Econometrica Supplementary Material

SUPPLEMENT TO "A MECHANISM DESIGN APPROACH TO RANKING ASYMMETRIC AUCTIONS"

(*Econometrica*, Vol. 80, No. 5, September 2012, 2349–2364)

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A1. OTHER AUCTION FORMATS

THE CONCLUSION IN Theorem 1 is made possible because the allocation in the SPA can be described precisely, while the possible allocations in the FPA can be narrowed down to a relatively small set. It is not necessary to know the exact allocation in the FPA.

Aside from the issue of how much rent is extracted from β_i types, Theorem 1 therefore really says that the SPA is a poor auction format if the objective is to generate high expected revenue. For instance, if $\beta_w = \beta_s$ and $u_w(\beta_w) = u_s(\beta_s) = 0$, any auction with $k(v) \in [v, r(v)]$ is more profitable than the SPA if condition (9) or (10) in the main paper is satisfied. In other words, it is profitable to design an auction that favors the weak bidder moderately.

It has long been understood that *optimal* auctions typically favor the weak bidder; see, for example, McAfee and McMillan (1989).[1] Based on this property, Klemperer (1999) argued that "it is plausible that a first-price auction may be more profitable [...] than a second-price auction." However, this paper establishes a bound on the amount of favoritism that can safely be extended to the weak bidder. Specifically, any mechanism where the weak bidder wins more often than is efficient but less often than he would in a counterfactual symmetric auction against another weak bidder is more profitable than a SPA.

To illustrate, define a winner-pay auction to be an auction in which the winner pays a proportion γ of his own bid and $(1-\gamma)$ of the losing bid, and the loser does not pay, $\gamma \in [0, 1]$. The FPA corresponds to $\gamma = 1$, the SPA to $\gamma = 0$.

PROPOSITION A1: *Assume that* (i) $F_w \leq_{rh} F_s$, (ii) *condition* (9) *or* (10) *holds*, *and* (iii) $\beta_s = \beta_w$. *Then*, *the SPA yields strictly the lowest expected revenue of all winner-pay auctions*.

PROOF: Consider $\gamma \in (0, 1]$, that is, an auction that is not a pure SPA. In this case, the two bidders must share the same maximal bid, $\overline{b}$. Let $\phi_i(b)$ denote bidder i's inverse bidding strategy, $i = s, w$, where $b \in [\beta_w, \overline{b}]$. Assume, for the

[1] In Maskin and Riley's (2000) one model where the SPA is superior to the FPA, the optimal auction would in fact discriminate *against* the weak bidder. In contrast, for the proof of Theorem 1 to work, it is necessary that F_s dominates F_w in terms of the hazard rate (see footnote 12 in the main paper). An optimal auction therefore favors the weak bidder.

 DOI: 10.3982/ECTA9859

moment, that the bidding strategy is strictly increasing and differentiable. If bidder i has type v, his problem is

$$\max_b \int_{\beta_w}^{b} \left[v-(\gamma b+(1-\gamma)x)\right] dF_j(\phi_j(x)),$$

where $j \neq i$ denotes bidder i's rival. The first order condition is

$$\frac{f_j(\phi_j(b))}{F_j(\phi_j(b))}\phi_j'(b)=\frac{\gamma}{v-b}.$$

In equilibrium, bidder i bids b if his type is $v=\phi_i(b)$. Substituting into the first order conditions produces the system of differential equations

$$\frac{f_w(\phi_w(b))}{F_w(\phi_w(b))}\phi_w'(b)=\frac{\gamma}{\phi_s(b)-b}, \quad \frac{f_s(\phi_s(b))}{F_s(\phi_s(b))}\phi_s'(b)=\frac{\gamma}{\phi_w(b)-b}.$$

The only difference from the FPA is that $\gamma \in (0,1]$ (the boundary conditions are the same). The proofs in Maskin and Riley (2000) can then be repeated to conclude that the auction has the same features as a FPA, $k_\gamma(v) \in [v, r(v)]$ for all $\gamma \in (0,1]$. Since bidders with type β_i earn zero rent for all $\gamma \in [0,1]$, Theorem 1 applies directly. *Q.E.D.*

Not all auctions have the property that $k(v) \in [v, r(v)]$. The most prominent example is probably the all-pay auction for which $k(v) < v$ when v is small. The reason is that a weak bidder with a low type is deterred from bidding (which is a sunk cost in an all-pay auction) when facing a rival he perceives as strong. Thus, it is not possible to rank the SPA and the all-pay auction using the method developed in this paper.

A2. TYPES WITH TWO COMPONENTS

Here, I first reexamine the "stochastic shift" model from Section 4.2 in the main paper. The condition that f_w is increasing is relaxed. Then, I consider a model where the strong bidder's type is obtained by multiplying two random variables. For both models, the following lemma is useful. Here, a function is said to be unimodal if it is monotonic or has an inverse-U shape (it may have regions where it is flat). If f_s is unimodal but not monotonic, let $\widehat{v}$ denote the smallest type at the peak, such that $f_s(\widehat{v}) > f_s(v)$ for all $v < \widehat{v}$.

LEMMA A1: *Condition* (9) *is satisfied if* $\beta_w = \beta_s$, f_s *and* f_w *are unimodal on* S_s *and* S_w, *respectively*, $f_w(v) \geq f_s(v)$ *and* $f_w(v) \geq f_s(r(v))$ *for all* $v \in S_w$, *and* $\widehat{v} \leq \alpha_w$.

PROOF: The lemma is straightforward if f_s is monotonic. Hence, assume f_s is not monotonic; $\widehat{v}$ is in the interior of S_s. By assumption, $\widehat{v} \in S_w$. Since F_s is more disperse than F_w, $f_w(v) \geq f_s(r(v))$ for all $v \in [\beta_w, r^{-1}(\widehat{v})]$, an interval on which f_s is increasing since $r^{-1}(\widehat{v}) < \widehat{v}$. Thus, condition (9) is satisfied for all $v \in [\beta_w, r^{-1}(\widehat{v})]$. Since $f_w(r^{-1}(\widehat{v})) \geq f_s(\widehat{v})$, $f_w(\widehat{v}) \geq f_s(\widehat{v})$, and f_w is itself unimodal, it must hold that $f_w(v) \geq f_s(\widehat{v})$ for all $v \in [r^{-1}(\widehat{v}), \widehat{v}]$, and since f_s attains its peak at $\widehat{v}$, condition (9) must also be satisfied on the interval $[r^{-1}(\widehat{v}), \widehat{v}]$. For $v > \widehat{v}$, $f_w(v) \geq f_s(v)$ combines with the monotonicity of f_s on that region to ensure that (9) is satisfied here as well. *Q.E.D.*

A2.1. *Stochastic Shifts*

To generalize Example 1 to permit G to be nondegenerate and obtain Proposition 1, the assumption that F_w is log-concave need only be replaced with the slightly stronger assumption that f_w is log-concave. By imposing more conditions on g, the assumption that f_w is increasing can also be relaxed. The intention is to use Lemma A1.

However, the convolution of two unimodal densities is not necessarily unimodal. Ibragimov (1956) has shown that the convolution of a log-concave density with any unimodal density is itself unimodal. Hence, a log-concave function is sometimes referred to as strongly unimodal. Since it has already been assumed that f_w is log-concave, unimodality of f_s is then guaranteed if g is unimodal.[2] It will also be assumed that $\beta = 0$, implying that $\beta_w = \beta_s$. Log-concavity of f_w also implies that F_s is more disperse than F_w, as mentioned in the main paper. However, to apply Lemma A1, it is also necessary that F_w is steeper than F_s on S_w. Unfortunately, a convolution may increase the density locally. Thus, additional assumptions are required.

PROPOSITION A2: *The FPA yields strictly higher expected revenue than the SPA in the additive model if f_w is log-concave, $\beta = 0$, and g is decreasing and satisfies*

$$f_w(\alpha_w) \geq \int_0^{\alpha} f_w(\alpha_w - z) g(z)\, dz. \tag{A1}$$

PROOF: For $v \in C = S_w$,

$$\frac{f_s(v)}{f_w(v)} = \frac{\int_0^{\alpha} f_w(v-z) g(z)\, dz}{f_w(v)} = \int_0^{\alpha} \frac{f_w(v-z)}{f_w(v)} g(z)\, dz,$$

[2]Miravete (2011) examined the properties of convolutions of two log-concave densities. He emphasized their relevance to models of asymmetric information, including multidimensional screening.

where $f_w(v-z)=0$ if $v-z\le\beta_w$. Thus,

$$\frac{d}{dv}\left(\frac{f_s(v)}{f_w(v)}\right)\ge\int_0^{\alpha}\left(\frac{f_w'(v-z)}{f_w(v-z)}-\frac{f_w'(v)}{f_w(v)}\right)\frac{f_w(v-z)}{f_w(v)}g(z)\,dz,$$

which is positive since f_w is log-concave. Hence, F_s dominates F_w in terms of the likelihood ratio (this conclusion relies on $\beta=0$). Condition (A1) is equivalent to $\frac{f_s(\alpha_w)}{f_w(\alpha_w)}\le 1$. Thus, since $\frac{f_s(v)}{f_w(v)}$ is increasing on S_w, $f_w(v)\ge f_s(v)$ for all $v\in S_w$. Since the convolution of f_w and g is unimodal, Lemma A1 applies if f_s peaks to the left of α_w. However, when $v>\alpha_w$,

$$f_s(v)=\int_{\max\{\beta_w,v-\alpha\}}^{\alpha_w}g(v-z)f_w(z)\,dz,$$

which is decreasing in v when g is decreasing. Consequently, f_s peaks at or before α_w. *Q.E.D.*

Condition (A1) requires that $f_w(\alpha_w)$ exceeds a weighted average of f_w over the interval $[\alpha_w-\alpha,\alpha_w]$ (on which f_w may be zero if $\alpha_w-\alpha\le\beta_w$). Thus, if α is small, it rules out that f_w is decreasing. However, f_w may be nonmonotonic as long as it does not "dip down" too much after it has passed its peak. The condition is less restrictive if α is large, such that the asymmetry between bidders is large.

A2.2. *A Multiplicative Model*

Assume that the strong bidder's type has two components, v and a. The former is drawn from F_w. The latter is drawn from G, which has support $[\beta,\alpha]$. The strong bidder's type is then $u(v,a)=va$.

Cuculescu and Theodorescu (1998) examined multiplication of random variables. For nonnegative random variables, they showed that log-concavity of f_w must be replaced by log-concavity of $f_w(e^v)$ to obtain results that mirror those for addition of random variables.[3] That is, if a random variable with this property is multiplied with another random variable with unimodal density, then the resulting variable also has unimodal density. Likewise, the multiplicative convolution of F_w and a nondegenerate random variable G is more "star disperse" than F_w itself (i.e., $\frac{r(v)}{v}$ is increasing).

Assume $\beta\ge 1$, such that $\beta_s\ge\beta_w$ and F_s (the multiplicative convolution of F_w and G) first order stochastically dominates F_w. Then, $\frac{r(v)}{v}$ increasing implies $r(v)-v$ increasing. In other words, F_s is more disperse than F_w.

[3]Jewitt (1987, footnote 15) made a related observation in a model of risk aversion with two sources of uncertainty. Jewitt (1987) used tools from total positivity, as in Miravete (2011).

Turning to reverse hazard rate dominance, note first that, for any $v \in C$,

$$\frac{F_s(v)}{F_w(v)} = \int_\beta^\alpha \frac{F_w\left(\frac{v}{z}\right)}{F_w(v)} g(z)\,dz.$$

Since $f_w(e^v)$ is log-concave, $F_w(e^v)$ is log-concave as well. Equivalently, $vf_w(v)/F_w(v)$ is decreasing. Thus, since $z \geq 1$,

$$\frac{d}{dv}\left(\frac{F_w\left(\frac{v}{z}\right)}{F_w(v)}\right) = \frac{1}{v}\frac{F_w\left(\frac{v}{z}\right)}{F_w(v)}\left(\frac{\left(\frac{v}{z}\right)f_w\left(\frac{v}{z}\right)}{F_w\left(\frac{v}{z}\right)} - \frac{vf_w(v)}{F_w(v)}\right) \geq 0,$$

and $F_w \leq_{rh} F_s$ follows. A counterpart to Proposition 1 in the main paper is now immediate.

PROPOSITION A3: *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if $f_w(e^v)$ is increasing and log-concave.*

The proof is identical to the proof of Proposition 1 in the main paper.

To relax the assumption that f_w is monotonic, it is necessary to impose more restrictions on g instead. As in the additive model, Lemma A1 is used. The proof of the following proposition is omitted since it is analogous to the proof of Proposition A2.

PROPOSITION A4: *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if $f_w(e^v)$ is log-concave, $\beta = 1$, and g is decreasing and satisfies*

$$f_w(\alpha_w) \geq \int_1^\alpha f_w\left(\frac{\alpha_w}{z}\right)\frac{g(z)}{z}\,dz. \tag{A2}$$

A3. AUCTIONS WITH MANY STRONG BIDDERS

This part of the Supplemental Material provides the details behind the claim in Section 4.4 of the main paper that the FPA is superior to the SPA if the asymmetry is large enough and $m \geq 2$. Let $\overline{b}_s = b_s(\alpha_s)$ and $\overline{b}_w = b_w(\alpha_w)$ denote the maximum bids of the strong and weak bidders, respectively. When $m \geq 2$, it is possible that $\overline{b}_s > \overline{b}_w$.

A3.1. *Small Overlap*

Assume that α_w is "close" to β_s such that there is little overlap between the supports. As a starting point, if $\beta_s = \alpha_w$, then (i) $J_w(\alpha_w) = \alpha_w = \beta_s > J_s(\beta_s)$, and (ii) $\overline{b}_s > \overline{b}_w$ in a FPA (a strong bidder with type β_s bids $\beta_s = \alpha_w \geq \overline{b}_w$, and his strategy is strictly increasing). If α_w is "slightly above" β_s, it must remain the case that $\overline{b}_s > \overline{b}_w$, or $k_1(\alpha_w) < \alpha_s$, with $J_w(\alpha_w) > J_s(x)$ for all $x \in [\beta_s, k_1(\alpha_w)]$. Moreover, by continuity,

$$\text{(A3)} \qquad J_w(v) > J_s(x) \quad \text{for all } v \in [\beta_s, \alpha_w] \text{ and } x \in [\beta_s, k_1(\alpha_w)],$$

when β_s and α_w are sufficiently close.[4] In the following, when the overlap is said to be "small," it should be taken to mean that (A3) is satisfied.

In this case, the FPA yields higher expected revenue than the SPA because the weak bidders are winning more often against strong bidders with inferior virtual valuation. Recall that the two are revenue equivalent if there is no overlap.

PROPOSITION A5: *Assume that $F_w \leq_{rh} F_s$ and the overlap is small. Then, the FPA generates strictly higher expected revenue than the SPA when $m \geq 2$, $n \geq 1$.*

PROOF: Both auctions ensure that $u_i^k(\beta_i) = 0$, $i = s, w$. A weak bidder with type below β_s loses both auctions (competition between the strong bidders ensures that any serious bid must be at least β_s). By (i), a weak bidder with type $v \in (\beta_s, \alpha_w]$ wins more often in the FPA than in the SPA. By (ii) or (A3), the winner's virtual valuation is no lower in the FPA, and may be higher. In other words, D_m is positive. *Q.E.D.*

EXAMPLE 1—Continued: Consider a many-bidder extension of Example 1, with $m \geq 2$. If F_s is shifted far to the right such that there is no overlap between supports, then the two auctions are revenue equivalent. The same is true if $a = 0$, in which case bidders are homogenous. Proposition A5 then states that the FPA is superior for large "interior" values of a. A comparison cannot be made for small values. Recall that Proposition A5 does not require F_s to be a "shifted" version of F_w.

A3.2. *Large Stretches*

Assume the asymmetry between bidders is so large that $\overline{b}_s > \overline{b}_w$. Define $\overline{\alpha}_s \equiv k_1(\alpha_w)$ as the highest strong type that competes with the weak bidders. A strong bidder outbids the weak bidders with probability 1 if his type exceeds $\overline{\alpha}_s$, $\overline{\alpha}_s < \alpha_s$.

Consider the consequences of "stretching" the strong bidder's distribution, transforming F_s with support $[\beta_s, \alpha_s]$ to F_s^λ with support $[\beta_s, \alpha_s^\lambda]$, $\alpha_s^\lambda > \alpha_s$, such that $F_s^\lambda = \lambda F_s$ on the subinterval $v \in [\beta_s, \alpha_s]$, with $\lambda \in (0, 1)$. More concisely,

[4] F_s need not be more disperse than F_w. For instance, the former could have a smaller support than the latter. It is a general property that $J_i(\alpha_i) = \alpha_i$ and $J_i(\beta_i) < \beta_i$, $i = s, w$.

F_s is a truncation of F_s^λ. Importantly, F_s and F_s^λ have the same reverse hazard rate on $[\beta_s, \alpha_s]$ and therefore on $[\beta_s, \overline{\alpha}_s]$. Thus, if F_s dominates F_w in terms of the reverse hazard rate, so does F_s^λ. Likewise, the system of first order conditions from the bidders' maximization problems is unchanged at bids below $\overline{b}_w$. This can be seen by examining the systems in Maskin and Riley (2000) or Lebrun (2006). The implication is that weak bidders regardless of type and strong bidders with type below $\overline{\alpha}_s$ use the exact same strategy in either case. Consequently, k_1 is the same in both environments.

For types in $[\beta_s, \overline{\alpha}_s]$, the strong bidders' virtual valuation is

$$J_s^\lambda(v) = v - \frac{1 - F_s^\lambda(v)}{f_s^\lambda(v)} = v - \frac{\dfrac{1}{\lambda} - F_s(v)}{f_s(v)}$$

as a function of λ. The important property is that J_s^λ decreases without bound as F_s is stretched more and more (i.e., as λ decreases and goes to zero). Thus,

$$J_w(v) \geq J_s^\lambda(x) \quad \text{for all } v \in [\beta_w, \alpha_w] \text{ and } x \in [\beta_s, \overline{\alpha}_s] \tag{A4}$$

when F_s is stretched sufficiently much. In the following, when F_s is said to be stretched "a lot," it should be taken to mean that (A4) is satisfied.

PROPOSITION A6: *Assume that $F_w \leq_{rh} F_s$ and F_s is stretched a lot. Then, the FPA generates strictly higher expected revenue than the SPA when $m \geq 2$, $n \geq 1$.*

The proof is identical to the proof of Proposition A5.

EXAMPLE 2—Continued: Proposition A6 applies directly if F_w is a truncation of F_s, in which case $F_s(v) = \lambda F_w(v)$ on $v \in [\beta_w, \alpha_w]$. As with Example 1, the two auctions are revenue equivalent if the bidders are homogenous, or $\lambda = 1$. A comparison cannot be made if λ is close to 1, or the asymmetry is small. By Proposition A6, however, the FPA is superior when λ is close to zero. Note that Proposition A6 does not require F_s and F_w to be related in any way other than through reverse hazard rate dominance (it does not imply one is a truncation of the other), nor does it require log-concavity.

A4. BOUNDING THE ALLOCATION

Recall from (3) in the main paper that the system of differential equations can be written

$$k_1'(v) = \frac{F_s(k_1(v))}{f_s(k_1(v))} \frac{f_w(v)}{F_w(v)} \frac{k_1(v) - b_w(v)}{v - b_w(v)}, \tag{A5}$$

$$b_w'(v) = \frac{f_w(v)}{F_w(v)}(k_1(v) - b_w(v))$$

for $v \in (b_*, \alpha_w]$, where $b_* \in [\beta_w, \beta_s]$ is the smallest serious bid (see Maskin and Riley (2000) for a characterization of b_*). The system must satisfy the condition $k_1(\alpha_w) = \alpha_s$. It is also the case that $k_1(b_*) = \beta_s$ and $b_w(b_*) = b_*$. Fix k_1 and $v \in (b_*, \alpha_w]$, with $k_1 \geq v$, and note that

$$\text{(A6)} \qquad \frac{d}{db_w}\left(\frac{k_1 - b_w}{v - b_w}\right) = \frac{k_1 - v}{(v - b_w)^2} \geq 0.$$

The implication is that if b_w can be bounded, then the last term in $k_1'(v)$ can be bounded as well.

To illustrate the usefulness of bounds on b_w, start with the crudest bound, namely, $b_w(v) > b_*$ for $v \in (b_*, \alpha_w]$. Consider the function $\overline{k}$, with domain $(b_*, \alpha_w]$. Assume $\overline{k}(\alpha_w) = \alpha_s$, with

$$\text{(A7)} \qquad \overline{k}'(v) = \frac{F_s(\overline{k}(v))}{f_s(\overline{k}(v))} \frac{f_w(v)}{F_w(v)} \frac{\overline{k}(v) - b_*}{v - b_*}$$

for $v \in (b_*, \alpha_w]$. Assuming $F_w \leq_{rh} F_s$, the same proof as in footnote 9 of the main paper proves that $\overline{k} \in [v, r(v)]$. Compare now k_1 and $\overline{k}$. Should they coincide at some $v \in (b_*, \alpha_w]$, then $k_1'(v) \geq \overline{k}'(v)$, by (A6). Hence, $\overline{k}$ can cross k_1 at most once, and then from above. If this occurs in the interior, then $k_1(\alpha_w) = \overline{k}(a_w)$ is violated because $k_1'(\alpha_w) > \overline{k}'(\alpha_w)$ due to the assumption that $k_1(\alpha_w) - \alpha_w = \alpha_s - \alpha_w > 0$. Consequently, $\overline{k}$ is an upper bound on k_1, on $(b_*, \alpha_w]$. The assumption that $\alpha_s > \alpha_w$ can be replaced by the joint assumption that $\alpha_s = \alpha_w$ and F_s log-concave.[5] Recall that $\overline{k}(v) \leq r(v)$, so the bound is tighter than $r(v)$. Clearly, $\overline{k}$ depends only on the primitives, F_s and F_w, and the fixed number b_* (which Maskin and Riley (2000) derived).

Like $r(v)$, $\overline{k}(v)$ can be implicitly characterized. Since (A7) is a separable differential equation, it is helpful to rewrite it as

$$\frac{f_s(\overline{k})}{F_s(\overline{k})} \frac{1}{\overline{k} - b_*} d\overline{k} = \frac{f_w(v)}{F_w(v)} \frac{1}{v - b_*} dv.$$

With the boundary condition $\overline{k}(\alpha_w) = \alpha_s$, $\overline{k}$ is then implicitly characterized by

$$\text{(A8)} \qquad \int_{\overline{k}(v)}^{\alpha_s} \frac{f_s(x)}{F_s(x)} \frac{1}{x - b_*} dx = \int_{v}^{\alpha_w} \frac{f_w(x)}{F_w(x)} \frac{1}{x - b_*} dx,$$

[5]In this case, k_1 and $\overline{k}$ coincide and are tangent at $v = \alpha_w$. Assume now that $k_1 > \overline{k}$ for some v. Then, by log-concavity, $\frac{F_s(k_1)}{f_s(k_1)} \geq \frac{F_s(\overline{k})}{f_s(\overline{k})}$. Thus, by (A6), k_1 is strictly steeper than $\overline{k}$. Hence, k_1 and $\overline{k}$ diverge, and $k_1(\alpha_w) = \overline{k}(\alpha_w)$ is impossible. Thus, $k_1(v) \leq \overline{k}(v)$ for all $v \in (b_*, \alpha_w]$.

for $v \in (b_*, \alpha_w]$. Incidentally, the tying function in an asymmetric all-pay auction can be characterized implicitly in much the same way; see Amann and Leininger (1996).

The upper bound can be tightened further, and a lower bound established as well, by exploiting tighter bounds on $b_w(v)$. To begin, let b_s denote the strong bidder's strategy in the asymmetric auction. Let b_{ii} denote bidders' strategy in a symmetric auction in which both bidders are of kind i, $i = s, w$. Then, Maskin and Riley's (2000) Proposition 3.5 implies that $b_{ww}(v) \le b_w(v)$, $v \in S_w$, and $b_{ss}(v) \ge b_s(v)$, $v \in S_s$. That is, bidders are more aggressive when they face a strong rather than a weak bidder. Let b^* denote the maximum bid in the asymmetric auction. Since the purpose is to bound b_w,

$$b_w(v) \ge \max\{b_*, b_{ww}(v)\} = \max\left\{b_*, v - \int_{\beta_w}^{v} \frac{F_w(x)}{F_w(v)}\, dv\right\}$$

is immediately useful. If $\beta_w = \beta_s$, then $b_{ww}(v) \ge b_* = \beta_w$ for all $v \in S_w$. An upper bound on k_1 can now be derived by replacing b_* in (A7) with $\max\{b_*, b_{ww}(v)\}$.

It is harder to derive a lower bound on k_1. To do so, an upper bound on b_w is needed. Recalling that $k_1(v) \le r(v)$ leads to the conclusion that $b_w(v) = b_s(k_1(v)) \le b_s(r(v))$. Hence,

$$\begin{aligned} b_w(v) &\le b_s(r(v)) \le \min\{b^*, b_{ss}(r(v))\} \\ &= \min\left\{b^*, r(v) - \int_{\beta_s}^{r(v)} \frac{F_s(x)}{F_s(r(v))}\, dv\right\}. \end{aligned}$$

There are two weaknesses here. First, b^* is endogenous and, unlike b_*, cannot generally be characterized. Second, it is entirely possible that $\min\{b^*, b_{ss}(r(v))\}$ exceeds v. An alternative is to begin with

$$\begin{aligned} (v - b_w(v))F_s(r(v)) &\ge (v - b_w(v))F_s(k_1(v)) \\ &= \int_{\beta_w}^{v} F_s(k_1(x))\, dx \\ &\ge \int_{\beta_w}^{v} F_s(x)\, dx, \end{aligned}$$

where the inequalities come from $r(v) \ge k_1(v) \ge v$. The equality is due to Myerson (1981). Hence,

$$b_w(v) \le v - \int_{\beta_w}^{v} \frac{F_s(x)}{F_w(v)}\, dx,$$

since $F_s(r(v)) = F_w(v)$. Clearly, this bound on b_w does not exceed v.

The next example illustrates the main points.

EXAMPLE: Assume bidder i, $i = s, w$, draws a type from F_i, where

$$F_i(v) = \left(\frac{v}{\alpha_i}\right)^{\gamma_i}, \quad v \in [0, \alpha_i],$$

with $\alpha_s = \frac{4}{3} > 1 = \alpha_w$ and $\gamma_s = 2 > 1 = \gamma_w$. These parameters fit with Cheng's (2006) assumptions. In this case, it is possible to solve the model analytically. Specifically, Cheng (2006) showed that bidding strategies are linear. Thus, $k_1(v) = \frac{4}{3}v$, which is also linear. The ex ante probability that bidder w wins can then be calculated to be $W(k_1) = \frac{1}{3}$. In contrast, if $r(v) = \frac{4}{3}\sqrt{v}$ is used as an upper bound, then it is possible to conclude only that bidder w wins (ex ante) with probability no greater than $W(r) = \frac{1}{2}$. This is a general conclusion when using $r(v)$ as the bound on $k_1(v)$, and shows just how loose a bound it is. Now, using the bound derived from $b_w(v) \geq b_* = 0$, (A8) yields $\overline{k}(v) = 4\frac{v}{v+2}$ and $W(\overline{k}) \approx 0.403$. If $b_w(v) \geq b_{ww}(v) = \frac{1}{2}v$ is used to derive an upper bound, the result is the solution to

$$\frac{dk_{ww}}{dv} = \frac{k_{ww}}{2}\frac{1}{v}\frac{k_{ww} - \frac{1}{2}v}{v - \frac{1}{2}v}, \quad k_{ww}(1) = \frac{4}{3},$$

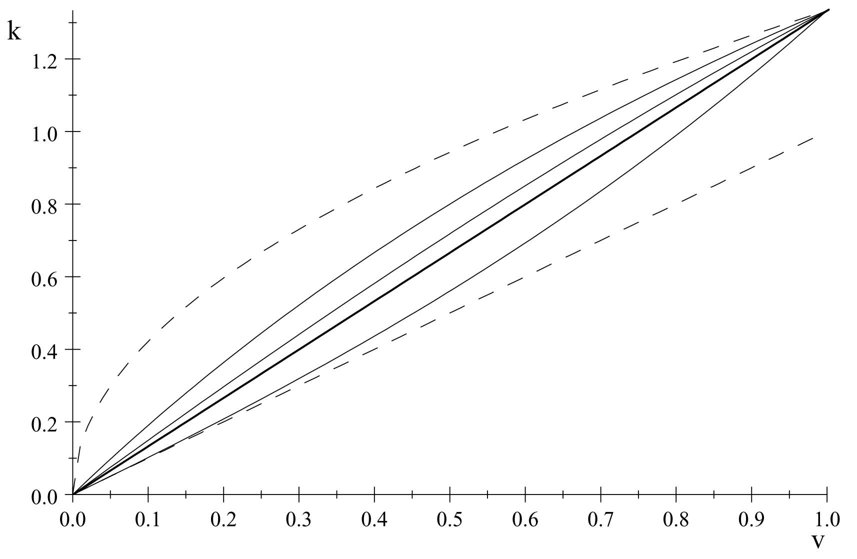

FIGURE A1.—Bounds on k_1.

which is $k_{ww}(v) = 12v(v^{3/2} + 8)^{-1}$, with $W(k_{ww}) \approx 0.360$. Turning to a lower bound, note that

$$b_w(v) \leq v - \int_0^v \frac{F_s(x)}{F_w(v)}\,dx = v - \frac{3}{16}v^2.$$

Using this in place of b_* in (A7) yields the less palatable lower bound

$$\underline{k}(v) = \left(\frac{3}{4\sqrt{v}}\frac{e^{-8/(3v)}}{e^{-8/3}} + \frac{1}{\sqrt{v}}e^{-8/(3v)}\int_v^1 \frac{8}{3x^{5/2}}e^{8/(3x)}\,dx\right)^{-1},$$

with $W(\underline{k}) \approx 0.281$. In comparison, using only $k_1(v) \geq v$ as the lower bound would yield $W(v) = \frac{3}{16} = 0.1875$. Figure A1 illustrates the bounds just derived. The dashed curves are the bounds on k_1 that exist in the current literature, namely, $r(v)$ and v, respectively. The fat line is the true k_1. The remaining curves are the two upper bounds and the lower bound just derived.

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Manuscript received February, 2011; final revision received December, 2011.