Econometrica Supplementary Material

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APPENDIX B: MULTIPLE OUTPUT LEVELS

IN THIS APPENDIX, we allow for an arbitrary finite number of outputs. Let $f(y|a)$ be the probability that output equals y given effort a and assume that $f(y|a)$ is twice differentiable with respect to a. Let $\bar{y}$ be the highest possible output and let $\underline{y}$ be the lowest possible output. We normalize the price of output to 1 and do not consider producer heterogeneity.[30] Equilibrium contracts are given by the solution to the problem:

$$\max_{(w(\cdot),p(\cdot),a,g)} \sum_y (y - w(y)) f(y|a) - \eta\chi(g)$$

subject to

$$(\mathrm{IR}_{\mathrm{B0}}) \qquad \sum_y (w(y) - p(y)) f(y|a) - c(a) \geq \bar{u} - g$$

and

$$(\mathrm{IC}_{\mathrm{B0}}) \qquad a \in \arg\max_{a\in[0,1]} \sum_y (w(y) - p(y)) f(y|a) - c(a).$$

We continue to focus on the case where equilibrium contracts involve $a > 0$.[31] Then, from Theorem 1 in Jewitt (1988), the first-order approach to this problem is valid provided the following statements hold:

(i) $\sum_{z\leq\bar{z}} \sum_{y\leq z} f(y|a)$ is nonincreasing and convex in a for each $\bar{z}$.
(ii) $\sum_{y\leq\bar{z}} yf(y|a)$ is nondecreasing and concave in a for each $\bar{z}$.
(iii) $\frac{f_a(y|a)}{f(y|a)}$ is nondecreasing and concave in y for every a.

Jewitt provided an interpretation of these conditions. Note that condition (iii) implies the usual (and relatively weak) monotone likelihood ratio property (MLRP). Therefore, MLRP holds throughout this appendix. Jewitt

[30] If producers differ according to productivity parameter x and a producer with productivity x produces output xy with probability $f(y|a)$, all supermodularity results from Section 3 will continue to hold.

[31] We do not spell out the assumptions on primatives under which equilibrium contracts involve $a > 0$. Assumption 2 suffices for this in the two-outcome case, and similar sufficient conditions can be developed for the case with multiple output levels, but this is orthogonal to our focus here.

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argued that the remainder of the third condition is the most restrictive; in addition to MLRP, "[it requires that] variations in output at higher levels are relatively less useful in providing 'information' on the agents effort than they are at lower levels of output" (Jewitt (1988, p. 1181)). Note that Jewitt's condition on utility functions is not needed when the agent is risk-neutral.

Given these three conditions we can apply the first-order approach and (writing $u(y)$ for $w(y) - p(y)$) rewrite the problem as

$$\max_{(w(\cdot),p(\cdot),a,g)} \sum_y \big(y - [u(y)]_+\big) f(y|a) - \eta\chi(g)$$

subject to

$$(\mathrm{IR}_{\mathrm{B1}}) \qquad \sum_y u(y) f(y|a) - c(a) \geq \bar{u} - g$$

and

$$(\mathrm{IC}_{\mathrm{B1}}) \qquad \sum_y u(y) f_a(y|a) = c'(a).$$

The associated Lagrangian is

$$\sum_y \Big[\big(y - [u(y)]_+\big) f(y|a) - \eta\chi(g) + \lambda\big(u(y) f(y|a) - c(a) - (\bar{u} - g)\big) + \mu\big(u(y) f_a(y|a) - c'(a)\big)\Big].$$

Differentiating under the sum with respect to $u(y)$ and rearranging implies that if $u(y) > 0$, then

$$(\text{B-1}) \qquad 1 = \lambda + \mu \frac{f_a(y|a)}{f(y|a)};$$

if $u(y) < 0$, then

$$(\text{B-2}) \qquad 0 = \lambda + \mu \frac{f_a(y|a)}{f(y|a)};$$

and if $u(y) = 0$, then

$$(\text{B-3}) \qquad 0 \leq \lambda + \mu \frac{f_a(y|a)}{f(y|a)} \leq 1.$$

By MLRP, (B-1), (B-2), and (B-3) imply that $u(y) = 0$ for all $y \notin \{\underline{y}, \bar{y}\}$, $u(\underline{y}) \leq 0$, and $u(\bar{y}) \geq 0$. This is a standard "bang–bang" result given MLRP and risk neutrality. To simplify notation, let $u^h \equiv u(\bar{y})$ and let $u^l \equiv u(\underline{y})$, paralleling the notation in Section 3.1. The producer's maximization problem can then be written as

$$\max_{(a,g,u^h,u^l)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}^2} \sum_y yf(y|a) - f(\bar{y}|a)[u^h]_+ - \eta\chi(g)$$

subject to

$$(\mathrm{IR_{B2}}) \qquad f(\bar{y}|a)u^h + f(\underline{y}|a)u^l - c(a) \geq \bar{u} - g$$

and

$$(\mathrm{IC_{B2}}) \qquad f_a(\bar{y}|a)u^h + f_a(\underline{y}|a)u^l = c'(a).$$

Equation ($\mathrm{IC_{B2}}$) can be rewritten as

$$(\text{B-4}) \qquad u^l = \frac{c'(a) - f_a(\bar{y}|a)u^h}{f_a(\underline{y}|a)}.$$

Substituting (B-4) into ($\mathrm{IR_{B2}}$) and using the fact that ($\mathrm{IR_{B2}}$) binds at the solution gives

$$\left(f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)} f(\underline{y}|a)\right)u^h + \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) - c(a) = \bar{u} - g,$$

which may be rewritten as

$$(\text{B-5}) \qquad u^h = \frac{c(a) - \dfrac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) + \bar{u} - g}{f(\bar{y}|a) - \dfrac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)} f(\underline{y}|a)}.$$

Finally, substituting (B-5) into the principal's objective gives

$$(\text{B-6}) \qquad \max_{(a,g)\in[0,1]\times\mathbb{R}_+} \sum_y yf(y|a) - f(\bar{y}|a)\left[\frac{c(a) - \dfrac{f(\underline{y}|a)}{f_a(\underline{y}|a)}c'(a) + \bar{u} - g}{f(\bar{y}|a) - \dfrac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)} f(\underline{y}|a)}\right]_+ - \eta\chi(g).$$

To establish the supermodularity of (B-6) in $(a, g, -\bar{u}, -\eta)$, it suffices to show that the cross-partial of the right-hand side of (B-6) with respect to a

and g is always nonnegative. This is immediate if $[u^h]_+ = 0$. If $[u^h]_+ > 0$, the right-hand side of (B-6) may be rewritten as

$$\text{(B-7)} \qquad \sum_y yf(y|a) - \left(\frac{1}{1 - \left(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)}\right) \Big/ \left(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)}\right)}\right) \times \left(c(a) - \frac{f(\underline{y}|a)}{f_a(\underline{y}|a)} c'(a) + \bar{u} - g\right) - \eta\chi(g),$$

where we have divided the numerator and denominator of the middle term by $f_a(\bar{y}|a)$. The cross-partial of (B-7) with respect to a and g is nonnegative if and only if the derivative of $(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)})/(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)})$ with respect to a is nonnegative. We have thus established the following proposition.

PROPOSITION 15: *Suppose that* $a > 0$, *Conditions* 1–3 *hold*, *and* $(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)})/(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)})$ *is increasing in* a. *Then equilibrium contracts are given by* (B-6), *and* (B-6) *is supermodular in* $(a, g, -\bar{u}, -\eta)$.

The condition that $(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)})/(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)})$ is increasing in a is not very restrictive. To see why it is sufficient for supermodularity of equilibrium contracts, note that, by (B-5) (which is determined by (IR_{B2}) and (IC_{B2})), increasing g by Δ allows the principal to reduce w^h (i.e., payment after the highest output level) by $\frac{\Delta}{f(\bar{y}|a)-(f_a(\bar{y}|a)/f_a(\underline{y}|a))f(\underline{y}|a)}$. Since the principal pays w^h with probability $f(\bar{y}|a)$, increasing g by Δ benefits the principal by $\frac{f(\bar{y}|a)}{f(\bar{y}|a)-(f_a(\bar{y}|a)/f_a(\underline{y}|a))f(\underline{y}|a)}$, which is increasing in a if $(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)})/(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)})$ is increasing. It is instructive to compare this with the following slight generalization of the two-outcome case discussed in Section 3.1: Suppose there are only two outcomes $\underline{y} < \bar{y}$, but that $f(\bar{y}|a)$ need not equal a. Then $f(\bar{y}|a) - \frac{f_a(\bar{y}|a)}{f_a(\underline{y}|a)} f(\underline{y}|a) = f(\bar{y}|a) + f(\underline{y}|a) = 1$ (as $f_a(\bar{y}|a) = -f_a(\underline{y}|a)$), so $\frac{\Delta}{f(\bar{y}|a)-(f_a(\bar{y}|a)/f_a(\underline{y}|a))f(\underline{y}|a)} = \Delta$. Therefore, increasing g by Δ benefits the principal by $f(\bar{y}|a)$, which is increasing in a under MLRP. In particular, we see that $(\frac{f_a(\bar{y}|a)}{f(\bar{y}|a)})/(\frac{f_a(\underline{y}|a)}{f(\underline{y}|a)})$ is increasing in a if $f(\bar{y}|a) + f(\underline{y}|a)$ does not depend on a, regardless of the number of outcomes.

REFERENCE

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