Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL AUCTION DESIGN WITH COMMON VALUES: AN INFORMATIONALLY ROBUST APPROACH"

(*Econometrica*, Vol. 89, No. 3, May 2021, 1313–1360)

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APPENDIX B: PROOFS FOR SECTION 5

B.1. *Proof of Proposition 5*

LET $\Delta = 1/K$ and recall that the message space for $\overline{\mathcal{M}}(\underline{m}, K)$ is

$$M_i = \{\underline{m}, \underline{m} + \Delta, \dots, \underline{m} + K\}.$$

Note that the highest message $\overline{m} = \underline{m} + K$ is at least Δ^{-1}. We shall extend the domain of the allocation and transfer rules to all of $\mathbb{R}^N_+$ for notational convenience. Given an allocation rule $q : M \to [0, 1]^N$ and transfer rule $t : M \to \mathbb{R}$, the discrete aggregate allocation sensitivity is

$$\mu(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} \big(q_i(m_i + \Delta, m_{-i}) - q_i(m) \big)$$

and the discrete aggregate excess growth is

$$\Xi(m) = \frac{1}{\Delta} \sum_{i=1}^{N} \mathbb{I}_{m_i < \overline{m}} \big(t_i(m_i + \Delta, m_{-i}) - t_i(m) \big) - \Sigma t(m).$$

Now define

$$\lambda(m; v) = v\mu(m) - \Xi(m) - c\overline{Q}(\Sigma m)$$

and let $\lambda(v) = \inf_{m \in M} \lambda(m; v)$.

LEMMA S1: *For any information structure $\mathcal{S}$ and equilibrium β of $(\mathcal{S}, \overline{\mathcal{M}}(\underline{m}, K))$, expected profit is at least $\int_V \lambda(v) H(dv)$.*

PROOF: The equilibrium hypothesis implies that for all i,

$$\int_S \sum_{m \in M} \big[w(s) \big(q_i\big(\min\{m_i + \Delta, \overline{m}\}, m_{-i}\big) - q_i(m) \big) \\ - \big(t_i\big(\min\{m_i + \Delta, \overline{m}\}, m_{-i}\big) - t_i(m) \big) \big] \beta(m|s) \pi(ds) \leq 0,$$

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 https://doi.org/10.3982/ECTA16297

which corresponds to the incentive constraint for deviating to $\min\{m_i+\Delta, \overline{m}\}$. Summing across bidders and dividing by Δ, we conclude that

$$\int_S \sum_{m\in M}\big[w(s)\mu(m) - \Xi(m) - \Sigma t(m)\big]\beta(m|s)\pi(ds) \le 0.$$

Hence, expected profit is

$$\begin{aligned}
&\int_S \sum_{m\in M}\big[\Sigma t(m) - cQ(\Sigma m)\big]\beta(m|s)\pi(ds)\\
&\quad\ge \int_S \sum_{m\in M}\big[\Sigma t(m) - cQ(\Sigma m) + w(s)\mu(m) - \Xi(m) - \Sigma t(m)\big]\beta(m|s)\pi(ds)\\
&\quad= \int_S \sum_{m\in M}\big[w(s)\mu(m) - \Xi(m) - cQ(\Sigma m)\big]\beta(m|s)\pi(ds)\\
&\quad\ge \int_S \lambda\big(w(s)\big)\pi(ds)\\
&\quad\ge \int_V \lambda(v)H(dv),
\end{aligned}$$

where the last line follows from the mean-preserving spread condition on $w(s)$ and the fact that λ is concave, being the infimum of linear functions. *Q.E.D.*

LEMMA S2: *For all $m \in M$,*

$$\mu(m) \ge \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{\mu}(\Sigma m + y)\,dy - \widehat{L}(\underline{m}, \Delta),$$

where

$$\widehat{L}(\underline{m}, \Delta) = N(N+1)\Delta + \frac{N(N-1)}{\Delta}\left(\log(N\underline{m}+\Delta) + \frac{N\underline{m}}{N\underline{m}+\Delta} - \log(N\underline{m}) - 1\right).$$

Moreover, for all $\underline{m} > 0$, $\widehat{L}(\underline{m}, \Delta) \to 0$ as $\Delta \to 0$.

PROOF: From Lemma 12, we know that

$$\begin{aligned}
\mu(m) &= \sum_{i=1}^{N} \frac{1}{\Delta}\big(q_i(m_i+\Delta, m_{-i}) - q_i(m)\big) - \sum_{i=1}^{N} \mathbb{I}_{m_i=\overline{m}}\frac{1}{\Delta}\big(q_i(m_i+\Delta, m_{-i}) - q_i(m)\big)\\
&\ge \sum_{i=1}^{N} \frac{1}{\Delta}\big(q_i(m_i+\Delta, m_{-i}) - q_i(m)\big) - N\frac{N+1}{\overline{m}}\\
&\ge \sum_{i=1}^{N} \frac{1}{\Delta}\big(q_i(m_i+\Delta, m_{-i}) - q_i(m)\big) - N(N+1)\Delta.
\end{aligned}$$

Recall that

$$\overline{\mu}(x) = \frac{N-1}{x}\overline{Q}(x) + \overline{Q}'(x).$$

Also recall that

$$\frac{\partial q_i(m)}{\partial m_i} = \frac{\Sigma m_{-i}}{(\Sigma m)^2}\overline{Q}(\Sigma m) + \frac{m_i}{\Sigma m}\overline{Q}'(\Sigma m).$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{\Delta}\big(q_i(m_i+\Delta, m_{-i}) - q_i(m)\big)\\
&\quad = \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta} \frac{\partial q_i(m_i+y, m_{-i})}{\partial m_i}\,dy\\
&\quad = \frac{1}{\Delta}\sum_{i=1}^{N}\int_{y=0}^{\Delta}\left(\frac{\Sigma m_{-i}}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{m_i+y}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right)dy\\
&\quad = \frac{1}{\Delta}\int_{y=0}^{\Delta}\left(\frac{(N-1)\Sigma m}{(\Sigma m+y)^2}\overline{Q}(\Sigma m+y) + \frac{\Sigma m+Ny}{\Sigma m+y}\overline{Q}'(\Sigma m+y)\right)dy\\
&\quad = \frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\mu}(\Sigma m+y)\,dy - \frac{N-1}{\Delta}\int_{y=0}^{\Delta}\frac{y}{\Sigma m+y}\left(\frac{\overline{Q}(\Sigma m+y)}{\Sigma m+y} - \overline{Q}'(\Sigma m+y)\right)dy.
\end{aligned}$$

We need to bound the last integral from above. If x is in a nongraded interval, then $\overline{Q}(x)/x - \overline{Q}'(x)$ is just $1/x$. If x is in a graded interval $[a, b]$, then

$$\frac{\overline{Q}(x)}{x} - \overline{Q}'(x) = \frac{C(a,b)}{N} + \frac{D(a,b)}{x^N} - \frac{C(a,b)}{N} + (N-1)\frac{D(a,b)}{x^N} = \frac{ND(a,b)}{x^N}.$$

From equation (33), $D(a,b) \le x^{N-1}$, so that the integrand in this case is at most N/x, and

$$\begin{aligned}
\int_{y=0}^{\Delta}\frac{y}{x+y}\left(\frac{\overline{Q}(x+y)}{x+y} - \overline{Q}'(x+y)\right)dy &\le N\int_{y=0}^{\Delta}\frac{y}{(x+y)^2}\,dy\\
&= N\int_{y=0}^{\Delta}\left(\frac{1}{x+y} - \frac{x}{(x+y)^2}\right)dy\\
&= N\left(\log(x+\Delta) + \frac{x}{x+\Delta} - \log(x) - 1\right).
\end{aligned}$$

The derivative with respect to x is

$$N\left(\frac{1}{x+\Delta} - \frac{1}{x} + \frac{\Delta}{(x+\Delta)^2}\right) = N\Delta\left(\frac{1}{(x+\Delta)^2} - \frac{1}{x(x+\Delta)}\right),$$

which is clearly negative. Thus, subject to $x \ge N\underline{m}$, the expression is maximized with $x = N\underline{m}$, which gives us the lower bound on μ.

Moreover, as $\Delta \to 0$, $N(N+1)\Delta \to 0$ and by l'Hôpital's rule,

$$\lim_{\Delta\to 0}\left(\frac{\log(N\underline{m}+\Delta)+\dfrac{N\underline{m}}{N\underline{m}+\Delta}-\log(N\underline{m})-1}{\Delta}\right)$$
$$=\lim_{\Delta\to 0}\left(\frac{1}{N\underline{m}+\Delta}-\frac{N\underline{m}}{(N\underline{m}+\Delta)^2}\right)=0. \qquad \text{Q.E.D.}$$

We define $\Xi^p(m)=\Xi(m)-\underline{v}(\mu(m)-Q(m))$. Recall that $\overline{\Xi}^p(x)=\overline{\Xi}(x)-\underline{v}(\overline{\mu}(x)-\overline{Q}(x))$. These are the excess growths for the "premium" transfers $t_i^p(m)=t_i(m)-\underline{v}\,q_i(m)$ and $\overline{t}_i^p(m)=\overline{t}_i(m)-\underline{v}\,\overline{q}_i(m)$, respectively. We similarly denote by $\overline{T}^p(x)=\overline{T}(x)-\underline{v}\,\overline{Q}(x)$ the aggregate premium transfer, and note that $\overline{T}^p$ satisfies the differential equation

$$\left(\frac{N-1}{x}-1\right)\overline{T}^p(x)+\frac{d}{dx}\overline{T}^p(x)=\overline{\Xi}^p(x),$$

with the boundary condition $\overline{T}^p(0)=0$.

LEMMA S3: *Let L_Ξ be an upper bound on $|\overline{\Xi}^p|$ and let L_T be an upper bound on $\overline{T}^p$. Then*

$$\Xi^p(m)\le\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\Xi}^p(\Sigma m+y)\,dy+\tilde{L}(\underline{m})\frac{\Delta}{2}+NL_p\underline{m}$$
$$-\frac{1}{\Delta}\sum_i\mathbb{I}_{m_i=\overline{m}}\big(\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\big),$$

where

$$\tilde{L}(\underline{m})=\left(1+\frac{N-1}{N\underline{m}}\right)L_p+\frac{N-1}{(N\underline{m})^2}L_T.$$

PROOF: Recall that $\overline{T}^p$ is Lipschitz with constant L_p. Furthermore, the function $\overline{T}^p(x)(N-1)/x$ is Lipschitz on $[N\underline{m},\infty)$ and

$$\left|\frac{d}{dx}\left(\frac{N-1}{x}\overline{T}^p(x)\right)\right|=\left|\frac{N-1}{x}\frac{d}{dx}\overline{T}^p(x)-\frac{N-1}{x^2}\overline{T}^p(x)\right|$$
$$\le\frac{N-1}{N\underline{m}}L_p+\frac{N-1}{(N\underline{m})^2}L_T=L_1(\underline{m}).$$

Using the differential equation for $\overline{T}^p$,

$$\frac{1}{\Delta}\int_{y=0}^{\Delta}\overline{\Xi}^p(\Sigma m+y)\,dy$$
$$=\frac{1}{\Delta}\int_{y=0}^{\Delta}\left[\left(\frac{N-1}{\Sigma m+y}-1\right)\overline{T}^p(\Sigma m+y)+\frac{d}{dx}\overline{T}^p(x)\bigg|_{x=\Sigma m+y}\right]dy$$

$$= \frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m+y}-1\right)\overline{T}^p(\Sigma m+y)\,dy+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$

$$\geq \frac{1}{\Delta}\left[\int_{y=0}^{\Delta}\left(\frac{N-1}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-L_1(\underline{m})(\Delta-y)-\overline{T}^p(\Sigma m)-L_p y\right)dy\right.$$
$$\left.+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$

$$= \frac{1}{\Delta}\left[\Delta\frac{N-1}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-\Delta\overline{T}^p(\Sigma m)-\big(L_1(\underline{m})+L_p\big)\frac{\Delta^2}{2}\right.$$
$$\left.+\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right]$$

$$= \frac{1}{\Delta}\left(\frac{\Sigma m+N\Delta}{\Sigma m+\Delta}\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right)-\overline{T}^p(\Sigma m)-\underbrace{\big(L_1(\underline{m})+L_p\big)}_{\equiv\tilde{L}(\underline{m})}\frac{\Delta}{2}.$$

We let $T^p(\Sigma m)$ denote the aggregate transfer when the messages are m. Thus,

$$\Xi^p(m)=\frac{1}{\Delta}\sum_{i=1}^{N}\big(t_i^p(m_i+\Delta,m_{-i})-t_i^p(m)\big)-T^p(\Sigma m)$$
$$-\frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\big(t_i^p(m_i+\Delta,m_{-i})-t_i^p(m)\big)$$
$$=\frac{1}{\Delta}\sum_{i=1}^{N}\big(\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\big)-T^p(\Sigma m)$$
$$-\frac{1}{\Delta}\sum_{i=1}^{N}\mathbb{I}_{m_i=\overline{m}}\big(\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\big)$$
$$\leq\frac{1}{\Delta}\left(\frac{\Sigma m+N\Delta}{(\Sigma m+\Delta)}\overline{T}^p(\Sigma m+\Delta)-\overline{T}^p(\Sigma m)\right)-T^p(\Sigma m)$$
$$-\frac{1}{\Delta}\sum_{i}\mathbb{I}_{m_i=\overline{m}}\big(\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\big).$$

The lemma follows from combining these two inequalities, with the observation that $T^p(x)=\overline{T}^p(x)-NL_p\underline{m}$. *Q.E.D.*

LEMMA S4: *For all $\epsilon>0$, there exists a K such that for all m such that $\Sigma m>K$ and for all i,*

$$\frac{1}{\Delta}\big|\overline{t}_i^p(m_i+\Delta,m_{-i})-\overline{t}_i^p(m)\big|<\epsilon.$$

PROOF: Since $\lim_{x\to\infty}\overline{T}^p(x)=-\overline{\Xi}^p(\infty)$, we can find a K large enough so that for $x>K$, $|\overline{T}^p(x)+\overline{\Xi}^p(\infty)|<\epsilon/4$ and $L_T/K<\epsilon/4$, and, thus, $|d\overline{T}^p(x)/dx|<\epsilon/2$. As a result, when $\Sigma m > K$, then using $\Delta = K^{-1}$,

$$\begin{aligned}
&\frac{1}{\Delta}\big(\bar{t}_i^p(m_i+\Delta, m_{-i}) - \bar{t}_i^p(m)\big) \\
&\quad = \frac{1}{\Delta}\int_{y=0}^{\Delta} \frac{\partial \bar{t}_i^p(m_i+y, m_{-i})}{\partial m_i}\,dy \\
&\quad = \frac{1}{\Delta}\int_{y=0}^{\Delta} \left(\frac{\Sigma m_{-i}}{(\Sigma m + y)^2}\overline{T}^p(\Sigma m + y) + \frac{m_i + y}{\Sigma m + y}\frac{d}{dx}\overline{T}^p(x)\bigg|_{x=\Sigma m + y}\right) dy \\
&\quad \leq \frac{L_T}{K} + \frac{\epsilon}{2} \\
&\quad < \epsilon.
\end{aligned}$$

Q.E.D.

PROOF OF PROPOSITION 5: We first argue that there exist $\underline{m}$ and a K such that $\lambda(m; v) \geq \inf_{m' \in \mathbb{R}^N} \overline{\lambda}(m'; v) - \epsilon$ for all $m \in M$ and $v \in [\underline{v}, \overline{v}]$, where

$$\overline{\lambda}(m; v) = (v - \underline{v})\overline{\mu}(\Sigma m) - \overline{\Xi}^p(\Sigma m) + (\underline{v} - c)\overline{Q}(\Sigma m).$$

From Lemma 12, we know that $|\overline{Q}(x+y) - \overline{Q}(x)| \leq y(N-1)/\underline{m}$. Thus,

$$\begin{aligned}
\left|\overline{Q}(x) - \frac{1}{\Delta}\int_{y=0}^{\Delta} \overline{Q}(x+y)\,dy\right| &\leq \frac{1}{\Delta}\int_{y=0}^{\Delta} \big|\overline{Q}(x+y) - Q(x)\big|\,dy \\
&\leq \frac{1}{\Delta}\int_{y=0}^{\Delta} y\frac{N-1}{\underline{m}}\,dy = \Delta\frac{N-1}{2\underline{m}}.
\end{aligned}$$

Combining this inequality with Lemmas S2 and S3, we get that

$$\begin{aligned}
\lambda(m; v) &= (v - \underline{v})\mu(m) - \Xi^p(m) + (\underline{v} - c)\overline{Q}(\Sigma m) \\
&\geq \frac{1}{\Delta}\int_{y=0}^{\Delta} \big[(v - \underline{v})\overline{\mu}(\Sigma m + \Delta) - \overline{\Xi}^p(\Sigma m + y) + (\underline{v} - c)\overline{Q}(\Sigma m + y)\big]\,dy \\
&\quad - (\overline{v} - \underline{v})\widehat{L}(\underline{m}, \Delta) - \overline{v}\Delta\frac{N-1}{2\underline{m}} - \frac{\Delta}{2}\tilde{L}(\underline{m}) - NL_p\underline{m} \\
&\quad - \frac{1}{\Delta}\sum_i \mathbb{I}_{m_i = \overline{m}}\big|\bar{t}_i^p(m_i+\Delta, m_{-i}) - \bar{t}_i^p(m)\big| \\
&\geq \inf_{\{m' | \Sigma m \leq \Sigma m' \leq \Sigma m + \Delta\}} \overline{\lambda}\big(m'; v\big) \\
&\quad - (\overline{v} - \underline{v})\widehat{L}(\underline{m}, \Delta) - \overline{v}\Delta\frac{N-1}{2\underline{m}} - \frac{\Delta}{2}\tilde{L}(\underline{m}) - NL_p\underline{m} \\
&\quad - \frac{1}{\Delta}\sum_i \mathbb{I}_{m_i = \overline{m}}\big|\bar{t}_i^p(m_i+\Delta, m_{-i}) - \bar{t}_i^p(m)\big|.
\end{aligned}$$

We first pick $\underline{m} > 0$ so that $NL_p\underline{m} < \epsilon/2$. We then pick K large enough (and Δ small enough) such that the remaining terms in the last two lines sum to less than $\epsilon/2$ (where for the first term in the middle line and last line, this follows from Lemmas S2 and S4, respectively). We then conclude that

$$\lambda(m; v) \geq \inf_{m' \in \mathbb{R}^+_N} \overline{\lambda}\big(m'; v\big) - \epsilon \geq \overline{\lambda}(v) - \epsilon.$$

Hence, $\lambda(v) \geq \overline{\lambda}(v) - \epsilon$, and Lemma S1 and Lemma 6 give the result. *Q.E.D.*

The preceding proof goes through verbatim with the maxmin must-sell mechanism $\widehat{\mathcal{M}}$.

B.2. *Proof of Proposition 6*

Recall the definition of $\overline{\mathcal{S}}(K)$. Let $\Delta = 1/K$. We subsequently choose K sufficiently large (and, equivalently, Δ sufficiently small) to attain the desired ϵ. Note that the signal space can be written

$$S_i = \{0, \Delta, \dots, K^2\Delta\}$$

and the highest message is simply Δ^{-1}. The probability mass function of s_i is

$$f_i(s_i) = \begin{cases} \big(1 - \exp(-\Delta)\big)\exp(-s_i) & \text{if } s_i < \Delta^{-1}, \\ \exp\big(-\Delta^{-1}\big) & \text{if } s_i = \Delta^{-1}. \end{cases}$$

As a result, s_i/Δ is a censored geometric random variable with arrival rate $1 - \exp(-\Delta)$. We write $f(s) = \bigtimes_{i=1}^N f_i(s_i)$ for the joint probability and write

$$F_i(s_i) = \sum_{s'_i \leq s_i} f_i\big(s'_i\big) = \begin{cases} 1 - \exp(-s_i - \Delta) & \text{if } s_i < \Delta^{-1}, \\ 1 & \text{otherwise} \end{cases}$$

for the cumulative distribution. The value function is

$$w(s) = \frac{1}{f(s)} \int_{\{s' \in \mathbb{R}^N_+ | \tau(s'_i) = s_i \forall i\}} \overline{w}\big(\Sigma s'\big) \exp\big(-\Sigma s'\big)\, ds',$$

where

$$\tau(x) = \begin{cases} \Delta\lfloor x/\Delta \rfloor & \text{if } x < \Delta^{-1}, \\ \Delta^{-1} & \text{otherwise.} \end{cases}$$

An interpretation is that we draw "true" signals s' for the bidders from $\overline{\mathcal{S}}$ and agent i observes $s_i = \min\{\Delta\lfloor\Delta^{-1}s'_i\rfloor, \Delta^{-1}\}$, that is, signals above Δ^{-1} are censored, signals below Δ^{-1} are rounded down to the nearest multiple of Δ, and w is the conditional expectation of $\overline{w}$ given the noisy observations s. It is immediate that the distribution of $\overline{w}$ is a mean-preserving spread of the distribution of w, so that H is a mean-preserving spread of the distribution of w as well.

LEMMA S5: *If $s_i < \Delta^{-1}$ for all i, then $w(s)$ only depends on the sum of the signals $l = \Sigma s$ and*

$$w(s) = \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\rho(x-l)\exp(-x)\,dx,$$

where $\rho(y)$ is the $(N-1)$-dimensional volume of the set $\{s \in [0,\Delta]^N | \Sigma s = y\}$.

PROOF: First observe that for a signal profile s such that $s_i < \Delta^{-1}$ for all i,

$$f(s) = \big(1-\exp(-\Delta)\big)^N \exp(-\Sigma s) = \big(1-\exp(-\Delta)\big)^N \exp(-l).$$

Thus,

$$\begin{aligned}
w(s) &= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{\{s' \in \mathbb{R}_+^N | \tau_i(s') = s_i\ \forall i\}} \overline{w}\big(\Sigma s'\big)\exp\big(-\Sigma s'\big)\,ds' \\
&= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l}^{l+N\Delta} \int_{\{s' \in \mathbb{R}_+^N | \tau_i(s') = s_i\ \forall i, \Sigma s' = x\}} \overline{w}\big(\Sigma s'\big)\exp\big(-\Sigma s'\big)\,ds'\,dx \\
&= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\exp(-x) \int_{\{s' \in \mathbb{R}_+^N | \tau_i(s'_i - s_i) = 0\ \forall i, \Sigma s' = x\}} ds'\,dx \\
&= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l}^{l+N\Delta} \overline{w}(x)\exp(-x) \int_{\{s' \in \mathbb{R}_+^N | \tau_i(s') = 0\ \forall i, \Sigma s' = x-l\}} ds'\,dx,
\end{aligned}$$

where the inner integral is just $\rho(x-l)$. *Q.E.D.*

We now abuse notation slightly by writing $w(l)$ for the value when $l = \Sigma s$, and we let $\gamma(l) = w(l) - c$.

LEMMA S6: *If $l > \Delta$, then $\gamma(l) \le \exp(\Delta)\gamma(l-\Delta)$.*

PROOF: From Lemma S5, we know that

$$\begin{aligned}
\gamma(l) &= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l}^{l+N\Delta} \overline{\gamma}(x)\exp(-x)\rho(x-l)\,dx \\
&= \frac{\exp(l)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x+\Delta)\exp(-x-\Delta)\rho(x-l+\Delta)\,dx \\
&\le \frac{\exp(l-\Delta)}{\big(1-\exp(-\Delta)\big)^N} \int_{x=l-\Delta}^{l+(N-1)\Delta} \overline{\gamma}(x)\exp(\Delta)\exp(-x)\rho(x-l+\Delta)\,dx \\
&= \exp(\Delta)\gamma(l-\Delta),
\end{aligned}$$

where the inequality follows from Lemma 2. *Q.E.D.*

LEMMA S7: *If the direct allocation $q_i(s)$ is incentive compatible and individually rational, profit is at most*

$$\sum_{s\in S} f(s)\sum_{i=1}^{N} q_i(s)\left(\gamma(\Sigma s) - \frac{1-F_i(s_i)}{f_i(s_i)}\big(\gamma(\Sigma s+\Delta)-\gamma(\Sigma s)\big)\right). \tag{S1}$$

PROOF: This follows from standard revenue equivalence arguments: We write $U_i(s_i, s_i')$ for the utility of a signal s_i that reports s_i', with $U_i(s_i) = U_i(s_i, s_i)$. Incentive compatibility implies that

$$U_i(s_i) \geq U_i(s_i, s_i') = U_i(s_i') + \sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(s_i', s_{-i})\big(\gamma(s_i+\Sigma s_{-i}) - \gamma(s_i'+\Sigma s_{-i})\big).$$

Thus, for $s_i \geq \Delta$,

$$U_i(s_i) \geq U_i(0) + \sum_{k=0}^{s_i/\Delta-1}\sum_{s_{-i}\in S_{-i}} f_{-i}(s_{-i})q_i(k\Delta, s_{-i})\big(\gamma\big((k+1)\Delta+\Sigma s_{-i}\big) - \gamma(k\Delta+\Sigma s_{-i})\big).$$

The expectation of $U_i(s_i)$ across s_i is therefore bounded below by

$$\begin{aligned}&\sum_{s\in S} f(s)\sum_{k=0}^{s_i/\Delta-1} q_i(k\Delta, s_{-i})\big(\gamma\big((k+1)\Delta+\Sigma s_{-i}\big) - \gamma(k\Delta+\Sigma s_{-i})\big)\\ &\quad = \sum_{s\in S} f(s)q_i(s)\big(\gamma(\Sigma s+\Delta)-\gamma(\Sigma s)\big)\frac{1-F_i(s_i)}{f_i(s_i)}.\end{aligned}$$

The formula then follows from subtracting the bound on bidder surplus from total surplus. *Q.E.D.*

Let $\tilde{\Pi}$ denote the profit bound when we set $q_1(s) = 1$ and $q_j(s) = 0$ for all $j \neq 1$.

LEMMA S8: *For any allocation q, the expression* (S1) *is at most* $\tilde{\Pi} + (1-(1-\exp(-\Delta^{-1}))^N)\overline{v}$.

PROOF: When signals are all less than Δ^{-1}, the bidder-independent virtual value is

$$\begin{aligned}&\gamma(l) - \frac{1}{\exp(\Delta)-1}\big(\gamma(l+\Delta)-\gamma(l)\big)\\ &\quad \geq \gamma(l) - \frac{\exp(-\Delta)}{1-\exp(-\Delta)}\big(\gamma(l)\exp(\Delta)-\gamma(l)\big) = 0,\end{aligned}$$

where the inequality follows from Lemma S6. Thus, the virtual value is maximized pointwise by allocating with probability 1 to, say, bidder 1. With probability $1-(1-\exp(-\Delta^{-1}))^N$, one of the signals is above Δ^{-1}, in which case $\overline{v}$ is an upper bound on the virtual value. *Q.E.D.*

LEMMA S9: *The limit of $\tilde{\Pi}$ as $\Delta \to 0$ is less than $\overline{\Pi}$.*

PROOF: Plugging in $q_1 = 1$, we find that

$$\begin{aligned}\tilde{\Pi} &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1\in S_1}\Big(f_1(s_1)\gamma(\Sigma s) - \sum_{s_1'>s_1} f_1(s_1')\big(\gamma(\Sigma s+\Delta)-\gamma(\Sigma s)\big)\Big)\\ &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1}) \sum_{s_1\in S_1}\Big(f_1(s_1)\Big[\gamma(\Sigma s) + \sum_{s_1'<s_1}\big(\gamma(s_1'+\Sigma s_{-1}) - \gamma(s_1'+\Sigma s_{-1}+\Delta)\big)\Big]\Big)\\ &= \sum_{s_{-1}\in S_{-1}} f_{-1}(s_{-1})\gamma(\Sigma s_{-1}).\end{aligned}$$

Using the definition of γ, this is

$$\begin{aligned}\tilde{\Pi} &= \frac{1}{1-\exp(-\Delta)}\int_{y=0}^{\Delta}\int_{x=0}^{\infty}\overline{\gamma}(x+y)g_{N-1}(x)\exp(-y)\,dx\,dy\\ &= \frac{1}{1-\exp(-\Delta)}\int_{x=0}^{\infty}\overline{\gamma}(x)\int_{y=0}^{\min\{x,\Delta\}} g_{N-1}(x-y)\exp(-y)\,dy\,dx\\ &\leq \frac{1}{1-\exp(-\Delta)}\bigg(\int_{x=\Delta}^{\infty}\overline{\gamma}(x)\int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)\,dy\,dx + G_N(\Delta)\overline{v}\bigg).\end{aligned}$$

Now observe that

$$\begin{aligned}\int_{y=0}^{\Delta} g_{N-1}(x-y)\exp(-y)\,dy &= \frac{x^{N-1}-(x-\Delta)^{N-1}}{(N-1)!}\exp(-x)\\ &\leq \frac{\Delta(N-1)x^{N-2}}{(N-1)!}\exp(-x) = \Delta g_{N-1}(x),\end{aligned}$$

where we have used convexity of x^{N-1}. Thus,

$$\tilde{\Pi} \leq \frac{\Delta}{1-\exp(-\Delta)}\int_{x=0}^{\infty}\overline{\gamma}(x)g_{N-1}(x)\,dx + \frac{G_N(\Delta)\overline{v}}{1-\exp(-\Delta)}.$$

An application of l'Hôpital's rule shows that the last term converges to zero as $\Delta \to 0$ and $\Delta/(1-\exp(-\Delta)) \to 1$; this implies the lemma. *Q.E.D.*

PROOF OF PROPOSITION 6: By Lemma S9, for any $\epsilon > 0$, we can pick $K = \Delta^{-1}$ sufficiently large that $\tilde{\Pi} \leq \overline{\Pi} + \epsilon/2$. Moreover, we can also take K large enough so that $(1-(1-\exp(-K))^N)\overline{v}$ is at most $\epsilon/2$. For any mechanism and equilibrium of $\overline{\mathcal{S}}(K)$, there is an incentive compatible and individually rational direct mechanism that has the same expected profit. By Lemmas S7 and S8, this expected profit is at most $\tilde{\Pi} + \epsilon/2$. Thus, we conclude that expected profit is at most $\overline{\Pi} + \epsilon$, which completes the proof of the proposition. *Q.E.D.*

Every step of the proof of Proposition 6 goes through in the must-sell case, where we replace $\overline{w}$ with $\widehat{w}$, except that we skip the step in Lemma S8 of proving that the discrete virtual value is nonnegative.

APPENDIX C: PROOFS FOR SECTION 6

PROOF OF LEMMA 9: The left-tail assumption is equivalently stated as follows: there exists some $\overline{\alpha} > 0$ and $\varphi > 1$ such that for all $0 \le \alpha' < \alpha \le \overline{\alpha}$,

$$H^{-1}(\alpha) - \underline{v} \le G_N^{-1}(\alpha)^{\varphi},$$

and if $\underline{v} > c$,

$$\frac{H^{-1}(\alpha) - c}{H^{-1}(\alpha') - c} \le \exp\bigl(G_N^{-1}(\alpha) - G_N^{-1}(\alpha')\bigr).$$

The following lemma implies that if the above two conditions hold for N, they hold for all $N' > N$ as well. *Q.E.D.*

LEMMA S10: *For any $N \ge 1$ and $N' > N$, there exists $\overline{\alpha} > 0$ such that $G_N^{-1}(\alpha) - G_N^{-1}(\alpha') \le G_{N'}^{-1}(\alpha) - G_{N'}^{-1}(\alpha')$ for all $0 \le \alpha' < \alpha \le \overline{\alpha}$.*

PROOF: Clearly it suffices to prove the lemma for $N' = N + 1$. Let us extend the definition of G_N to any real number N,

$$G_N(x) = \int_{y=0}^{x} e^{-y} \frac{y^{N-1}}{\Gamma(N)} \, dy,$$

where

$$\Gamma(N) = \int_{y=0}^{\infty} e^{-y} y^{N-1} \, dy.$$

(We have $\Gamma(N) = (N-1)!$ when $N \ge 1$ is an integer.)

By definition, we have

$$\int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{x^{N-1}}{\Gamma(N)} \, dx = \alpha.$$

Differentiating the above equation with respect to N gives

$$\frac{\partial G_N^{-1}(\alpha)}{\partial N} \frac{e^{-G_N^{-1}(\alpha)} G_N^{-1}(\alpha)^{N-1}}{\Gamma(N)} + \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\dfrac{x^{N-1}}{\Gamma(N)} \right)}{\partial N} \, dx = 0,$$

that is,

$$\begin{aligned}
\frac{\partial G_N^{-1}(\alpha)}{\partial N} &= \frac{\Gamma(N) e^{G_N^{-1}(\alpha)}}{G_N^{-1}(\alpha)^{N-1}} \left(- \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \frac{\partial \left(\dfrac{x^{N-1}}{\Gamma(N)} \right)}{\partial N} \, dx \right) \\
&= \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N) G_N^{-1}(\alpha)^{N-1}} \int_{x=0}^{G_N^{-1}(\alpha)} e^{-x} \bigl(-x^{N-1} \log(x) \Gamma(N) + x^{N-1} \Gamma'(N) \bigr) \, dx \\
&= \frac{e^{G_N^{-1}(\alpha)}}{\Gamma(N)} f\bigl(G_N^{-1}(\alpha), N\bigr),
\end{aligned}$$

where

$$f(z,N)=\frac{1}{z^{N-1}}\int_{x=0}^{z}e^{-x}\big(-x^{N-1}\log(x)\Gamma(N)+x^{N-1}\Gamma'(N)\big)\,dx.$$

Next, we compute

$$\begin{aligned}&\frac{\partial f(z,N)}{\partial z}\\&=\frac{1}{z^{2(N-1)}}\bigg(z^{N-1}e^{-z}\big(-z^{N-1}\log(z)\Gamma(N)+z^{N-1}\Gamma'(N)\big)\\&\quad-(N-1)z^{N-2}\int_{x=0}^{z}e^{-x}\big(-x^{N-1}\log(x)\Gamma(N)+x^{N-1}\Gamma'(N)\big)\,dx\bigg)\\&=e^{-z}\big(-\log(z)\Gamma(N)+\Gamma'(N)\big)\\&\quad-(N-1)z^{-N}\int_{x=0}^{z}e^{-x}\big(-x^{N-1}\log(x)\Gamma(N)+x^{N-1}\Gamma'(N)\big)\,dx.\end{aligned}$$

For any $z\le 1$, we have

$$\begin{aligned}&\frac{\partial f(z,N)}{\partial z}\\&\ge e^{-z}\big(-\log(z)\Gamma(N)+\Gamma'(N)\big)-(N-1)z^{-N}\int_{x=0}^{z}\big(-x^{N-1}\log(x)\Gamma(N)+x^{N-1}\Gamma'(N)\big)\,dx\\&=e^{-z}\big(-\log(z)\Gamma(N)+\Gamma'(N)\big)-(N-1)z^{-N}\bigg(\Gamma(N)\bigg(\frac{z^N}{N^2}-\frac{z^N\log z}{N}\bigg)+\Gamma'(N)\frac{z^N}{N}\bigg)\\&=e^{-z}\big(-\log(z)\Gamma(N)+\Gamma'(N)\big(-\frac{N-1}{N}\bigg(\Gamma(N)\bigg(\frac{1}{N}-\log z\bigg)+\Gamma'(N)\bigg)\\&=\bigg(e^{-z}-\frac{N-1}{N}\bigg)\big(-\log(z)\Gamma(N)+\Gamma'(N)\big)-\frac{N-1}{N^2}\Gamma(N).\end{aligned}$$

Since the last line goes to infinity as z goes to zero, for any fixed $N\ge 1$, we can choose $\overline{z}\in(0,1]$ such that $\partial f(z,\widehat{N})/\partial z\ge 0$ for all $z\in[0,\overline{z}]$ and $\widehat{N}\in[N,N+1]$. Let $\overline{\alpha}=G_{N+1}(\overline{z})$.

Suppose $0\le\alpha'<\alpha\le\overline{\alpha}$. We have

$$\big(G_{N+1}^{-1}(\alpha)-G_{N+1}^{-1}(\alpha')\big)-\big(G_N^{-1}(\alpha)-G_N^{-1}(\alpha')\big)=\int_{\widehat{N}=N}^{N+1}\bigg(\frac{\partial G_{\widehat{N}}^{-1}(\alpha)}{\partial\widehat{N}}-\frac{\partial G_{\widehat{N}}^{-1}(\alpha')}{\partial\widehat{N}}\bigg)\,d\widehat{N}.$$

Since $d(e^z f(z,\widehat{N})/\Gamma(\widehat{N}))/dz\ge 0$ for all $z\in[0,\overline{z}]$ and $\widehat{N}\in[N,N+1]$, we have $\partial G_{\widehat{N}}^{-1}(\alpha)/\partial\widehat{N}-\partial G_{\widehat{N}}^{-1}(\alpha')/\partial\widehat{N}\ge 0$, which proves the lemma. *Q.E.D.*

Recall that

$$\begin{aligned}G_N^C(x)&=G_N(\sqrt{N-1}x+N-1),\\g_N^C(x)&=\sqrt{N-1}g_N(\sqrt{N-1}x+N-1).\end{aligned}$$

To prove Proposition 7, we first need a number of technical results.

LEMMA S11: *As N goes to infinity, g_N^C and G_N^C converge pointwise to ϕ and Φ, respectively.*

PROOF: Note that

$$\begin{aligned} g_{N+1}^C(x) &= \sqrt{N} g_{N+1}(\sqrt{N}x + N) \\ &= \sqrt{N} \frac{(\sqrt{N}x + N)^N}{N!} \exp(-\sqrt{N}x - N). \end{aligned}$$

Stirling's approximation says that

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N} = 1.$$

Moreover, for all N, the ratio inside the limit is greater than 1.

Thus, when N is large, $g_{N+1}^C(x)$ is approximately

$$\frac{1}{\sqrt{2\pi}}\left(1 + \frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x)$$

and, hence,

$$\log\bigl(g_{N+1}^C(x)\bigr) \approx \log(1/\sqrt{2\pi}) + N \log\left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N}x.$$

Using the mean-value formulation of Taylor's theorem centered around 0, for every y, there exists a $z \in [0, y]$ such that

$$\log(1 + y) = y - \frac{y^2}{2} + \frac{1}{(1 + z)^3} y^3.$$

Plugging in $y = x/\sqrt{N}$, we conclude that

$$\begin{aligned} \log\bigl(g_{N+1}^C(x)\bigr) &\approx \log(1/\sqrt{2\pi}) + N\frac{x}{\sqrt{N}} - N\frac{1}{2}\left(\frac{x}{\sqrt{N}}\right)^2 + N\frac{1}{(1+z)^3}\left(\frac{x}{\sqrt{N}}\right)^3 - \sqrt{N}x \\ &= \log(1/\sqrt{2\pi}) - \frac{1}{2}x^2 + \frac{1}{(1+z)^3}\frac{x^3}{\sqrt{N}}, \end{aligned}$$

which converges to $\log(1/\sqrt{2\pi}) - \frac{1}{2}x^2$ as N goes to infinity, so $g_{N+1}^C(x)$ converges to $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Pointwise convergence of G_N^C to Φ follows from Scheffé's lemma. *Q.E.D.*

Let us define

$$\tilde{g}(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{x^2}{2}\right) & \text{if } x < 0, \\ \dfrac{1}{\sqrt{2\pi}}(1 + x)\exp(-x) & \text{otherwise.} \end{cases}$$

LEMMA S12: *The function $\tilde{g}(x)|x|$ is integrable, and for all N and x, $|g_N^C(x)| \leq \tilde{g}(x)$.*

PROOF: Note that

$$\int_{x=-\infty}^{\infty} \tilde{g}(x)|x|\,dx = \int_{x=-\infty}^{0} \phi(x)|x|\,dx + \frac{1}{\sqrt{2\pi}} \int_{x=0}^{\infty} (1+x)x\exp(-x)\,dx,$$

which is clearly finite, since the half-normal distribution has finite expectation.

Next, Stirling's approximation implies that

$$g_{N+1}^C(x) \leq \frac{1}{\sqrt{2\pi}}\left(1+\frac{x}{\sqrt{N}}\right)^N \exp(-\sqrt{N}x) \equiv \tilde{g}_N(x).$$

Now

$$\frac{d}{dN}\log\bigl(\tilde{g}_N(x)\bigr) = \log\left(1+\frac{x}{\sqrt{N}}\right) - \frac{1}{2}\frac{x}{\sqrt{N}+x} - \frac{x}{2\sqrt{N}},$$

which is clearly zero when $x=0$, and

$$\begin{aligned}\frac{d}{dx}\frac{d}{dN}\log\bigl(\tilde{g}_N(x)\bigr) &= \frac{1}{\sqrt{N}+x} - \frac{\sqrt{N}}{2(\sqrt{N}+x)^2} - \frac{1}{2\sqrt{N}} \\ &= \frac{2N+2\sqrt{N}x}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N}{2\sqrt{N}(\sqrt{N}+x)^2} - \frac{N+2\sqrt{N}x+x^2}{2\sqrt{N}(\sqrt{N}+x)^2} \\ &= \frac{-x^2}{2\sqrt{N}(\sqrt{N}+x)^2},\end{aligned}$$

which is nonpositive and strictly negative when $x \neq 0$. As a result, $\tilde{g}_N(x)$ is increasing in N when $x<0$ and decreasing in N when $x>0$. Since it converges to $\phi(x)$ in the limit as N goes to infinity, we conclude that for $x<0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \phi(x) = \tilde{g}(x)$, and for $x>0$, $g_{N+1}^C(x) \leq \tilde{g}_N(x) \leq \tilde{g}_1(x) = \tilde{g}(x)$ as desired. *Q.E.D.*

LEMMA S13: *As N goes to infinity, $\widehat{\gamma}_N^C$ converges almost surely to $\widehat{\gamma}_\infty^C(x) = H^{-1}(\Phi(x))$ and $\widehat{\Gamma}_N^C$ converges pointwise to*

$$\widehat{\Gamma}_\infty^C(x) = \int_{y=-\infty}^{x} \widehat{\gamma}_\infty^C(y)\phi(y)\,dy.$$

The latter convergence is uniform on any bounded interval.

PROOF: Note that $\widehat{\gamma}_N^C(x) = H^{-1}(G_N^C(x)) - c$. By Lemma S11, $G_N^C(x)$ converges to $\Phi(x)$ pointwise. Thus, if H^{-1} is continuous at $\Phi(x)$, then as N goes to infinity, we must have $\widehat{\gamma}_N^C(x) \to H^{-1}(\Phi(x)) - c = \widehat{\gamma}_\infty^C(x)$. Since H^{-1} is monotonic, the set of discontinuities has Lebesgue measure zero, so that the pointwise convergence is almost everywhere.

Pointwise convergence of $\widehat{\Gamma}_N^C$ follows from almost sure convergence of $\widehat{\gamma}_N^C$, combined with the fact that $\widehat{\gamma}_N^C$ is uniformly bounded by $|\overline{v}|$, so that we can apply the dominated convergence theorem. Moreover, $\widehat{\Gamma}_N^C(x)$ is uniformly Lipschitz continuous across N and x. As a result, the family $\{\widehat{\Gamma}_N^C(\cdot)\}_{N=2}^{\infty}$ is uniformly bounded and uniformly equicontinuous.

The conclusion about uniform convergence is then a consequence of the Arzela–Ascoli theorem. *Q.E.D.*

Recall that x^* is the largest solution to $\widehat{\Gamma}^C_\infty(x^*) = 0$ (which may be $-\infty$). Also, let us define x_N so that $\overline{\Gamma}^C_N$ has a graded interval $[-\sqrt{N-1}, x_N]$. (If there is no graded interval with left endpoint $-\sqrt{N-1}$, then we let $x_N = -\sqrt{N-1}$.)

LEMMA S14: *As N goes to infinity, x_N converges to x^*.*

PROOF: By a change of variables $y = (G^C_N)^{-1}(\Phi(x))$, we conclude that

$$\widehat{\Gamma}^C_\infty(x^*) = \int_{x=-\infty}^{x^*} \widehat{\gamma}^C_\infty(x)\phi(x)\,dx = \int_{x=-\sqrt{N-1}}^{(G^C_N)^{-1}(\Phi(x^*))} \widehat{\gamma}^C_N(x) g^C_N(x)\,dx = \widehat{\Gamma}^C_N\big((G^C_N)^{-1}(\Phi(x^*))\big).$$

This integral must be zero by the definition of x^*, so that $x_N \geq (G^C_N)^{-1}(\Phi(x^*))$. Since the latter converges to x^* as $N \to \infty$, we conclude that $\liminf_{N\to\infty} x_N \geq x^*$.

Next recall that x_{N+1} solves the equation

$$\begin{aligned}
\widehat{\Gamma}^C_{N+1}(x_{N+1}) &= \widehat{\gamma}^C_{N+1}(x_{N+1}) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp\big(\sqrt{N}(x - x_{N+1})\big) g^C_{N+1}(x)\,dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \exp(\sqrt{N}x + N) g^C_{N+1}(x)\,dx \\
&= \widehat{\gamma}^C_{N+1}(x_{N+1}) \exp(-\sqrt{N}x_{N+1} - N) \int_{x=-\sqrt{N}}^{x_{N+1}} \sqrt{N}\frac{(\sqrt{N}x + N)^N}{N!}\,dx \\
&\leq \overline{v}\exp(-\sqrt{N}x_{N+1} - N)\frac{(\sqrt{N}x_{N+1} + N)^{N+1}}{(N+1)!} \\
&= \overline{v} g^C_{N+2}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}} \\
&\leq \overline{v}\tilde{g}\left(\sqrt{\frac{N}{N+1}}x_{N+1} - \frac{1}{\sqrt{N+1}}\right)\frac{1}{\sqrt{N+1}},
\end{aligned}$$

where we have used Lemma S12. The last line converges to zero pointwise, so $\widehat{\Gamma}^C_N(x_N)$ must converge to zero as well.

Now, if $z = \limsup_{N\to\infty} x_N > x^*$, then since $\widehat{\Gamma}^C_\infty(z) > \widehat{\Gamma}^C_\infty(x^*) = 0$, we would contradict our earlier finding that $\widehat{\Gamma}^C_N(x_N) \to 0$. Thus, $\limsup_{N\to\infty} x_N \leq x^*$, so x_N must converge to x^* as N goes to ∞. *Q.E.D.*

LEMMA S15: *For every $\epsilon > 0$, there exists $\widehat{N}$ such that for all $N > \widehat{N}$, there exists an $x \in [x^* + \epsilon, x^* + 2\epsilon]$ at which $\overline{\gamma}^C_N$ is not graded.*

PROOF: Suppose not. Then there exist infinitely many N such that for every $x \in [x^* + \epsilon, x^* + 2\epsilon]$, $\overline{\gamma}^C_{N+1}(x) = \exp(\sqrt{N}(x - \tilde{x}))\widehat{\gamma}^C_{N+1}(\tilde{x})$ for some $\tilde{x} \geq x^* + 2\epsilon$. Thus, for all $x \leq x^* + \epsilon$, we conclude that

$$\overline{\gamma}^C_{N+1}(x) \leq \overline{\gamma}^C_{N+1}\big(x^* + \epsilon\big) \leq \exp(-\sqrt{N}\epsilon)\overline{v},$$

which converges to zero as N goes to infinity. This implies that $\liminf_{N\to\infty} \overline{\Gamma}^C_{N+1}(x^*+\epsilon) = 0$. But $\overline{\Gamma}^C_{N+1}(x^*+\epsilon)$ must be weakly larger than $\widehat{\Gamma}^C_{N+1}(x^*+\epsilon)$, so

$$0 = \lim\inf_{N\to\infty} \overline{\Gamma}^C_{N+1}\big(x^*+\epsilon\big) \geq \lim\inf_{N\to\infty} \widehat{\Gamma}^C_{N+1}\big(x^*+\epsilon\big) = \widehat{\Gamma}^C_{\infty}\big(x^*+\epsilon\big) > 0,$$

a contradiction. *Q.E.D.*

LEMMA S16: *As N goes to infinity,* $\overline{\gamma}^C_N$ *converges almost surely to*

$$\overline{\gamma}^C_\infty(x) = \begin{cases} 0 & \text{if } x < x^*, \\ \widehat{\gamma}^C_\infty(x) & \text{if } x \geq x^*. \end{cases}$$

PROOF: Let $x < x^*$. Since $x_N \to x^*$ by Lemma S14, for N sufficiently large, $x_N > (x^*+x)/2$. Since $\overline{\gamma}^C_N(x)$ is graded on $(-\infty, x_N]$, it is graded at x, and

$$\begin{aligned} \overline{\gamma}^C_N(x) &= \exp\big(\sqrt{N-1}(x-x_N)\big)\widehat{\gamma}^C_N(x_N) \\ &\leq \exp\big(\sqrt{N-1}\big(x-x^*\big)/2\big)\overline{v}. \end{aligned}$$

The last line clearly converges to zero pointwise. Since $\overline{\gamma}^C_N(x) \geq 0$ for all N, we conclude that $\overline{\gamma}^C_N(x) \to 0$.

Now consider $x > x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Take ϵ so that $x > x^* + 2\epsilon$ and so that $\widehat{\gamma}^C_\infty$ is continuous at $x^*+\epsilon$. Lemma S15 says that there is a $\widehat{N}$ such that for all $N > \widehat{N}$, there exists a point in $[x^*+\epsilon, x^*+2\epsilon]$ at which the gains function is not graded. Moreover, since $\widehat{\gamma}^C_N(x^*+\epsilon)$ converges to $\widehat{\gamma}^C_\infty(x^*+\epsilon)$, we can pick $\widehat{N}$ large enough and find a constant $\underline{\gamma} > 0$ such that for $N > \widehat{N}$, $\widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$.

Now suppose that $\overline{\gamma}^C_N$ is graded at x, with x in a graded interval $[a, b]$. Then $a \geq x^* + \epsilon$ and, hence, $\widehat{\gamma}^C_N(a) \geq \widehat{\gamma}^C_N(x^*+\epsilon) \geq \underline{\gamma}$. Recall that on $[a, b]$,

$$\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(a)\exp\big(\sqrt{N-1}(x-a)\big).$$

Since $\widehat{\gamma}^C_N$ is bounded above by $\overline{v}$, it must be that $\widehat{\gamma}^C_N(a)\exp(\sqrt{N-1}(b-a)) \leq \overline{v}$, so

$$\begin{aligned} b-a &\leq \frac{1}{\sqrt{N-1}}\log\left(\frac{\overline{v}}{\widehat{\gamma}^C_N(a)}\right) \\ &\leq \frac{1}{\sqrt{N-1}}\log\left(\frac{\overline{v}}{\underline{\gamma}}\right) = \epsilon_N. \end{aligned}$$

Thus,

$$\widehat{\gamma}^C_N(x-\epsilon_N) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_N(x+\epsilon_N).$$

This inequality holds if $\overline{\gamma}^C_N$ is graded at x, but clearly the inequality is also true if $\overline{\gamma}^C_N$ is not graded at x, in which case $\overline{\gamma}^C_N(x) = \widehat{\gamma}^C_N(x)$. Now, $\widehat{\gamma}^C_N(x) = \widehat{\gamma}^C_\infty(\Phi^{-1}(G^C_N(x)))$, so

$$\widehat{\gamma}^C_\infty\big(\Phi^{-1}\big(G^C_N(x-\epsilon_N)\big)\big) \leq \overline{\gamma}^C_N(x) \leq \widehat{\gamma}^C_\infty\big(\Phi^{-1}\big(G^C_N(x+\epsilon_N)\big)\big).$$

As $N \to \infty$, the left- and right-hand sides converge to $\widehat{\gamma}_\infty^C(x)$ from the left and right, respectively. Since $\widehat{\gamma}_\infty^C$ is continuous at x, we conclude that $\overline{\gamma}_N^C(x) \to \widehat{\gamma}_\infty^C(x)$. The lemma follows from the fact that the monotonic function $\widehat{\gamma}_\infty^C$ is continuous almost everywhere. *Q.E.D.*

PROOF OF PROPOSITION 7: We argue that

$$Z_{N+1} = \sqrt{N} \int_{x=0}^{\infty} \overline{\gamma}_{N+1}(x)\big(g_{N+1}(x) - g_N(x)\big)\, dx$$

converges to a positive constant as N goes to infinity. Since this is $\sqrt{N}$ times the difference between ex ante gains from trade and profit, this proves the result.

To that end, observe that

$$\begin{aligned} Z_{N+1} = {} & \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)\big(g_{N+1}(x) - g_N(x)\big)\, dx \\ & + \int_{x=-\sqrt{N}/2}^{\infty} \overline{\gamma}_{N+1}^C(x) g_{N+1}^C(x) \frac{Nx}{\sqrt{N}x + N}\, dx. \end{aligned}$$

We claim that the first integral converges to zero as $N \to \infty$. Note that $g_{N+1}(x) \le g_N(x)$ if and only if $x \le N$. Therefore,

$$\begin{aligned} \left| \sqrt{N} \int_{x=0}^{N/2} \overline{\gamma}_{N+1}(x)\big(g_{N+1}(x) - g_N(x)\big)\, dx \right| & \le (\overline{v} + c)\sqrt{N} \int_{x=0}^{N/2} \big(g_N(x) - g_{N+1}(x)\big)\, dx \\ & = (\overline{v} + c)\sqrt{N}\big(G_N(N/2) - G_{N+1}(N/2)\big) \\ & = (\overline{v} + c)\sqrt{N} g_{N+1}(N/2) \\ & = (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{N!} \\ & \approx (\overline{v} + c)\sqrt{N} \frac{(N/2)^N \exp(-N/2)}{\sqrt{2\pi N}(N/e)^N} \\ & = (\overline{v} + c) \frac{1}{\sqrt{2\pi}} \exp\big(-N\big(\log(2) - 1/2\big)\big), \end{aligned}$$

where we have again used Stirling's approximation between the third-to-last and second-to-last lines. The last line converges to zero as N goes to infinity.

Now consider the second integral in the formula for Z_{N+1}. By Lemma S12, the integrand is bounded above in absolute value by the integrable function $\overline{v}\tilde{g}(x)|x|$. Moreover, from Lemmas S11 and S16, we know that the integrand converges pointwise to $\overline{\gamma}_\infty^C(x)\phi(x)x$. The dominated convergence theorem then implies that as N goes to infinity, Z_N converges to

$$\int_{x=-\infty}^{\infty} \overline{\gamma}_\infty^C(x)\phi(x)x\, dx,$$

which is strictly positive because $\overline{\gamma}_\infty^C$ is strictly increasing. *Q.E.D.*

The preceding proof remains valid for the must-sell case if we replace $\overline{\gamma}_N^C$ with $\widehat{\gamma}_N^C$.

To prove Proposition 9, we need a few more intermediate results. Let $\overline{G}_N(x) = G_N(Nx)$ be the cumulative distribution for the mean of N independent standard exponential random variables. Define $\overline{F}_N(x) = \exp(N(1-x+\log(x)))$. Clearly, $\overline{F}_N(x)$ is a cumulative distribution for $x \in [0,1]$: $\overline{F}_N(0) = 0$ and $\overline{F}_N(1) = 1$. Finally, define the function

$$D_N(\alpha) = \begin{cases} \dfrac{1}{\overline{F}_N^{-1}(\alpha)} & \text{if } \alpha \in [0, 0.4], \\ 1.1 & \text{if } \alpha \in (0.4, 1]. \end{cases}$$

The choices of 0.4 and 1.1 in $D_N(\alpha)$ are arbitrary: any numbers work that are less than 1/2 and more than 1, respectively.

LEMMA S17: *There exists a $\widehat{N}$ such that for all $N \geq \hat{N}$ and $\alpha \in [0,1]$, $\overline{\mu}_N(G_N^{-1}(\alpha)) \leq D_{\widehat{N}}(\alpha)$.*

PROOF: We first apply the theory of large deviations to the exponential distribution. Let $\Lambda(t)$ be the logarithmic moment generating function for the exponential distribution:

$$\Lambda(t) = \log\left(\int_{x=0}^{\infty} \exp(xt - x)\,dx\right) = \begin{cases} \infty & \text{if } t \geq 1, \\ -\log(1-t) & \text{if } t < 1. \end{cases}$$

Let $\Lambda^*(x)$ be the Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}}\bigl\{xt - \Lambda(t)\bigr\} = \begin{cases} \infty, & x \leq 0, \\ x - 1 - \log x, & x > 0. \end{cases}$$

Cramér's theorem (or the Chernoff bound; see Theorem 1.3.12 in Stroock (2011)) implies that for any N,

$$\overline{G}_N(x) \leq \exp\bigl(-N\Lambda^*(x)\bigr) = \overline{F}_N(x)$$

for every $x \in [0,1]$, or, equivalently, $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for every $\alpha \in [0, \overline{G}_N(1)]$.

By the law of large numbers, when $\widehat{N}$ is sufficiently large, we have $\overline{G}_N(1) \geq 0.4$ and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ for all $N \geq \widehat{N}$. The claim of the lemma then follows from two cases.

If $\alpha \in [0, 0.4]$, then we have

$$\overline{\mu}_N\bigl(G_N^{-1}(\alpha)\bigr) \leq \frac{N}{G_N^{-1}(\alpha)} = \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_N^{-1}(\alpha)} \leq \frac{1}{\overline{F}_{\widehat{N}}^{-1}(\alpha)} = D_{\widehat{N}}(\alpha),$$

where we have used the bound $\overline{\mu}_N(x) \leq N/x$ (equation (21)), and the facts that $\overline{G}_N(1) \geq 0.4$ when $N \geq \widehat{N}$ (so $\overline{F}_N^{-1}(\alpha) \leq \overline{G}_N^{-1}(\alpha)$ for $\alpha \leq 0.4 \leq \overline{G}_N(1)$) and that $\overline{F}_N(x) \leq \overline{F}_{\widehat{N}}(x)$ for all $N \geq \widehat{N}$ and $x \in [0,1]$ (so $\overline{F}_{\widehat{N}}^{-1}(\alpha) \leq \overline{F}_N^{-1}(\alpha)$ for all α).

If $\alpha \in (0.4, 1]$, then

$$\overline{\mu}_N\bigl(G_N^{-1}(\alpha)\bigr) \leq \frac{1}{\overline{G}_N^{-1}(\alpha)} \leq \frac{1}{\overline{G}_N^{-1}(0.4)} \leq 1.1 = D_{\widehat{N}}(\alpha),$$

since $\overline{G}_N^{-1}(\alpha)$ is increasing in α, and $1/\overline{G}_N^{-1}(0.4) \leq 1.1$ when $N \geq \widehat{N}$. *Q.E.D.*

LEMMA S18: *There exists a* $\hat{N}$ *such that for all* $N \geq \hat{N}$,

$$\int_{\alpha=0}^{1} D_N(\alpha)\, dH^{-1}(\alpha) < \infty.$$

PROOF: Since $G_N(x) = 1 - \sum_{k=1}^{N} g_k(x)$, we have

$$\begin{aligned}\overline{G}_N(x) &= 1 - \sum_{k=1}^{N} \exp(-Nx)\frac{(Nx)^{k-1}}{(k-1)!} \\ &= 1 - \exp(-Nx)\left(\exp(Nx) - \sum_{k=N}^{\infty}\frac{(Nx)^k}{k!}\right) \geq \exp(-Nx)\frac{(Nx)^N}{N!}.\end{aligned}$$

Clearly, there exists an $\overline{x} \in (0, 1)$ such that

$$\overline{F}_{N+1}(x) = \exp\bigl((N+1)(1-x)\bigr)x^{N+1} \leq \exp(-Nx)\frac{(Nx)^N}{N!} \leq \overline{G}_N(x)$$

for all $x \in [0, \overline{x}]$. We therefore have $D_{N+1}(\alpha) = 1/\overline{F}_{N+1}^{-1}(\alpha) \leq 1/\overline{G}_N^{-1}(\alpha)$ for all $\alpha \in [0, \overline{\alpha}]$, where $\overline{\alpha} = \min\{\overline{F}_{N+1}(\overline{x}), 0.4\}$. As a result,

$$\int_{\alpha=0}^{1} D_{N+1}(\alpha)\, dH^{-1}(\alpha) \leq \int_{\alpha=0}^{\overline{\alpha}} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) + \int_{\alpha=\overline{\alpha}}^{1} \max\left(\frac{1}{\overline{F}_{N+1}^{-1}(\overline{\alpha})}, 1.1\right) dH^{-1}(\alpha) < \infty$$

whenever we have

$$\int_{\alpha=0}^{1} \frac{1}{\overline{G}_N^{-1}(\alpha)}\, dH^{-1}(\alpha) = \int_{x=0}^{\infty} \frac{N}{x}\, d\widehat{w}_N(x) < \infty.$$

Finiteness of the last integral follows from the first part of the left-tail assumption. *Q.E.D.*

LEMMA S19: *Suppose* $\lim_{N\to\infty} y_N \in (-\infty, \infty)$. *Then* $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N + N) = 1$.

PROOF: We first argue that for almost every y, $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ tends to 1 as $N \to \infty$. For this we recall x^* and x_N from Lemmas S14–S16.

Consider first $y < x^*$. By Lemma S14, for N sufficiently large, the gains function is graded at y and, hence,

$$\overline{\mu}_{N+1}(\sqrt{N}y + N) = C(0, \sqrt{N}x_{N+1} + N) = \frac{N+1}{\sqrt{N}x_{N+1} + N}.$$

Since we have already shown that $x_N \to x^*$ (Lemma S14), we conclude that $\overline{\mu}_{N+1}(\sqrt{N}y + N)$ goes to 1.

Now consider $y > x^*$ at which $\widehat{\gamma}_{\infty}^{C}$ is continuous. If the gains function is not graded at y, then $\overline{\mu}_{N+1}(\sqrt{N}y + N) = N/(\sqrt{N}y + N)$. If the gains function is graded at y, then the length of the graded interval $[a, b] \ni y$ in central limit units is less than $\epsilon_N = \overline{v}/(\underline{\gamma}\sqrt{N})$ for some $\underline{\gamma} > 0$ independent of N (see Lemma S16). Since $\overline{\mu}$ is decreasing (Lemma 3), we have

$$\frac{N}{\sqrt{N}(y+\epsilon_N)+N} \leq \overline{\mu}_{N+1}(\sqrt{N}y+N) \leq \frac{N}{\sqrt{N}(y-\epsilon_N)+N},$$

since $\lim_{z\nearrow a} \overline{\mu}_{N+1}(\sqrt{N}z+N) = N/(\sqrt{N}a+N)$ and $\lim_{z\searrow b} \overline{\mu}_{N+1}(\sqrt{N}z+N) = N/(\sqrt{N}b+N)$. As a result, $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ is squeezed to 1 as N goes to infinity.

We conclude that $\overline{\mu}_{N+1}(\sqrt{N}y+N)$ goes to 1 for $y > x^*$ at which $\widehat{\gamma}^C_\infty$ is continuous. Since $\widehat{\gamma}^C_\infty(y)$ is a monotone function of y, it is continuous at almost every y, so the convergence $\overline{\mu}_N \to 1$ is almost everywhere.

Finally, suppose $\lim_{N\to\infty} y_N = y \in (-\infty, \infty)$. Choose y' and y'' such that $y \in (y', y'')$ and such that

$$\lim_{N\to\infty} \overline{\mu}_{N+1}\big(\sqrt{N}y'+N\big) = 1 = \lim_{N\to\infty} \overline{\mu}_{N+1}\big(\sqrt{N}y''+N\big).$$

When N is sufficiently large, we have $y_N \in (y', y'')$, so

$$\overline{\mu}_{N+1}\big(\sqrt{N}y''+N\big) \leq \overline{\mu}_{N+1}(\sqrt{N}y_N+N) \leq \overline{\mu}_{N+1}\big(\sqrt{N}y'+N\big).$$

Taking the limit as $N \to \infty$, we conclude $\lim_{N\to\infty} \overline{\mu}_{N+1}(\sqrt{N}y_N+N) = 1$. *Q.E.D.*

PROOF OF PROPOSITION 9: We first prove that

$$\lim_{N\to\infty} \overline{\lambda}_N(v; H) \to v - c \tag{S2}$$

for every $v \in [\underline{v}, \overline{v}]$.

Replacing $\overline{\mu}_N$ by 1 in equation (18), the definition of $\overline{\lambda}_N(v; H)$, we have

$$\begin{aligned}
&\overline{\Pi}_N(H) + \int_{y=0}^{\infty} G_N(y)\, d\widehat{w}_N(y) - \int_{\nu=v}^{\overline{v}} d\nu \\
&= \overline{\Pi}_N(H) + \left(\overline{v} - \int_{y=0}^{\infty} g_N(y)\widehat{w}_N(y)\, dy\right) - (\overline{v} - v) \\
&= \overline{\Pi}_N(H) - \int_{v'=\underline{v}}^{\overline{v}} v'\, dH\big(v'\big) + v.
\end{aligned}$$

Since by Proposition 7, $\lim_{N\to\infty} \overline{\Pi}_N(H) \to \int_{v'=\underline{v}}^{\overline{v}} v'\, dH(v') - c$, to prove (S2), it suffices to prove that

$$\lim_{N\to\infty} \int_{y=0}^{\infty} \big|1 - \overline{\mu}_N(y)\big|\, d\widehat{w}_N(y) = 0.$$

Changing variables, we can rewrite the above equation as

$$\lim_{N\to\infty} \int_{\alpha=0}^{1} \big|1 - \overline{\mu}_N\big(G_N^{-1}(\alpha)\big)\big|\, dH^{-1}(\alpha) = 0. \tag{S3}$$

We note that Stieltjes integration with respect to $dH^{-1}(\alpha)$ is equivalent to a Lebesgue integration with respect to the finite measure ω on $[0, 1]$ satisfying $\omega([s, t)) = H^{-1}(t) -$

$H^{-1}(s)$, $0 \le s \le t \le 1$, and $\omega(\{1\}) = 0$. The first part of the left-tail assumption implies that

$$\omega\big(\{0\}\big) = \lim_{\alpha\to 0} \omega\big([0,\alpha)\big) = \lim_{\alpha\to 0} H^{-1}(\alpha) - H^{-1}(0) \le \lim_{\alpha\to 0} G_N^{-1}(\alpha)^{\varphi} = 0$$

for some $\varphi > 1$. Therefore, $\omega(\{0, 1\}) = 0$.

The central limit theorem implies that $\lim_{N\to\infty}(G_N^{-1}(\alpha) - (N-1))/\sqrt{N-1} = \Phi^{-1}(\alpha)$ for every $\alpha \in (0, 1)$. Therefore, Lemma S19 implies $\lim_{N\to\infty} \overline{\mu}_N(G_N^{-1}(\alpha)) = 1$ for every $\alpha \in (0, 1)$. Moreover, Lemmas S17 and S18 imply that there exists a $\widehat{N}$ such that for all $N \ge \widehat{N}$, the integrand $|1 - \overline{\mu}_N(G_N^{-1}(\alpha))|$ in (S3) is dominated by $1 + D_{\widehat{N}}(\alpha)$ which is integrable with respect to ω. Therefore, equation (S3) follows from the dominated convergence theorem, from which equation (S2) follows.

Finally, using the definition of $\overline{\lambda}_N(v; H)$, we have

$$\begin{aligned}\overline{\lambda}_N(v; H) &\le \overline{\Pi}_N(H) + \int_{y=0}^{\infty} \overline{\mu}_N(y)\big(1 + G_N(y)\big)\, d\widehat{w}_N(y) \\ &\le (\overline{v} - c) + 2\int_{\alpha=0}^{1} D_{\widehat{N}}(\alpha)\, dH^{-1}(\alpha) < \infty\end{aligned}$$

for all $v \in [\underline{v}, \overline{v}]$ and $N \ge \widehat{N}$, where the last two inequalities follow from Lemmas S17 and S18, respectively. Thus,

$$\lim_{N\to\infty} \int_V \overline{\lambda}_N(v; H)\, dH'(v) = \int_V v\, dH'(v) - c$$

follows from the dominated convergence theorem using (S2).

The preceding proof remains valid for the must-sell case, if we replace $\overline{\mu}_N(x)$ with $\widehat{\mu}_N(x) = (N-1)/x$ and $\overline{\Pi}_N(H)$ with $\widehat{\Pi}_N(H)$. *Q.E.D.*

LEMMA S20: *Suppose the condition on H in Lemma 10 holds. For any $\epsilon > 0$, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, we have*

$$\widehat{\gamma}_N(x) \le \widehat{\gamma}_N(y)\exp(x - y)$$

for all $x \ge y$ such that $\widehat{\gamma}_N(y) \ge \epsilon$.

PROOF: The condition on H implies that the support of H has no gap on $[\underline{v}, \overline{v}]$, so H^{-1} is continuous on $[0, 1]$. We can partition $[0, 1]$ into a countable collection of intervals $\{[\alpha_i, \beta_i] : i \in I\}$ such that $\alpha_i < \beta_i$ and either H^{-1} is strictly increasing on $[\alpha_i, \beta_i]$ or H^{-1} is constant on $[\alpha_i, \beta_i]$ (i.e., H has a mass point at v, where $v = H^{-1}(p)$ for all $p \in [\alpha_i, \beta_i]$). If H^{-1} is strictly increasing on $[\alpha_i, \beta_i]$, then

$$H^{-1}(q) - H^{-1}(p) \le \frac{q - p}{C} \tag{S4}$$

for any $p, q \in (\alpha_i, \beta_i)$ such that $p \le q$, since in this case we have $H(H^{-1}(q)) = q$ and $H(H^{-1}(p)) = p$. By continuity of H^{-1} we can extend (S4) to any $p, q \in [\alpha_i, \beta_i]$ such that $p \le q$.

If H^{-1} is constant on $[\alpha_i, \beta_i]$, then clearly (S4) also holds for any $p, q \in [\alpha_i, \beta_i]$ such that $p \le q$. Since $\{[\alpha_i, \beta_i] : i \in I\}$ is a partition of $[0, 1]$, we conclude that (S4) holds for any $p, q \in [0, 1]$ such that $p < q$.

With the substitution $q = G_N^C(x)$ and $p = G_N^C(y)$, with $x > y$, equation (S4) becomes

$$\widehat{\gamma}_N^C(x) - \widehat{\gamma}_N^C(y) \leq \frac{G_N^C(x) - G_N^C(y)}{C}.$$

Thus,

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq 1 + \frac{1}{\widehat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C}.$$

The log-1 Lipschitz condition that we want to prove is equivalent to

$$\frac{\widehat{\gamma}_N^C(x)}{\widehat{\gamma}_N^C(y)} \leq \exp\bigl(G_N^{-1}\bigl(G_N^C(x)\bigr) - G_N^{-1}\bigl(G_N^C(y)\bigr)\bigr).$$

Thus, it is sufficient to show that for large N,

$$1 + \frac{1}{\widehat{\gamma}_N^C(y)} \frac{G_N^C(x) - G_N^C(y)}{C} \leq \exp\bigl(G_N^{-1}\bigl(G_N^C(x)\bigr) - G_N^{-1}\bigl(G_N^C(y)\bigr)\bigr).$$

Both sides are equal to 1 when $x = y$, and the derivatives of the left- and right-hand sides with respect to x are, respectively

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C} \tag{S5}$$

and

$$\begin{aligned} &\frac{g_N^C(x)}{g_N\bigl(G_N^{-1}\bigl(G_N^C(x)\bigr)\bigr)} \exp\bigl(G_N^{-1}\bigl(G_N^C(x)\bigr) - G_N^{-1}\bigl(G_N^C(y)\bigr)\bigr) \\ &\quad = \sqrt{N-1} \exp\bigl(G_N^{-1}\bigl(G_N^C(x)\bigr) - G_N^{-1}\bigl(G_N^C(y)\bigr)\bigr) \geq \sqrt{N-1}. \end{aligned} \tag{S6}$$

We now show that (S5) is always less than (S6). Note that g_N attains its maximum when $g_N = g_{N-1}$, i.e., when $x = N-1$, at a value of $\frac{(N-1)^{N-1}}{(N-1)!} \exp(-(N-1))$. Multiplied by $\sqrt{N-1}$, this upper bound converges to $\phi(0)$. Hence, when N is sufficiently large, $g_N^C(x) \leq 2\phi(0)$ for all x. Since $\widehat{\gamma}_N^C(z) > 0$, then there is an N large enough such that

$$\frac{g_N^C(x)}{\widehat{\gamma}_N^C(y)C} \leq \frac{2\phi(0)}{\epsilon C} \leq \sqrt{N-1},$$

which proves the lemma. *Q.E.D.*

PROOF OF LEMMA 10: If $\underline{v} > c$, then we can take $\epsilon = \underline{v} - c$ in the statement of Lemma S20, in which case the statement of the lemma follows immediately.

If $\underline{v} < c$, then $\widehat{\gamma}_N^C(-\sqrt{N-1}) < 0$, so that $\widehat{\Gamma}_N^C(x)$ is nonpositive for x close to $-\sqrt{N-1}$. Hence, there must be a graded interval at the bottom of the form $[-\sqrt{N-1}, x_N]$. By Lemma S14, x_N converges to x^*. Moreover, by Lemma S16, $\overline{\gamma}_N^C$ converges almost surely to $\overline{\gamma}_\infty^C$. Thus, there exists an $\widehat{N}$ such that for all $N > \widehat{N}$, $\widehat{\gamma}_N^C(x_N) \geq \epsilon$. If we take $\epsilon = \widehat{\gamma}_\infty^C(x^*)/2$

in Lemma S20, then there exists a $\widehat{N}' \geq \widehat{N}$ so that for all $N > \widehat{N}'$, the log-1 Lipschitz condition is satisfied for all $x \geq x_N$. This implies that there is exactly one graded interval and the conclusion of the lemma follows. *Q.E.D.*

PROOF OF PROPOSITION 10: We first derive the allocation. When $\underline{v} > c$, we have $x^* = -\infty$ and the gains function $\overline{\gamma}$ is not graded when N is sufficiently large. In this case, $\overline{Q}_N^C(x)$ is always exactly 1.

When $\underline{v} < c$, $x^* \in (-\infty, \infty)$, and the gains function $\overline{\gamma}$ is single crossing (Section 4.4) when N is sufficiently large. Then $\overline{Q}_N^C(x) = \min((x\sqrt{N} + N)/(x_N\sqrt{N} + N), 1)$. Since x_N converges to x^* as defined by equation (29), $\overline{Q}_N^C(x)$ converges to 1 as $N \to \infty$.

We now derive the transfer. From Lemma 10, we know that there is at most one graded interval of the form $[-\sqrt{N}, x_N]$, where $x_N = -\sqrt{N}$ if $\underline{v} > c$ and $x_N > -\sqrt{N}$ if $\underline{v} < c$.

Recall that

$$\overline{T}_N(x) = \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}_N(y) g_N(y)\, dy,$$

$$\overline{\Xi}_N(x) = \overline{\mu}_N(x)\widehat{w}_N(x) - \overline{\lambda}_N\big(\widehat{w}_N(x)\big) - c\overline{Q}_N(x),$$

$$\begin{aligned}
&\overline{\lambda}_N\big(\widehat{w}_N(x)\big) \\
&\quad = \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y)\, dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y)\, d\widehat{w}_N(y) - \int_{y=x}^{\infty} \overline{\mu}_N(y)\, d\widehat{w}_N(y) \\
&\quad = \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y)\, dy + \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y)\, d\widehat{w}_N(y) \\
&\qquad + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y)\, d\widehat{\mu}_N(y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y)\, d\widehat{w}_N(y) \\
&\quad = \int_{y=0}^{\infty} \overline{\mu}_N(y) G_N(y)\, d\widehat{\gamma}_N(y) \\
&\quad = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y)\, d\big(\overline{\mu}_N(y) G_N(y)\big) \\
&\quad = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y)\, d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \widehat{\gamma}_N(y)\overline{\mu}(y) g_N(y)\, dy \\
&\quad = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y)\, d\overline{\mu}_N(y) - \int_{y=0}^{\infty} \overline{\gamma}_N(y) g_{N-1}(y)\, dy,
\end{aligned}$$

where the last inequality comes from equation (32). Thus,

$$\overline{\lambda}_N\big(\widehat{w}_N(x)\big) = -\int_{y=0}^{\infty} \widehat{\gamma}_N(y) G_N(y)\, d\overline{\mu}_N(y) + \overline{\mu}_N(x)\widehat{w}_N(x) + \int_{y=x}^{\infty} \widehat{w}_N(y)\, d\overline{\mu}_N(y)$$

and

$$\begin{aligned}\overline{\Xi}_N(x) &= \int_{y=0}^{x} \widehat{\gamma}_N(y)G_N(y)\,d\overline{\mu}_N(y) \\ &\quad + \int_{y=x}^{\infty} \big(\widehat{\gamma}_N(y)G_N(y) - \widehat{w}_N(y)\big)\,d\overline{\mu}_N(y) - c\overline{Q}_N(x) \\ &= \int_{y=0}^{x} \widehat{\gamma}_N(y)G_N(y)\,d\overline{\mu}_N(y) \\ &\quad - \int_{y=x}^{\infty} \widehat{\gamma}_N(y)\big(1 - G_N(y)\big)\,d\overline{\mu}_N(y) - c\big(\overline{Q}_N(x) - \overline{\mu}_N(x)\big).\end{aligned}$$

Let us now switch to central limit units:

$$\begin{aligned}\Xi_N^C(x) &= \overline{\Xi}_N(\sqrt{N-1}x + N - 1) \\ &= \int_{y=-\sqrt{N}}^{x} \widehat{\gamma}_N^C(y)G_N^C(y)\,d\overline{\mu}_N^C(y) \\ &\quad - \int_{y=x}^{\infty} \widehat{\gamma}_N^C(y)\big(1 - G_N^C(y)\big)\,d\overline{\mu}_N^C(y) - c\big(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)\big).\end{aligned}$$

By Lemmas S11 and S13, $\widehat{\gamma}_N^C(y) \to \widehat{\gamma}_\infty^C(y) = H^{-1}(\Phi(y)) - c$ and $G_N^C(y) \to \Phi(y)$ as $N \to \infty$.

Moreover, we have

$$\begin{aligned}&\sqrt{N-1}\,d\overline{\mu}_N^C(y) \\ &= \begin{cases} 0 & \text{if } y < x_N, \\ (N-1)\Big(\dfrac{N-1}{x_N\sqrt{N-1}+N-1} - \dfrac{N}{x_N\sqrt{N-1}+N-1}\Big) \to -1 & \text{if } y = x_N, \\ -(N-1)\dfrac{N-1}{(y\sqrt{N-1}+N-1)^2}\,dy \to -dy & \text{if } y > x_N, \end{cases}\end{aligned}$$

where the mass point on x_N is derived by comparing $\overline{\mu}_N^C$ to the left and right of x_N, and

$$\begin{aligned}&\sqrt{N-1}\big(\overline{Q}_N^C(x) - \overline{\mu}_N^C(x)\big) \\ &= \begin{cases} \sqrt{N-1}\Big(\dfrac{x\sqrt{N-1}+N-1}{x_N\sqrt{N-1}+N-1} - \dfrac{N}{x_N\sqrt{N-1}+N-1}\Big) & \text{if } x < x_N, \\ \sqrt{N-1}\Big(1 - \dfrac{N-1}{x\sqrt{N-1}+N-1}\Big) & \text{if } x > x_N, \end{cases}\end{aligned}$$

which converges to x in both cases.

Define $F(x) = \lim_{N\to\infty} \sqrt{N-1}\overline{\Xi}_N^C(x)$. We have

$$F(x) = \begin{cases} -cx + \widehat{\gamma}_\infty^C(x^*)\big(1-\Phi(x^*)\big) + \int_{y=x^*}^{\infty} \widehat{\gamma}_\infty^C(y)\big(1-\Phi(y)\big)\,dy, \\ \quad x < x^*, \\ -cx - \widehat{\gamma}_\infty^C(x^*)\Phi(x^*) - \int_{y=x^*}^{x} \widehat{\gamma}_\infty^C(y)\Phi(y)\,dy + \int_{y=x}^{\infty} \widehat{\gamma}_\infty^C(y)\big(1-\Phi(y)\big)\,dy, \\ \quad x > x^*. \end{cases}$$

Therefore,

$$\lim_{N\to\infty} \overline{T}_N^C(x) = \frac{1}{\phi(x)} \int_{y=0}^{x} F(y)\phi(y)\,dy. \qquad \textit{Q.E.D.}$$

APPENDIX D: Derivation of the Aggregate Transfer for the Uniform Distribution

Suppose the prior H is the standard uniform distribution, so that $\widehat{w}(x) = G_N(x)$, and that $c = 0$.

D.1. *The Must-Sell Case*

We have

$$\begin{aligned} \widehat{\lambda}\big(G_N(x)\big) &= \int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy + \int_{y=0}^{\infty} \frac{N-1}{y} G_N(y)g_N(y)\,dy - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\,dy \\ &= 2\int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy - \big(1 - G_{N-1}(x)\big) \\ &= 2\widehat{\Pi} - \big(1 - G_{N-1}(x)\big), \\ \widehat{\Xi}(x) &= \frac{N-1}{x} G_N(x) - G_{N-1}(x) + 1 - 2\widehat{\Pi}. \end{aligned}$$

Next,

$$\begin{aligned} &\int_{y=0}^{x} \widehat{\Xi}(y)g_N(y)\,dy \\ &\quad= \int_{y=0}^{x} \left(\frac{N-1}{y} G_N(y) - G_{N-1}(y) + 1 - 2\widehat{\Pi}\right) g_N(y)\,dy \\ &\quad= 2\int_{y=0}^{x} G_N(y)g_{N-1}(y)\,dy - G_N(x)G_{N-1}(x) + (1-2\widehat{\Pi})G_N(x) \\ &\quad= G_{N-1}(x)^2 - 2\int_{y=0}^{x} g_N(y)g_{N-1}(y)\,dy - G_N(x)G_{N-1}(x) + (1-2\widehat{\Pi})G_N(x) \\ &\quad= G_{N-1}(x)g_N(x) - 2\int_{y=0}^{x} g_N(y)g_{N-1}(y)\,dy + (1-2\widehat{\Pi})G_N(x) \end{aligned}$$

$$= G_{N-1}(x)g_N(x) - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!}G_{2N-2}(2x) + (1-2\widehat{\Pi})G_N(x)$$
$$= G_{N-1}(x)g_N(x) + \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!}\big(G_N(x) - G_{2N-2}(2x)\big),$$

where the second line follows from integration by parts, the third and fourth lines use $G_N = G_{N-1} - g_N$, the fifth line is a direct computation using the formula for g_N in (14), and the last line follows from

$$\widehat{\Pi} = \int_{y=0}^{\infty} G_N(y)g_{N-1}(y)\,dy = \frac{1}{2} - \int_{y=0}^{\infty} g_N(y)g_{N-1}(y)\,dy$$
$$= \frac{1}{2}\left(1 - \frac{(2N-3)!}{2^{2N-3}(N-1)!(N-2)!}\right).$$

Therefore, when $x > 0$,

$$\widehat{T}(x) = G_{N-1}(x) + \frac{\binom{2N-3}{N-1}}{2^{2N-3}}\frac{G_N(x) - G_{2N-2}(2x)}{g_N(x)}.$$

In the central limit normalization, we define

$$\widehat{T}^C(x) = \widehat{T}(N-1+\sqrt{N-1}x).$$

Lemma S11 shows that $G_N(N-1+\sqrt{N-1}x) \to \Phi(x)$ and $g_N(N-1+\sqrt{N-1}x) \times \sqrt{N-1} \to \phi(x)$ as $N \to \infty$, where Φ and ϕ are, respectively, the cumulative distribution and the density of a standard Normal; this also implies that $G_{2N-2}(2(N-1+\sqrt{N-1}x)) \to \Phi(x\sqrt{2})$. Finally, using Stirling's approximation, it is easy to check that $\frac{\binom{2N-3}{N-1}}{2^{2N-3}}\sqrt{N-1} \to \frac{1}{\sqrt{\pi}}$ as $N \to \infty$. Therefore,

$$\lim_{N\to\infty} \widehat{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\phi(x)}$$

for a fixed x.

D.2. *The Can-Keep Case*

We have shown in Section 4.4 that the uniform distribution is single crossing. Let $[0, x^*]$ denote the graded interval. The cutoff x^* satisfies (cf. (28))

$$\frac{G_N(x^*)}{2} = g_{N+1}(x^*). \tag{S7}$$

This equation implies that $G_{N+1}(x^*) = G_N(x^*) - g_{N+1}(x^*) = g_{N+1}(x^*) = G_N(x^*)/2$.

Define the constants

$$C = \int_{x=0}^{\infty} \overline{\gamma}(x)g_{N-1}(x)\,dx + \int_{x=0}^{\infty} \overline{\mu}(x)G_N(x)g_N(x)\,dx$$

$$= \underbrace{\int_{x=0}^{x^*} \exp(x - x^*) G_N(x^*) g_{N-1}(x)\, dx + \int_{x=0}^{x^*} \frac{N}{x^*} G_N(x) g_N(x)\, dx}_{C_1}$$

$$+ \underbrace{\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\, dx + \int_{x=x^*}^{\infty} \frac{N-1}{x} G_N(x) g_N(x)\, dx}_{C_2}.$$

We next simplify the constants:

$$\begin{aligned}
C_1 &= 2\int_{x=0}^{x^*} \exp(x - x^*) G_N(x^*) g_{N-1}(x)\, dx \\
&= 2G_N(x^*) g_N(x^*), \\
C_2 &= 2\int_{x=x^*}^{\infty} G_N(x) g_{N-1}(x)\, dx \\
&= 1 - G_{N-1}(x^*)^2 - 2\int_{x=x^*}^{\infty} g_N(x) g_{N-1}(x)\, dx \\
&= 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}} \big(1 - G_{2N-2}(2x^*)\big),
\end{aligned}$$

$$C = 2G_N(x^*) g_N(x^*) + 1 - G_{N-1}(x^*)^2 - \frac{\binom{2N-3}{N-1}}{2^{2N-3}} \big(1 - G_{2N-2}(2x^*)\big).$$

Then

$$\begin{aligned}
\overline{\lambda}\big(G_N(x)\big) &= C - \int_{y=x}^{\infty} \overline{\mu}(y) g_N(y)\, dy \\
&= \begin{cases} C - \int_{y=x}^{x^*} \frac{N}{x^*} g_N(y)\, dy - \int_{y=x^*}^{\infty} \frac{N-1}{y} g_N(y)\, dy, & x \le x^*, \\ C - \int_{y=x}^{\infty} \frac{N-1}{y} g_N(y)\, dy, & x > x^* \end{cases} \\
&= \begin{cases} C - \big(G_N(x^*) - G_N(x)\big)\frac{N}{x^*} - \big(1 - G_{N-1}(x^*)\big), & x \le x^*, \\ C - \big(1 - G_{N-1}(x)\big), & x > x^* \end{cases}
\end{aligned}$$

and

$$\overline{\Xi}(x) = \begin{cases} G_N(x)\frac{N}{x^*} - C + \big(G_N(x^*) - G_N(x)\big)\frac{N}{x^*} + (1 - G_{N-1}(x^*) \\ \quad = -C + G_N(x^*)\frac{N}{x^*} + 1 - G_{N-1}(x^*) & x \le x^*, \\ G_N(x)\frac{N-1}{x} - C + 1 - G_{N-1}(x), & x > x^*. \end{cases}$$

For $x \le x^*$, we have

$$\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\,dy = \int_{y=0}^{x} \Big(-C + G_N\big(x^*\big)\frac{N}{x^*} + 1 - G_{N-1}\big(x^*\big)\Big) g_N(y)\,dy$$
$$= \Big(-C + G_N\big(x^*\big)\frac{N}{x^*} + 1 - G_{N-1}\big(x^*\big)\Big) G_N(x).$$

For $x > x^*$, we have

$$\int_{y=0}^{x} \overline{\Xi}(y) g_N(y)\,dy = \Big(-C + G_N\big(x^*\big)\frac{N}{x^*} + 1 - G_{N-1}\big(x^*\big)\Big) G_N\big(x^*\big)$$
$$+ \underbrace{\int_{x^*}^{x} \Big(G_N(y)\frac{N-1}{y} - C + 1 - G_{N-1}(y)\Big) g_N(y)\,dy}_{X}.$$

Simplifying the second term, we get

$$\begin{aligned} X &= (1-C)\big(G_N(x) - G_N\big(x^*\big)\big) \\ &\quad + 2\int_{y=x^*}^{x} G_N(y) g_{N-1}(y)\,dy - \big(G_N(x)G_{N-1}(x) - G_N\big(x^*\big)G_{N-1}\big(x^*\big)\big) \\ &= (1-C)\big(G_N(x) - G_N\big(x^*\big)\big) \\ &\quad - 2\int_{y=x^*}^{x} g_N(y) g_{N-1}(y)\,dy + g_N(x)G_{N-1}(x) - g_N\big(x^*\big)G_{N-1}\big(x^*\big) \\ &= (1-C)\big(G_N(x) - G_N\big(x^*\big)\big) \\ &\quad - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}\big(G_{2N-2}(2x) - G_{2N-2}\big(2x^*\big)\big) + g_N(x)G_{N-1}(x) - g_N\big(x^*\big)G_{N-1}\big(x^*\big). \end{aligned}$$

Therefore, for $x \le x^*$, we have

$$\overline{T}(x) = \Big(-C + G_N\big(x^*\big)\frac{N}{x^*} + 1 - G_{N-1}\big(x^*\big)\Big)\frac{G_N(x)}{g_N(x)}.$$

For $x > x^*$, we have

$$\begin{aligned} \overline{T}(x) = \Bigg[& G_N\big(x^*\big)^2 \frac{N}{x^*} - G_{N-1}\big(x^*\big)^2 + (1-C)G_N(x) \\ & - \frac{\binom{2N-3}{N-1}}{2^{2N-3}}\big(G_{2N-2}(2x) - G_{2N-2}\big(2x^*\big)\big)\Bigg]\frac{1}{g_N(x)} + G_{N-1}(x). \end{aligned}$$

Finally, we take the limit as $N \to \infty$ for the central limit normalization:

$$\overline{T}^C(x) = \overline{T}(N - 1 + \sqrt{N-1}\,x).$$

Since $G_N(x^*)/2 = G_{N+1}(x^*)$ by the discussion following equation (S7), we must have $(x^* - (N-1))/\sqrt{N-1} \to -\infty$, $G_N(x^*) \to 0$, and $g_N(x^*) \to 0$ as $N \to \infty$. Moreover, by equation (S7), $NG_N(x^*)/x^* = 2Ng_{N+1}(x^*)/x^* = 2g_N(x^*) \to 0$ as $N \to \infty$. Substituting these into the expressions of C and $\overline{T}$, and simplifying as in the must-sell case, we get

$$\lim_{N\to\infty} \overline{T}^C(x) = \Phi(x) + \frac{\Phi(x) - \Phi(x\sqrt{2})}{\sqrt{\pi}\phi(x)}.$$

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Co-editor Itzhak Gilboa handled this manuscript.

Manuscript received 26 April, 2018; final version accepted 2 October, 2020; available online 11 November, 2020.