SUPPLEMENT TO "TESTING HYPOTHESES ABOUT THE NUMBER OF FACTORS IN LARGE FACTOR MODELS": SUPPLEMENTARY APPENDIX

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DETAILED PROOF OF LEMMA 5

LEMMA 5: *Let* $\hat{e} \equiv [\hat{e}_1(n), \ldots, \hat{e}_m(n)]$. *Then, under the assumptions of Theorem* 1, *there exists an* $n \times m$ *matrix* $\tilde{e}$ *with independent* $N_n^{\mathbb{C}}(0, 2\pi S_n^e(\omega_0))$ *columns, independent from* $\hat{F}$, *and such that* $\sigma_1^2(\hat{e} - \tilde{e}) = o_p(n^{-1/3})$.

PROOF: First, suppose that Assumption 2(ii) holds and $n \sim m = o(T^{3/8})$. Define $\eta \equiv ((\operatorname{Re}\hat{e}_1)', (\operatorname{Im}\hat{e}_1)', \ldots, (\operatorname{Re}\hat{e}_m)', (\operatorname{Im}\hat{e}_m)')'$. First, let us show that $E\eta\eta' = V + R$ with a block diagonal

$$V = \pi I_m \otimes \begin{pmatrix} \operatorname{Re} S_n^e(\omega_0) & -\operatorname{Im} S_n^e(\omega_0) \\ \operatorname{Im} S_n^e(\omega_0) & \operatorname{Re} S_n^e(\omega_0) \end{pmatrix}$$

and $R_{ij} = \delta_{[i/2n],[j/2n]} O(m/T) + O(T^{-1})$, where δ_{st} is the Kronecker delta, and $O(m/T)$ and $O(T^{-1})$ are uniform in i and j running from 1 to $2nm$.

By the definition of the discrete Fourier transform (d.f.t.), we have

$$E\hat{e}_{js}\hat{e}_{rl} \equiv \frac{1}{T} E\left[\left(\sum_{t=1}^{T} e_{jt} e^{-i\omega_s t}\right)\left(\sum_{t=1}^{T} e_{rt} e^{-i\omega_l t}\right)\right].$$

Hence, we can write

$$E\hat{e}_{js}\hat{e}_{rl} = \frac{1}{T} \sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) \sum_{t=1}^{T} h(t+u) e^{-i(\omega_s+\omega_l)t},$$

where $h(\tau) = 1$ for $1 \le \tau \le T$ and $h(\tau) = 0$ otherwise. Denote

$$\sum_{t=1}^{T} h(t+u) e^{-i(\omega_s+\omega_l)t} - \sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t}$$

as U_1 and denote

$$\sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) - 2\pi [S_n^e(\omega_s)]_{jr}$$

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as U_2. Then

$$E\hat{e}_{js}\hat{e}_{rl} - \frac{2\pi}{T}\sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t}[S_n^e(\omega_s)]_{jr} = \frac{1}{T}\sum_{u=1-T}^{T-1} e^{-i\omega_s u}c_{jr}(u)U_1 + U_2\sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t}.$$

But $|U_1| \le |u|$ and

$$|U_2| = \left|\sum_{|u|\ge T} e^{-i\omega_s u}c_{jr}(u)\right| \le \sum_{|u|\ge T}\frac{|u|}{T}|c_{jr}(u)|.$$

Hence,

$$\left|E\hat{e}_{js}\hat{e}_{rl} - \frac{2\pi}{T}\sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t}[S_n^e(\omega_s)]_{jr}\right| \le \frac{1}{T}\sum_{u=1-T}^{T-1}|u||c_{jr}(u)| + \sum_{|u|\ge T}\frac{|u|}{T}|c_{jr}(u)|\left|\sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t}\right|.$$

Note that by the definition of ω_s and ω_l, $\omega_s+\omega_l = (2\pi(s_s+s_l))/T \neq 0$ for all s and l. Therefore,

$$\frac{1}{T}\sum_{t=1}^{T} e^{-i(\omega_s+\omega_l)t} = \frac{e^{-i(\omega_s+\omega_l)}}{T}\frac{e^{-i(\omega_s+\omega_l)T}-1}{e^{-i(\omega_s+\omega_l)}-1} = 0$$

for all s and l, and we have $|E\hat{e}_{js}\hat{e}_{rl}| \le \frac{1}{T}\sum_{u=1-T}^{T-1}|u||c_{jr}(u)| = O(T^{-1})$ uniformly in s and l, but also in j and r by Assumption 2(ii). Similarly, $|E\hat{e}'_{js}\hat{e}'_{rl}| = O(T^{-1})$ uniformly in s, l, j and r.

Consider now $E\hat{e}_{js}\hat{e}'_{rl}$. Similar to above,

$$\left|E\hat{e}_{js}\hat{e}'_{rl} - \frac{2\pi}{T}\sum_{t=1}^{T} e^{-i(\omega_s-\omega_l)t}[S_n^e(\omega_s)]_{jr}\right| \le \frac{1}{T}\sum_{u=1-T}^{T-1}|u||c_{jr}(u)| + \sum_{|u|\ge T}\frac{|u|}{T}|c_{jr}(u)|\left|\sum_{t=1}^{T} e^{-i(\omega_s-\omega_l)t}\right|,$$

and if $s \neq l$, we have $|E\hat{e}_{js}\hat{e}'_{rl}| = O(T^{-1})$ uniformly in s, l, j, and r. However, if $s = l$, then $\omega_s - \omega_l = 0$ and we have

$$E\hat{e}_{js}\hat{e}'_{rl} - 2\pi[S_n^e(\omega_s)]_{jr} = \frac{1}{T}\sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u)(T-u) - 2\pi[S_n^e(\omega_s)]_{jr}$$
$$= -\frac{1}{T}\sum_{u=1-T}^{T-1} u e^{-i\omega_s u} c_{jr}(u) - \sum_{|u|\geq T} e^{-i\omega_s u} c_{jr}(u)$$

so that $|E\hat{e}_{js}\hat{e}'_{rl} - 2\pi[S_n^e(\omega_s)]_{jr}| \leq \frac{1}{T}\sum |u||c_{jr}(u)| = O(T^{-1})$ uniformly in s, l, j, and r. To summarize,

(S1) $\quad E\hat{e}_{js}\hat{e}_{rl} = O(T^{-1}), \quad E\hat{e}'_{js}\hat{e}'_{rl} = O(T^{-1}),$

(S2) $\quad E\hat{e}_{js}\hat{e}'_{rl} = \delta_{sl} 2\pi[S_n^e(\omega_s)]_{jr} + O(T^{-1}),$

where $O(T^{-1})$ is uniform in s, l, j, and r.

This result is very similar to Theorem 4.3.2 of Brillinger (1981), which is more general in that it gets estimates for higher order cumulants of d.f.t.'s in addition to the second-order cumulants, but which is less general in that it only considers situations when j and r are bounded so that uniformity of $O(T^{-1})$ in j and r is trivial.

Note that

$$\begin{aligned} E\hat{e}_{js}\hat{e}_{rl} &= E(\operatorname{Re}\hat{e}_{js} + i\operatorname{Im}\hat{e}_{js})(\operatorname{Re}\hat{e}_{rl} + i\operatorname{Im}\hat{e}_{rl}) \\ &= E(\operatorname{Re}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl} - \operatorname{Im}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) \\ &\quad + iE(\operatorname{Re}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl} + \operatorname{Im}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) \end{aligned}$$

and

$$\begin{aligned} E\hat{e}_{js}\hat{e}'_{rl} &= E(\operatorname{Re}\hat{e}_{js} + i\operatorname{Im}\hat{e}_{js})(\operatorname{Re}\hat{e}_{rl} - i\operatorname{Im}\hat{e}_{rl}) \\ &= E(\operatorname{Re}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl} + \operatorname{Im}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) \\ &\quad + iE(-\operatorname{Re}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl} + \operatorname{Im}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}). \end{aligned}$$

Therefore,

$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \frac{1}{4}(E\hat{e}_{js}\hat{e}_{rl} + E\hat{e}'_{js}\hat{e}'_{rl} + E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl}),$$
$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = \frac{1}{4}(E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl} - E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl}),$$
$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = \frac{1}{4i}(E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl} - E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl}),$$

$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \frac{1}{2i}(E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl} + E\hat{e}_{js}\hat{e}'_{rl} - E\hat{e}'_{js}\hat{e}_{rl}).$$

Using formulas (S1) and (S2), we finally get

$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \delta_{sl}\frac{\pi}{2}\big([S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj}\big) + O(T^{-1}),$$
$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = \delta_{sl}\frac{\pi}{2}\big([S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj}\big) + O(T^{-1}),$$
$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = \delta_{sl}\frac{\pi}{2i}\big(-[S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj}\big) + O(T^{-1}),$$
$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \delta_{sl}\frac{\pi}{2i}\big([S_n^e(\omega_s)]_{jr} - [S_n^e(\omega_s)]_{rj}\big) + O(T^{-1}).$$

Since $S_n^e(\omega_s)$ is a Hermitian matrix, we have

$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \delta_{sl}\pi\operatorname{Re}[S_n^e(\omega_s)]_{jr} + O(T^{-1}),$$
$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = \delta_{sl}\pi\operatorname{Re}[S_n^e(\omega_s)]_{jr} + O(T^{-1}),$$
$$E(\operatorname{Re}\hat{e}_{js}\operatorname{Im}\hat{e}_{rl}) = -\delta_{sl}\pi\operatorname{Im}[S_n^e(\omega_s)]_{jr} + O(T^{-1}),$$
$$E(\operatorname{Im}\hat{e}_{js}\operatorname{Re}\hat{e}_{rl}) = \delta_{sl}\pi\operatorname{Im}[S_n^e(\omega_s)]_{jr} + O(T^{-1}).$$

Further, by the definition of the spectrum and by Assumption 2(ii), $[S_n^e(\omega_s)]_{jr} - [S_n^e(\omega_0)]_{jr} = O(m/T)$ uniformly in j, r, and s. Hence, the above covariance formulas for the real and imaginary parts of $\hat{e}_{js}$ and $\hat{e}_{rl}$ imply that the i, jth entries of R equal $\delta_{[i/2n],[j/2n]}O(m/T) + O(T^{-1})$, where $O(m/T)$ and $O(T^{-1})$ are uniform in i and j running from 1 to $2nm$.

Construct $\tilde{\eta} = V^{1/2}(V+R)^{-1/2}\eta$ and define an $n \times m$ matrix $\tilde{e}$ with the sth columns $\tilde{e}_s$ so that $((\operatorname{Re}\tilde{e}_1)', (\operatorname{Im}\tilde{e}_1)', \dots, (\operatorname{Re}\tilde{e}_m)', (\operatorname{Im}\tilde{e}_m)')' = \tilde{\eta}$. Note that $\tilde{e}$ has independent $N_n^{\mathbb{C}}(0, 2\pi S_n^e(\omega_0))$ columns by construction.

Using inequalities $\|BA\|_2 \le \|B\|\|A\|_2$ and $\|AB\|_2 \le \|A\|_2\|B\|$ (see, for example, Horn and Johnson (1985, Problem 20, p. 313)), we obtain $E\|\eta - \tilde{\eta}\|^2 = \|(V+R)^{1/2} - V^{1/2}\|_2^2 \le \|V^{1/4}\|^4\|(I + V^{-1/2}RV^{-1/2})^{1/2} - I\|_2^2$. Denote the ith largest eigenvalue of $V^{-1/2}RV^{-1/2}$ as μ_i and note that $|\mu_i| \le 1$ for large enough T. Since $|(1+\mu_i)^{1/2} - 1| \le |\mu_i|$ for any $|\mu_i| \le 1$, the ith eigenvalue of $(I + V^{-1/2}RV^{-1/2})^{1/2} - I$ is no larger by absolute value than the ith eigenvalue of $V^{-1/2}RV^{-1/2}$ for large enough T. Therefore, $E\|\eta - \tilde{\eta}\|^2 \le \|V^{1/4}\|^4\|V^{-1/2}RV^{-1/2}\|_2^2 \le \|V^{1/4}\|^4\|V^{-1/2}\|^4\|R\|_2^2$. But $\|V^{1/4}\| = (\pi l_{1n})^{1/4}$ and $\|V^{-1/2}\| = (\pi l_{nn})^{-1/2}$ by construction, and

$$\|R\|_2^2 = \sum_{i,j=1}^{2nm}\big(\delta_{[i/2n],[j/2n]}O(m/T) + O(T^{-1})\big)^2$$
$$= m(2n)^2 O(m^2/T^2) + (2mn)^2 O(T^{-2}) = o\big(n^{-1/3}\big)$$

because $n \sim m = o(T^{3/8})$. Hence,

$$E\|\eta - \tilde{\eta}\|^2 \leq (\pi l_{1n})(\pi l_{nn})^{-2} o\big(n^{-1/3}\big) = o\big(n^{-1/3}\big),$$

where the last equality holds because l_{1n} and l_{nn}^{-1} remain bounded as $n, m \to \infty$ by Assumption 3. Finally, Lemma 2 and Markov's inequality imply that $\sigma_1^2(\hat{e} - \tilde{e}) = o_p(n^{-1/3})$.

Now, suppose that Assumption 2(ii)(a) holds and $m = o(T^{1/2-1/p} \log^{-1} T)^{6/13}$. In this case, $\hat{e}_{is} = \sum_{j=1}^{\infty} A_{ij} \hat{u}_{js}$, where $\hat{u}_{js}$ is the d.f.t. of u_{jt} at frequency ω_s. For fixed j and $\omega_0 = 0$, Phillips (2007) showed that there exist independent and identically distributed (i.i.d.) complex normal variables ξ_{js}, $s = 1, \ldots, m$, such that $\hat{u}_{js} - \xi_{js} = o_p(m/T^{1/2-1/p})$ uniformly over $s \leq m$. Lemmas S1, S2, and S3 below extend Phillips' proof to the case $\omega_0 \neq 0$ and show that there exist Gaussian processes u_{jt}^G with the same autocovariance structure as u_{jt} and independent over $j \in \mathbb{N}$ such that the differences between the d.f.t.'s $\hat{u}_{js} - \hat{u}_{js}^G \equiv r_{js}$ satisfy $\sup_{j>0} E(\max_{s\leq m} |r_{js}|)^2 \leq K m^2 T^{2/p-1} \log^2 T$ for large enough T, where $K > 0$ depends only on p, μ_p, $\sup_{j\geq 1}(\sum_{k=0}^{\infty} k|c_{jk}|)^p$, and $\sup_{j\geq 1} |C_j(e^{-i\omega_0})|$.

Note that the process $e_{it}^G = \sum_{j=1}^{\infty} A_{ij} u_{jt}^G$ satisfies Assumption 2(ii). Indeed, let $c_{ij}^G(u) \equiv E e_{i,t+u}^G e_{jt}^G$. Then, since u_{jt}^G are independent over $j \in \mathbb{N}$ and have the same autocovariance structure as $u_{jt,}$ we have $c_{ij}^G(u) = \sum_{r=1}^{\infty} A_{ir} A_{jr} E u_{r,t+u} u_{rt}$. Therefore,

$$\sum_u (1 + |u|)|c_{ij}^G(u)| \leq \sum_{r=1}^{\infty} |A_{ir} A_{jr}| \sum_u (1 + |u|)|E u_{r,t+u} u_{rt}|.$$

On the other hand,

$$\begin{aligned}
\sum_u (1 + |u|)|E u_{r,t+u} u_{rt}| &\leq \sum_u \sum_{k=|u|}^{\infty} (1 + |u|)|c_{rk}| \big|c_{rk-|u|}\big| \\
&= \sum_{k=0}^{\infty} \sum_{u:|u|\leq k} (1 + |u|)|c_{rk}| \big|c_{rk-|u|}\big| \\
&\leq \sum_{k=0}^{\infty} \sum_{u:|u|\leq k} (1 + k)|c_{rk}| \big|c_{rk-|u|}\big| \\
&\leq \left(\sum_{k=0}^{\infty} (1 + k)|c_{rk}| \right)^2.
\end{aligned}$$

Hence,

$$\sup_{i,j}\sum_{u}(1+|u|)|c^G_{ij}(u)| \leq \sup_{i}\sum_{r=1}^{\infty}A^2_{ir}\sup_{r>0}\left(\sum_{k=0}^{\infty}(1+k)|c_{rk}|\right)^2 < \infty$$

by Assumption 2(ii)(a).

Thus, the problem reduces to the Gaussian case analyzed above if we show that $\sigma^2_1(\hat{e}-\hat{e}^G) = o_p(n^{-1/3})$. But we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{s=1}^{m}E|(\hat{e}-\hat{e}^G)_{is}|^2 \\
&\quad = \sum_{i=1}^{n}\sum_{s=1}^{m}\sum_{j=1}^{\infty}A^2_{ij}E|r_{js}|^2 \leq m\sum_{i=1}^{n}\sum_{j=1}^{\infty}A^2_{ij}E\Bigl(\max_{s\leq m}|r_{js}|\Bigr)^2 \\
&\quad \leq mn\left(\sup_{i>0}\sum_{j=1}^{\infty}A^2_{ij}\right)Km^2T^{2/p-1}\log^2 T = o\bigl(n^{-1/3}\bigr)
\end{aligned}$$

if $n \sim m = o(T^{1/2-1/p}\log^{-1}T)^{6/13}$ as has been assumed. Therefore, Lemma 2 and Markov's inequality imply that $\sigma^2_1(\hat{e}-\hat{e}^G) = o_p(n^{-1/3})$. *Q.E.D.*

LEMMA S1—Zaitsev (2006): *Suppose that $x_1, \dots, x_T$ are independent zero-mean random vectors in $\mathbb{R}^d$ such that $L_p \equiv \sum_{t=1}^{T}E\|x_t\|^p < \infty$ with $p \geq 2$ and there exists a sequence $0 = m_0 < m_1 < \cdots < m_\tau = T$ such that for $D_k \equiv \mathrm{Var}[x_{m_{k-1}+1} + \cdots + x_{m_k}]$, $k = 1, \dots, \tau$, we have $I_d \leq \gamma^{-2}D_k \leq C\cdot I_d$ with $C \geq 1$ and $\gamma = 2eL_p^{1/p}$. Then there exists a probability space that supports both a sequence distributionally equivalent to $x_1, \dots, x_T$ and a sequence of independent $N(0, \mathrm{Var}(x_t))$ vectors y_t, $t = 1, \dots, T$, such that*

$$\Pr\left(\max_{1\leq t\leq T}\left\|\sum_{s=1}^{t}(x_s-y_s)\right\| > 5z\right) \leq 2L_pz^{-p} + \exp\left(-\frac{a_1 z}{\gamma d^{9/2}\log^* d}\right)$$

for any $z > a_2(d^8\log^* d)\gamma\log^*\tau$, where $a_1, a_2 > 0$ depend only on C and where $\log^* a \equiv \max(1, \log a)$.

This lemma is a slightly weakened version of Corollary 3 in Zaitsev (2006).

LEMMA S2: *Under Assumption 2(ii)(a), there exists a probability space that supports a process distributionally equivalent to ε_{it} and a process $\varepsilon^G_{it} \sim$ i.i.d.*

$N(0,1)$ such that $E(\max_{1\le t\le T}(|R_{jt}|/\sqrt{T}))^2 \le bT^{2/p-1}\log^2 T$ for large enough T, where $R_{jt}\equiv\sum_{l=1}^{t}e^{-i\omega_0 l}(\varepsilon_{jl}-\varepsilon_{jl}^G)$ and $b>0$ depends only on p and μ_p.

PROOF: In Lemma S1, take $x_t = v_t\varepsilon_{jt}$, where $v_t\equiv(\cos\omega_0 t, -\sin\omega_0 t)'$, and assume that $\omega_0\neq 0\,\mathrm{mod}(\pi)$. Then, for any l, the two singular values of $\mathrm{Var}(x_{2l-1}+x_{2l})$ are $\sigma_{1,2}\equiv 1\pm|\cos\omega_0|$. Therefore, for $k=1,\dots,\tau$, $(\sigma_2(D_k))/\gamma^2\ge\sigma_2/\gamma^2[(m_k-m_{k-1})/2]$ and $(\sigma_1(D_k))/\gamma^2\le\sigma_1/\gamma^2[(m_k-m_{k-1}+1)/2]$ for any positive γ and, hence, for $\gamma=2eL_p^{1/p}$ with $L_p=T\mu_p$. In particular, if we choose $m_k=k([2\gamma^2/\sigma_2]+1)$ for $k\le\tau-1$ with $\tau=[T/m_1]$, we have $\min_{k\le\tau}[(m_k-m_{k-1})/2]=[m_1/2]\ge\gamma^2/\sigma_2$ and $\max_{k\le\tau}[(m_k-m_{k-1}+1)/2]\le m_1\le 3\gamma^2/\sigma_2$, where the latter inequality holds because $\mu_p^{2/p}\equiv(E|\varepsilon_{jt}|^p)^{2/p}\ge E\varepsilon_{jt}^2\equiv 1$; thus, $\gamma^2/\sigma_2\ge 1$. Summarizing the above inequalities, we get $I_2\le\gamma^{-2}D_k\le 3(1+|\cos\omega_0|)/(1-|\cos\omega_0|)I_2$ for $k=1,\dots,\tau$. Hence, for each j, Zaitsev's inequality for the tail probability of $\max_{1\le t\le T}\|\sum_{s=1}^{t}\{v_s\varepsilon_{js}-y_{js}\}\|$ is satisfied for independent $N(0,v_s v_s')$ vectors y_{js}, $s=1,\dots,T$. By expanding the probability space, we can choose y_{js}'s independent across different j's and embed the finite sequences y_{js}, $s=1,\dots,T$, into the infinite ones y_{js}, $s\in\mathbb{Z}$.

Now, define independent $N(0,1)$ variables $\varepsilon_{js}^G\equiv y_{1,js}\cos\omega_0 s-y_{2,js}\sin\omega_0 s$, where $y_{1,js}$ and $y_{2,js}$ are the two components of vector $y_{js}\equiv(y_{1,js},y_{2,js})'$. Note that

$$\mathrm{Re}(e^{-i\omega_0 s}\varepsilon_{js}^G)=y_{1,js}\cos^2\omega_0 s-y_{2,js}\cos\omega_0 s\sin\omega_0 s, \tag{S3}$$

$$\mathrm{Im}(e^{-i\omega_0 s}\varepsilon_{js}^G)=-y_{1,js}\sin\omega_0 s\cos\omega_0 s+y_{2,js}\sin^2\omega_0 s. \tag{S4}$$

Further, note that

$$y_{1,js}\sin(\omega_0 s)+y_{2,js}\cos(\omega_0 s)\equiv 0 \tag{S5}$$

because $E((\sin(\omega_0 s),\cos(\omega_0 s))y_{js})^2=(\sin(\omega_0 s),\cos(\omega_0 s))v_s v_s'(\sin(\omega_0 s),\cos(\omega_0 s))'\equiv 0$. Multiplying the left hand side of (S5) by $\sin(\omega_0 s)$ and by $\cos(\omega_0 s)$, and adding the results to the right hand sides of (S3) and (S4), respectively, we find that the components of y_{js} equal the real and the imaginary parts of $e^{-i\omega_0 s}\varepsilon_{js}^G$. Therefore, we have

$$\Pr\Big(\max_{1\le t\le T}|R_{jt}|>5z\Big)\le 2L_p z^{-p}+\exp\left(-\frac{a_1 z}{\gamma 2^{9/2}\log 2}\right)$$

for any $z>a_2(2^8\log 2)\gamma\log^*\tau$, where $R_{jt}\equiv\sum_{l=1}^{t}e^{-i\omega_0 l}(\varepsilon_{jl}-\varepsilon_{jl}^G)$ and $a_1,a_2>0$ depend only on $\omega_0\neq 0\,\mathrm{mod}(\pi)$. For $\omega_0=0\,\mathrm{mod}(\pi)$, defining $x_t=e^{-i\omega_0 t}\varepsilon_{jt}$ and repeating a simplified scalar version of the above argument, we obtain the same tail probability estimate (with different a_1 and a_2, and $\tau=[T/m_1]$ with $m_1=[\gamma^2]+1$).

Now, we have

$$E\left(\max_{1\le t\le T}\frac{|R_{jt}|}{\sqrt{T}}\right)^2$$
$$=2\int_0^\infty x\,\Pr\left(\max_{1\le t\le T}\frac{|R_{jt}|}{\sqrt{T}}>x\right)dx$$
$$\le \bar{x}^2+2\int_{\bar{x}}^\infty x\left(2L_p\left(\frac{x\sqrt{T}}{5}\right)^{-p}+\exp\left(-\frac{a_1x\sqrt{T}}{5\gamma 2^{9/2}\log 2}\right)\right)dx$$

for $\bar{x}>5T^{-1/2}a_2(2^8\log 2)\gamma\log^*\tau$. Recall that $\tau=[T/m_1]=[T/([2\gamma^2/\sigma_2]+1)]$, $\gamma=2eL_p^{1/p}$ with $L_p=T\mu_p$, and $\sigma_2=1-|\cos\omega_0|$ (or, alternatively, $\tau=[T/m_1]=[T/([\gamma^2]+1)]$ for $\omega_0=0\operatorname{mod}(\pi)$). As $T\to\infty$, $\gamma\sim T^{1/p}$ and $\tau\sim T^{1-2/p}$ so that there exists a constant $b_2>0$ such that $\bar{x}\ge b_2T^{1/p-1/2}\log T$ implies that $\bar{x}>5T^{-1/2}a_2(2^8\log 2)\gamma\log^*\tau$ for large enough T. Furthermore, since the inequality $x\ge b_2T^{1/p-1/2}\log T$ implies that $xT^{1/2-1/p}\to\infty$ as $T\to\infty$ and since $2L_p(x\sqrt{T}/5)^{-p}\sim(xT^{1/2-1/p})^{-p}$ and $a_1x\sqrt{T}/(5\gamma 2^{9/2}\log 2)\sim xT^{1/2-1/p}$, we have $2L_p(x\sqrt{T}/5)^{-p}>\exp(-a_1x\sqrt{T}/(5\gamma 2^{9/2}\log 2))$ for large enough T. To summarize, for large enough T and for $\bar{x}\ge b_2T^{1/p-1/2}\log T$ with some positive constant b_2,

$$E\left(\max_{1\le t\le T}\frac{|R_{jt}|}{\sqrt{T}}\right)^2\le \bar{x}^2+2\int_{\bar{x}}^\infty 4xL_p\left(\frac{x\sqrt{T}}{5}\right)^{-p}dx$$
$$=\bar{x}^2+b_1T^{1-p/2}\bar{x}^{2-p},$$

where $b_1>0$ depends only on μ_p and p. Setting $\bar{x}=b_2T^{1/p-1/2}\log T$, we get $E(\max_{1\le t\le T}(|R_{jt}|/\sqrt{T}))^2\le bT^{2/p-1}\log^2 T$ for large enough T, where $b>0$ depends only on μ_p and p. *Q.E.D.*

LEMMA S3: *Let Assumption* 2(ii)(a) *hold and let* ε_{it}^G, $j\in\mathbb{N}, t\in\mathbb{Z}$, *be the i.i.d.* $N(0,1)$ *variables described in Lemma* S2. *Define* $u_{jt}^G\equiv C_j(L)\varepsilon_{jt}^G$ *and consider the differences* $r_{js}\equiv\hat{u}_{js}-\hat{u}_{js}^G$ *between the d.f.t.'s of* u_{jt} *and* u_{jt}^G *at frequencies* ω_s *with* $s=1,\dots,m$. *Then* $\sup_{j>0}E(\max_{s\le m}|r_{js}|)^2\le Km^2T^{2/p-1}\log^2 T$ *for large enough* T, *where* $K>0$ *depends only on* p, μ_p, $\sup_{j\ge1}(\sum_{k=0}^\infty k|c_{jk}|)^p$, *and* $\sup_{j\ge1}|C_j(e^{-i\omega_0})|$.

PROOF: Consider the representation for $r_{js}\equiv\hat{u}_{js}-\hat{u}_{js}^G$,

$$\sqrt{T}r_{js}=e^{-i(\omega_s-\omega_0)T}\tilde{R}_{jT}-\sum_{t=1}^{T-1}\tilde{R}_{jt}e^{-i(\omega_s-\omega_0)t}\left(e^{-i(\omega_s-\omega_0)}-1\right),\tag{S6}$$

where $\tilde{R}_{jt} \equiv \sum_{l=1}^{t} e^{-i\omega_0 l}(u_{jl} - u_{jl}^G)$. Using a modified Beveridge–Nelson decomposition, $C_j(L) = C_j(e^{-i\omega_0}) + \tilde{C}_j(L)(L - e^{-i\omega_0})$, where $\tilde{C}_j(L) = \sum_{k=0}^{\infty} \tilde{c}_{jk} L^k$ with $\tilde{c}_{jk} = \sum_{s=k+1}^{\infty} e^{-i\omega_0(s-k-1)} c_{js}$, we get $e^{-i\omega_0 l}(u_{jl} - u_{jl}^G) = C_j(e^{-i\omega_0}) e^{-i\omega_0 l}(\varepsilon_{jl} - \varepsilon_{jl}^G) + (\tilde{\varepsilon}_{j,l-1} - \tilde{\varepsilon}_{j,l-1}^G) - (\tilde{\varepsilon}_{jl} - \tilde{\varepsilon}_{jl}^G)$ with $\tilde{\varepsilon}_{jl} = e^{-i\omega_0(l+1)} \tilde{C}_j(L) \varepsilon_{jl}$ and $\tilde{\varepsilon}_{jl}^G = e^{-i\omega_0(l+1)} \times \tilde{C}_j(L) \varepsilon_{jl}^G$. Therefore,

$$\text{(S7)} \qquad \tilde{R}_{jt} = C_j(e^{-i\omega_0}) R_{jt} + (\tilde{\varepsilon}_{j,0} - \tilde{\varepsilon}_{j,0}^G) - (\tilde{\varepsilon}_{jt} - \tilde{\varepsilon}_{jt}^G),$$

where $R_{jt} \equiv \sum_{l=1}^{t} e^{-i\omega_0 l}(\varepsilon_{jl} - \varepsilon_{jl}^G)$. Substituting (S7) in (S6) and using the fact that $|e^{-i(\omega_s - \omega_0)} - 1| \leq \frac{2\pi(m+1)}{T}$, we obtain

$$\text{(S8)} \qquad \max_{1 \leq s \leq m} |r_{js}| \leq 2\pi(m+1) \times \left(|C_j(e^{-i\omega_0})| \max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}} + 2 \max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} + 2 \max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}^G|}{\sqrt{T}} \right).$$

By Lemma S2, $E(\max_{1 \leq t \leq T}(|R_{jt}|/\sqrt{T}))^2 \leq bT^{2/p-1} \log^2 T$ for some $b > 0$, which depends only on p and μ_p for large enough T. Furthermore,

$$\Pr\left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > \delta \right) \leq \Pr\left(\sum_{t=0}^{T} \frac{|\tilde{\varepsilon}_{jt}|^p}{T^{p/2}} > \delta^p \right) \leq 2T^{1-p/2} E|\tilde{\varepsilon}_{jt}|^p / \delta^p.$$

But by Minkowski's inequality,

$$\text{(S9)} \qquad E|\tilde{\varepsilon}_{jt}|^p = E\left(\left| \sum_{k=0}^{\infty} \tilde{c}_{jk} \varepsilon_{jt-k} \right|^p \right) < \left(\sum_{k=0}^{\infty} |\tilde{c}_{jk}| (E|\varepsilon_{jt-k}|^p)^{1/p} \right)^p \leq \left(\sum_{k=0}^{\infty} k|c_{jk}| \right)^p \mu_p.$$

Hence,

$$\Pr\left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > \delta \right) \leq 2T^{1-p/2} \left(\sum_{k=0}^{\infty} k|c_{jk}| \right)^p \mu_p / \delta^p$$

and, therefore,

$$\begin{aligned} E\left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} \right)^2 &= 2 \int_0^{\infty} x \Pr\left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > x \right) dx \\ &\leq T^{2/p-1} + 4 \int_{T^{1/p-1/2}}^{\infty} x^{1-p} T^{1-p/2} \left(\sum_{k=0}^{\infty} k|c_{jk}| \right)^p \mu_p \, dx \\ &\leq aT^{2/p-1} \end{aligned}$$

for some $a>0$ which depends only on p, μ_p, and $\sup_{j\geq 1}(\sum_{k=0}^{\infty} k|c_{jk}|)^p$. Using a similar argument, we can show that $E(\max_{0\leq t\leq T}(|\tilde{\varepsilon}_{jt}^G|/\sqrt{T}))^2 \leq cT^{2/p-1}$ for some $c>0$ which depends only on p and $\sup_{j\geq 1}(\sum_{k=0}^{\infty} k|c_{jk}|)^p$. Using the above estimates for the second moments together with (S9), we get $\sup_{j\geq 1} E(\max_{1\leq s\leq m}|r_{js}|)^2 \leq Km^2T^{2/p-1}\log^2 T$ for large enough T and for $K>0$ which depends only on p, μ_p, $\sup_{j\geq 1}(\sum_{k=0}^{\infty} k|c_{jk}|)^p$, and $\sup_{j\geq 1}|C_j(e^{-i\omega_0})|$.

Q.E.D.

A DETAIL OF THE PROOF OF THEOREM 1

In the proof of Theorem 1, we mention that we can establish the fact that $\bar{c}_{m,n} - c_{m,n} = O(1/n)$ by finding bounds on function $\bar{f}(c) \equiv \int (\frac{\lambda c}{1-\lambda c})^2\, d\bar{H}_n(\lambda)$ in terms of function $f(c) \equiv \int (\frac{\lambda c}{1-\lambda c})^2\, dH_n(\lambda)$. Here we derive such bounds and use them to prove that $\bar{c}_{m,n} - c_{m,n} = O(1/n)$.

Since $(\frac{\lambda c}{1-\lambda c})^2$ is an increasing function of λ for $\lambda c<1$, inequalities $l_{k+i,n} \leq \bar{l}_{in} \leq l_{in}$ for $n-k\leq i\leq 1$ imply that

$$f(c) - \frac{k}{n}\left(\frac{l_{1n}c}{1-l_{1n}c}\right)^2 \leq \frac{n-k}{n}\bar{f}(c) \leq f(c) \tag{S9}$$

for $c\in[0, l_{1n}^{-1})$. By definition, $\bar{c}_{m,n}$ and $c_{m,n}$ are the solutions to equations $\bar{f}(c) = \frac{m-k}{n-k}$ and $f(c) = \frac{m}{n}$, respectively. Furthermore, $\bar{f}(c)$ and $f(c)$ are increasing functions of c on $c\in[0, \bar{l}_{1n}^{-1})$ and on $c\in[0, l_{1n}^{-1})$, respectively. Hence, inequalities (S9) would imply $\bar{c}_{m,n} - c_{m,n} = O(1/n)$ if we show that for any $n>N$,

$$f\left(c_{m,n} + \frac{M}{n}\right) - \frac{k}{n}\left(\frac{l_{1n}(c_{m,n}+M/n)}{1-l_{1n}(c_{m,n}+M/n)}\right)^2 \geq \frac{m-k}{n}$$

and

$$f\left(c_{m,n} - \frac{M}{n}\right) \leq \frac{m-k}{n},$$

where $N>0$ and $M>0$ are constants yet to be chosen.

Since $f(c_{m,n}) = \frac{m}{n}$, we have $f(c_{m,n} \pm \frac{M}{n}) \gtrless \frac{m}{n} \pm \frac{M}{n}\min_{|c-c_{m,n}|\leq M/n} f'(c)$ for any $n>N_1(M)$, where $N_1(M)$ is so large that $(c_{m,n}+\frac{M}{n})l_{1n} < 1$ and $c_{m,n} - \frac{M}{n} > 0$ for any $n>N_1(M)$. That such an $N_1(M)$ exists follows from the assumption that $\limsup l_{1n}c_{m,n} < 1$ and from inequality $\liminf c_{m,n} > 0$. The latter inequality holds because $\frac{m}{n} = \int(\lambda c_{m,n}/(1-\lambda c_{m,n}))^2\, dH_n(\lambda) \leq (l_{1n}c_{m,n}/(1-l_{1n}c_{m,n}))^2$ so that $\liminf c_{m,n} \geq \liminf(\sqrt{\frac{m}{n}})/(1-\limsup l_{1n}c_{m,n})/\limsup l_{1n}$, where $\liminf(\sqrt{\frac{m}{n}}) > 0$ by assumption that $\frac{m}{n}$ remains in a compact subset of $(0,\infty)$, and $\limsup l_{1n}c_{m,n} < 1$ and $\limsup l_{1n} < \infty$ by Assumption 3.

Further, note that

$$f''(c) \equiv \int \frac{2\lambda^2 + 4\lambda^3 c}{(1-\lambda c)^4}\, dH_n(\lambda) > 0$$

for $c \in [0, l_{1n}^{-1})$. Therefore, $f(c)$ is convex on $[0, l_{1n}^{-1})$ and we have

$$\begin{aligned}\min_{|c-c_{m,n}|\le M/n} f'(c) = f'\left(c_{m,n} - \frac{M}{n}\right) &\equiv \int \frac{2\lambda^2 c}{(1-\lambda c)^3}\, dH_n(\lambda)\bigg|_{c=c_{m,n}-M/n} \\ &\ge \frac{2l_{nn}^2(c_{m,n}-M/n)}{(1-l_{nn}(c_{m,n}-M/n))^3} > 2l_{nn}^2\left(c_{m,n}-\frac{M}{n}\right).\end{aligned}$$

But by Assumption 3, $\liminf l_{nn} > 0$. Therefore, there exist $N_2(M)$ and $\gamma > 0$ such that $\min_{|c-c_{m,n}|\le M/n} f'(c) \ge \gamma$ for any $n > N_2(M)$, and, hence, $f(c_{m,n} \pm \frac{M}{n}) \gtrless \frac{m}{n} \pm \frac{M}{n}\gamma$ for any $n > \max(N_1(M), N_2(M))$.

Finally, let $N_3(M)$ and $C \ge k$ be such that for any $n > N_3(M)$,

$$\left(\frac{l_{1n}(c_{m,n}+M/n)}{1-l_{1n}(c_{m,n}+M/n)}\right)^2 \le \frac{C}{k}.$$

Choose $M > \frac{C}{\gamma}$ and $N = \max(N_1(M), N_2(M), N_3(M))$. Then, for any $n > N$,

$$\begin{aligned}f\left(c_{m,n}+\frac{M}{n}\right) - \frac{k}{n}\left(\frac{l_{1n}(c_{m,n}+M/n)}{1-l_{1n}(c_{m,n}+M/n)}\right)^2 &> \frac{m}{n} + \frac{M}{n}\gamma - \frac{C}{n} \\ &\ge \frac{m-k}{n}\end{aligned}$$

and

$$f\left(c_{m,n}-\frac{M}{n}\right) < \frac{m}{n} - \frac{M}{n}\gamma < \frac{m}{n} - \frac{C}{n} \le \frac{m-k}{n}$$

as desired, and thus, $\bar{c}_{m,n} - c_{m,n} = O(1/n)$. *Q.E.D.*

PROOF OF THEOREM 3

Let $\lambda_j(A)$ denote the ith largest eigenvalue of a Hermitian matrix A. Let $\tilde{F}_t = F_t + \sqrt{-1}F_{t+T/2}$. Below, we will assume that T is an even number. If it is not, we will redefine T as $T-1$. We have the following lemma:

LEMMA S4: *Suppose Assumption* 1m *holds. Then there exists a constant* $B > 0$ *such that* $\Pr(\lambda_k(\frac{2}{T}\sum_{t=1}^{T/2}\tilde{F}_t\tilde{F}_t') < B) \to 0$ *as* $T \to \infty$.

PROOF: Let us denote $1/(t_2-t_1)\sum_{t=t_1+1}^{t_2} F_tF'_{t+u}$ as $\hat{\Gamma}^{(t_1,t_2)}(u)$ and denote the i,jth entry of matrix $\hat{\Gamma}^{(t_1,t_2)}(u)$ as $\hat{\Gamma}_{ij}^{(t_1,t_2)}(u)$. Note that, by definition of $\tilde{F}_j$, $\frac{2}{T}\sum_{j=1}^{T/2}\tilde{F}_j\tilde{F}'_j = 2\hat{\Gamma}^{(0,T)}(0)+\sqrt{-1}(\hat{\Gamma}^{(T/2,T)}(-T/2)-\hat{\Gamma}^{(0,T/2)}(T/2))$. Using Weyl's inequalities for eigenvalues of a sum of Hermitian matrices (see Horn and Johnson (1985, Theorem 4.3.7)), we obtain

$$\text{(S10)}\qquad \left|\lambda_k\left(\frac{2}{T}\sum_{t=1}^{T/2}\tilde{F}_t\tilde{F}'_t\right)-\lambda_k\big(2\hat{\Gamma}^{(0,T)}(0)\big)\right| \leq \left\|\hat{\Gamma}^{(T/2,T)}\left(-\frac{T}{2}\right)-\hat{\Gamma}^{(0,T/2)}\left(\frac{T}{2}\right)\right\|.$$

According to Formula 3.3 of Hannan (1970, p. 209), the variance of $\hat{\Gamma}_{ij}^{(t_1,t_2)}(s)$ equals

$$\frac{1}{t_2-t_1}\sum_{u=-t_2+t_1+1}^{t_2-t_1-1}\left(1-\frac{|u|}{t_2-t_1}\right) \times\big\{\Gamma_{ii}(u)\Gamma_{jj}(u)+\Gamma_{ij}(u+s)\Gamma_{ji}(u-s)+\mathrm{cum}(F_{i0},F_{j,s},F_{i,u},F_{j,u+s})\big\}.$$

Since by Assumption 1m, for any i and j, $\Gamma_{ij}(v)\to 0$ as $v\to\infty$ and $\mathrm{cum}(F_{i0}, F_{j,s},F_{i,u},F_{j,u+s})\to 0$ as $\max(|s|,|u|,|s+u|)\to\infty$, the variances of $2\hat{\Gamma}_{ij}^{(0,T)}(0)$, of $\hat{\Gamma}_{ij}^{(T/2,T)}(-T/2)$, and of $\hat{\Gamma}_{ij}^{(0,T/2)}(T/2)$ converge to zero as $T\to\infty$. Therefore, $2\hat{\Gamma}_{ij}^{(0,T)}(0)$ converges in probability to its mean $2\Gamma_{ij}(0)$ and, since by Assumption 1m, $\Gamma_{ij}(-T/2)-\Gamma_{ij}(T/2)\to 0$, $\hat{\Gamma}_{ij}^{(T/2,T)}(-T/2)-\hat{\Gamma}_{ij}^{(0,T/2)}(T/2)$ converges in probability to zero. Since the eigenvalues are continuous functions of the entries of the matrix, $\lambda_k(2\hat{\Gamma}^{(0,T)}(0))$ converges in probability to $2\lambda_k(\Gamma(0))>0$. Further, $\|\hat{\Gamma}_{ij}^{(T/2,T)}(-T/2)-\hat{\Gamma}_{ij}^{(0,T/2)}(T/2)\|$ converges in probability to zero. The latter two convergence results and inequality (S10) imply that the statement of the lemma holds with $B=\lambda_k(\Gamma(0))$. *Q.E.D.*

LEMMA S5: *Let Assumptions* 1m–4m *hold, and let n and T go to infinity so that n/T remains in a compact subset of* $(0,\infty)$. *Then, for any positive integer r, the joint distribution of $\sigma_{T/2,n}^{-1}(\tilde{\gamma}_{k+1}-\mu_{T/2,n}),\dots,\sigma_{T/2,n}^{-1}(\tilde{\gamma}_{k+r}-\mu_{T/2,n})$ weakly converges to the r-dimensional TW_2 distribution.*

PROOF: The proof of this lemma is almost identical to the proof of Theorem 1 in the Appendix. We introduce the following notation to minimize the discrepancies. Let $m=T/2$, $\tilde{X}=\sqrt{2\pi}[\tilde{X}_1,\dots,\tilde{X}_m]$, $\hat{F}=\sqrt{2\pi}[\tilde{F}_1,\dots,\tilde{F}_m]$, and $\tilde{e}=\sqrt{2\pi}[\tilde{e}_1,\dots,\tilde{e}_m]$. Then, by definition, $\tilde{\gamma}_i=\lambda_i(\tilde{X}\tilde{X}'/(2\pi m))$ for all

$i = 1, \dots, n$. The remaining proof of Lemma S5 repeats the proof of Theorem 1, starting from the second paragraph of that proof with the following changes: matrices $S_n^e(\omega_0)$ and $\bar{S}_n^e(\omega_0)$ must be replaced by Σ_n^e and $\bar{\Sigma}_n^e$, where $\Sigma_n^e \equiv Ee_t(n)e_t'(n)$; the word "Assumption 3" must be replaced by "Assumption 3m" the words "by Assumptions 1 and 4" must be replaced by "by Lemma S4 and Assumption 4m." *Q.E.D.*

The convergence of $\tilde{R}$ to $\max_{0<i\le k_1-k_0}((\lambda_i - \lambda_{i+1})/(\lambda_{i+1} - \lambda_{i+2}))$ when $k = k_0$ follows from Lemma S5. When $k_0 < k \le k_1$, $\tilde{R} \ge (\tilde{\gamma}_k - \tilde{\gamma}_{k+1})/(\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2})$. Therefore, we only need to show that $(\tilde{\gamma}_k - \tilde{\gamma}_{k+1})/(\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2}) \xrightarrow{p} \infty$. Using the notation of Lemma S5, we have $\tilde{\gamma}_i = \lambda_i(\tilde{X}\tilde{X}'/2\pi m)$ for all $i = 1, \dots, n$. Using Weyl's inequalities for singular values (see Lemma 3), we obtain

$$\left|\lambda_i^{1/2}\left(\frac{\tilde{X}\tilde{X}'}{2\pi m}\right) - \lambda_i^{1/2}\left(\frac{\hat{\Lambda}_0\hat{F}\hat{F}'\hat{\Lambda}_0'}{2\pi m}\right)\right| \le \lambda_1^{1/2}\left(\frac{\tilde{e}\tilde{e}'}{2\pi m}\right)$$

for $i = 1, \dots, n$, where $\lambda_1(\frac{\tilde{e}\tilde{e}'}{2\pi m}) = O_p(1)$ by Lemma 1. Take $i = k$. By Assumption 4m and Lemma S4, $\lambda_k(\hat{\Lambda}_0\hat{F}\hat{F}'\hat{\Lambda}_0'/(2\pi m)) \xrightarrow{p} \infty$. Therefore, $\lambda_k(\tilde{X}\tilde{X}'/(2\pi m)) \xrightarrow{p} \infty$ and, hence, $\tilde{\gamma}_k \xrightarrow{p} \infty$. Now, take $i > k$. Then $\lambda_i^{1/2}(\hat{\Lambda}_0\hat{F}\hat{F}'\hat{\Lambda}_0'/(2\pi m)) = 0$. Therefore, $\lambda_i(\tilde{X}\tilde{X}'/(2\pi m)) = O_p(1)$ and, hence, $\tilde{\gamma}_i = O_p(1)$. Summing up, $\tilde{\gamma}_k - \tilde{\gamma}_{k+1} \xrightarrow{p} \infty$, while $\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2} = O_p(1)$. Hence $(\tilde{\gamma}_k - \tilde{\gamma}_{k+1})/(\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2}) \xrightarrow{p} \infty$. *Q.E.D.*

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