Econometrica Supplementary Material

SUPPLEMENT TO "ROBUST BAYESIAN INFERENCE FOR SET-IDENTIFIED MODELS"

(*Econometrica*, Vol. 89, No. 4, July 2021, 1519–1556)

RAFFAELLA GIACOMINI
Department of Economics, University College London

TORU KITAGAWA
Department of Economics, University College London

APPENDIX B: CONVEXITY, CONTINUITY, AND DIFFERENTIABILITY OF THE IMPULSE-RESPONSE IDENTIFIED SET

Notation: THE PROOFS BELOW USE the following notation. For given $\phi \in \Phi$ and $i = 1, \ldots, n$, let $\tilde{f}_i(\phi) \equiv \text{rank}(F_i(\phi))$. Since the rank of $F_i(\phi)$ is determined by its row rank, $\tilde{f}_i(\phi) \leq f_i(\phi)$ holds. Let $\mathcal{F}_i^{\perp}(\phi)$ be the linear subspace of $\mathbb{R}^n$ that is orthogonal to the row vectors of $F_i(\phi)$. If no zero restrictions are placed on q_i, we interpret $\mathcal{F}_i^{\perp}(\phi)$ to be $\mathbb{R}^n$. Note that the dimension of $\mathcal{F}_i^{\perp}(\phi)$ is equal to $n - \tilde{f}_i(\phi)$. We let $\mathcal{H}_i(\phi)$ be the half-space in $\mathbb{R}^n$ defined by the sign normalization restriction $\{z \in \mathbb{R}^n : (\sigma^i)'z \geq 0\}$, where σ^i is the ith column vector of Σ_{tr}^{-1}. Given linearly independent vectors, $A = [a_1, \ldots, a_j] \in \mathbb{R}^{n \times j}$, denote the linear subspace in $\mathbb{R}^n$ that is orthogonal to the column vectors of A by $\mathcal{P}(A)$. Note that the dimension of $\mathcal{P}(A)$ is $n - j$.

B.1. *Convexity*

The next proposition shows conditions for the convexity of the impulse-response identified set.

PROPOSITION B.1—Convexity: *Let the object of interest be $\eta = c'_{ih}(\phi)q_{j^*}$, the hth-horizon impulse response of ith variable to the j^*th structural shock, $i \in \{1, 2, \ldots, n\}$, $h \in \{0, 1, 2, \ldots\}$, where the variables are ordered according to Definition* 3.

(I) *Suppose there are only zero restrictions of the form* (4.9). *Assume $f_i \leq n - i$ for all $i = 1, \ldots, n$. Then, for every i and h, and almost every $\phi \in \Phi$, the identified set of η is non-empty and bounded, and it is convex if any of the following mutually exclusive conditions holds*:

  (i) $j^* = 1$ *and* $f_1 < n - 1$;
  (ii) $j^* \geq 2$, *and $f_i < n - i$ for all $i = 1, \ldots, j^* - 1$*;
  (iii) $j^* \geq 2$ *and there exists $1 \leq i^* \leq j^* - 1$ such that $f_i < n - i$ for all $i = i^* + 1, \ldots, j^*$ and $[q_1, \ldots, q_{i^*}]$ is exactly identified, meaning that, for almost every $\phi \in \Phi$, the constraints $F_i(\phi)q_i = \mathbf{0}$, $i = 1, \ldots, i^*$, and the sign-normalizations $(\sigma^i)'q_i \geq 0$, $i = 1, \ldots, i^*$, pin down a unique $[q_1, \ldots, q_{i^*}]$.*[1]

Raffaella Giacomini: r.giacomini@ucl.ac.uk
Toru Kitagawa: t.kitagawa@ucl.ac.uk

[1] If $\text{rank}(F_i(\phi)) = f_i$ for all $i = 1, \ldots, i^*$ and for almost every $\phi \in \Phi$, a necessary condition for exact identification of $[q_1, \ldots, q_{i^*}]$ is that $f_i = n - i$ for all $i = 1, 2, \ldots, i^*$. One can check if the condition is also sufficient by checking that Algorithm 1 of Rubio-Ramírez, Waggoner, and Zha (2010) yields a unique set of orthonormal vectors $[q_1, \ldots, q_{i^*}]$ for every ϕ randomly drawn from a prior supporting the entire Φ. This corresponds to rank conditions (B.3) in Algorithm B.1 below.

(II) *Consider the case with both zero and sign restrictions, and suppose that sign restrictions are placed only on the responses to the j^*th structural shock, that is,* $\mathcal{I}_S = \{j^*\}$.

(iv) *Suppose the zero restrictions satisfy one of conditions* (i) *and* (ii) *in the current proposition. If there exists a unit-length vector* $q \in \mathbb{R}^n$ *such that*

$$F_{j^*}(\phi)q = 0 \quad \text{and} \quad \begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*})' \end{pmatrix} q > \mathbf{0}, \tag{B.1}$$

then the identified set of η, $\mathrm{IS}_\eta(\phi|F, S)$, *is non-empty and convex for every* i *and* h.

(v) *Suppose that the zero restrictions satisfy condition* (iii) *in the current proposition. Let* $[q_1(\phi), \dots, q_{i^*}(\phi)]$ *be the first* i^**th orthonormal vectors that are exactly identified* (*see condition* (iii)). *If there exists a unit length vector* $q \in \mathbb{R}^n$ *such that*

$$F_{j^*}(\phi)q = \mathbf{0}, \qquad q_i'(\phi)q = 0 \quad \text{for } i = 1, \dots, i^*, \quad \text{and}$$
$$\begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*})' \end{pmatrix} q > \mathbf{0}, \tag{B.2}$$

then the identified set of η, $\mathrm{IS}_\eta(\phi|F, S)$, *is non-empty and convex for every* i *and* h.

PROOF OF PROPOSITION B.1: The proof builds on Algorithm 1 of Rubio-Ramírez, Waggoner, and Zha (2010). We present its slightly modified version here.

ALGORITHM B.1: Consider a collection of zero restrictions of the form given by (4.9), where the order of the variables is consistent with $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. Assume $f_i = n - i$ for all $i = 1, \dots, i^*$, and $\mathrm{rank}(F_i(\phi)) = f_i$ for all $i = 1, \dots, i^*$, ϕ-a.s. Let q_1 be a unit-length vector satisfying $F_1(\phi)q_1 = 0$, which is unique up to sign since $\mathrm{rank}(F_1(\phi)) = n - 1$ by assumption. Given q_1, find orthonormal vectors $q_2, \dots, q_{i^*}$, by solving $F_i(\phi)q_i = \mathbf{0}$ and $q_j'q_i = 0$, $j = 1, \dots, i - 1$, successively for $i = 2, 3, \dots, i^*$. If

$$\mathrm{rank}\begin{pmatrix} F_i'(\phi) & q_1 & \cdots & q_{i-1} \end{pmatrix} = n - 1 \quad \text{for } i = 2, \dots, i^*, \tag{B.3}$$

and q_i, $i = 1, \dots, i^*$, obtained by this algorithm satisfies $(\sigma^i)'q_i \neq 0$ for almost all $\phi \in \Phi$, that is, the sign normalization restrictions determine a unique sign for the $q_i's$, then $[q_1, \dots, q_{i^*}]$ is exactly identified.[2]

Consider first the case with only zero restrictions (Case (I)). Fix $\phi \in \Phi$. Let $Q_{1:i} = [q_1, \dots, q_i]$, $i = 2, \dots, (n-1)$, be an $n \times i$ matrix of orthonormal vectors in $\mathbb{R}^n$. The set of feasible Q's satisfying the zero restrictions and the sign normalizations, $\mathcal{Q}(\phi|F)$, can be

[2] A special situation where the rank conditions (B.3) are guaranteed at almost every ϕ is when σ^i is linearly independent of the row vectors in $F_i(\phi)$ for all $i = 1, \dots, n$, and the row vectors of $F_i(\phi)$ are spanned by the row vectors of $F_{i-1}(\phi)$ for all $i = 2, \dots, i^*$. This condition holds in the recursive identification scheme, which imposes a triangularity restriction on A_0^{-1}. See Example B.2 in Appendix B.

written in the following recursive manner:

$$\begin{aligned}
&Q = [q_1, \dots, q_n] \in \mathcal{Q}(\phi|F) \\
&\quad \text{if and only if} \quad Q = [q_1, \dots, q_n] \text{ satisfies} \\
&q_1 \in D_1(\phi) \equiv \mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^{n-1}, \\
&q_2 \in D_2(\phi, q_1) \equiv \mathcal{F}_2^{\perp}(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1) \cap \mathcal{S}^{n-1}, \\
&q_3 \in D_3(\phi, Q_{1:2}) \equiv \mathcal{F}_3^{\perp}(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}) \cap \mathcal{S}^{n-1}, \\
&\vdots \\
&q_j \in D_j(\phi, Q_{1:(j-1)}) \equiv \mathcal{F}_j^{\perp}(\phi) \cap \mathcal{H}_j(\phi) \cap \mathcal{P}(Q_{1:(j-1)}) \cap \mathcal{S}^{n-1}, \\
&\vdots \\
&q_n \in D_n(\phi, Q_{1:(n-1)}) \equiv \mathcal{F}_n^{\perp}(\phi) \cap \mathcal{H}_n(\phi) \cap \mathcal{P}(Q_{1:(n-1)}) \cap \mathcal{S}^{n-1}.
\end{aligned} \tag{B.4}$$

where $D_i(\phi, Q_{1:(i-1)}) \subset \mathbb{R}^n$ denotes the set of feasible q_i's given $Q_{1:(i-1)} = [q_1, \dots, q_{i-1}]$, the set of $(i-1)$ orthonormal vectors in $\mathbb{R}^n$ preceding i. Non-emptiness of the identified set for $\eta = c_{ih}(\phi)q_j$ follows if the feasible domain of the orthogonal vector $D_i(\phi, Q_{1:(i-1)})$ is non-empty at every $i = 1, \dots, n$.

Note that by the assumption $f_1 \le n-1$, $\mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi)$ is the half-space of the linear subspace of $\mathbb{R}^n$ with dimension $n - \tilde{f}_1(\phi) \ge n - f_1 \ge 1$. Hence, $D_1(\phi)$ is non-empty for every $\phi \in \Phi$. For $i = 2, \dots, n$, $\mathcal{F}_i^{\perp}(\phi) \cap \mathcal{H}_i(\phi) \cap \mathcal{P}(Q_{1:(i-1)})$ is the half-space of the linear subspace of $\mathbb{R}^n$ with dimension at least

$$n - \tilde{f}_i(\phi) - \dim\bigl(\mathcal{P}(Q_{1:(n-1)})\bigr) \ge n - f_i - (i-1) \ge 1,$$

where the last inequality follows from the assumption $f_i \le n - i$. Hence, $D_i(\phi, Q_{1:(i-1)})$ is non-empty for every $\phi \in \Phi$. We thus conclude that $\mathcal{Q}(\phi|F)$ is non-empty, and this implies non-emptiness of the impulse-response identified sets for every $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\}$, and $h = 0, 1, 2, \dots$. The boundedness of the identified sets follows since $|c_{ih}(\phi)q_j| \le \|c_{ih}(\phi)\| < \infty$ for any $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n\}$, and $h = 0, 1, 2, \dots$, where the boundedness of $\|c_{ih}(\phi)\|$ is ensured by the restriction on ϕ that the reduced-form VAR is invertible to VMA(∞).

Next, we show convexity of the identified set of the impulse response to the j^*th shock under each one of conditions (i)–(iii). Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i)). Since $\tilde{f}_1(\phi) < n-1$ for all $\phi \in \Phi$, $D_1(\phi)$ is a path-connected set because it is an intersection of the half-space with dimension at least 2 and the unit sphere. Since the impulse response is a continuous function of q_1, the identified set of $\eta = c_{ih}(\phi)q_1$ is an interval, as the range of a continuous function with a path-connected domain is always an interval (see, e.g., Propositions 12.11 and 12.23 in Sutherland (2009)).

Suppose $j^* \ge 2$ and assume condition (ii) holds. Denote the set of feasible q_{j^*}'s by $\mathcal{E}_{j^*}(\phi) \equiv \{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F)\}$. The next lemma provides a specific expression for $\mathcal{E}_{j^*}(\phi)$.

LEMMA B.1: *Suppose $j^* \ge 2$ and assume condition* (ii) *of Proposition* B.1 *holds*. *Then* $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{S}^{n-1}$.

PROOF OF LEMMA B.1: Given zero restrictions $F(\phi, Q) = \mathbf{0}$ and the set of feasible orthogonal matrices $\mathcal{Q}(\phi|F)$, define the projection of $\mathcal{Q}(\phi|F)$ with respect to the first i column vectors,

$$\mathcal{Q}_{1:i}(\phi|F) \equiv \big\{[q_1, \dots, q_i] : Q \in \mathcal{Q}(\phi|F)\big\}.$$

Following the recursive representation of (B.4), $\mathcal{E}_{j^*}(\phi) \equiv \{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F)\}$ can be written as

$$\begin{aligned}\mathcal{E}_{j^*}(\phi) &= \bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \big[\mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:(j^*-1)}) \cap \mathcal{S}^{n-1}\big] \\ &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)})\right] \cap \mathcal{S}^{n-1}.\end{aligned}$$

Hence, the conclusion follows if we can show $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. To show this claim, let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we construct $Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)$ such that $q \in \mathcal{P}(Q_{1:(j^*-1)})$ holds. Specifically, construct q_i, $i = 1, \dots, (j^* - 1)$, successively, by solving $F_i(\phi)q_i = \mathbf{0}$, $q_\ell' q_i = 0$, $\ell = 1, \dots, i-1$, and $q' q_i = 0$, jointly, and choose the sign of q_i to satisfy its sign normalization. Under condition (ii) of Proposition B.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exists since the rank of the coefficient matrix is at most $f_i + i < n$. The obtained $Q_{1:(j^*-1)} = [q_1, \dots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{1:(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{1:(j^*-1)})$. Since q is arbitrary, we obtain $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. *Q.E.D.*

Lemma B.1 shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} \geq j^* \geq 2$ with the unit sphere. Hence, $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$ and convexity of $\text{IS}_\eta(\phi|F)$ follows.

Next, suppose condition (iii) holds. Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \dots, q_{i^*}(\phi)]$ be the first i^* columns of feasible $Q \in \mathcal{Q}(\phi|F)$ that are common for all $Q \in \mathcal{Q}(\phi|F)$ by the assumption of exact identification of the first i^* columns. In this case, the set of feasible q_{j^*}'s can be expressed as in the next lemma.

LEMMA B.2: *Suppose $j^* \geq 2$ and assume condition* (iii) *of Proposition* B.1 *holds. Then, whenever $Q_{1:i^*}(\phi) = (q_1(\phi), \dots, q_{i^*}(\phi))$ is uniquely determined as a function of ϕ* (*this is the case for almost every $\phi \in \Phi$ by the assumption of exact identification*), $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:i^*}(\phi)) \cap \mathcal{S}^{n-1}$.

PROOF OF LEMMA B.2: Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \dots, q_{i^*}(\phi)]$ be the first i^* columns of feasible $Q \in \mathcal{Q}(\phi|F)$, that are common for all $Q \in \mathcal{Q}(\phi|F)$, ϕ-a.s., by exact identification of the first i^* columns. As in the proof of Lemma A.1, $\mathcal{E}_{j^*}(\phi)$ can be written as

$$\begin{aligned}\mathcal{E}_{j^*}(\phi) &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)})\right] \cap \mathcal{S}^{n-1} \\ &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}\big(Q_{1:i^*}(\phi)\big) \\ &\quad \cap \left[\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)})\right] \cap \mathcal{S}^{n-1},\end{aligned}$$

where $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F) = \{Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \dots, q_{j^*-1}] : Q \in \mathcal{Q}(\phi|F)\}$ is the projection of $\mathcal{Q}(\phi|F)$ with respect to the (i^*+1)th to (j^*-1)th columns of Q. We now show that, under condition (iii) of Proposition B.1, $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$ holds. Let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we consider constructing $Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ such that $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$ holds. For $i = (i^*+1), \dots, (j^*-1)$, we recursively obtain q_i by solving

$$q_i' \begin{pmatrix} F_i'(\phi) & q_1(\phi) & \cdots & q_{i^*}(\phi) & q_{i^*+1} & \cdots & q_{i-1} & q \end{pmatrix} = \mathbf{0}',$$

and choose the sign of q_i to be consistent with the sign normalization. Under condition (iii) of Proposition B.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exists since the rank of the coefficient matrix is at most $f_i + i < n$ for all $i = (i^*+1), \dots, (j^*-1)$. The obtained $Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \dots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$. Since q is arbitrary, we have that $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$. *Q.E.D.*

Lemma B.2 shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} - i^* \geq j^* + 1 - i^* \geq 2$ with the unit sphere. Hence, $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$ and convexity of $\text{IS}_\eta(\phi|F)$ follows.

For the cases under condition (i) or (ii), since $\phi \in \Phi$ is arbitrary, the convexity of the impulse-response identified set holds for every $\phi \in \Phi$. As for the case of condition (iii), the exact identification of $[q_1(\phi), \dots, q_{i^*}(\phi)]$ assumes its unique determination up to almost every $\phi \in \Phi$, so convexity of the identified set holds for almost every $\phi \in \Phi$.

Next, consider the case with both zero and sign restrictions (Case (II)). Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i)). Following (B.4), the set of feasible q_1's can be denoted by $D_1(\phi) \cap \{x \in \mathbb{R}^n : S_1(\phi)x \geq 0\}$. Let $\tilde{q}_1 \in D_1(\phi)$ be a unit-length vector that satisfies $\binom{S_1(\phi)}{(\sigma^1)'}\tilde{q}_1 > \mathbf{0}$. Such $\tilde{q}_1$ is guaranteed to exist by the assumption stated in the current proposition. Let $q_1 \in D_1(\phi) \cap \{x \in \mathbb{R}^n : S_1(\phi)x \geq 0\}$ be arbitrary. Note that $q_1 \neq -\tilde{q}_1$ must hold, since otherwise some of the sign restrictions are violated. Consider

$$q_1(\lambda) = \frac{\lambda q_1 + (1-\lambda)\tilde{q}_1}{\|\lambda q_1 + (1-\lambda)\tilde{q}_1\|}, \quad \lambda \in [0, 1],$$

which is a connected path in $D_1(\phi) \cap \{x \in \mathbb{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ since the denominator is nonzero for all $\lambda \in [0, 1]$ by the fact that $q_1 \neq -\tilde{q}_1$. Since q_1 is arbitrary, we can connect any points in $D_1(\phi) \cap \{x \in \mathbb{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ by connected paths via $\tilde{q}_1$. Hence, $D_1(\phi) \cap \{x \in \mathbb{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ is path-connected, and convexity of the impulse-response identified set follows.

Suppose $j^* \geq 2$ and assume that the imposed zero restrictions satisfy condition (ii). Let $\mathcal{E}_{j^*}(\phi) \equiv \{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F, S)\}$, and let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\binom{S_{j^*}(\phi)}{[\sigma^{j^*}(\phi)]'}\tilde{q}_{j^*} > \mathbf{0}$. Such $\tilde{q}_{j^*}$ exists by the assumption stated in the current proposition. For any $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$, $q_{j^*} \neq -\tilde{q}_{j^*}$ must be true, since otherwise q_{j^*} would violate some of the imposed sign restrictions. Consider constructing a path between q_{j^*} and $\tilde{q}_{j^*}$ as follows. For $\lambda \in [0, 1]$, let

$$q_{j^*}(\lambda) = \frac{\lambda \tilde{q}_{j^*} + (1-\lambda)q_{j^*}}{\|\lambda \tilde{q}_{j^*} + (1-\lambda)q_{j^*}\|}, \tag{B.5}$$

which is a continuous path on the unit sphere since the denominator is nonzero for all $\lambda \in [0, 1]$ by the construction of $\tilde{q}_{j^*}$. Along this path, $F_{j^*}(\phi)q_{j^*}(\lambda) = \mathbf{0}$ and the sign restrictions hold. Hence, for every $\lambda \in [0,1]$, if there exists $Q(\lambda) \equiv [q_1(\lambda), \dots, q_{j^*}(\lambda), \dots, q_n(\lambda)] \in \mathcal{Q}(\phi|F,S)$, then the path-connectedness of $\mathcal{E}_{j^*}(\phi)$ follows. A recursive construction similar to Algorithm B.1 can be used to construct such $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$. For $i = 1, \dots, (j^* - 1)$, we recursively obtain $q_i(\lambda)$ that solves

$$q_i'(\lambda)\begin{pmatrix} F_i'(\phi) & q_1(\lambda) & \cdots & q_{i-1}(\lambda) & q_{j^*}(\lambda)\end{pmatrix} = \mathbf{0}', \tag{B.6}$$

and satisfies $[\sigma^i(\phi)]'q_i(\lambda) \geq 0$. Such a $q_i(\lambda)$ always exists since the rank of the matrix multiplied to $q_i(\lambda)$ is at most $f_i + i$, which is less than n under condition (ii). For $i = (j^*+1), \dots, n$, a direct application of Algorithm B.1 yields a feasible $q_i(\lambda)$. Thus, existence of $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$, $\lambda \in [0,1]$, is established. We therefore conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected under condition (ii), and the convexity of impulse-response identified sets holds for every variable and every horizon. This completes the proof for Case (iv) of the current proposition.

Last, we consider Case (v). Suppose that the imposed zero restrictions satisfy condition (iii) of the current proposition. Let $[q_1(\phi), \dots, q_{i^*}(\phi)]$ be the first i^*th columns of feasible Q's, that are common for all $Q \in \mathcal{Q}(\phi|F,S)$, ϕ-a.s., by exact identification of the first i^* columns. Let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\binom{S_{j^*}(\phi)}{(\sigma^{j^*})'}\tilde{q}_{j^*} > \mathbf{0}$, and $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be arbitrary. Consider $q_{j^*}(\lambda)$ in (B.5) and construct $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ as follows. The first i^*th column of $Q(\lambda)$ must be $[q_1(\phi), \dots, q_{i^*}(\phi)]$, ϕ-a.s., by the assumption of exact identification. For $i = (i^*+1), \dots, (j^*-1)$, we can recursively obtain $q_i(\lambda)$ that solves

$$\begin{gathered} F_i(\phi)q_i(\lambda) = \mathbf{0} \quad \text{and} \\ q_i'(\lambda)\begin{pmatrix} q_1(\phi) & \cdots & q_{i^*}(\phi) & q_{i^*+1}(\lambda) & \cdots & q_{i-1}(\lambda) & q_{j^*}(\lambda)\end{pmatrix} = \mathbf{0}', \end{gathered} \tag{B.7}$$

and satisfies $[\sigma^i(\phi)]'q_i(\lambda) \geq 0$. There always exists such $q_i(\lambda)$ because $f_i < n - i$ for all $i = (i^*+1), \dots, (j^*-1)$. The rest of the column vectors $q_i(\lambda)$, $i = j^*+1, \dots, n$, of $Q(\lambda)$ are obtained successively by applying Algorithm B.1. Having shown a feasible construction of $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ for $\lambda \in [0,1]$, we conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected, and convexity of the impulse-response identified sets follows for every variable and every horizon. *Q.E.D.*

Proposition B.1 shows that, when a set of zero restrictions satisfies $f_i \leq n - i$ for all $i = 1, 2, \dots, n$, the identified set for the impulse response is never empty, so the zero restrictions cannot be refuted by data. In this case, the plausibility of the identifying restrictions defined in Section 2.6.1 is always equal to 1. When there are also sign restrictions, we can have an empty identified set and a non-trivial value for the plausibility of the identifying restrictions.

Lemma 1 of Granziera, Moon, and Schorfheide (2018) shows convexity of the impulse-response identified set for the special case where zero and sign restrictions are imposed only on responses to the j^*th shock, that is, $j^* = 1$, $f_i = 0$ for all $i = 2, \dots, n$, and $\mathcal{I}_S = \{1\}$ in our notation. Proposition B.1 extends their result to the case where zero restrictions are placed on the columns of Q other than q_{j^*}. The inequality conditions (iv) and (v) of Proposition B.1 imply that the set of feasible q's does not collapse to a one-dimensional subspace in $\mathbb{R}^n$. If the set of feasible q's becomes degenerate, non-convexity arises since the intersection of a one-dimensional subspace in $\mathbb{R}^n$ with the unit sphere consists of two disconnected points.[3]

To gain some intuition for Proposition B.1, consider the case of equality restrictions that restrict a single column q_j by linear constraints of the form (4.9). Convexity of the identified set for η then follows if the subspace of constrained q_j's has dimension greater than 1. This is because the set of feasible q_j's is a subset on the unit sphere in $\mathbb{R}^n$ where any two elements q_j and $q_{j'}$ are path-connected, which implies a convex identified set for the impulse response because the impulse response is a continuous function of q_j. When the subspace has dimension 1, non-convexity can occur if, for example, the identified set consists of two disconnected points and the sign normalization restriction fails to select one, which means that the impulse response is locally, but not globally, identified. This argument implies that for almost every $\phi \in \Phi$, we can guarantee convexity of the identified set by finding a condition on the number of zero restrictions that yields a subspace of q_j's with dimension greater than 1.

The following examples show how to verify the conditions of Proposition B.1.

EXAMPLE B.1: Consider Example 1 in Section 4 in the main text. If the object of interest is an impulse response to the monetary policy shock $\epsilon_{i,t}$, we order the variables as $(i_t, m_t, \pi_t, y_t)'$ and have $(f_1, f_2, f_3, f_4) = (2, 2, 0, 0)$ with $j^* = 1$. Since $f_1 = 2$, condition (i) of Proposition B.1 guarantees that the impulse-response identified set is ϕ-a.s. convex. If the object of interest is an impulse response to a demand shock $\epsilon_{y,t}$, we order the variables as (i_t, m_t, y_t, π_t), and $j^* = 3$. None of the conditions of Proposition B.1 apply in this case, so Proposition B.1 does not gurantee convexity of the impulse-response identified set.

EXAMPLE B.2: Consider adding to Example 1 in Section 4 of the main text a long-run money neutrality restriction, which sets the long-run impulse response of output to a monetary policy shock ($\epsilon_{i,t}$) to zero. This adds a zero restriction on the $(2, 4)$th element of the long-run cumulative impulse-response matrix CIR^∞ and implies one more restriction on q^i. We can order the variables as $(i_t, m_t, \pi_t, y_t)'$ and we have $(f_1, f_2, f_3, f_4) = (3, 2, 0, 0)$. It can be shown that, in this case, the first two columns $[q_1, q_2]$ are exactly identified,[4] which implies that the impulse responses to $\epsilon_{i,t}$ and $\epsilon_{m,t}$ are point-identified. The impulse responses to $\epsilon_{y,t}$ are instead set-identified and their identified sets are convex, as condition (iii) of Proposition B.1 applies to $(i_t, m_t, y_t, \pi_t)'$ with $j^* = 3$.

The next corollary presents a formal result to establish whether the addition of identifying restrictions tightens the identified set.

COROLLARY B.1: *Let a set of zero restrictions, an ordering of variables* $(1, \dots, j^*, \dots, n)$, *and the corresponding number of zero restrictions* $(f_1, \dots, f_n)$ *satisfy* $f_i \le n - i$ *for all* i, $f_1 \ge \dots \ge f_n \ge 0$, *and* $f_{j^*-1} > f_{j^*}$, *as in Definition* 3. *Consider imposing additional zero restrictions. Let* $\pi(\cdot) : \{1, \dots, n\} \to \{1, \dots, n\}$ *be a permutation that reorders the variables to be consistent with Definition* 3 *after adding the new restrictions, and let* $(\tilde{f}_{\pi(1)}, \dots, \tilde{f}_{\pi(n)})$ *be the new number*

[3]If the set of ϕ's that leads to such degeneracy has measure zero in Φ, then, as a corollary of Proposition B.1, we can claim that the impulse-response identified set is convex for almost all ϕ conditional on it being nonempty.

[4]In this case, $F_2(\phi)$ is a submatrix of $F_1(\phi)$, which implies that the vector space spanned by the rows of $F_1(\phi)$ contains the vector space spanned by the rows of $F_2(\phi)$ for every $\phi \in \Phi$. Hence, the rank condition for exact identification (B.3) holds,

of restrictions. If $\tilde{f}_{\pi(i)} \leq n - \pi(i)$ for all $i = 1, \dots, n$, $(\pi(1), \dots, \pi(j^*)) = (1, \dots, j^*)$, and $(f_1, \dots, f_{j^*}) = (\tilde{f}_1, \dots, \tilde{f}_{j^*})$, that is, adding the zero restrictions does not change the order of the variables and the number of restrictions for the first j^* variables, then the additional restrictions do not tighten the identified sets for the impulse response to the j^*th shock for every $\phi \in \Phi$.

PROOF OF COROLLARY B.1: The successive construction of the feasible column vectors q_i, $i = 1, \dots, n$, shows that the additional zero restrictions that do not change the order of variables and the zero restrictions for those preceding j^* do not constrain the set of feasible q_{j^*}'s. *Q.E.D.*

EXAMPLE B.3: Consider adding to Example 1 in Section 4 of the main text the restriction $a^{12} = 0$. Then, an ordering of the variables when the objects of interest are the impulse responses to $\epsilon_{i,t}$ is $(i_t, m_t, y_t, \pi_t)'$ with $j^* = 1$ and $(f_1, f_2, f_3, f_4) = (2, 2, 1, 0)$. Compared to Example 1 in Section 4 of the main text, imposing $a^{12} = 0$ does not change j^*. Corollary B.1 then implies that the restriction does not bring any additional identifying information for the impulse responses.

The next corollary shows invariance of the identified sets when relaxing the zero restrictions, which partially overlaps with the implications of Corollary B.1.

COROLLARY B.2: *Let a set of zero restrictions, an ordering of variables $(1, \dots, j^*, \dots, n)$, and the corresponding number of zero restrictions $(f_1, \dots, f_n)$ satisfy $f_i \leq n - i$ for all i, $f_1 \geq \dots \geq f_n \geq 0$, and $f_{j^*-1} > f_{j^*}$, as in Definition 3 in the main text. Under any of the conditions* (i)–(iii) *of Proposition B.1, the identified set for the impulse responses to the j^*th structural shock does not change when relaxing any or all of the zero restrictions on $q_{j^*+1}, \dots, q_{n-1}$. Furthermore, if condition* (ii) *of Proposition B.1 is satisfied, the identified set for the impulse responses to the j^*th structural shock does not change when relaxing any or all of the zero restrictions on $q_1, \dots, q_{j^*-1}$. When condition* (iii) *of Proposition B.1 is satisfied, the identified set for the impulse responses to the j^*th shock does not change when relaxing any or all of the zero restrictions on $q_{i^*+1}, \dots, q_{j^*-1}$.*

PROOF OF COROLLARY B.2: Dropping the zero restrictions imposed for those following the j^*th variable does not change the order of variables nor the construction of the set of feasible q_{j^*}'s. Under condition (ii) of Proposition B.1, Lemma A.1 in Appendix A shows that the set of feasible q_{j^*}'s does not depend on any of $F_i(\phi)$, $i = 1, \dots, (j^* - 1)$. Hence, removing or altering them (as long as condition (ii) of Proposition B.1 holds) does not affect the set of feasible q_{j^*}'s. Under condition (iii) of Proposition B.1, Lemma B.2 shows that the set of feasible q_{j^*}'s does not depend on any $F_i(\phi)$, $i = (i^* + 1), \dots, (j^* - 1)$. Hence, relaxing the zero restrictions constraining $[q_{i^*+1}, \dots, q_{j^*-1}]$ does not affect the set of feasible q_{j^*}'s. *Q.E.D.*

EXAMPLE B.4: Consider relaxing one of the zero restrictions in (4.11),

$$\begin{pmatrix} u_{\pi,t} \\ u_{y,t} \\ u_{m,t} \\ u_{i,t} \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & 0 & 0 \\ a^{21} & a^{22} & 0 & a^{24} \\ a^{31} & a^{32} & a^{33} & a^{34} \\ a^{41} & a^{42} & a^{43} & a^{44} \end{pmatrix} \begin{pmatrix} \epsilon_{\pi,t} \\ \epsilon_{y,t} \\ \epsilon_{m,t} \\ \epsilon_{i,t} \end{pmatrix},$$

where the $(2,4)$th element of A_0^{-1} is now unconstrained, that is, the aggregate demand equation is allowed to respond contemporaneously to the monetary policy shock. If the interest is on the impulse responses to the monetary policy shock $\epsilon_{i,t}$, the variables can be ordered as $(m_t, i_t, \pi_t, y_t)'$ with $j^* = 2$. Condition (ii) of Proposition B.1 is satisfied and the impulse-response identified sets are convex. In fact, Lemma A.1 in Appendix A implies that in situations where condition (ii) of Proposition B.1 applies, the zero restrictions imposed on the preceding shocks to the j^*th structural shocks do not tighten the identified sets for the j^*th shock impulse responses compared to the case with no zero restrictions. In the current context, this means that dropping the two zero restrictions on q_m does not change the identified sets for the impulse responses to $\epsilon_{i,t}$.

If sign restrictions are imposed on impulse responses to a shock other than the j^*th shock, the identified set can become non-convex, as we show in the next example.[5]

EXAMPLE B.5: Consider an SVAR(0) model,

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = A_0^{-1} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

Let $\Sigma_{\text{tr}} = \left(\begin{smallmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{smallmatrix}\right)$, where $\sigma_{11} \geq 0$ and $\sigma_{22} \geq 0$. Note that positive semidefiniteness of $\Sigma = \Sigma_{\text{tr}}\Sigma_{\text{tr}}'$ does not impose other constraints on the elements of Σ_{tr}. Denoting an orthonormal matrix by $Q = \left(\begin{smallmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{smallmatrix}\right)$, we can express the contemporaneous impulse-response matrix as

$$\text{IR}^0 = \begin{pmatrix} \sigma_{11}q_{11} & \sigma_{11}q_{12} \\ \sigma_{21}q_{11} + \sigma_{22}q_{21} & \sigma_{21}q_{12} + \sigma_{22}q_{22} \end{pmatrix}.$$

Consider restricting the sign of the $(1,2)$th element of IR^0 to being positive, $\sigma_{11}q_{12} \geq 0$. Since $\Sigma_{\text{tr}}^{-1} = (\sigma_{11}\sigma_{22})^{-1}\left(\begin{smallmatrix} \sigma_{22} & 0 \\ -\sigma_{21} & \sigma_{11} \end{smallmatrix}\right)$, the sign normalization restrictions give $\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0$ and $\sigma_{11}q_{22} \geq 0$. We now show that the identified set for the $(1,1)$th element of IR^0 is non-convex for a set of Σ with a positive measure. Note first that the second column vector of Q is constrained to $\{q_{12} \geq 0, q_{22} \geq 0\}$, so that the set of $(q_{11}, q_{21})'$ orthogonal to $(q_{12}, q_{22})'$ is constrained to

$$\{q_{11} \geq 0, q_{21} \leq 0\} \cup \{q_{11} \leq 0, q_{21} \geq 0\}.$$

When $\sigma_{21} < 0$, intersecting this union set with the half-space defined by the first sign normalization restriction $\{\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0\}$ yields two disconnected arcs,

$$\left\{ \begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : \theta \in \left(\left[\frac{1}{2}\pi, \frac{1}{2}\pi + \psi\right] \cup \left[\frac{3}{2}\pi + \psi, 2\pi\right] \right) \right\},$$

[5]See also the example in Section 4.4 of Rubio-Ramírez, Waggoner, and Zha (2010), where $n = 3$ and the zero restrictions satisfy $f_1 = f_2 = f_3 = 1$. Their paper shows that the identified set for an impulse response consists of two distinct points. If we interpret the zero restrictions on the second and third variables as pairs of linear inequality restrictions for q_2 and q_3 with opposite signs, convexity of the impulse-response identified set fails. In this example, the assumption that sign restrictions are only placed on q_j fails.

where $\psi = \arccos(\frac{\sigma_{22}}{\sqrt{\sigma_{22}^2+\sigma_{21}^2}}) \in [0, \frac{1}{2}\pi]$. Accordingly, the identified set for $r = \sigma_{11}q_{11}$ is given by the union of two disconnected intervals

$$\left[\sigma_{11}\cos\left(\frac{1}{2}\pi + \psi\right), 0\right] \cup \left[\sigma_{11}\cos\left(\frac{3}{2}\pi + \psi\right), \sigma_{11}\right].$$

Since $\{\sigma_{21} < 0\}$ has a positive measure in the space of Σ, the identified set is non-convex with a positive measure.

B.2. *Continuity*

One of the key assumptions for Theorem 3 is the continuity of $\mathrm{IS}_\eta(\phi)$ at $\phi = \phi_0$ (Assumption 2(i)).[6] The next proposition shows that in SVARs, this continuity property is ensured by mild regularity conditions on the coefficient matrices of the zero and sign restrictions.

PROPOSITION B.2—Continuity: *Let $\eta = c'_{ih}(\phi)q_{j^*}$, $i \in \{1, \dots, n\}$, $h \in \{0, 1, 2, \dots\}$, be the impulse response of interest. Suppose that the variables are ordered according to Definition 3 and sign restrictions are placed only on the responses to the j^*th structural shock, that is, $\mathcal{I}_S = \{j^*\}$.*

(i) *Suppose that the zero restrictions satisfy one of conditions* (i) *and* (ii) *of Proposition* B.1. *If there exists an open neighborhood of ϕ_0, $G \subset \Phi$, such that* $\mathrm{rank}(F_{j^*}(\phi)) = f_{j^*}$ *for all $\phi \in G$, and if there exists a unit-length vector $q \in \mathbb{R}^n$ such that*

$$F_{j^*}(\phi_0)q = 0 \quad \textit{and} \quad \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q \gg \mathbf{0},$$

then the identified set correspondence $\mathrm{IS}_\eta(\phi|F, S)$ *is continuous at $\phi = \phi_0$ for every i and h.*[7] (ii) *Suppose that the zero restrictions satisfy condition* (iii) *of Proposition* B.1, *and let $[q_1(\phi), \dots, q_{i^*}(\phi)]$ be the first i^* column vectors of Q that are exactly identified. If there exists an open neighborhood of ϕ_0, $G \subset \Phi$, such that $\begin{pmatrix} F'_{j^*}(\phi) & q_1(\phi) & \cdots & q_{i^*}(\phi) \end{pmatrix}$ is a full column-rank matrix for all $\phi \in G$, and if there exists a unit-length vector $q \in \mathbb{R}^n$ such that*

$$q'\begin{pmatrix} F'_{j^*}(\phi_0) & q_1(\phi_0) & \cdots & q_{i^*}(\phi_0) \end{pmatrix} = \mathbf{0}' \quad \textit{and} \quad \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q \gg \mathbf{0},$$

then the identified-set correspondence $\mathrm{IS}_\eta(\phi|F, S)$ *is continuous at $\phi = \phi_0$ for every i and h.*

PROOF OF PROPOSITION B.2: (i) Following the notation introduced in the proof of Proposition B.1, the upper and lower bounds of the impulse-response identified set are written as

$$\begin{aligned} u(\phi)/\ell(\phi) &= \max/\min_{q_{j^*}} c'_{ih}(\phi)q_{j^*}, \\ &\text{s.t.} \quad q_{j^*} \in \mathcal{E}_{j^*}(\phi) \quad \text{and} \quad S_{j^*}(\phi)q_{j^*} \geq \mathbf{0}. \end{aligned} \tag{B.8}$$

When $j^* = 1$ (Case (i) of Proposition B.1), $\mathcal{E}_1(\phi)$ is given by $D_1(\phi)$ defined in (B.4). On the other hand, when $j^* \geq 2$ and Case (ii) of Proposition B.1 applies, Lemma B.1 provides

[6] Proposition B.1 shows boundedness of $\mathrm{IS}_\eta(\phi|F, S)$ for all ϕ so that Assumption 2(iii) also holds.

[7] For a vector $y = (y_1, \dots, y_m)'$, $y \gg \mathbf{0}$ means $y_i > 0$ for all $i = 1, \dots, m$.

a concrete expression for $\mathcal{E}_{j^*}(\phi)$. Accordingly, in either case, the constrained set of q_{j^*} in (B.8) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi) \equiv \left\{ q \in \mathcal{S}^{n-1} : F_{j^*}(\phi) q = \mathbf{0}, \begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*}(\phi))' \end{pmatrix} q \geq \mathbf{0} \right\}.$$

The objective function of (B.8) is continuous in q_{j^*}, so, by the Theorem of Maximum (see, e.g., Theorem 9.14 of Sundaram (1996)), the continuity of $u(\phi)$ and $\ell(\phi)$ is obtained if $\tilde{\mathcal{E}}_{j^*}(\phi)$ is shown to be a continuous correspondence at $\phi = \phi_0$.

To show continuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$, note first that $\tilde{\mathcal{E}}_{j^*}(\phi)$ is a closed and bounded correspondence, so upper-semicontinuity and lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be defined in terms of sequences (see, e.g., Propositions 21 of Border (2013)):

- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is upper-semicontinuous (*usc*) at $\phi = \phi_0$ if and only if, for any sequence $\phi^v \to \phi_0$, $v = 1, 2, \dots$, and any $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$, there is a subsequence of $q^v_{j^*}$ with limit in $\tilde{\mathcal{E}}_{j^*}(\phi_0)$.
- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lower-semicontinuous (*lsc*) at $\phi = \phi_0$ if and only if $\phi^v \to \phi_0$, $v = 1, 2, \dots$, and $q^0_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$ imply that there is a sequence $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ with $q^v_{j^*} \to q^0_{j^*}$.

In the proofs below, we use the same index v to denote a subsequence for brevity of notation.

Usc: Since $q^v_{j^*}$ is a sequence on the unit sphere, it has a convergent subsequence $q^v_{j^*} \to q_{j^*}$. Since $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$, $F_{j^*}(\phi^v) q^v_{j^*} = \mathbf{0}$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} q^v_{j^*} \geq \mathbf{0}$ hold for all v. Since $F_{j^*}(\cdot)$ and $\begin{pmatrix} S_{j^*}(\cdot) \\ (\sigma^{j^*}(\cdot))' \end{pmatrix}$ are continuous in ϕ, these equality and sign restrictions hold at the limit as well. Hence, $q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Our proof of lsc proceeds similarly to the proof of Lemma 3 in the 2013 working paper version of Granziera, Moon, and Schorfheide (2018). Let $\phi^v \to \phi_0$ be arbitrary. Let $q^0_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$, and define $\mathbf{P}^0 = F_{j^*}(\phi_0)'[F_{j^*}(\phi_0) F_{j^*}(\phi_0)']^{-1} F_{j^*}(\phi_0)$ be the projection matrix onto the space spanned by the row vectors of $F_{j^*}(\phi_0)$. By the assumption of the current proposition, $F_{j^*}(\phi)$ has full row-rank in the open neighborhood of ϕ_0, so $\mathbf{P}^0$ and $\mathbf{P}^v = F_{j^*}(\phi^v)'[F_{j^*}(\phi^v) F_{j^*}(\phi^v)']^{-1} F_{j^*}(\phi^v)$ are well-defined for all large v. Let $\boldsymbol{\xi}^* \in \mathbb{R}^n$ be a vector satisfying $\begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} [I_n - \mathbf{P}^0] \boldsymbol{\xi}^* \gg \mathbf{0}$, which exists by the assumption. Let

$$\zeta = \min \left\{ \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} [I_n - \mathbf{P}^0] \boldsymbol{\xi}^* \right\} > 0,$$

and define

$$\boldsymbol{\xi} = \frac{2}{\zeta} \boldsymbol{\xi}^*,$$

$$\boldsymbol{\epsilon}^v = \left\| \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] - \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} [I_n - \mathbf{P}^0] \right\|,$$

$$q^v_{j^*} = \frac{[I_n - \mathbf{P}^v][q^0_{j^*} + \boldsymbol{\epsilon}^v \boldsymbol{\xi}]}{\| [I_n - \mathbf{P}^v][q^0_{j^*} + \boldsymbol{\epsilon}^v \boldsymbol{\xi}] \|}.$$

Since $\mathbf{P}^v$ converges to $\mathbf{P}^0$, $\boldsymbol{\epsilon}^v \to 0$. Furthermore, $[I_n - \mathbf{P}^0] q^0_{j^*} = q^0_{j^*}$ implies that $q^v_{j^*}$ converges to $q^0_{j^*}$ as $v \to \infty$. Note that $q^v_{j^*}$ is orthogonal to the row vectors of $F_{j^*}(\phi^v)$ by construction. Furthermore, note that

$$\begin{aligned}
&\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} q_{j^*}^v \\
&\quad = \frac{1}{\|[I_n - \mathbf{P}^v][q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}]\|} \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [[I_n - \mathbf{P}^v][q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}]] \\
&\quad \geq \frac{1}{\|[I_n - \mathbf{P}^v][q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}]\|} \left(\left(\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] - \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} [I_n - \mathbf{P}^0] \right) q_{j^*}^0 \right. \\
&\qquad \left. + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] \boldsymbol{\xi} \right) \\
&\quad \geq \frac{1}{\|[I_n - \mathbf{P}^v][q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}]\|} \left(-\epsilon^v \|q_{j^*}^0\| \mathbf{1} + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] \boldsymbol{\xi} \right) \\
&\quad = \frac{\epsilon^v}{\|[I_n - \mathbf{P}^v][q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}]\|} \left(\frac{2}{\zeta} \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] \boldsymbol{\xi}^* - \mathbf{1} \right),
\end{aligned}$$

where the third line follows by $\begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} [I_n - \mathbf{P}^0] q_{j^*}^0 = \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q_{j^*}^0 \geq \mathbf{0}$. By the construction of $\boldsymbol{\xi}^*$ and ζ, $\frac{2}{\zeta} \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} [I_n - \mathbf{P}^v] \boldsymbol{\xi}^* > \mathbf{1}$ holds for all large v. This implies that $\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} q_{j^*}^v \geq 0$ holds for all large v, implying that $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ for all large v. Hence, $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lsc at $\phi = \phi_0$.

(ii) *Usc*: Under Case (iii) of Proposition B.1, Lemma B.2 implies that the constraint set of q_{j^*} in (B.8) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi) \equiv \left\{ q \in \mathcal{S}^{n-1} : q' \begin{pmatrix} F'_{j^*}(\phi) & q_1(\phi) & \cdots & q_{i^*}(\phi) \end{pmatrix} = \mathbf{0}', \begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*}(\phi))' \end{pmatrix} q \geq \mathbf{0} \right\}.$$

Let $q_{j^*}^v$, $v = 1, 2, \ldots$, be a sequence on the unit sphere, such that $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ holds for all v. This has a convergent subsequence $q_{j^*}^v \to q_{j^*}$. Since $F_i(\phi)$ are continuous in ϕ for all $i = 1, \ldots, i^*$, $q_i(\phi)$, $i = 1, \ldots, i^*$, are continuous in ϕ as well, implying that the equality restrictions and the sign restrictions, $q_{j^*}^{v\prime} \begin{pmatrix} F'_{j^*}(\phi^v) & q_1(\phi^v) & \cdots & q_{i^*}(\phi^v) \end{pmatrix} = \mathbf{0}'$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} q_{j^*}^v \geq \mathbf{0}$, must hold at the limit $v \to \infty$. Hence, $q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Define $\mathbf{P}^0$ and $\mathbf{P}^v$ as the projection matrices onto the column vectors of $\begin{pmatrix} F'_{j^*}(\phi_0) & q_1(\phi_0) & \cdots & q_{i^*}(\phi_0) \end{pmatrix}$ and $\begin{pmatrix} F'_{j^*}(\phi^v) & q_1(\phi^v) & \cdots & q_{i^*}(\phi^v) \end{pmatrix}$, respectively. The imposed assumptions imply that $\mathbf{P}^v$ and $\mathbf{P}^0$ are well-defined for all large v, and $\mathbf{P}^v \to \mathbf{P}^0$. With the current definition of $\mathbf{P}^v$ and $\mathbf{P}^0$, lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be shown by repeating the same argument as in the proof of part (i) of the current proposition. We omit details for brevity. *Q.E.D.*

B.3. *Differentiability*

In the development of a delta method for the endpoints of the impulse-response identified set, Theorem 2 in Gafarov, Meier, and Montiel-Olea (2018) shows their directional differentiability. We restrict our analysis to the settings of Proposition B.1 where the identified set is convex. The following proposition extends Theorem 2 of Gafarov, Meier, and Montiel-Olea (2018) and obtains a sufficient condition for differentiability of $\ell(\phi)$ and $u(\phi)$.

PROPOSITION B.3—Differentiability: *Let* $\eta = c'_{ih}(\phi)q_{j^*}$, $i \in \{1, \dots, n\}$, $h \in \{0, 1, 2, \dots\}$, *be the impulse response of interest. Suppose that the variables are ordered according to Definition* 3 *and sign restrictions are placed only on the responses to the* j^**th structural shock, that is*, $\mathcal{I}_S = \{j^*\}$.

(i) *Suppose that the zero restrictions satisfy one of conditions* (i) *and* (ii) *of Proposition* B.1 *and the column vectors of* $[F'_{j^*}(\phi_0), S'_{j^*}(\phi_0), \sigma^{j^*}(\phi_0)]$ *are linearly independent in the sense that for any* $n \times k$ *matrix* B_k, $0 \leq k \leq 1 + \sum_{h=0}^{\bar{h}} s_{hj^*}$, *formed by selecting* k *column vectors from* $[S'_{j^*}(\phi_0), \sigma^{j^*}(\phi_0)]$, $n \times (f_{j^*} + k)$ *matrix* $[F'_{j^*}(\phi_0), B_k]$ *is full-rank. If the set of solutions of the following optimization problem*:

$$
\begin{aligned}
&\min_{q \in \mathcal{S}^{n-1}} \{c'_{ih}(\phi_0)q\} \quad \Big(\textit{resp.} \max_{q \in \mathcal{S}^{n-1}} \{c'_{ih}(\phi_0)q\}\Big) \\
&\textit{s.t.} \quad F_{j^*}(\phi_0)q = \mathbf{0} \quad \textit{and} \quad \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q \geq \mathbf{0},
\end{aligned}
\tag{B.9}
$$

is singleton, the optimized value $\ell(\phi_0)$ (*resp.* $u(\phi_0)$) *is nonzero, and the number of binding sign restrictions at the optimum is less than or equal to* $n - f_{j^*} - 1$, *then* $\ell(\phi)$ (*resp.* $u(\phi)$) *is differentiable at* $\phi = \phi_0$.

(ii) *Suppose that the zero restrictions satisfy conditions* (iii) *of Proposition* B.1. *Let* $[q_1(\phi_0), \dots, q_{i^*}(\phi_0)]$ *be the first* i^**th column vectors of* Q *that are exactly identified at* $\phi = \phi_0$. *Assume that the column vectors of* $[F_{j^*}(\phi_0)', S_{j^*}(\phi_0)', \sigma^{j^*}(\phi_0), q_1(\phi_0), \dots, q_{i^*}(\phi_0)]$ *are linearly independent in the sense that for any* $n \times k$ *matrix* B_k, $0 \leq k \leq 1 + \sum_{h=0}^{\bar{h}} s_{hj^*}$, *formed by selecting* k *column vectors from* $[S'_{j^*}(\phi_0), \sigma^{j^*}(\phi_0)]$, $n \times (f_{j^*} + i^* + k)$ *matrix* $[F'_{j^*}(\phi_0), B_k, q_1(\phi_0), \dots, q_{i^*}(\phi_0)]$ *is full-rank. If the set of solutions of the following optimization problem*:

$$
\begin{aligned}
&\min_{q \in \mathcal{S}^{n-1}} \{c'_{ih}(\phi_0)q\} \quad \Big(\textit{resp.} \max_{q \in \mathcal{S}^{n-1}} \{c'_{ih}(\phi_0)q\}\Big) \\
&\textit{s.t.} \quad \begin{pmatrix} F_{j^*}(\phi_0) \\ q_1(\phi_0)' \\ \vdots \\ q_{i^*}(\phi_0)' \end{pmatrix} q = 0 \quad \textit{and} \quad \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q \geq \mathbf{0},
\end{aligned}
\tag{B.10}
$$

is singleton, the optimized value $\ell(\phi_0)$ (*resp.* $u(\phi_0)$) *is nonzero, and the number of binding sign restrictions at the optimum is less than or equal to* $n - f_{j^*} - i^* - 1$, *then* $\ell(\phi)$ (*resp.* $u(\phi)$) *is differentiable at* $\phi = \phi_0$.

PROOF OF PROPOSITION B.3: We show that in each of Cases (i)–(iii) of Proposition B.1 with the sign restrictions imposed on the j^*th shock only, the optimization problem to be solved for the endpoints of the identified set $(\ell(\phi), u(\phi))$ is reduced to the optimization problem that Gafarov, Meier, and Montiel-Olea (2018) analyzed. The differentiability of the endpoints in ϕ then follows by directly applying Theorem 2 of Gafarov, Meier, and

Montiel-Olea (2018). Our proof focuses on the lower bound $\ell(\phi_0)$ only, as the conclusion for the upper bound can be proved similarly.

To show claim (i) of this proposition, assume $j^* = 1$ and $f_1 < n - 1$ (i.e., Case (i) of Proposition B.1 with $\mathcal{I}_S = \{1\}$). The choice set of q_1 is given by $D_1(\phi_0) \cap \{q \in \mathcal{S}^{n-1} : S_1(\phi_0)q \geq 0\}$, where $D_1(\phi)$ is as defined in (B.4), and the optimization problem to obtain $\ell(\phi)$ can be written as (B.9) with $j^* = 1$. One-to-one differentiable reparameterization of $q \in \mathcal{S}^{n-1}$ by $x = \Sigma_{tr}q$ leads to the optimization problem in equation (2.5) of Gafarov, Meier, and Montiel-Olea (2018). Hence, under the assumptions stated in claim (i) of the current proposition, their Theorem 2 proves differentiability of $\ell(\phi_0)$.

Assume that the imposed zero restrictions satisfy Case (ii) of Proposition B.1 with $\mathcal{I}_S = \{j^*\}$. By applying Lemma B.1, the choice set of q_{j^*} is given by $\mathcal{F}_{j^*}^{\perp}(\phi_0) \cap \mathcal{H}_{j^*}(\phi) \cap \{q \in \mathcal{S}^{n-1} : S_{j^*}(\phi_0)q \geq 0\}$, and the optimization problem to obtain $\ell(\phi_0)$ can be written as (B.10). One-to-one differentiable reparameterization of $q \in \mathcal{S}^{n-1}$ by $x = \Sigma_{tr}q$ leads to the optimization problem in equation (2.5) of Gafarov, Meier, and Montiel-Olea (2018), so the conclusion follows by their Theorem 2.

Last, assume that the imposed zero restrictions satisfy Case (iii) of Proposition B.1 with $\mathcal{I}_S = \{j^*\}$. By applying Lemma B.2, the choice set of q_{j^*} is given by $\mathcal{F}_{j^*}^{\perp}(\phi_0) \cap \mathcal{H}_{j^*}(\phi_0) \cap \mathcal{P}(Q_{1:i^*}(\phi_0)) \cap \{q \in \mathcal{S}^{n-1} : S_{j^*}(\phi_0)q \geq 0\}$ with $Q_{1:i^*}(\phi_0) = [q_1(\phi_0), \dots, q_{i^*}(\phi_0)]$ pinned down uniquely by the assumption of exact identification. Accordingly, the optimization problem to obtain $\ell(\phi_0)$ can be written as (B.10). One-to-one differentiable reparameterization of $q \in \mathcal{S}^{n-1}$ by $x = \Sigma_{tr}q$ leads to the optimization problem in equation (2.5) of Gafarov, Meier, and Montiel-Olea (2018) with the expanded set of equality restrictions consisting of $F_{j^*}(\phi_0)(\Sigma_{tr})^{-1}x = 0$ and $Q_{1:i^*}(\phi_0)'(\Sigma_{tr})^{-1}x = 0$. Hence, under the assumptions stated in claim (ii) of the current proposition, their Theorem 2 implies differentiability of $\ell(\phi_0)$. *Q.E.D.*

Theorem 2 in Gafarov, Meier, and Montiel-Olea (2018) concerns Case (i) of Proposition B.1 with sign restrictions placed on $\mathcal{I}_S = \{1\}$ and no zero restrictions on the other shocks, $f_2 = \cdots = f_n = 0$. Proposition B.3 extends Theorem 2 in Gafarov, Meier, and Montiel-Olea (2018) to the setting where we impose the zero restrictions on the column vectors of Q other than j^* subject to the conditions for convexity of the identified set characterized in Proposition B.1.[8]

Among the sufficient conditions for differentiability of the bounds shown in Proposition B.3, the two key conditions are uniqueness of the optimizer and the maximal number of binding constraints.

The uniqueness condition seems mild in SVARs. Since the feasible set for q is the intersection of the unit sphere and the polyhedron formed by the sign restrictions, it has vertices and the manifolds connecting the vertices are arcs or a subset on the sphere rather than lines or planes. Hence, if $c_{ih}(\phi_0)$ is linearly independent of any of the linear inequality constraints—which holds if the sign of the impulse response of interest is not restricted—we can rule out that the continuum of optimizers lies on one of the arcs connecting neighboring vertices. Another possible failure of the condition is when the linear objective functions are optimized at multiple vertices of the (curved) feasible set. In theory, we cannot exclude such possibility, but we believe this to be a pathological case in practice. In the other cases, the solution is unique, attained at either one of the vertices,

[8]The statement of Theorem 2 of Gafarov, Meier, and Montiel-Olea (2018) does not explicitly constrain the maximal number of binding inequality restrictions at the optimum (cf. Proposition B.3 in this paper), while their proof implicitly does so.

an interior point on one of the edge arcs, or a strict interior of the surface subset of the sphere.

The condition on the maximal number of binding constraints implies that if the optimum is attained at one of the vertices on the constrained surface of $\mathcal{S}^{n-1}$, it has to be a basic solution, that is, exactly $n-1$ equality and inequality restrictions (excluding the unit sphere constraint) are binding at the vertex. Otherwise, a local perturbation of ϕ around ϕ_0 can create additional vertices and generate non-smooth changes of the value function depending on which of the new vertices becomes a new solution. We want to rule out this case to ensure differentiability of $\ell(\phi)$ and $u(\phi)$. If the vertex attaining the optimum is not a basic solution, $\ell(\phi)$ or $u(\phi)$ that involves the max or min operator has multiple entries attaining the maximum or minimum at $\phi = \phi_0$. This type of non-differentiable $\ell(\phi)$ and $u(\phi)$ generally leads to failure of Assumptions 4(i) and 4(ii). In frequentist inference in moment inequality models, this issue translates to non-pivotal asymptotic distributions of the test statistics depending on which moment inequalities are binding. The frequentist approaches to uniformly valid inference studied in Andrews and Soares (2010), Bugni, Canay, and Shi (2017), and Kaido, Molinari, and Stoye (2019), among others, involve generalized moment selection. A Bayesian or robust Bayesian interpretation or justification for such moment selection procedures is not yet known and thus left for future research.

REFERENCES

ANDREWS, D., AND G. SOARES (2010): "Inference for Parameters Defined by Moment Inequalities Using Generalized Moment Selection," *Econometrica*, 78, 119–157. [15]

BORDER, K. (2013): "Introduction to Correspondences," unpublished manuscript. [11]

BUGNI, F., I. CANAY, AND X. SHI (2017): "Inference for Subvectors and Other Functions of Partially Identified Parameters in Moment Inequality Models," *Quantitative Economics*, 8, 1–38. [15]

GAFAROV, B., M. MEIER, AND J. MONTIEL-OLEA (2018): "Delta-Method Inference for a Class of Set-Identified SVARs," *Journal of Econometrics*, 203, 316–327. [12-14]

GRANZIERA, E., H. MOON, AND F. SCHORFHEIDE (2018): "Inference for VARs Identified With Sign Restrictions," *Quantitative Economics*, 9 (3), 1087–1121. [6,11]

KAIDO, H., F. MOLINARI, AND J. STOYE (2019): "Confidence Intervals for Projections of Partially Identified Parameters," *Econometrica*, 87, 1397–1432. [15]

RUBIO-RAMÍREZ, J., D. WAGGONER, AND T. ZHA (2010): "Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference," *Review of Economic Studies*, 77, 665–696. [1,2,9]

SUNDARAM, R. K. (1996): *A First Course in Optimization Theory*. Cambridge, MA: Cambridge University Press. [11]

SUTHERLAND, W. A. (2009): *Introduction to Metric and Topological Spaces* (Second Ed.). Oxford University Press. [3]

Co-editor Ulrich K. Müller handled this manuscript.

Manuscript received 25 October, 2018; final version accepted 11 September, 2020; available online 29 September, 2020.