}(c) &= E\big[(c'x_t)^2 P_1'R_tR_t'P_1\big] && \text{(S.10)}\\ &= P_1'E\big[(c'x_t)^2\varepsilon_t\varepsilon_t'\big]P_1 + P_1'BE\big[(c'x_t)^2 x_tx_t'\big]B'P_1 && \text{(S.11)}\\ &= \big\{(1+\kappa)c'E\big[x_tx_t'\big]c - \kappa\big(c'E[x_t]\big)^2\big\}P_1'\Sigma P_1 + P_1'BE\big[(c'x_t)^2 x_tx_t'\big]B'P_1 && \text{(S.12)}\\ &\equiv \mathcal{V}_1(c) + \mathcal{V}_2(c). && \text{(S.13)}\end{aligned}$$

A consistent estimator of $\mathcal{V}_1(c)$ can be obtained as

$$\mathcal{A}_1(c) = c'\left[(1+\kappa)\left(\frac{X'X}{T}\right) - \kappa\bar{x}\bar{x}'\right]cP_1'\hat{\Sigma}P_1, \tag{S.14}$$

where $\bar{x} = \sum_{t=1}^T x_t/T$. Similarly, a consistent estimator of $\mathcal{V}_2(c)$ is given by

$$\mathcal{A}_2(c) = P_1'\hat{B}\left[\frac{1}{T}\sum_{t=1}^T (c'x_t)^2 x_tx_t'\right]\hat{B}'P_1. \tag{S.15}$$

Let $\mathcal{A}(c) = \mathcal{A}_1(c) + \mathcal{A}_2(c)$. A similar proof as in the paper allows us to show that we can obtain an asymptotically equivalent $\mathcal{J}$ test by dropping $\mathcal{A}_2(c)$ so that

$$\begin{aligned}\mathcal{J} &= T\min_{c:c'c=1} c'\hat{D}'P_1\mathcal{A}(c)^{-1}P_1'\hat{D}c = T\min_{c:c'c=1} c'\hat{D}'P_1\mathcal{A}_1(c)^{-1}P_1'\hat{D}c + o_p(1) && \text{(S.16)}\\ &= T\min_{\tilde{c}:\tilde{c}'\tilde{c}=1} \frac{\tilde{c}'\hat{B}'P_1\big(P_1'\hat{\Sigma}P_1\big)^{-1}P_1'\hat{B}\tilde{c}}{\tilde{c}'\left[(1+\kappa)\left(\frac{X'X}{T}\right)^{-1} - \kappa e_1e_1'\right]\tilde{c}} + o_p(1), && \text{(S.17)}\end{aligned}$$

where $e_1 = [1, 0_{K-1}']'$. Since

$$\sqrt{T}\big(P_1'\hat{B}\tilde{c} - P_1'B\tilde{c}\big) \xrightarrow{d} \mathcal{N}\big(0_{N-1}, \tilde{c}'\big[(1+\kappa)E\big[x_tx_t'\big]^{-1} - \kappa e_1e_1'\big]\tilde{c}P_1'\Sigma P\big), \tag{S.18}$$

we can proceed as in the proof of Theorem 2 in the paper to obtain the limiting distribution of $\mathcal{J}$. Specifically, let L be a lower triangular matrix such that

$$LL' = \big[(1+\kappa)E\big[x_tx_t'\big]^{-1} - \kappa e_1e_1'\big]^{-1}. \tag{S.19}$$

Then the $\mathcal{J}$ test has the same distribution as the smallest eigenvalue of

$$W = TL'\hat{B}'P_1\big(P_1'\Sigma P_1\big)^{-1}P_1'\hat{B}L = TZ'Z, \tag{S.20}$$

where $Z = (P_1'\Sigma P_1)^{-\frac{1}{2}}P_1'\hat{B}L$. The multivariate normality case is obtained by setting $\kappa = 0$.

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