Econometrica Supplementary Material

SUPPLEMENT TO "BOOTSTRAP TESTING OF HYPOTHESES ON CO-INTEGRATION RELATIONS IN VECTOR AUTOREGRESSIVE MODELS"

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A. INTRODUCTION

THIS SUPPLEMENT CONTAINS proofs of the theoretical results stated in Cavaliere, Nielsen, and Rahbek (2015), and also contains the bootstrap theory (as well as some additional asymptotic results) for the co-integrated VAR model with an intercept. In addition, extended details and discussions (also covering models with intercept) are given for the Monte Carlo results reported in Section 4 of Cavaliere, Nielsen, and Rahbek (2015).

The supplement is organized as follows. Section B contains the extended Monte Carlo results for processes of different dimensions p, and different values of the co-integration rank r and of the lag length k. Section C contains proofs of Lemma 1, Proposition 1, and Theorem 1 of Cavaliere, Nielsen, and Rahbek (2015). Section D reports the additional theoretical results and proofs for the bootstrap test in the case of an intercept.

B. EXTENDED NUMERICAL RESULTS

In this section, we give a full presentation of the Monte Carlo results and comparisons introduced in Section 4 of Cavaliere, Nielsen, and Rahbek (2015). Accordingly, we consider the bootstrap test based on restricted parameter estimates (*bootstrap* in the following), the asymptotic likelihood ratio (LR) test (*asymptotic*), the Bartlett-corrected test (*Bartlett*), and the bootstrap test based on unrestricted parameter estimates (*unrestricted bootstrap*); see Cavaliere, Nielsen, and Rahbek (2015) for further details.

Together with the VAR(k) considered in Cavaliere, Nielsen, and Rahbek (2015), where the p-dimensional data generating process (DGP) and the statistical model are both given by

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \tag{B.1}$$

with $\varepsilon_t \sim N(0, \Omega)$, we also present results for the model with an intercept,

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t. \tag{B.2}$$

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We consider cases with lag length $k \in \{1, 2\}$, co-integration rank $r \in \{1, 2\}$, and parameter values α, β, Γ_i ($i = 1, \dots, k-1$), and μ varying as specified in Sections B.1, B.2, and B.3 below. Moreover, so as to evaluate how the dimension p of the system affects the finite-sample properties of the tests, in addition to $p = 4$, cases where $p = 2$ are also discussed. All results are reported for a 10% nominal significance level. For further details on the simulation design, see Cavaliere, Nielsen, and Rahbek (2015).

This section is organized as follows. Section B.1 considers the case of $r = 1$ and $k = 1$ in the VAR model with and without an intercept. Next, Section B.2 considers the case with $r = 2$ and explores the role of the pseudo-true rank r^* with $r^* \in \{0, 1\}$ so as to assess the behavior of the test when the null hypothesis is not true. Section B.3 considers cases with more general dynamic structures. Finally, in Section B.4, we summarize the results, compare them with what was reported in previous literature, and briefly discuss two further bootstrap implementations.

B.1. *Model With $k = 1$ and $r = 1$*

The design considered here is identical to Section 4 of Cavaliere, Nielsen, and Rahbek (2015), except that here models with an intercept term and models of dimension $p = 2$ are also covered. Accordingly, we set $\alpha = (a_1, a_2, 0, 0)'$, $\beta = (1, b_1, 0, 0)$, and $\Omega = I_4$ for $p = 4$, and $\alpha = (a_1, a_2)'$, $\beta = (1, b_1)$, and $\Omega = I_2$ for $p = 2$. In all cases, a_1, a_2, and b_1 are chosen such that process is $I(1)$ and co-integrated. Tests are considered for the hypothesis $\mathsf{H}_0 : \beta = \tau$, with $\tau = (1, 0, 0, 0)'$ for $p = 4$ and $\tau = (1, 0)'$ for $p = 2$.

We initially take the case where $p = 4$ and an intercept is added in the estimation of the model (see (B.2)), while $\mu = 0$ in the DGP, such that the generated time series do not contain linear deterministic trends; see Section 5 of Cavaliere, Nielsen, and Rahbek (2015). The results are reported in Figure B.1. We note that the problem of severe size distortions of the asymptotic test is marginally worse than for the test in the basic model (B.1) with no intercept reported in Section 4 of Cavaliere, Nielsen, and Rahbek (2015), but the relative performance of the two bootstrap tests and the Bartlett-corrected test are unchanged. That is, as for the case of no intercept, in terms of empirical size, the bootstrap test based on restricted parameter estimates is the only method that allows for a proper size control, with the other approaches showing severe size distortions. We also considered DGPs that generate linear deterministic trends in the data, using $\mu = (0, 0, c, c)'$ with $c > 0$ such that $\alpha'_\perp \mu \neq 0$. We report here the results obtained for $c = 1$, such that the simulated series contain pronounced linear trends. The results are very similar to the results in Figure B.1, hence showing that the presence of a deterministic linear trend in the DGP does not deteriorate the finite-sample size of the bootstrap test, provided an intercept is included in estimation. The same conclusion was reached when other values of c were considered.

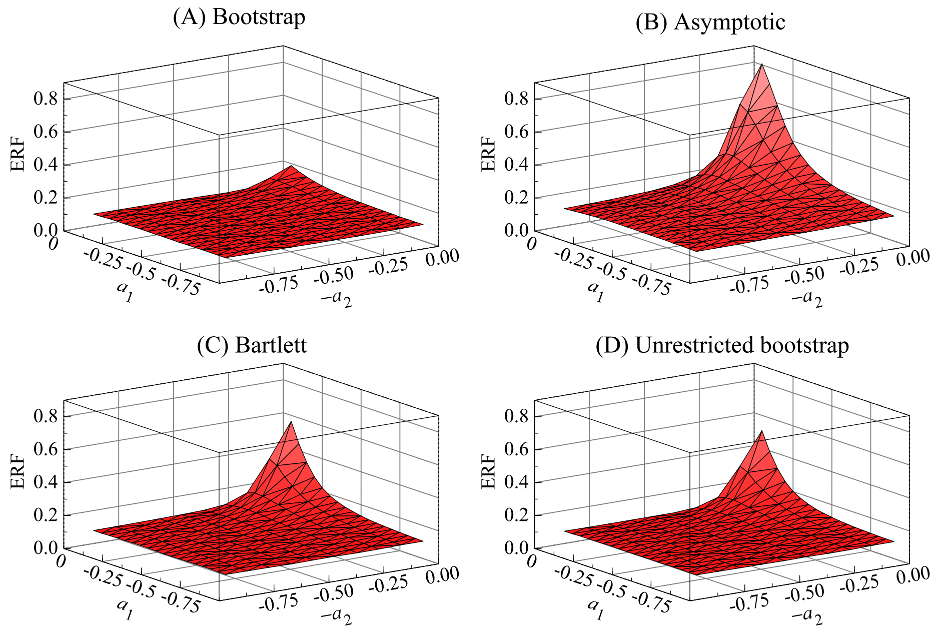

FIGURE B.1.—Empirical rejection frequencies under the null hypothesis for the different tests with $T = 100$ observations. Results are based on a 10% nominal level. The model has $p = 4$ and includes an intercept.

In terms of finite-sample power, we also report, in addition to the usual empirical rejection frequencies (ERFs), ERFs obtained after size-adjusting the tests pointwise; see Cavaliere, Nielsen, and Rahbek (2015) for further details. Figure B.2 shows the rejection frequencies for tests of the hypothesis H_0 against a sequence of DGPs (of dimension $p = 4$) with $b_1 > 0$, for three different combinations of (a_1, a_2, T): Graphs A and B for $(a_1, a_2, T) = (-0.4, 0, 60)$, Graphs C and D for $(-0.8, 0.8, 60)$, and Graphs E and F for $(-0.4, 0, 100)$. As before, an intercept is included in estimation. The left hand column reports the ERF of the tests for a nominal level of 10%, while the right hand column shows the pointwise size-adjusted ERFs. The results illustrate that the suggested bootstrap test is very close, in terms of ERFs under the alternative, to the infeasible size-adjusted asymptotic test. As for the case of no intercept discussed in Cavaliere, Nielsen, and Rahbek (2015), the reason for marginally lower (size-adjusted) ERFs of the bootstrap test under the alternative seems to be that the distribution of $Q_T^*(\tau)$ under the alternative is shifted to the right with respect to the asymptotic (χ^2) null distribution; see Theorem 1 and Remark 3.4. Finally, it is worth noting that the size-adjusted power in the case of an intercept is overall lower than for the basic model discussed in Cavaliere, Nielsen, and Rahbek (2015), but the relative performance of the proposed bootstrap test,

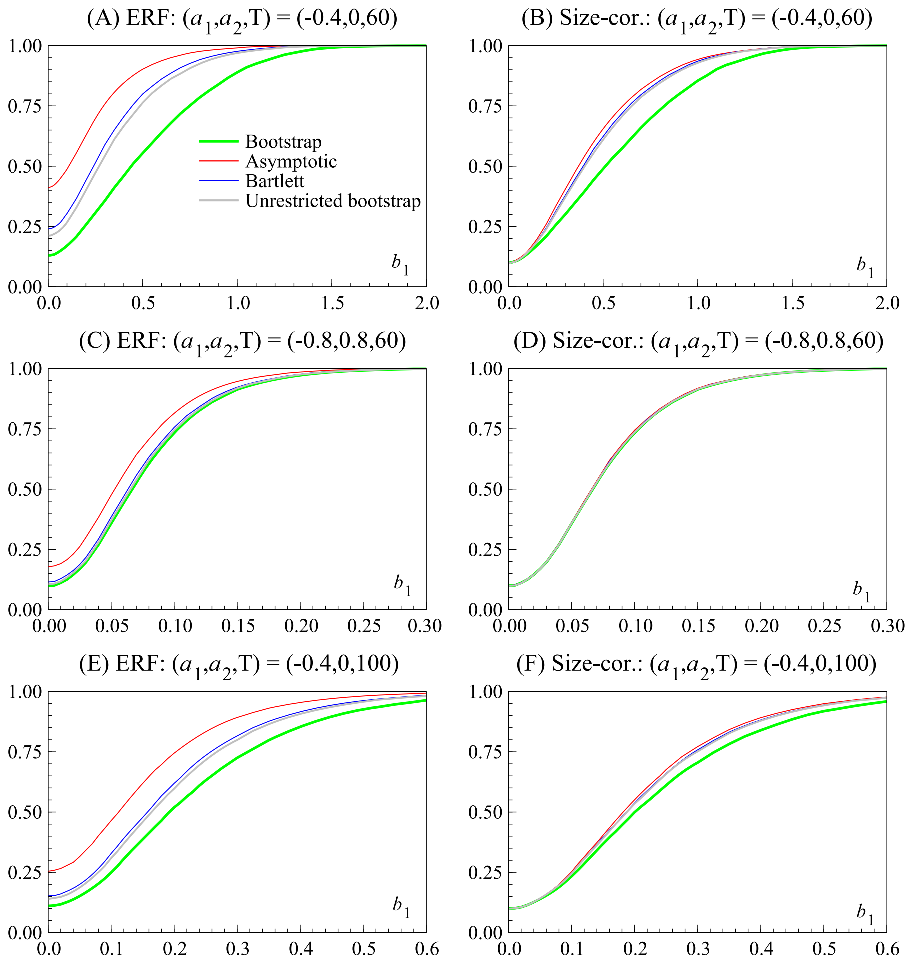

FIGURE B.2.—Rejection frequencies for the hypothesis $\mathsf{H}_0 : \beta = \tau := (1, 0, 0, 0)'$ for a sequence of DGPs defined by $\beta = (1, b_1, 0, 0)'$, with $b_1 > 0$. (A), (C), and (E) show empirical rejection frequencies for a nominal level of 10%, whereas (B), (D), and (F) show rejection frequencies that are pointwise size corrected. The model has $p = 4$ and includes an intercept.

as compared to the other approaches, is unchanged. Almost identical results prevail for the trending case where $c > 0$.

BEHAVIOR AS A FUNCTION OF T: To illustrate the finite-sample behavior as a function of the number of observations T, Figure B.3(A) shows the ERFs of the four tests under the null hypothesis for T ranging from 40 to 1000. We consider the case $(a_1, a_2, b_1) = (-0.4, 0, 0)$. As before, we report the case of $p = 4$ and intercept included in estimation (see (B.2)), but not in the DGP ($\mu = 0$);

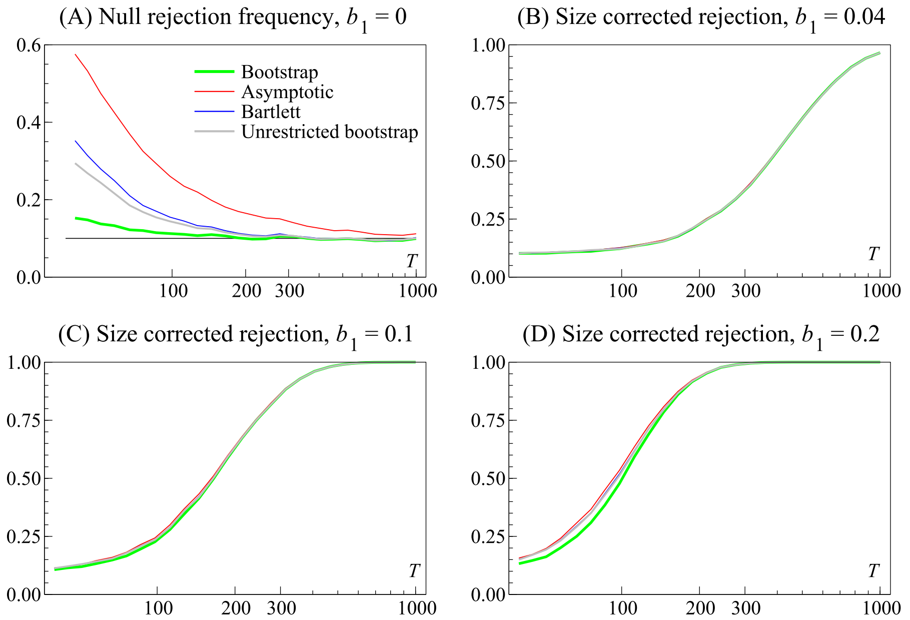

FIGURE B.3.—Empirical rejection frequencies as a function of T under the null and under the alternative hypotheses. (A) shows rejection frequencies under the null. (B), (C), and (D) show pointwise size-corrected rejection frequencies under the alternative. In all simulations $(a_1, a_2) = (-0.4, 0.0)$ and T varies between 40 and 1000. The model has $p = 4$ and includes an intercept.

results for the case $\mu \neq 0$ are identical. As before, the proposed bootstrap displays excellent size control, while the asymptotic test, the Bartlett-corrected test, and the unrestricted bootstrap are all subject to severe size distortions for samples of small and even moderate sizes.

Figure B.3(B)–(D) shows rejection frequencies under H_1, that is, when the DGP has $\beta = (1, b_1, 0, 0)'$ for $b_1 \in \{0.04, 0.1, 0.2\}$ (as before, the ERFs are pointwise size adjusted). For small deviations from the null, the rejection frequencies of the proposed bootstrap are indistinguishable from the asymptotic test, while they are only marginally lower for larger deviations from the null.

IMPACT OF THE VAR DIMENSION: Results obtained for $p = 2$ do not contrast with those obtained for $p = 4$ discussed above. In terms of ERF under the null hypothesis (not reported), size distortions for $p = 2$ are less pronounced than for $p = 4$, as expected. Despite this, our bootstrap allows for a proper size control over the entire parameter space, with the other approaches still showing large size distortions.

So as to evaluate the implication for power of the VAR dimension p, we show in Figure B.4 the same results reported earlier in Figure B.2, Graphs

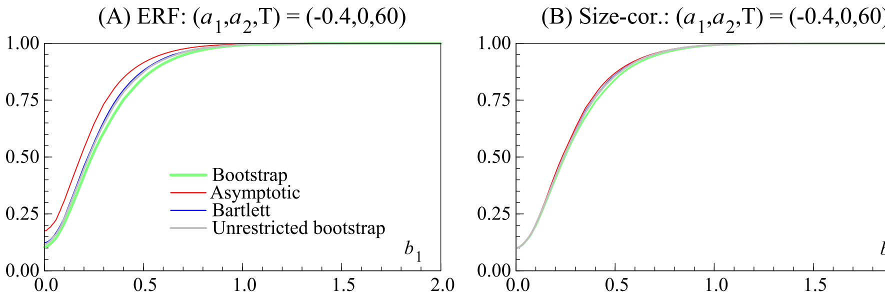

FIGURE B.4.—Rejection frequencies for the case with $p = 2$, for the hypothesis $\mathsf{H}_0 : \beta = \tau := (1, 0)'$ for a sequence of DGPs defined by $\beta = (1, b_1)'$, with $b_1 > 0$. (A) shows empirical rejection frequencies for a nominal level of 10%. (B) shows rejection frequencies that are pointwise size corrected. The model includes an intercept.

A and B, but now setting $p = 2$. In this case, the power loss is smaller than for $p = 4$, and the power of the proposed bootstrap test virtually coincides with the infeasible size-corrected power of the asymptotic test. Given this evidence, we may conjecture that under the alternative, the distribution of $Q_T^*(\tau)$ is more shifted to the right for large values of p relative to small values of p. Nonetheless, in both cases considered, the effect on power of such shifts seems negligible.

B.2. *Model With $k = 1$ and $r = 2$*

To discuss the case with $r = 2$, and the importance of r^* in particular, we consider the DGP in (B.1) with $k = 1$,

$$\text{(B.3)} \qquad \alpha' = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \end{pmatrix}, \quad \beta' = \begin{pmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \end{pmatrix}, \quad \Omega = I_4,$$

and investigate the hypothesis

$$\text{(B.4)} \qquad \mathsf{H}_0 : \beta = \tau = (\tau_1, \tau_2), \quad \tau' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Letting $T = 100$, we first consider the case $(a_1, a_2, b_1, b_2) = (-0.1, -0.1, 0, b)$ for various values of b. The null is true if $b = 0$, while $b \neq 0$ corresponds to a point in the alternative. In this case, $\tau_1 \in \text{span}(\beta)$ and $r^* = 1$. The ERFs and the corresponding pointwise size-corrected rejections are shown in Figure B.5(A) and (B). First, note that the size properties of the asymptotic test are unreliable, with ERFs around 50%. The proposed bootstrap test offers an excellent size control, whereas the Bartlett correction and the unrestricted bootstrap are also unreliable, having ERFs around 25%. As for the case with $r = 1$, the size-corrected results for ERFs under the alternative hypotheses in Figure B.5(B) indicate only a minor loss of power.

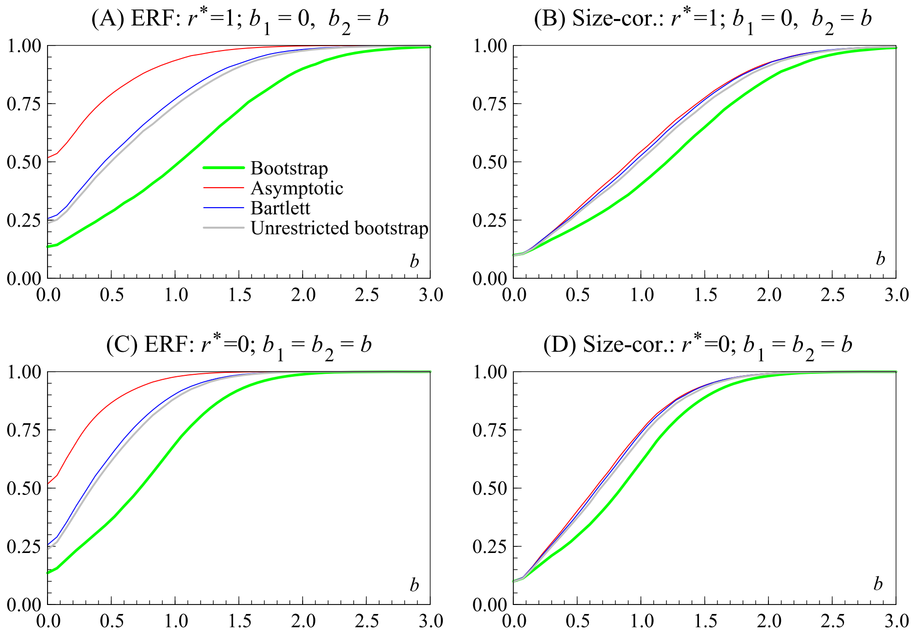

FIGURE B.5.—Empirical rejection frequencies for the case $r = 2$. (A) and (C) show rejection frequencies for a nominal level of 10%. (B) and (D) show pointwise size-corrected rejection frequencies. The model has $p = 4$.

Figure B.5(C) and (D) shows similar results for $(a_1, a_2, b_1, b_2) = (-0.1, -0.1, b, b)$. Here r^* equals 0 for $b \neq 0$. First, with respect to the case where $r^* = 1$, the power of all tests is now higher. This is reasonably expected, since when $r^* = 0$, the true β is now completely orthogonal to τ. Second, the conclusions regarding the power of the bootstrap test relatively to the asymptotic test appear to be identical to the previous case of $r^* = 1$, hence indicating that the presence of extra (local-to-) unit roots in the bootstrap sample makes little or no difference in the performance of the bootstrap test.

We conclude this section by noticing that the results for the case of intercept (see (B.2)) do not substantially differ from those reported here.

B.3. *Model With $k = 2$ and $r = 1$*

We finally consider the case $k = 2$, with the aim of assessing the behavior of the tests under a more general dynamic structure. We focus on the DGP in (B.1) with $p = 4$, $r = 1$, and $\alpha' = (a_1, a_2, 0, 0)$, $\beta' = (1, 0, 0, 0)$, and $\Omega = I_4$, with $(a_1, a_2) = (-0.2, 0.2)$. We consider 100 randomly chosen points in the parameter space. Specifically, each entry in Γ_1 is drawn from a uniform random variable on $[-1, 1]$. If the $I(1, r)$ rank conditions are satisfied for the chosen configuration of parameters, we proceed to examine the test behavior. Notice

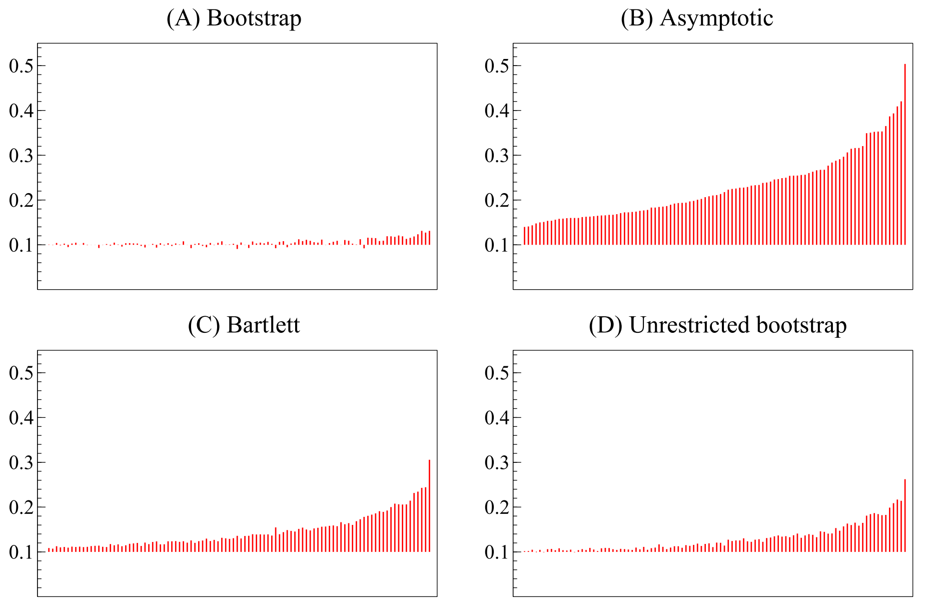

FIGURE B.6.—Empirical rejection frequencies for the case $r = 1$ and $k = 2$. The horizontal axis indexes 100 different DGPs with randomly chosen coefficients, all satisfying the $I(1, r)$ conditions. Nominal level of 10%. The model has $p = 4$.

that, as in the previous section, the intercept term does not affect the results; hence, we only report results for the case of no intercept.

The ERFs under the null hypothesis are reported in Figure B.6, where the results are sorted by the ERF of the asymptotic test. We note again that the proposed bootstrap test has an excellent size control in all cases, with ERF close to the nominal 10%. The asymptotic test, on the other hand, shows ERFs between 15% and 50%. The Bartlett-corrected test and the unrestricted bootstrap test reduce the size distortion but remain oversized.

B.4. *Summary of Results and Relation to Existing Literature*

Previous simulation studies of bootstrap tests on co-integrating relations in VAR models include Fachin (2000), Gredenhoff and Jacobson (2001), and Omtzigt and Fachin (2006). Compared to these, the Monte Carlo simulation study reported here differs substantially. First, we provide an exhaustive and detailed systematic comparison of the bootstrap tests based on restricted parameter estimates with the bootstrap based on unrestricted estimates, with Bartlett-corrected tests, and with the asymptotic tests. So as to discuss and compare power or, more generally, the properties of the tests under the alternative hypothesis, we also consider—in addition to the usual empirical rejection frequencies—*size-adjusted* power. Most important, with respect to the previous studies, our simulation design made it possible to consider a much larger portion of the parameter space. Finally, our study is the first where a comparison between the cases of models with no deterministic components and models with an intercept term is considered.

More specifically, Fachin (2000) considers empirical size and power of a bootstrap version of the Wald test, using a bootstrap generating process (BGP) based on restricted estimates and i.i.d. resampling of unrestricted residuals. Gredenhoff and Jacobson (2001) consider size properties for a bootstrap test based on restricted parameter estimates, and with bootstrap innovations ε_t^* not based on i.i.d. resampling, but instead drawn from a Gaussian distribution with covariance matrix $\tilde{\Omega}$. Finally, Omtzigt and Fachin (2006) compare the aforementioned tests with focus on the *unrestricted* bootstrap.

Although the simulations are not fully comparable, partly because the bootstrap algorithms considered differ and partly because the simulation designs do not overlap, our findings in terms of size properties seem to *reinforce* previous results. In particular, (i) the bootstrap offers a clear improvement over the asymptotic test, (ii) the size control of the bootstrap based on restricted estimates is very satisfactory, and (iii) the unrestricted bootstrap and the Bartlett correction do not correct the large finite-sample distortions documented for the asymptotic test.

In terms of power, previous results were mostly based on very specific points in the alternative and, moreover, did not consider size-adjusted power. Conversely, in our simulation study, we were able to show the key fact that the empirical power of our bootstrap test coincides with—or is only slightly lower than—the power of the infeasible size-adjusted asymptotic test.

Overall, these results complement the theory in Theorem 1 of Cavaliere, Nielsen, and Rahbek (2015), where the asymptotic validity of our proposed bootstrap is established.

We conclude this section by briefly discussing two further bootstrap algorithms that were considered in this study but are not reported here (these supplementary results are available from the authors upon request). The first is the *hybrid bootstrap* algorithm mentioned in Remark 3.13 of Cavaliere, Nielsen, and Rahbek (2015). Although this algorithm is not valid in general in the sense that it may, for example, generate (limiting) explosive roots for the bootstrap process, we investigated its finite-sample properties in those cases where the algorithm is valid; that is, for specific regions of the parameter space where indeed explosive roots can be excluded. We found that this bootstrap has properties analogous to the unrestricted bootstrap (with only marginally better size and marginally worse power).

The second algorithm combines our suggested bootstrap algorithm (based on restricted parameter estimates) with i.i.d. resampling from the *unrestricted* residuals. That is, in Step (ii) of Algorithm 1 in Cavaliere, Nielsen, and Rahbek (2015), the T bootstrap errors ε_t^* are instead obtained by i.i.d. resampling of the re-centered *unrestricted* residuals, $\hat{\varepsilon}_t^c := \hat{\varepsilon}_t - T^{-1} \sum_{i=1}^{T} \hat{\varepsilon}_t$, with $\hat{\varepsilon}_t$ as defined in Section 2 of Cavaliere, Nielsen, and Rahbek (2015). The idea behind this bootstrap scheme is that the restricted residuals $\tilde{\varepsilon}_t$ are expected to have larger variation than the unrestricted residuals $\hat{\varepsilon}_t$ when the null does not hold. However, since Algorithm 1 is based on i.i.d. resampling and the likelihood ratio statistic is invariant to scaling, one would expect the two bootstrap implementations to lead to similar results. Indeed, unreported simulations showed that there are no discernible differences in the finite-sample properties of the two approaches.

C. PROOFS OF LEMMA 1, PROPOSITION 1, AND THEOREM 1

Sections C.1–C.3 provide the proofs of Lemma 1, Proposition 1, and Theorem 1 of Cavaliere, Nielsen, and Rahbek (2015).

ADDITIONAL NOTATION: In addition to notation introduced in Cavaliere, Nielsen, and Rahbek (2015), the following notation is used. We use wlim and plim to denote weak convergence and convergence in probability, respectively, as $T \to \infty$. For any $m \times n$ matrix a, we define $a^{\otimes 2} := aa'$; if $m = n$, $\rho(a)$ denotes its spectral radius (that is, the maximal modulus of the eigenvalues of a). We shall also use $K_n^{(m)} := (I_m, 0_{m\times(n-m)})'$ for $n \geq m$, which acts as a selection matrix. Finally we use the definitions $\Sigma_{\beta\beta} := \operatorname{plim} \beta_0' S_{11}\beta_0$, $\Sigma_{0\beta} := \operatorname{plim} S_{01}\beta_0$, and $\Sigma_{00} := \operatorname{plim} S_{00}$.

C.1. *Proof of Lemma 1*

To prove the lemma we proceed as follows. First, we derive explicit expressions for $\operatorname{plim} \tilde{\Pi} =: \Pi_0^* = \alpha_0^*\beta_0^{*\prime}$ (showing that α_0^* and β_0^* are $p \times r^*$-dimensional matrices of rank r^*), $\operatorname{plim} \tilde{\Psi} =: \Psi_0^*$, and $\operatorname{plim} \tilde{\Omega} =: \Omega_0^*$. Next, we show that the DGP for X_t can be rewritten as $\Delta X_t = \alpha_0^*\beta_0^{*\prime} X_{t-1} + \Psi_0^* \Delta\mathbb{X}_{2t} + e_t$, with the key property being that the pseudo-innovations e_t are uncorrelated with both $\beta_0^{*\prime} X_{t-1}$ and $\Delta\mathbb{X}_{2t}$. This is then explored further to establish that $\{\alpha_0^*, \beta_0^*, \Psi_0^*\}$ satisfy the $I(1, r^*)$ conditions.

Observe that as $\beta_0\phi \in \operatorname{span}(\tau)$, then $\tau\xi = \beta_0\phi$ for some ξ of dimension $(r_0 \times r^*)$. Thus the r^* linear combinations $\xi'\tau' X_t$ are stationary, while the remaining combinations, $\xi_\perp'\tau' X_t$, are integrated of order 1, or $I(1)$. With $D_T := \operatorname{diag}(\xi, \bar{\xi}_\perp T^{-1/2})$, then

$$D_T'\tau' S_{11}\tau D_T = \operatorname{diag}\left(\phi'\Sigma_{\beta\beta}\phi, \int_0^1 GG'\,du\right) + o_p(1)$$

as $T \to \infty$, and where $G := \bar{\xi}_\perp'\tau' C_g \operatorname{wlim}(T^{-1/2}\sum_{t=1}^{\lfloor T\cdot\rfloor} \alpha_{0\perp}'\varepsilon_t)$ and

$$C_g := \beta_{0\perp}\left(\alpha_{0\perp}'\Gamma_0\beta_{0\perp}\right)^{-1}. \tag{C.1}$$

Likewise, $S_{01}\tau D_T \xrightarrow{p} (\Sigma_{0\beta}\phi, 0_{p\times(r_0-r^*)})$. Collecting terms yields

$$\begin{aligned}\text{(C.2)}\qquad \tilde{\alpha}\tau' &= \tilde{\Pi} = S_{01}\tau D_T\big(D_T'\tau' S_{11}\tau D_T\big)^{-1} D_T'\tau' \\ &\xrightarrow{p} \Pi_0^* = \alpha_0^*\beta_0^{*\prime} := \Sigma_{0\beta}\phi\big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1}\phi'\beta_0',\end{aligned}$$

as desired. Since $\Sigma_{0\beta} = \alpha_0\Sigma_{\beta\beta}$ under the $I(1, r_0)$ conditions, this implies that we can choose the pseudo-true co-integration parameters as

$$\text{(C.3)}\qquad \beta_0^* = \beta_0\phi \quad \text{and} \quad \alpha_0^* = \alpha_0\Sigma_{\beta\beta}\phi\big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1}.$$

Let $\kappa_\phi := \Sigma_{\beta\beta}\phi(\phi'\Sigma_{\beta\beta}\phi)^{-1}$ and $\kappa_{\phi_\perp} := \phi_\perp(\phi_\perp'\Sigma_{\beta\beta}^{-1}\phi_\perp)^{-1}$, and define the skew projection

$$\text{(C.4)}\qquad I_{r_0} = \kappa_\phi\phi' + \big(I_{r_0} - \kappa_\phi\phi'\big) = \kappa_\phi\phi' + \kappa_{\phi_\perp}\phi_\perp'\Sigma_{\beta\beta}^{-1}.$$

Next, we show that $\tilde{\Psi}$ and $\tilde{\Omega}$ converge, respectively, to the pseudo-true parameters Ψ_0^* and Ω_0^* given by

$$\text{(C.5)}\qquad \Psi_0^* = \Psi_0 + \alpha_0\kappa_{\phi_\perp}\phi_\perp'\Sigma_{\beta\beta}^{-1}Y_{\beta 2}Y_{22}^{-1} \quad \text{and} \quad \Omega_0^* = \Omega_0 + \alpha_0\kappa_{\phi_\perp}\phi_\perp'\alpha_0',$$

with $Y_{\beta 2} := \operatorname{plim}\beta_0' M_{12}$ and $Y_{22} := \operatorname{plim} M_{22}$. Observe that

$$\begin{aligned}\tilde{\Psi} &= M_{02}M_{22}^{-1} - \tilde{\alpha}\tau' M_{12}M_{22}^{-1} \\ &\xrightarrow{p} \Psi_0^* := Y_{02}Y_{22}^{-1} - \alpha_0\Sigma_{\beta\beta}\phi\big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1}\phi' Y_{\beta 2}Y_{22}^{-1},\end{aligned}$$

such that (C.5) holds by using the identity $\Psi_0 = Y_{02}Y_{22}^{-1} - \alpha_0 Y_{\beta 2}Y_{22}^{-1}$. Using the pseudo-true parameters α_0^*, β_0^*, and Ψ_0^*, we can rewrite the equation for ΔX_t as

$$\text{(C.6)}\qquad \Delta X_t = \alpha_0^*\beta_0^{*\prime}X_{t-1} + \Psi_0^*\Delta\mathbb{X}_{2t} + e_t,$$

and the pseudo-innovations e_t are defined by

$$\text{(C.7)}\qquad e_t := \varepsilon_t + \alpha_0\kappa_{\phi_\perp}\phi_\perp'\Sigma_{\beta\beta}^{-1}\big(\beta_0' X_{t-1} - Y_{\beta 2}Y_{22}^{-1}\Delta\mathbb{X}_{2t}\big).$$

Observe that by definition, $\operatorname{Var}(e_t) = \Omega_0^*$ with also

$$\begin{aligned}\text{(C.8)}\qquad \tilde{\Omega} &= \frac{1}{T}\sum_{t=1}^{T}\big(\Delta X_t - \tilde{\alpha}\tau' X_{t-1} - \tilde{\Psi}\Delta\mathbb{X}_{2t}\big)^{\otimes 2} \\ &\xrightarrow{p} \Omega_0 + \alpha_0\phi_\perp\big(\phi_\perp'\Sigma_{\beta\beta}^{-1}\phi_\perp\big)^{-1}\phi_\perp'\alpha_0' = \Omega_0^*.\end{aligned}$$

Here it has been used that e_t is uncorrelated with $\beta_0^{*\prime} X_{t-1}$ and ΔX_{2t}. To see this, observe that by (C.7) and the definition of $\kappa_{\phi_\perp}$ in (C.4),

$$E\big(e_t X_{t-1}' \beta_0^*\big) = E\big(\big(\varepsilon_t + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1}\big(\beta_0' X_{t-1} - \Upsilon_{\beta 2} \Upsilon_{22}^{-1} \Delta \mathbb{X}_{2t}\big)\big) X_{t-1}' \beta_0^*\big)$$
$$= \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1}\big(\Upsilon_{\beta\beta} - \Upsilon_{\beta 2} \Upsilon_{22}^{-1} \Upsilon_{2\beta}\big)\phi = 0,$$

where $\Upsilon_{\beta\beta} = \operatorname{plim} \beta_0' M_{11} \beta_0$ and we have used $\Sigma_{\beta\beta} = \Upsilon_{\beta\beta} - \Upsilon_{\beta 2} \Upsilon_{22}^{-1} \Upsilon_{2\beta}$. Likewise, $E(e_t \Delta \mathbb{X}_{2t}') = \alpha_0 (\Upsilon_{\beta 2} - \Upsilon_{\beta 2}) = 0$.

To see that the pseudo-true parameters $\{\alpha_0^* \beta_0^{*\prime}, \Psi_0^*\}$ satisfy the $I(1, r^*)$ conditions, observe first that, by definition, $\Pi_0^* = \alpha_0^* \beta_0^{*\prime}$ has rank r^*. Next, with $\alpha_0^* = \alpha_0 \kappa_\phi$ and $\beta_0^* = \beta_0 \phi$, we can set

$$\alpha_{0\perp}^* := \big(\alpha_{0\perp}, \bar{\alpha}_0 \Sigma_{\beta\beta}^{-1} \phi_\perp\big), \quad \beta_{0\perp}^* := (\beta_{0\perp}, \bar{\beta}_0 \phi_\perp). \tag{C.9}$$

With $\mathbb{X}_t := (X_t', X_{t-1}', \ldots, X_{t-k+1}')'$, rewrite the system in companion form as

$$\Delta \mathbb{X}_t = \mathbb{A}^* \mathbb{B}^{*\prime} \mathbb{X}_{t-1} + \mathbb{E}_t, \tag{C.10}$$

where $\mathbb{E}_t := (e_t', 0, \ldots, 0)'$, $\mathbb{X}_0$ is fixed, and

$$\mathbb{A}^* := \begin{pmatrix} \alpha_0^* & \Psi_0^* \\ 0 & I_{p(k-1)} \end{pmatrix}, \quad \mathbb{B}^* := \begin{pmatrix} \beta_0^* & I_p & 0 & \cdots & 0 \\ 0 & -I_p & I_p & \cdots & 0 \\ 0 & 0 & -I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ 0 & 0 & 0 & \cdots & -I_p \end{pmatrix}.$$

By assumption, $\mathbb{Y}_t := \mathbb{B}^{*\prime} \mathbb{X}_t = (X_t' \beta_0^*, \Delta X_t', \ldots, \Delta X_{t-k+1}')'$ is covariance stationary with covariance $\Sigma_{YY}^* > 0$ and solves

$$\Sigma_{YY}^* = \Phi^* \Sigma_{YY}^* \Phi^{*\prime} + \Sigma_{EE}^*, \tag{C.11}$$

where $\Phi^* = (I_{r^* + p(k-1)} + \mathbb{B}^{*\prime} \mathbb{A}^*)$ and $\Sigma_{EE}^* := \operatorname{Var}(\mathbb{B}^{*\prime} \mathbb{E}_t)$. Now, by definition $\Sigma_{EE}^* \geq 0$ and as Φ^* satisfies (C.11), it follows that $\rho(\Phi^*) < 1$ as in Cavaliere, Rahbek, and Taylor (2012, p. 1735). Finally, the roots of the characteristic polynomial $A^*(z)$, $z \in C$, that correspond to (C.10) are found by solving $\det(A^*(z)) = 0$, with

$$A^*(z) := (1 - z) I_{pk} - \mathbb{A}^* \mathbb{B}^{*\prime} z.$$

Now $\mathbb{B}^{*\prime} \mathbb{A}^* = \Phi^* - I_{r^* + p(k-1)}$ and, hence, $\det(\mathbb{B}^{*\prime} \mathbb{A}^*) \neq 0$ since $\rho(\Phi^*) < 1$. With $\mathbb{A}_\perp^* = (I_p, -\Psi_0^*)' \alpha_{0\perp}^*$, this implies that $M_L := (\mathbb{B}^*, \mathbb{A}_\perp^*)$ and, hence, $M_R := (\mathbb{A}^*, \mathbb{B}_\perp^*)$, have full rank, where $\mathbb{B}_\perp^* = (\beta_{0\perp}^{*\prime}, \ldots, \beta_{0\perp}^{*\prime})'$. Multiplying from the left by $\det(M_L')$ and from the right by $\det(M_R)$ shows that the roots of $A^*(z)$ satisfy

$$\det\big(I_{r^*+p(k-1)} - \Phi^* z\big)\det\big(I_{p-r^*}(1-z)\big) = 0,$$

such that the $I(1, r^*)$ conditions apply. *Q.E.D.*

C.2. *Proof of Proposition 1*

According to Algorithm 1, the bootstrap generating process for X_t^* is given by

$$\Delta X_t^* = \tilde{\alpha}\tau' X_{t-1}^* + \tilde{\Psi}\Delta\mathbb{X}_{2t}^* + \varepsilon_t^*,$$

where $\Delta\mathbb{X}_{2t}^* = (\Delta X_{t-1}^{*\prime}, \dots, \Delta X_{t-k+1}^{*\prime})'$ and $\tilde{\Psi} = (\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_{k-1})$. Similar to the companion form in the proof of Lemma 1 (see (C.10)), we may write this as

$$\text{(C.12)} \qquad \Delta\mathbb{X}_t^* = \tilde{A}\tilde{B}'\mathbb{X}_{t-1}^* + \mathbb{E}_{xt}^*,$$

where $\Delta\mathbb{X}_t^* := (\Delta X_t^{*\prime}, \dots, \Delta X_{t-k+1}^{*\prime})'$, $\mathbb{X}_{t-1}^* := (X_{t-1}^{*\prime}, \dots, X_{t-k}^{*\prime})'$, $E_{xt}^* = (\varepsilon_t^{*\prime}, 0, \dots, 0)'$, and

$$\tilde{A} := \begin{pmatrix} \tilde{\alpha} & \tilde{\Psi} \\ 0 & I_{p(k-1)} \end{pmatrix}, \qquad \tilde{B} := \begin{pmatrix} \tau & I_p & 0 & \cdots & 0 \\ 0 & -I_p & I_p & \cdots & 0 \\ 0 & 0 & -I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ 0 & 0 & 0 & \cdots & -I_p \end{pmatrix}.$$

Next, use that by (C.10) we have, from the proof of Lemma 1,

$$\mathbb{A}_\perp^* := \big(I_p, -\Psi_0^*\big)'\alpha_{0\perp}^*, \qquad \mathbb{B}_\perp^* := \big(\beta_{0\perp}^{*\prime}, \dots, \beta_{0\perp}^{*\prime}\big)',$$

with $\alpha_{0\perp}^*$ and $\beta_{0\perp}^*$ as defined in (C.9). In terms of these companion form parameters we next rotate X_t^* and define Z_t^* by

$$\text{(C.13)} \qquad \mathbb{Z}_t^* := Q_z'\mathbb{X}_t^*, \qquad Q_z := \big(\mathbb{B}^*, \mathbb{A}_\perp^*\big).$$

By (C.12), Z_t^* satisfies, with $E_{zt}^* = Q_z' E_{xt}^*$,

$$\Delta\mathbb{Z}_t^* = \left(a_T b' + \frac{1}{T}c_T b_\perp'\right)\mathbb{Z}_{t-1}^* + \mathbb{E}_{zt}^*.$$

Here $b := K_{pk}^{(r^*+(k-1)p)}$ such that $b_\perp = (0_{(p-r^*)\times(r^*+(k-1)p)}, I_{p-r^*})'$ and

$$a_T := Q_z'\tilde{A}\tilde{B}'\mathbb{A}^*\big(\mathbb{B}^{*\prime}\mathbb{A}^*\big)^{-1}, \qquad \frac{1}{T}c_T := Q_z'\tilde{A}\tilde{B}'\mathbb{B}_\perp^*\big(\mathbb{A}_\perp^{*\prime}\mathbb{B}_\perp^*\big)^{-1}.$$

Observe that $a_T b'$ is of rank $r^* + (k-1)p$, corresponding to $\mathbb{B}^*$, while $c_T b'_\perp$ is of rank $(p - r^*)$, corresponding to $\mathbb{A}^*_\perp$. Also $\mathbb{A}^{*\prime}_\perp \mathbb{B}^*_\perp = \alpha^{*\prime}_{0\perp} \Gamma^*_0 \beta^*_{0\perp}$, with $\Gamma^*_0 := (I_p - \sum_{i=0}^{k-1} \Gamma^*_{0,i})$.

Moreover, as will be shown next, as $T \to \infty$, $a_T, c_T = O_p(1)$ with

$$a_T \xrightarrow{p} a := \begin{pmatrix} \mathbb{B}^{*\prime}\mathbb{A}^* \\ 0_{(p-r^*)\times(r^*+(k-1)p)} \end{pmatrix}, \tag{C.14}$$

$$c_T \xrightarrow{w} c := Q'_z \begin{pmatrix} N\bar{\xi}'_\perp \tau' \beta^*_{0\perp} \\ 0_{(k-1)p\times(p-r^*)} \end{pmatrix} \big(\alpha^{*\prime}_{0\perp} \Gamma^*_0 \beta^*_{0\perp}\big)^{-1}, \tag{C.15}$$

where N is defined below in (C.17).

Consider first a_T. From Lemma 1 and by definition of $\tilde{A}$ and $\tilde{B}$, it follows that $\tilde{A}\tilde{B}' \xrightarrow{p} \mathbb{A}^*\mathbb{B}^{*\prime}$ and the result follows by simple insertion.

Consider next c_T. Observe initially that

$$\tilde{A}\tilde{B}'\mathbb{B}^*_\perp = \begin{pmatrix} \tilde{\alpha}\tau'\beta^*_{0\perp} \\ 0_{(k-1)p\times(p-r^*)} \end{pmatrix},$$

such that we can focus on the limiting behavior of $T\tilde{\alpha}\tau'\beta^*_{0\perp}$. As $\beta^*_0 = \beta_0\phi = \tau\xi$ and $\tilde{\alpha} = S_{01}\tau S^{-1}_{\tau\tau}$, we find with $D_T := (\xi, \bar{\xi}_\perp/\sqrt{T})$ and $\beta^*_{0\perp} = (\beta_{0\perp}, \bar{\beta}_0\phi_\perp)$ that

$$\begin{aligned} T\tilde{\alpha}\tau'\beta^*_{0\perp} &= TS_{01}\tau D_T\big(D'_T\tau' S_{11}\tau D_T\big)^{-1} D'_T \tau'\beta^*_{0\perp} \\ &= \sqrt{T} S_{01}\tau D_T \begin{pmatrix} \big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1} & 0 \\ 0 & \left(\int_0^1 GG'\,du\right)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \bar{\xi}'_\perp \tau'\beta^*_{0\perp} \end{pmatrix} \\ &\quad + o_p(1) \\ &= \sqrt{T} S_{01}\tau D_T \left(0_{(p-r^*)\times r^*}, \beta^{*\prime}_{0\perp}\tau\bar{\xi}_\perp \left(\int_0^1 GG'\,du\right)^{-1}\right)' + o_p(1), \end{aligned} \tag{C.16}$$

where the $(r_0 - r^*)$-dimensional process G is defined in the proof of Lemma 1. We thus find

$$\begin{aligned} \operatorname{wlim}\big(T\tilde{\alpha}\tau'\beta^*_{0\perp}\big) &= \operatorname{wlim}(S_{01}\tau\bar{\xi}_\perp)\left(\int_0^1 GG'\,du\right)^{-1} \bar{\xi}'_\perp\tau'\beta^*_{0\perp} \\ &= N\big(\beta^{*\prime}_{0\perp}\tau\bar{\xi}_\perp\big)', \end{aligned} \tag{C.17}$$

where $N := \operatorname{wlim}((S_{\varepsilon\tau} + \alpha_0\beta'_0 S_{10}\tau)\bar{\xi}_\perp)(\int_0^1 GG'\,du)^{-1}$ of dimension $p \times (r_0 - r^*)$ is well defined as $\bar{\xi}'_\perp\tau' X_t$ is integrated of order 1 and classic convergence to stochastic integrals as in Hansen (1992) applies. Observe that as $\beta^{*\prime}_{0\perp}\tau\bar{\xi}_\perp$ is $(p - r^*) \times (r_0 - r^*)$, then the $p \times (p - r^*)$-dimensional $\operatorname{wlim}(T\tilde{\alpha}\tau'\beta^*_{0\perp})$ is of reduced rank $(r_0 - r^*)$. Finally, by simple insertion, we find the desired expression for c.

Turn again to the error correction process Z_t^* in (C.13), which, with $Z_0^* = 0$ without loss of generality, we can write as

$$\mathbb{Z}_t^* = \sum_{j=1}^{t} \left(I_{pk} + a_T b' + \frac{1}{T} c_T b'_\perp \right)^{t-j} \mathbb{E}_{zj}^*, \tag{C.18}$$

such that with $V_T(\cdot) := \sum_{j=1}^{\lfloor T\cdot \rfloor} \mathbb{E}_{zj}^*$,

$$\mathbb{Z}_{\lfloor Tu \rfloor}^* = \int_0^u \left(I_{pk} + a_T b' + \frac{1}{T} c_T b'_\perp \right)^{\lfloor Tu \rfloor - \lfloor Ts \rfloor} dV_T(s).$$

By Theorem A.14 in Johansen (1995) combined with the convergence of a_T and b_T established above,

$$\left(I_{pk} + a_T b' + \frac{1}{T} c_T b'_\perp \right)^T \xrightarrow{w} \exp\bigl(Bcb'_\perp\bigr)B, \tag{C.19}$$

where $B = b_\perp (a'_\perp b_\perp)^{-1} a'_\perp$ with $a_\perp = b_\perp$.[1] Hence, by definition the expression for B simplifies to $B = b_\perp b'_\perp$. Also, with $\pi^* := b'_\perp c$, using (C.15) and (C.17),

$$\pi^* := \alpha_{0\perp}^{*\prime} N \bar{\xi}'_\perp \tau' \beta_{0\perp}^* \bigl(\alpha_{0\perp}^{*\prime} \Gamma_0^* \beta_{0\perp}^*\bigr)^{-1} = \alpha_{0\perp}^{*\prime} N \bar{\xi}'_\perp \tau' C_z^*. \tag{C.20}$$

Moreover, by definition, $b'_\perp V_T(\cdot) = \alpha_{0\perp}^{*\prime} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t^*$. As in Cavaliere, Rahbek, and Taylor (2012, proof of Proposition 1), we have that $T^{-1/2} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t^* \xrightarrow{w^*}_p W^*(\cdot)$ on $\mathcal{D}^p$, which trivially implies the weak convergence in probability $T^{-1/2} \alpha_{0\perp}^{*\prime} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t^* \xrightarrow{w^*}_p \alpha_{0\perp}^{*\prime} W^*(\cdot)$ on $\mathcal{D}^{p-r^*}$. Hence, by Basawa, Mallik, McCormick, Reeves, and Taylor (1991), we find

$$\begin{aligned} T^{-1/2} \mathbb{Z}_{\lfloor Tu \rfloor}^* &\xrightarrow{w^*}_p \int_0^u \exp\bigl(b_\perp \pi^* b'_\perp (u-s)\bigr) b_\perp \, d\alpha_{0\perp}^{*\prime} W^*(s) \\ &= b_\perp \int_0^u \exp\bigl(\pi^*(u-s)\bigr) \alpha_{0\perp}^{*\prime} \, dW^*(s). \end{aligned}$$

Multiplying by $b'_\perp$, we get

$$\begin{aligned} T^{-1/2} \mathbb{A}_\perp^{*\prime} \mathbb{X}_{\lfloor T\cdot \rfloor}^* &= T^{-1/2} \alpha_{0\perp}^{*\prime} \Gamma_0^* X_{\lfloor T\cdot \rfloor}^* + T^{-1/2} \alpha_{0\perp}^{*\prime} \Psi_0^* \Delta \mathbb{X}_{2\lfloor T\cdot \rfloor}^* \\ &= T^{-1/2} b'_\perp \mathbb{Z}_{\lfloor T\cdot \rfloor}^* \xrightarrow{w^*}_p Z(\cdot) \end{aligned} \tag{C.21}$$

[1]Note that the proof of Theorem A.14 applies Lemma A.1 in Johansen (1995), where a misprint occurs in (A.23), in which the last T should be omitted.

on $\mathcal{D}^{p-r^*}$, with $Z(u) := \int_0^u \exp(\pi^*(u-s))\alpha_{0\perp}^{*\prime}\, dW^*(s)$. That is, Z satisfies the stochastic differential equation

$$dZ(u) = \pi^* Z(u)\, du + \alpha_{0\perp}^{*\prime}\, dW^*(u).$$

Next, consider $\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger$, where $\mathbb{X}_t^\dagger$ is the companion form of $X_t^\dagger$ defined in (6) of Cavaliere, Nielsen, and Rahbek (2015). Using (C.13) and (C.18), we obtain

$$\begin{aligned}\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger &= b'\mathbb{Z}_t^* - b'\mathbb{Z}_t^\dagger \\ &= \sum_{i=0}^{t-1} b'\left(\left(I_{pk} + a_T b' + \frac{1}{T}cb'_\perp\right)^i - \left(I_{pk} + ab'\right)^i\right)\mathbb{E}_{zt-i}^*.\end{aligned}$$

As in the proof of Theorem 14.1 in Johansen (1995), we find, with $\rho_i := b'((I_{pk} + a_T b' + \frac{1}{T}cb'_\perp)^i - (I_{pk} + ab')^i)$ and $\mathbb{E}_{zt} := (\mathbb{B}^*, \mathbb{A}_\perp^*)'(\varepsilon_t', 0, \dots, 0)'$,

$$\mathrm{Var}^*\left(\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger\right) = \sum_{i=0}^{t-1} \rho_i \mathbb{E}_{zt}\mathbb{E}_{zt}'\rho_i'.$$

By Lemma A.1 and (A.22) in Johansen (1995), we conclude that $\|\rho_t\| \le \rho = O_p(\|T^{-1}c_T\|)$ and, hence, as $c_T = O_p(1)$,

$$\max_t \left\| \mathrm{Var}^*\left(\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger\right)\right\| \le \|\rho\|^2 \sum_{i=0}^{T} \mathbb{E}_{zt}'\mathbb{E}_{zt} = O_p\left(T^{-1}\right).$$

Now the result in (10) of Cavaliere, Nielsen, and Rahbek (2015) follows by observing that by definition, we have $\mathbb{B}^{*\prime}\mathbb{X}_t^* = (X_t^{*\prime}\beta_0^*, \Delta X_t^{*\prime}, \Delta X_{t-1}^{*\prime}, \dots, \Delta X_{t-k}^{*\prime})'$. Observe that (C.21) has the immediate implication $T^{-1/2}\alpha_{0\perp}^{*\prime}\Gamma_0^* X_{\lfloor T\cdot\rfloor}^* \overset{w^*}{\to}_p Z(\cdot)$, as the last term on the right hand side asymptotically vanishes.

Collecting terms, using the skew projection $I_{pk} = \mathbb{A}^*(\mathbb{B}^{*\prime}\mathbb{A}^*)^{-1}\mathbb{B}^{*\prime} + \mathbb{B}_\perp^*(\mathbb{A}_\perp^{*\prime}\mathbb{B}_\perp^*)^{-1}\mathbb{A}_\perp^{*\prime}$, and $X_t^* = (I_p, 0, \dots, 0)\mathbb{X}_t^*$, we find

$$\begin{aligned} X_t^* &= \beta_{0\perp}^*\left(\alpha_{0\perp}^{*\prime}\Gamma_0^*\beta_{0\perp}^*\right)^{-1}\mathbb{A}_\perp^{*\prime}\mathbb{X}_t^* + \left(\alpha_0^*, \Psi_0^*\right)\left(\mathbb{B}^{*\prime}\mathbb{A}^*\right)^{-1}\mathbb{B}^{*\prime}\mathbb{X}_t^* \\ &=: C_z^* Z_t^* + S_t^*. \end{aligned} \tag{C.22}$$

Finally, to show $\max_t \|S_t^*\| = o_p^*(T^{1/2})$, rewrite as

$$S_t^* = \left(\alpha_0^*, \Psi_0^*\right)\left(\mathbb{B}^{*\prime}\mathbb{A}^*\right)^{-1}\left(\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger\right) + \left(\alpha_0^*, \Psi_0^*\right)\left(\mathbb{B}^{*\prime}\mathbb{A}^*\right)^{-1}\mathbb{B}^{*\prime}\mathbb{X}_t^\dagger. \tag{C.23}$$

The first part on the right hand side was just considered, and the last term on the right hand side converges as desired from the proof of Proposition 1 in Cavaliere, Rahbek, and Taylor (2012), since the parameters for $X_t^\dagger$ satisfy the $I(1, r^*)$ conditions (see Lemma 1). This completes the proof of Proposition 1. *Q.E.D.*

C.3. *Proof of Theorem 1*

First we introduce and prove a lemma based on the results of Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015), which establishes the asymptotic behavior of bootstrap (cross-) product moment matrices. The lemma is next used for the proof of Theorem 1.

LEMMA C.1: *Consider the product moment matrices in terms of* $\{X_t^*\}$. *Under Assumptions* 1 *and* 2, *as* $T \to \infty$,

$$\beta_0^{*\prime} S_{11}^* \beta_0^* \xrightarrow{p^*}_p \Sigma_{\beta\beta}^\dagger, \quad S_{00}^* \xrightarrow{p^*}_p \Sigma_{00}^\dagger, \quad \text{and} \quad \beta_0^{*\prime} S_{10}^* \xrightarrow{p^*}_p \Sigma_{\beta 0}^\dagger,$$

where $\Sigma_{\beta\beta}^\dagger$, $\Sigma_{\beta 0}^\dagger$, *and* $\Sigma_{00}^\dagger$ *are defined after equation* (C.29) *below*. *Moreover*, *with the subscript* ε *referring to the bootstrap innovations* ε_t^*,

$$T^{-1} \beta_{0\perp}^{*\prime} S_{11}^* \beta_{0\perp}^* \xrightarrow{w^*}_p \int_0^1 G_{\beta_\perp}^* G_{\beta_\perp}^{*\prime}\, du, \tag{C.24}$$

$$\beta_{0\perp}^{*\prime} S_{1\varepsilon}^* \xrightarrow{w^*}_p \int_0^1 G_{\beta_\perp}^*\, dW^{*\prime}, \tag{C.25}$$

$$\beta_0^{*\prime} S_{1\varepsilon}^* \xrightarrow{w^*}_p N_{r^* \times p}\big(0, \Sigma_{\beta\beta}^\dagger \otimes \alpha_{0\perp}^{*\prime} \Omega_0^* \alpha_{0\perp}^*\big), \tag{C.26}$$

where $G_{\beta_\perp}^* := \beta_{0\perp}^{*\prime} G^*$ *and* $G^* := C_z^* Z$, *with* $Z(\cdot)$ *as defined in Proposition* 1 *of Cavaliere, Nielsen, and Rahbek* (2015). *Finally*, $\beta_{0\perp}^{*\prime} S_{11}^* \beta_0^*$, $\beta_{0\perp}^{*\prime} S_{10}^* = O_p^*(1)$.

PROOF: In the proof, notation and quantities introduced in the proofs of Lemma 1 and Proposition 1 will be applied. Specifically, in the following discussion, we shall repeatedly apply the companion form $\mathbb{X}_t^*$ of X_t^* (see (C.12)) and $\mathbb{X}_t^\dagger$ of $X_t^\dagger$ defined in (6). Also we use the notation $\mathbb{M}_{ij}^*$ and $\mathbb{M}_{ij}^\dagger$ to denote the usual product moment matrices M_{ij} in terms of $\mathbb{X}_t^*$, $\Delta\mathbb{X}_t^*$ and $\mathbb{X}_t^\dagger$, $\Delta\mathbb{X}_t^\dagger$, respectively, and likewise for M_{ij}^* and $M_{ij}^\dagger$.

First, consider $(X_t^{*\prime}\beta_0^*, \Delta X_t^{*\prime}, \ldots, \Delta X_{t-k+1}^{*\prime})' = \mathbb{B}^{*\prime}\mathbb{X}_t^*$ and the corresponding product moment $\mathbb{B}^{*\prime}\mathbb{M}_{11}^*\mathbb{B}^*$, which can be rewritten as

$$\mathbb{B}^{*\prime}\mathbb{M}_{11}^*\mathbb{B}^* = \big(\mathbb{B}^{*\prime}\mathbb{M}_{11}^*\mathbb{B}^* - \mathbb{B}^{*\prime}\mathbb{M}_{11}^\dagger\mathbb{B}^*\big) + \mathbb{B}^{*\prime}\mathbb{M}_{11}^\dagger\mathbb{B}^*.$$

By Lemma A.7 in Cavaliere, Rahbek, and Taylor (2010a),

$$\mathbb{B}^{*\prime}\mathbb{M}_{11}^\dagger\mathbb{B}^* \xrightarrow{p^*}_p \Upsilon^\dagger := \begin{pmatrix} \Upsilon_{\beta\beta}^\dagger & \Upsilon_{\beta 2}^\dagger \\ \Upsilon_{2\beta}^\dagger & \Upsilon_{22}^\dagger \end{pmatrix}, \tag{C.27}$$

where $\Upsilon_{ij}^\dagger$ and, hence, $\Upsilon^\dagger$ are all well defined by Lemma 1. Next,

$$\mathbb{B}^{*\prime}\mathbb{M}_{11}^*\mathbb{B}^* - \mathbb{B}^{*\prime}\mathbb{M}_{11}^\dagger\mathbb{B}^* = \mathbb{Q}_{\beta\beta} + \mathbb{Q}_{\beta\beta}' + \mathbb{R}_{\beta\beta}, \tag{C.28}$$

where $Q_{\beta\beta} := \mathbb{B}^{*\prime}(\frac{1}{T}\sum_{t=1}^{T}(\mathbb{X}_{t-1}^{*}-\mathbb{X}_{t-1}^{\dagger})\mathbb{X}_{t-1}^{\dagger\prime})\mathbb{B}^{*}$ and, moreover,

$$\mathbb{R}_{\beta\beta} := \mathbb{B}^{*\prime}\left(\frac{1}{T}\sum_{t=1}^{T}\big(\mathbb{X}_{t-1}^{*}-\mathbb{X}_{t-1}^{\dagger}\big)\big(\mathbb{X}_{t-1}^{*}-\mathbb{X}_{t-1}^{\dagger}\big)'\right)\mathbb{B}^{*}.$$

Applying Hölder's inequality, $\|\mathbb{Q}_{\beta\beta}\|^{2} \leq \|\mathbb{R}_{\beta\beta}\|\|\mathbb{B}^{*\prime}\mathbb{M}_{11}^{\dagger}\mathbb{B}^{*}\|$ and, hence, as $\mathbb{R}_{\beta\beta}$ is positive semidefinite, the desired result holds by establishing $E^{*}(\mathbb{R}_{\beta\beta}) \xrightarrow{p} 0$. Now $E^{*}(\mathbb{R}_{\beta\beta}) = \frac{1}{T}\sum_{t=1}^{T}\operatorname{Var}^{*}(\mathbb{B}^{*\prime}(\mathbb{X}_{t-1}^{*}-\mathbb{X}_{t-1}^{\dagger}))$ and we find

$$\text{(C.29)}\qquad \big\|E^{*}(\mathbb{R}_{\beta\beta})\big\| \leq \max_{t}\big\|\operatorname{Var}^{*}\big(\mathbb{B}^{*\prime}\big(\mathbb{X}_{t}^{*}-\mathbb{X}_{t}^{\dagger}\big)\big)\big\| \xrightarrow{p} 0$$

by Proposition 1, and can, therefore, conclude $\beta_{0}^{*\prime}S_{11}^{*}\beta_{0}^{*} \xrightarrow{p^{*}}_{p} \Sigma_{\beta\beta}^{\dagger} := \Upsilon_{\beta\beta}^{\dagger} - \Upsilon_{\beta2}^{\dagger}\Upsilon_{22}^{\dagger-1}\Upsilon_{2\beta}^{\dagger}$. Regarding S_{00}^{*}, use $\Delta\mathbb{X}_{t}^{*} = (\Delta\mathbb{X}_{t}^{*}-\Delta\mathbb{X}_{t}^{\dagger})+\Delta\mathbb{X}_{t}^{\dagger}$, and, as before, with $\mathbb{M}_{00}^{*} = (\mathbb{M}_{00}^{*}-\mathbb{M}_{00}^{\dagger})+\mathbb{M}_{00}^{\dagger}$, the result follows by Lemma A.7 in Cavaliere, Rahbek, and Taylor (2010a) and from the fact that

$$\left\|\frac{1}{T}\sum_{t=1}^{T}\operatorname{Var}^{*}\big(\Delta\mathbb{X}_{t}^{*}-\Delta\mathbb{X}_{t}^{\dagger}\big)\right\|$$

tends to zero in probability. Again the latter is implied by Proposition 1, and $S_{00}^{*} \xrightarrow{p^{*}}_{p} \Sigma_{00}^{\dagger} := \Upsilon_{00}^{\dagger} - \Upsilon_{02}^{\dagger}\Upsilon_{22}^{\dagger-1}\Upsilon_{20}^{\dagger}$ is established. Likewise, $\beta_{0}^{*\prime}S_{10}^{*} \xrightarrow{p^{*}}_{p} \Sigma_{\beta0}^{\dagger} := \Upsilon_{\beta0}^{\dagger} - \Upsilon_{\beta2}^{\dagger}\Upsilon_{22}^{\dagger-1}\Upsilon_{20}^{\dagger}$.

Next, (C.24) holds by observing first that by definition,

$$\beta_{0\perp}^{*\prime}S_{11}^{*}\beta_{0\perp}^{*} = \beta_{0\perp}^{*\prime}M_{11}^{*}\beta_{0\perp}^{*} - \beta_{0\perp}^{*\prime}M_{12}^{*}M_{22}^{*-1}M_{21}^{*}\beta_{0\perp}^{*}.$$

For the first term, we find $\frac{1}{T}\beta_{0\perp}^{*\prime}M_{11}^{*}\beta_{0\perp}^{*} \xrightarrow{w^{*}}_{p} \int_{0}^{1}G_{\beta_{\perp}}^{*}G_{\beta_{\perp}}^{*\prime}\,du$ by the continuous mapping theorem and as by Proposition 1, $\beta_{0\perp}^{*\prime}X_{\lfloor T\cdot\rfloor}^{*} \xrightarrow{w^{*}}_{p} G_{\beta_{\perp}}^{*}(\cdot)$. Next, $M_{22}^{*} = O_{p}^{*}(1)$ as just established and $\beta_{0\perp}^{*\prime}M_{12}^{*} = O_{p}^{*}(T^{-1/2})$ using Theorem 2.1 of Hansen (1992). Likewise (C.25) and $\beta_{0\perp}^{*\prime}S_{11}^{*}\beta_{0}^{*} = O_{p}^{*}(1)$ hold.

Consider now $\beta_{0}^{*\prime}S_{1\varepsilon}^{*}$, which by definition satisfies

$$\sqrt{T}\beta_{0}^{*\prime}S_{1\varepsilon}^{*} = \big(I_{r}, -\beta_{0}^{*\prime}M_{12}^{*}M_{22}^{*-1}\big)\sqrt{T}\mathbb{B}^{*\prime}\mathbb{N}_{1\varepsilon},$$

where $\mathbb{N}_{1\varepsilon} := T^{-1}\sum_{t=1}^{T}X_{t-1}^{*}\varepsilon_{t}^{*\prime}$. Rewrite $\mathbb{B}^{*\prime}\mathbb{N}_{1\varepsilon}^{*}$ in terms of $\mathbb{X}_{t-1}^{\dagger}$ as

$$\text{(C.30)}\qquad \mathbb{B}^{*\prime}\mathbb{N}_{1\varepsilon}^{*} = \mathbb{B}^{*\prime}\mathbb{N}_{1\varepsilon}^{\dagger} + \mathbb{B}^{*\prime}\big(\mathbb{N}_{1\varepsilon}^{*}-\mathbb{N}_{1\varepsilon}^{\dagger}\big).$$

By Lemma A.6 in Cavaliere, Rahbek, and Taylor (2010b),

$$\sqrt{T}\mathbb{B}^{*\prime}\mathbb{N}_{1\varepsilon}^{\dagger} \xrightarrow{w^{*}}_{p} N\big(0, \Upsilon^{\dagger}\otimes\alpha_{0\perp}^{*\prime}\Omega_{0}^{*}\alpha_{0\perp}^{*}\big),$$

where $Y^\dagger$ is defined in (C.27). Hence, as $\beta_0^{*\prime} M_{12}^* M_{22}^{*-1} \xrightarrow{p^*}_p Y_{\beta 2}^\dagger Y_{22}^{\dagger -1}$ and with $\Sigma_{\beta\beta}^\dagger := Y_{\beta\beta}^\dagger - Y_{\beta 2}^\dagger Y_{22}^{\dagger -1} Y_{2\beta}^\dagger$, the results follows by showing the second term in (C.30) is $o_p^*(T^{-1/2})$. As for (C.28), this holds by $\mathbb{R}_{\beta\beta} = o_p^*(T^{-1/2})$. Likewise, $\beta_{0\perp}^{*\prime} S_{11}^* \beta_0^* = O_p^*(1)$, which completes the proof. *Q.E.D.*

PROOF OF THEOREM 1: We here consider the bootstrap LR test statistic of $\mathsf{H}_0 : \beta = \tau$. We present the proof for the case of $k = 1$; extension to the general case is straightforward and can be done exactly as for the previous proofs of Lemma 1 and Proposition 1 using the companion form representation.

Let H_1 refer to estimation when β is unrestricted. On the original data, the estimators are denoted by $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\Omega}$, while on the bootstrap data generated as in (4), we denote the estimators by $\hat{\alpha}^*$, $\hat{\beta}^*$, and $\hat{\Omega}^*$. Likewise, $\tilde{\alpha}$, $\tilde{\Omega}$ and $\tilde{\alpha}^*$, $\tilde{\Omega}^*$ denote the restricted ML estimators under $\mathsf{H}_0 : \beta = \tau$, computed on the original data and on the bootstrap sample, respectively.

So as to show that $Q_T^*(\tau) = -2 \log Q^*(\mathsf{H}_0|\mathsf{H}_1)$ is $O_p^*(1)$, we introduce the auxiliary hypothesis $\mathsf{H}_{\text{aux}} : \alpha\beta' = \alpha_0^* \beta_0^{*\prime}$, such that

$$
\text{(C.31)} \quad \begin{aligned} -2\log Q^*(\mathsf{H}_0|\mathsf{H}_1) &= -2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_1) - \left(-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_0)\right) \\ &= -T\log\det\left(\hat{\Omega}_{\text{aux}}^{*-1}\hat{\Omega}^*\right) - \left(-T\log\det\left(\hat{\Omega}_{\text{aux}}^{*-1}\tilde{\Omega}^*\right)\right), \end{aligned}
$$

where $\hat{\Omega}^* = S_{\hat{\varepsilon}\hat{\varepsilon}}^* := T^{-1}\sum_{t=1}^T \hat{\varepsilon}_t^* \hat{\varepsilon}_t^{*\prime}$ and $\tilde{\Omega}^* = S_{\tilde{\varepsilon}\tilde{\varepsilon}}^* := T^{-1}\sum_{t=1}^T \tilde{\varepsilon}_t^* \tilde{\varepsilon}_t^{*\prime}$ with $\hat{\varepsilon}_t^* = \Delta X_t^* - \hat{\alpha}^* \hat{\beta}^{*\prime} X_{t-1}^*$ and $\tilde{\varepsilon}_t^* = \Delta X_t^* - \tilde{\alpha} X_{\tau,t-1}^*$. Moreover, $\hat{\Omega}_{\text{aux}}^* = S_{ee}^* := T^{-1} \times \sum_{t=1}^T e_t^* e_t^{*\prime}$ with $e_t^* = \Delta X_t^* - \alpha_0^* \beta_0^{*\prime} X_{t-1}^*$. A similar decomposition was applied for the likelihood ratio test in Nielsen and Rahbek (2007), with the notable difference that here the auxiliary hypothesis does not correspond to the (bootstrap) generating process (see Proposition 1). Note in particular that, as shown in Lemma 1 and Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015), the bootstrap sample has exactly $p - r_0$ unit roots under the null hypothesis, while in general it has $p - r_0$ unit roots and $r_0 - r^*$ additional near-unit roots as reflected in the derivations below.

The proof is structured as follows. We consider first the asymptotic behavior of the unrestricted bootstrap estimators and next establish that the first term in (C.31) is bounded. Thereafter, we consider the restricted bootstrap estimators and show that the corresponding second term in (C.31) is also bounded. Finally, the proof is completed by showing that the asymptotic distribution of the bootstrap LR statistic $-2\log Q^*(\mathsf{H}_0|\mathsf{H}_1)$ is χ^2 under the null hypothesis.

Asymptotic theory for the bootstrap unrestricted estimators.

Under H_1, the eigenvalue problem to be solved, $\det(\lambda^* S_{11}^* - S_{10}^* S_{00}^{*-1} S_{01}^*) = 0$, implies, using the basis $(\beta_0^*, \bar{\beta}_{0\perp}^*/\sqrt{T})$ and Lemma C.1, that in the limit $(\lambda_i^*)_{i=1,\dots,r^*}$ are nonzero and solve

$$
\det\left(\lambda^* \Sigma_{\beta\beta}^\dagger - \Sigma_{\beta 0}^\dagger \Sigma_{00}^{\dagger -1} \Sigma_{0\beta}^\dagger\right) = 0.
$$

On the other hand, $(\hat{\lambda}_i^*)_{i=r^*+1,\dots,p}$ tend to zero at the rate of T. Recall that $r^* \in \{(2r_0 - p)^+, \dots, r_0\}$ such that $r_0 - r^*$ additional near-unit roots appear asymptotically. More precisely, with $\hat{\rho}_i^* := T\hat{\lambda}_i^*$ for $i = r^* + 1, \dots, p$, $\hat{\rho}_i^*$ solve in the limit, using the results in Lemma C.1 and standard arguments,

$$\det\left(\rho^* \int_0^1 G_{\beta_\perp}^* G_{\beta_\perp}^{*\prime} \, du - \int_0^1 G_{\beta_\perp}^* \, dZ' \left(\alpha_{0\perp}^{*\prime} \Omega_0^* \alpha_{0\perp}^*\right)^{-1} \int_0^1 dZ G_{\beta_\perp}^{*\prime}\right) = 0,$$

where the convergence $\beta_{0\perp}^{*\prime} S_{10}^* \alpha_{0\perp}^* \overset{w^*}{\to}_p \int_0^1 G_{\beta_\perp}^* \, dZ'$ has been used, with Z defined in Proposition 1, equation (9) of Cavaliere, Nielsen, and Rahbek (2015). To see this, observe that by definition, $\beta_{0\perp}^{*\prime} S_{10}^* \alpha_{0\perp}^* = \beta_{0\perp}^{*\prime} S_{1\varepsilon}^* \alpha_{0\perp}^* + \beta_{0\perp}^{*\prime} S_{11}^* \tau \tilde{\alpha}' \alpha_{0\perp}^*$ and, hence, by Lemma C.1 and (C.20),

$$\beta_{0\perp}^{*\prime} S_{10}^* \alpha_{0\perp}^* \overset{w^*}{\to}_p \int_0^1 G_{\beta_\perp}^* \left[\alpha_{0\perp}^{*\prime} \, dW^* + \pi^* Z \, du\right]' = \int_0^1 G_{\beta_\perp}^* \, dZ'.$$

Using the definitions of $G_{\beta_\perp}^*$ and Z, we may conclude

$$\begin{aligned} \sum_{i=r^*+1}^{p} T\hat{\lambda}_i^* &= \sum_{i=r^*+1}^{p} \hat{\rho}_i^* \\ &\overset{w^*}{\to}_p \operatorname{tr}\left\{\int_0^1 dZ^* Z^{*\prime} \left(\int_0^1 Z^* Z^{*\prime} \, du\right)^{-1} \int_0^1 Z^* \, dZ^{*\prime}\right\}, \end{aligned}$$

where $Z^* := (\alpha_{0\perp}^{*\prime} \Omega_0^* \alpha_{0\perp}^*)^{-1/2} Z$.

To find the limiting behavior in terms of rates of convergence of $\hat{\Pi}^*$, we begin by rewriting it in terms of the eigenvectors corresponding to the limiting nonzero and zero eigenvalues, respectively. That is,

$$\hat{\Pi}^* = \hat{\alpha}^* \hat{\beta}^{*\prime} = \hat{\alpha}^* \left(bb' + b_\perp b_\perp'\right) \hat{\beta}^{*\prime} =: \hat{\Pi}_n^* + \hat{\Pi}_z^*, \tag{C.32}$$

where $\hat{\Pi}_n^* := \hat{\alpha}_n^* \hat{\beta}_n^{*\prime}$, $\hat{\Pi}_z^* := \hat{\alpha}_z^* \hat{\beta}_z^{*\prime}$, and $b := K_{r_0}^{(r^*)}$ such that $\hat{\beta}_n^* = \hat{\beta}^* b$ is of rank r^*; see also the proof of Proposition 1. Define the normalized version $\check{\beta}_n^* := \hat{\beta}_n^* (\bar{\beta}_0^{*\prime} \hat{\beta}_n^*)^{-1}$ such that

$$\check{\beta}_n^* = \beta_0^* + \beta_{0\perp}^* u_T^*, \quad u_T^* = \bar{\beta}_{0\perp}^{*\prime} \hat{\beta}_n^* \left(\bar{\beta}_0^{*\prime} \hat{\beta}_n^*\right)^{-1}, \tag{C.33}$$

where $\bar{\beta}_0^{*\prime}(\check{\beta}_n^* - \beta_0^*) = 0$ by definition and $\bar{\beta}_{0\perp}^{*\prime}(\check{\beta}_n^* - \beta_0^*) = u_T^* = o_p^*(T^{-1/2})$. Define correspondingly $\check{\alpha}_n^* := \hat{\alpha}_n^*(\hat{\beta}_n^{*\prime} \bar{\beta}_0^*)$, where $\check{\alpha}_n^* \overset{p^*}{\to}_p \alpha_0^*$.

At the same time, $\|\hat{\Pi}_z^*\|^2 = o_p^*(1)$. To see this, first observe that $\hat{\alpha}_z^* = S_{01}^*\hat{\beta}_z^*$ such that

$$\|\hat{\alpha}_z^*\|^2 = \|(S_{00}^*)^{1/2}(S_{00}^*)^{-1/2}S_{01}^*\hat{\beta}_z^*\|^2 \leq \operatorname{tr}\{S_{00}^*\}\operatorname{tr}\{\hat{\beta}_z^{*\prime}S_{10}^*(S_{00}^*)^{-1}S_{01}^*\hat{\beta}_z^*\} \leq T^{-1}\operatorname{tr}\{S_{00}^*\}\sum_{i=r^*+1}^{p}\hat{\rho}_i^* = O_p^*(T^{-1}). \tag{C.34}$$

Second,

$$\|\hat{\beta}_z^*\|^2 \leq \operatorname{tr}\{S_{11}^{*-1}\}\operatorname{tr}\{\hat{\beta}_z^{*\prime}S_{11}^*\hat{\beta}_z^*\} = O_p^*(1), \tag{C.35}$$

since $\operatorname{tr}\{\hat{\beta}_z' S_{11}^*\hat{\beta}_z\} = p - r^*$ and $S_{11}^{*-1} = O_p^*(1)$ from Lemma C.1.

Next, from the Gaussian likelihood function, it follows that the score in the direction of β, evaluated at $\check{\beta}_n^*$, $\check{\alpha}_n^*$, $\hat{\beta}_z^*$, and $\hat{\alpha}_z^*$, satisfies

$$0 = (\check{\alpha}_n^*, \hat{\alpha}_z^*)'\hat{\Omega}^{*-1}(S_{01}^* - \check{\alpha}_n^*\check{\beta}_n^{*\prime}S_{11}^* - \hat{\alpha}_z^*\hat{\beta}_z^{*\prime}S_{11}^*). \tag{C.36}$$

Rewrite S_{01}^* as $S_{01}^* = S_{\varepsilon 1}^* + \tilde{\alpha}\tau' S_{11}^*$ and postmultiply by $\beta_{0\perp}^*$ such that the term on the right hand side of (C.36) can be written as

$$\Big[\underbrace{S_{\varepsilon 1}^*}_{\text{(a)}} + \underbrace{(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime})S_{11}^*}_{\text{(b)}} - \check{\alpha}_n^* u_T^* \underbrace{\beta_{0\perp}^{*\prime}S_{11}^*}_{\text{(c)}} - \underbrace{(\check{\alpha}_n^* - \alpha_0^*)\beta_0^{*\prime}S_{11}^*}_{\text{(d)}} - \underbrace{\hat{\alpha}_z^*\hat{\beta}_z^{*\prime}S_{11}^*}_{\text{(e)}}\Big]\beta_{0\perp}^*. \tag{C.37}$$

Using (C.34), (C.35), $(\check{\alpha}_n^* - \alpha_0^*) = o_p^*(1)$, and Lemma C.1, the terms (a) and (d) are $O_p^*(1)$ while (c) is $O_p^*(T)$ and (e) is $o_p^*(1)$, in probability. For the term (b), note that $\beta_0^{*\prime}S_{11}\beta_{0\perp}^* = O_p^*(1)$ and, as used for (c), $\beta_{0\perp}^{*\prime}S_{11}\beta_{0\perp}^* = O_p^*(T)$ from Lemma C.1 and $(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime})\bar{\beta}_{0\perp}^* = O_p(T^{-1})$; see (C.16). Also $(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime})\bar{\beta}_0^* = O_p(T^{-1/2})$, which holds as from (C.2) and (C.6),

$$\sqrt{T}(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime})\bar{\beta}_0^* = \sqrt{T}S_{e\tau}S_{\tau\tau}^{-1}\tau'\bar{\beta}_0^* = \sqrt{T}S_{e1}\beta_0^*(\beta_0^{*\prime}S_{11}\beta_0^*)^{-1} + o_p(1). \tag{C.38}$$

Collecting terms, we conclude that

$$u_T^* = \bar{\beta}_{0\perp}^{*\prime}(\hat{\beta}_n^* - \beta_0^*) = O_p^*(T^{-1}). \tag{C.39}$$

Finally, we also have $\sqrt{T}(\check{\alpha}_n^* - \alpha_0^*) = O_p^*(1)$. To see this, rewrite as

$$(\check{\alpha}_n^* - \alpha_0^*)(\check{\beta}_n^{*\prime}S_{11}^*\check{\beta}_n^*) = \underbrace{S_{\varepsilon 1}^*\check{\beta}_n^*}_{\text{(a)}} + \underbrace{(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime})S_{11}^*\check{\beta}_n^*}_{\text{(b)}} - \underbrace{\alpha_0^*(\check{\beta}_n^* - \beta_0^*)'S_{11}^*\check{\beta}_n^*}_{\text{(c)}}, \tag{C.40}$$

where (a) is $O_p^*(T^{-1/2})$, (b) is $O_p^*(T^{-1/2})$ using (C.16) and (C.38), and, finally, (c) is $o_p^*(T^{-1/2})$.

The test statistic $-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_1)$.

As $-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_1) = -T\log\det(\hat{\Omega}_{\text{aux}}^{*-1}\hat{\Omega}^*)$, rewrite $\hat{\Omega}^*$ as $\hat{\Omega}^* = \hat{\Omega}_{\text{aux}} + \hat{Z}_T$, where

$$
\begin{aligned}
Z_T &= \hat{\Omega}^* - \hat{\Omega}_{\text{aux}} = Z_{\pi\pi} - Z_{e\pi} - Z_{e\pi}' \\
&:= \big(\hat{\alpha}^*\hat{\beta}^{*\prime} - \Pi_0^*\big)S_{11}^*\big(\hat{\alpha}^*\hat{\beta}^{*\prime} - \Pi_0^*\big)' \\
&\quad - S_{e1}^*\big(\hat{\alpha}^*\hat{\beta}^{*\prime} - \Pi_0^*\big)' - \big(\hat{\alpha}^*\hat{\beta}^{*\prime} - \Pi_0^*\big)S_{1e}^*.
\end{aligned}
$$

Observe first, using (C.32) and Lemma C.1, that $Z_{\pi\pi} = O_p^*(T^{-1})$ as

$$
\begin{aligned}
TZ_{\pi\pi} &= \sqrt{T}\big(\tilde{\alpha}_n^* - \alpha_0^*\big)\beta_0^{*\prime}S_{11}^*\beta_0^*\sqrt{T}\big(\tilde{\alpha}_n^* - \alpha_0^*\big)' \\
&\quad + \big(T\hat{\alpha}_n^*u_T^{*\prime}\big)\left(\frac{1}{T}\beta_{0\perp}^{*\prime}S_{11}^*\beta_{0\perp}^*\right)\big(T\hat{\alpha}_n^*u_T^{*\prime}\big)' \\
&\quad + \big(\sqrt{T}\hat{\alpha}_z^*\hat{\beta}_z^{*\prime}\big)S_{11}^*\big(\sqrt{T}\hat{\alpha}_z^*\hat{\beta}_z^{*\prime}\big)' + F_T + F_T',
\end{aligned}
$$

with

$$
\begin{aligned}
F_T &= \sqrt{T}\big(\tilde{\alpha}_n^* - \alpha_0^*\big)\beta_0^{*\prime}S_{11}^*\beta_{0\perp}^*\big(T\hat{\alpha}_n^*u_T^{*\prime}\big)' \\
&\quad + \sqrt{T}\big(\tilde{\alpha}_n^* - \alpha_0^*\big)\beta_0^{*\prime}S_{11}^*\big(\sqrt{T}\hat{\alpha}_z\hat{\beta}_z'\big)' \\
&\quad + \sqrt{T}\hat{\alpha}_z\hat{\beta}_z'S_{11}^*\beta_{0\perp}^*\big(T\hat{\alpha}_n^*u_T^{*\prime}\big)'.
\end{aligned}
$$

We conclude that $Z_{\pi\pi} = O_p^*(T^{-1})$, using (C.34), (C.35), (C.39), and (C.40) together with Lemma C.1. Next, consider $Z_{e\pi}$. Using Lemma C.1 and Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015), as well as

$$
e_t^* = \Delta X_t^* - \alpha_0^*\beta_0^{*\prime}X_{t-1}^* = \varepsilon_t^* + \big(\tilde{\alpha}\tau' - \alpha_0^*\beta_0^{*\prime}\big)X_{t-1}^*
$$

(again see (C.40)), we have that

$$
\begin{aligned}
TZ_{e\pi} &= TS_{e1}^*\big(\hat{\alpha}^*\hat{\beta}^{*\prime} - \Pi_0^*\big) \\
&= \sqrt{T}S_{e1}^*\beta_0^*\big(\sqrt{T}\big(\hat{\alpha}_n^* - \alpha_0^*\big)'\big) + S_{e1}^*\beta_{0\perp}^*\big(T\big(\tilde{\alpha}_n^*u_T^*\big)' + S_{e1}^*\hat{\beta}_z\hat{\alpha}_z'\big) \\
&= O_p^*(1).
\end{aligned}
$$

Collecting terms, $Z_T = O_p^*(T^{-1})$, and as

$$
\begin{aligned}
\hat{\Omega}_{\text{aux}}^* &= S_{00}^* + \alpha_0^*\beta_0^{*\prime}S_{11}^*\beta_0^*\alpha_0^{*\prime} - S_{01}^*\beta_0^*\alpha_0^{*\prime} - \alpha_0^*\beta_0^{*\prime}S_{10}^* \\
&\overset{p^*}{\to}_p \Sigma_{00}^\dagger + \alpha_0^*\Sigma_{\beta\beta}^\dagger\alpha_0^{*\prime} - \Sigma_{0\beta}^\dagger\alpha_0^{*\prime} - \alpha_0^*\Sigma_{\beta\beta}^\dagger = \Omega_0^*,
\end{aligned}
$$

by Proposition 1, we find, by a Taylor expansion,

$$-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_1) = -T\log\det\bigl(\hat{\Omega}^{*-1}_{\text{aux}}\hat{\Omega}^*\bigr) = O_p^*(1),$$

as desired.

Asymptotic theory for the bootstrap restricted estimators.

Using $D_T = (\xi, \bar{\xi}_\perp/\sqrt{T})$, then as in the proof of Lemma 1 and using Lemma C.1,

$$\tilde{\Pi}^* = \tilde{\alpha}^*\tau' = S_{01}^*\tau S_{\tau\tau}^{*-1}\tau' = S_{01}^*\tau D_T\bigl(D_T' S_{\tau\tau}^* D_T\bigr)^{-1}D_T'\tau' \xrightarrow{p}{}_p^* \Pi_0^* := \alpha_0^*\beta_0^{*\prime}.$$

Moreover, by direct insertion,

$$\tilde{\Pi}^* - \Pi_0^* = \bigl(\tilde{\alpha}^*\bar{\xi} - \alpha_0^*\bigr)\beta_0^{*\prime} + \tilde{\alpha}^*\xi_\perp\bar{\xi}_\perp'\tau',$$

such that we need to find the asymptotic behavior of $(\tilde{\alpha}^*\bar{\xi} - \alpha_0^*)$ and $\tilde{\alpha}^*\xi_\perp$, respectively. As in (C.16) and applying Lemma C.1, we find, with $G_\xi^* := \bar{\xi}_\perp'\tau' G^*$,

$$\begin{aligned} T\tilde{\alpha}^*\xi_\perp &= TS_{01}^*\tau D_T\bigl(D_T' S_{\tau\tau}^* D_T\bigr)^{-1}D_T'\xi_\perp \\ &= S_{01}^*\tau\bar{\xi}_\perp\left(\int_0^1 G_\xi^* G_\xi^{*\prime}\,du\right)^{-1} + o_p^*(1). \end{aligned}$$

As $\bar{\xi}_\perp' S_{\tau 0}^* = \bar{\xi}_\perp'\tau' S_{10}^* = \bar{\xi}_\perp'\tau'\bar{\beta}_{0\perp}^*\beta_{0\perp}^{*\prime}S_{10}^* + \bar{\xi}_\perp'\tau'\bar{\beta}_0^*\beta_0^{*\prime}S_{10}^* = O_p^*(1)$ by Lemma C.1, we therefore have that $\tilde{\alpha}^*\xi_\perp = O_p^*(T^{-1})$. Likewise, $\sqrt{T}(\tilde{\alpha}^*\bar{\xi} - \alpha_0^*)\beta_0^{*\prime} = O_p^*(1)$ and by collecting terms, we finally find

$$\tilde{\Pi}^* = \tilde{\alpha}\tau' = \tilde{\alpha}\bar{\xi}\beta_0^{*\prime} + \tilde{\alpha}\bar{\xi}_\perp\xi_\perp'\tau' =: \tilde{\Pi}_\xi^* + \tilde{\Pi}_{\xi_\perp}^*, \tag{C.41}$$

with $\tilde{\Pi}_\xi^* - \Pi_0^* = O_p^*(T^{-1/2})$ and $\tilde{\Pi}_{\xi_\perp}^* = O_p^*(T^{-1})$.

The test statistic $-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_0)$.

Consider $-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_0) = -T\log\det(\hat{\Omega}_{\text{aux}}^{-1}\tilde{\Omega}^*)$, where $\hat{\Omega}_{\text{aux}}$ is as above and

$$\tilde{\Omega}^* = S_{\tilde{\varepsilon}\tilde{\varepsilon}}^*, \quad \text{with} \quad \tilde{\varepsilon}_t^* = \Delta X_t^* - \tilde{\alpha}^*\tau' X_{t-1}^*.$$

Then rewrite $\tilde{\Omega}^*$ as $\tilde{\Omega}^* = \hat{\Omega}_{\text{aux}} + Z_T^*$, where

$$\begin{aligned} Z_T^* &= \tilde{\Omega}^* - \hat{\Omega}_{\text{aux}} = Z_{\pi\pi}^* - Z_{e\pi}^* - Z_{e\pi}^{*\prime} \\ &:= \bigl(\tilde{\alpha}^*\tau' - \Pi_0^*\bigr)S_{11}^*\bigl(\tilde{\alpha}^*\tau' - \Pi_0^*\bigr)' \\ &\quad - S_{e1}^*\bigl(\tilde{\alpha}^*\tau' - \Pi_0^*\bigr)' - \bigl(\tilde{\alpha}^*\tau' - \Pi_0^*\bigr)S_{1e}^*. \end{aligned}$$

Observe first that by (C.41) and Lemma C.1, it holds that $Z^*_{\pi\pi} = O^*_p(T^{-1})$, as

$$TZ^*_{\pi\pi} = \big(\sqrt{T}\big(\tilde{\alpha}^*\bar{\xi} - \alpha^*_0\big)\big)\beta^{*\prime}_0 S^*_{11}\beta^*_0\big(\sqrt{T}\big(\tilde{\alpha}^*\bar{\xi} - \alpha^*_0\big)\big)'$$
$$+ \big(T\tilde{\alpha}^*\xi_\perp\big)\left(\frac{1}{T}\bar{\xi}'_\perp\tau' S^*_{11}\tau\bar{\xi}_\perp\right)\big(T\tilde{\alpha}^*\xi_\perp\big)'$$
$$+ \frac{1}{\sqrt{T}}\big(\sqrt{T}\big(\tilde{\alpha}^*\bar{\xi} - \alpha^*_0\big)\beta^{*\prime}_0 S^*_{11}\tau\bar{\xi}_\perp\big(T\big(\tilde{\alpha}^*\xi_\perp\big)'\big)$$
$$+ \big(T\tilde{\alpha}^*\xi_\perp\big)\big(\bar{\xi}'_\perp\tau' S^*_{11}\big)\beta^*_0\big(\sqrt{T}\big(\tilde{\alpha}^*\bar{\xi} - \alpha^*_0\big)'\big)\big).$$

Consider $Z^*_{e\pi}$ next. By using Lemma C.1 and Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015),

$$TZ^*_{e\pi} = TS^*_{e1}\big(\tilde{\alpha}^*\tau' - \Pi^*_0\big)$$
$$= \sqrt{T}S^*_{e1}\beta^*_0\sqrt{T}\big(\tilde{\alpha}^*\bar{\xi} - \alpha^*_0\big)' + S^*_{e1}\tau\bar{\xi}_\perp\big(T\tilde{\alpha}^*\xi_\perp\big)' = O^*_p(1).$$

Collecting terms, $Z^*_T = O^*_p(T^{-1})$, and we find by a Taylor expansion that

$$-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_0) = -T\log\det\big(\hat{\Omega}^{*-1}_{\text{aux}}\tilde{\Omega}^*\big) = O^*_p(1),$$

as desired, which shows that $Q^*_T(\tau) = O^*_p(1)$.

The asymptotic distribution of the bootstrap LR statistic under the null hypothesis.

To show that $Q^*_T(\tau) \xrightarrow{w^*}_p \chi^2(r_0(p - r_0))$ when $r^* = r_0$ or H_0 holds, we apply the same expansions with a few simplifications due to the fact that, under H_0, it holds that $\alpha^*_0 = \alpha_0$ and $\beta^*_0 = \beta_0$. Specifically, we omit the auxiliary hypothesis, and consider directly the statistic $-2\log Q^*(\mathsf{H}_0|\mathsf{H}_1) = -T\log\det(\tilde{\Omega}^{*-1}\hat{\Omega}^*)$. Recall that $\hat{\Omega}^* = S^*_{\hat{\varepsilon}\hat{\varepsilon}}$, where

$$\hat{\varepsilon}^*_t = \Delta X^*_t - \hat{\alpha}^*\hat{\beta}^{*\prime}X^*_{t-1} = \Delta X^*_t - \check{\alpha}^*\beta'_0 X^*_{t-1} - \check{\alpha}^* u^*_T\beta'_{0\perp}X^*_{t-1},$$

using the definition of u^*_T in (C.33) and we set $\check{\alpha}^* = \check{\alpha}^*_n$ since $\hat{\alpha}^*_z = 0$ by definition. Moreover, from (C.36) and (C.37), we find

$$Tu^{*\prime}_T \xrightarrow{w^*}_p u^{*\prime} := \big(\alpha'_0\Omega_0^{-1}\alpha_0\big)^{-1}\alpha'_0\Omega_0^{-1}\int_0^1 dW^* G^{*\prime}_{\beta_\perp}\left(\int_0^1 G^*_{\beta_\perp}G^{*\prime}_{\beta_\perp}\,ds\right)^{-1},$$

where, as $r^* = r_0$, $G^*_{\beta_\perp}(s) = \beta'_{0\perp}\beta_{0\perp}(\alpha'_{0\perp}\beta_{0\perp})^{-1}\alpha'_{0\perp}W^*(s)$. Next, similar to the expansion used for $-2\log Q^*(\mathsf{H}_{\text{aux}}|\mathsf{H}_1)$, write $\hat{\Omega}^* = S^*_{\check{\varepsilon}\check{\varepsilon}} + Z_{\alpha\alpha} - Z_{\alpha\check{\varepsilon}} - Z'_{\alpha\check{\varepsilon}}$, where $\check{\varepsilon}^*_t := \Delta X^*_t - \check{\alpha}^*\beta'_0 X^*_{t-1}$. We find $S^*_{\check{\varepsilon}\check{\varepsilon}} \xrightarrow{p^*}_p \Omega_0$ and

$$TZ_{\alpha\alpha} := T\check{\alpha}^* u^*_T\beta'_{0\perp}S^*_{11}\beta_{0\perp}u^{*\prime}_T\check{\alpha}^{*\prime} \xrightarrow{w^*}_p \alpha_0 u^*\int_0^1 G^*_{\beta_\perp}G^{*\prime}_{\beta_\perp}\,ds\,u^{*\prime}\alpha'_0.$$

Moreover,

$$TZ_{\alpha\breve{\varepsilon}} := T\breve{\alpha}^* u_T^* \beta_{0\perp}' S_{1\breve{\varepsilon}}^* = T\breve{\alpha}^* u_T^* \big(\beta_{0\perp}' S_{1\varepsilon}^* + \beta_{0\perp}' S_{11}^* \beta_0 (\tilde{\alpha} - \breve{\alpha}^*)\big)$$
$$\overset{w^*}{\to}_p \alpha_0 u^* \int_0^1 G_{\beta_\perp}^* \, dW^{*\prime}.$$

Hence, collecting terms, we obtain that $-2\log Q^*(\mathsf{H}_0|\mathsf{H}_1) = T\,\mathrm{tr}\{\Omega_0^{-1}(Z_{\alpha\breve{\varepsilon}} + Z_{\alpha\breve{\varepsilon}}' - Z_{\alpha\alpha})\} + o_p^*(1)$ and, finally,

$$Q_T^*(\tau) \overset{w^*}{\to}_p \mathrm{tr}\bigg\{\Omega_0^{-1}\alpha_0\big(\alpha_0'\Omega_0^{-1}\alpha_0\big)^{-1}\alpha_0'\Omega_0^{-1}$$
$$\times \int_0^1 dW^* G_{\beta_\perp}^{*\prime} \bigg(\int_0^1 G_{\beta_\perp}^* G_{\beta_\perp}^{*\prime} \, ds\bigg)^{-1} \int_0^1 G_{\beta_\perp}^* \, dW^{*\prime}\bigg\},$$

which is $\chi^2(r_0(p - r_0))$ as desired. This completes the proof of Theorem 1. *Q.E.D.*

D. MODEL WITH AN INTERCEPT

In Section D.1 we state and provide the proofs of the equivalents of Lemma 1 and Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015) for the model with an intercept. Section D.2 contains additional lemmas applied in the proofs.

ADDITIONAL NOTATION: Due to the inclusion of the intercept, introduce the following notation: with $Z_{0t} := \Delta X_t$, $Z_{1t} := X_{t-1}$, and $Z_{2t} := \Delta\mathbb{X}_{2t}$, for $i, j = 0, 1, 2$, set $M_{ic} := T^{-1}\sum_{t=1}^T Z_{it}$ and $M_{ij\cdot c} := M_{ij} - M_{ic}M_{jc}'$.

D.1. *Bootstrap Asymptotic Theory*

Lemma D.1 and Proposition D.1 below respectively generalize Lemma 1 and Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015) to the case of an intercept included in the model and in the DGP.

LEMMA D.1: *With $\tilde{\Pi} = \tilde{\alpha}\tau'$, $\tilde{\Psi}$, $\tilde{\mu}$, and $\tilde{\Omega}$ the restricted QML estimators for the parameters of the model in* (11) *of* Cavaliere, Nielsen, and Rahbek (2015), *it follows that under Assumptions* 1 *and* 2, *as $T \to \infty$,*

$$\tilde{\Pi} \overset{p}{\to} \Pi_0^* = \alpha_0^*\beta_0^{*\prime}, \quad \tilde{\Psi} \overset{p}{\to} \Psi_0^* = \big(\Gamma_{0,1}^*, \ldots, \Gamma_{0,k-1}^*\big), \quad \tilde{\mu} \overset{p}{\to} \mu_0^*, \quad \textit{and}$$
$$\tilde{\Omega} \overset{p}{\to} \Omega_0^* > \Omega_0,$$

where the pseudo-true parameters α_0^*, β_0^*, and Ψ_0^* satisfy the $I(1, r^*)$ conditions.

PROPOSITION D.1: *Consider the bootstrap process X_t^* as defined in Section 5 of Cavaliere, Nielsen, and Rahbek (2015). With $\varphi := \alpha_{0\perp}^{*\prime}\mu_0^*$, then if $\varphi \neq 0$, X_t^* has the representation*

$$X_t^* = C_z^* \bar{\varphi}_\perp Z_t^* + C_z^* \varphi \varsigma_t + S_t^*, \tag{D.1}$$

where $C_z^* = \beta_{0\perp}^*(\alpha_{0\perp}^{*\prime}\Gamma_0^*\beta_{0\perp}^*)^{-1}$, $\varsigma_{\lfloor Tu \rfloor}/T \overset{w^*}{\to}_p u$ on $\mathcal{D}^1$, and $\max_{t=1,\dots,T}\|S_t^*\| = o_p^*(T^{1/2})$. Moreover, the $(p - r^* - 1)$-dimensional Z_t^* satisfies $T^{-1/2}Z_{\lfloor T\cdot \rfloor}^* \overset{w^*}{\to}_p Z$ on $\mathcal{D}^{p-r^*-1}$, where Z is the Ornstein–Uhlenbeck process with random drift parameters defined in (D.21). If $\varphi = 0$, the limiting Ornstein–Uhlenbeck process Z is $(p - r^*)$ dimensional.

REMARK D.1: Notice that by definition, $\varphi := \alpha_{0\perp}^{*\prime}\mu_0^* = 0$ if and only if $C\mu_0 = C_g\alpha_{0\perp}'\mu_0 = 0$. Consequently, the deterministic trend component (of order T) $C_z^*\varphi\varsigma_t$ appearing in the BGP (D.1) is nonzero if and only if the original DGP also has a deterministic trend component (of order T). That is, if the condition $C\mu_0 = 0$ holds, see Section 5 of Cavaliere, Nielsen, and Rahbek (2015). In this respect, the BGP mimics the original DGP in terms of deterministic components of order T.

PROOF OF LEMMA D.1: Proceeding as in the proof of Lemma 1, observe initially that X_t in the case of an intercept has the representation

$$X_t = C\left(\sum_{i=1}^{t} \varepsilon_i + \mu_0 t\right) + \eta_t, \tag{D.2}$$

where η_t is a stationary linear process with exponentially decaying coefficients, $E\eta_t := \eta$, and $C = C_g\alpha_{0\perp}'$, with C_g as defined in (C.1). Thus, X_t is nonstationary with a linear trend, which vanishes if $C\mu_0 = 0$.

Assume first $C\mu_0 \neq 0$. It holds that the r^* linear combinations $\xi'\tau'X_t$ are stationary. Next, define the $(r^* - r_0)$-dimensional vector $\gamma := \xi_\perp'\bar{\tau}'C\mu_0$ and its orthogonal complement $\gamma_\perp$, which is $(r^* - r_0) \times (r^* - r_0 - 1)$ dimensional. By (D.2), $\bar{\gamma}'\xi_\perp'\bar{\tau}'X_t$ is (dominated by) a linear trend t, while $\gamma_\perp'\xi_\perp'\bar{\tau}'X_t$ is integrated of order 1. With the basis for $\mathbb{R}^{r_0}$ defined by

$$D_T^\mu := \operatorname{diag}\left(\xi, T^{-1/2}(\tau'\tau)^{-1}\xi_\perp\gamma_\perp, (\tau'\tau)^{-1}\xi_\perp\bar{\gamma}T^{-3/2}\right), \tag{D.3}$$

then

$$D_T^{\mu\prime}\tau' S_{11}\tau D_T^\mu = \operatorname{diag}\left(\phi'\Sigma_{\beta\beta}\phi, \int_0^1 G_\mu^c G_\mu^{c\prime}\, du\right) + o_p(1)$$

as $T \to \infty$ and where $G_\mu^c(\cdot) = G_\mu(\cdot) - \int_0^1 G_\mu(s)\,ds$ (that is, G_μ corrected for a constant), where

$$G_\mu(\cdot) := \big(G(\cdot)', \cdot\big)' \quad \text{and} \quad G(\cdot) := \gamma_\perp' \xi_\perp' \bar{\tau}' C_g \,\mathrm{wlim}\left(T^{-1/2}\alpha_{0\perp}' \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t\right).$$

Likewise, $S_{01}\tau D_T^\mu \xrightarrow{p} (\Sigma_{0\beta}\phi, 0_{p\times(r_0-r^*)})$. Collecting terms yields

$$\tilde{\alpha}\tau' = S_{01}\tau D_T^\mu \big(D_T^{\mu\prime} S_{\tau\tau} D_T^\mu\big)^{-1} D_T^{\mu\prime}\tau' \xrightarrow{p} \Pi_0^* = \alpha_0^* \beta_0^{*\prime},$$

as desired. With α_0^*, β_0^*, κ_ϕ, and $\kappa_{\phi_\perp}$ as defined in (C.3) and (C.4), $\tilde{\Psi}$, $\tilde{\mu}$, and $\tilde{\Omega}$ converge, respectively, to the pseudo-true parameters Ψ_0^*, μ_0^*, and Ω_0^* given by

(D.4) $$\Psi_0^* = \Psi_0 + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1} Y_{\beta 2} Y_{22}^{-1},$$

(D.5) $$\mu_0^* = \mu_0 + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1} \big(\eta_\beta - Y_{\beta 2} Y_{22}^{-1} \eta_2\big),$$

(D.6) $$\Omega_0^* = \Omega_0 + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \alpha_0'.$$

Here $Y_{\beta 2} := \operatorname{plim} \beta_0' M_{12\cdot c}$, $Y_{22} := \operatorname{plim} M_{22\cdot c}$, $\eta_2 := \operatorname{plim} M_{2c} = E\Delta\mathbb{X}_{2t}$, and $\eta_\beta := \operatorname{plim} \beta_0' M_{1c} = \beta_0'\eta$ (see (D.2)). It follows that

$$\begin{aligned} \tilde{\Psi} &= M_{02\cdot c} M_{22\cdot c}^{-1} - \tilde{\alpha}\tau' M_{12\cdot c} M_{22\cdot c}^{-1} \\ &\xrightarrow{p} \Psi_0^* := Y_{02} Y_{22}^{-1} - \alpha_0 \Sigma_{\beta\beta}\phi\big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1}\phi' Y_{\beta 2} Y_{22}^{-1} \end{aligned}$$

and $\Psi_0 = Y_{02} Y_{22}^{-1} - \alpha_0 Y_{\beta 2} Y_{22}^{-1}$, and (D.4) holds. Next, for $\tilde{\mu}$,

$$\tilde{\mu} = M_{0c} - \tilde{\alpha}\tau' M_{1c} - \tilde{\Psi} M_{2c} \xrightarrow{p} E\Delta X_t - \alpha_0^* E\beta_0^{*\prime} X_t - \Psi_0^* E\Delta\mathbb{X}_{2t}$$

and the result (D.5) holds by simple rewriting, using that by (D.2), $E\Delta X_t = C\mu_0$, as well as (C.4) and (D.4). Next, rewrite the equation for ΔX_t in terms of the pseudo-true parameters as

$$\Delta X_t = \alpha_0^* \beta_0^{*\prime} X_{t-1} + \Psi_0^* \Delta\mathbb{X}_{2t} + \mu_0^* + e_t,$$

where the pseudo-innovations e_t are defined by

(D.7) $$e_t := \varepsilon_t + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1} \big(\beta_0'(X_{t-1} - \eta) - Y_{\beta 2} Y_{22}^{-1} (\Delta\mathbb{X}_{2t} - \eta_2)\big).$$

As e_t is uncorrelated with $\beta_0^{*\prime} X_{t-1}$ and $\Delta\mathbb{X}_{2t}$, we find again that $\tilde{\Omega} \xrightarrow{p} \Omega_0^*$. Specifically, by (D.7),

$$\begin{aligned} E\big(e_t X_{t-1}' \beta_0^*\big) &= E\big(\big(\varepsilon_t + \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1} \\ &\qquad \times \big(\beta_0'(X_{t-1} - \eta) - Y_{\beta 2} Y_{22}^{-1} (\Delta\mathbb{X}_{2t} - \eta_2)\big)\big) X_{t-1}' \beta_0^*\big) \\ &= \alpha_0 \kappa_{\phi_\perp} \phi_\perp' \Sigma_{\beta\beta}^{-1} \big(Y_{\beta\beta} - Y_{\beta 2} Y_{22}^{-1} Y_{2\beta}\big)\phi = 0, \end{aligned}$$

where $Y_{\beta\beta} = \text{plim}\,\beta_0' M_{11\cdot c}\beta_0$ and we have used $\Sigma_{\beta\beta} = Y_{\beta\beta} - Y_{\beta 2}Y_{22}^{-1}Y_{2\beta}$. Likewise, $E(e_t(\Delta\mathbb{X}_{2t} - \eta_2)') = \alpha_0(Y_{\beta 2} - Y_{\beta 2}) = 0$.

To see that the pseudo-true parameters $(\alpha_0^*\beta_0^{*\prime}, \Psi_0^*)$ satisfy the $I(1, r^*)$ conditions, it suffices to proceed as in the proof of Lemma 1 in Cavaliere, Nielsen, and Rahbek (2015), after rewriting the system in companion form as

$$\Delta\mathbb{X}_t = \mathbb{A}^*\mathbb{B}^{*\prime}\mathbb{X}_{t-1} + \mathbb{M} + \mathbb{E}_t,$$

where $\mathbb{M} := (\mu_0^{*\prime}, 0, \dots, 0)'$ and $\mathbb{E}_t$ is defined in terms of e_t in (D.7).

Finally, turn to the case of $C\mu_0 = 0$. In this case, no linear trend is present and the results above hold by redefining D_T^μ as $D_T^\mu := D_T$ and setting $G_\mu^c := G - \int_0^1 G(s)\,ds$, with D_T and G as defined in the proof of Lemma 1. *Q.E.D.*

PROOF OF PROPOSITION D.1: The proof mimics the proof of Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015) for the case of no intercept, and we state here the main steps and results sufficient for extending the arguments to the bootstrap in the intercept case.

By definition, the BGP for X_t^* is given by

$$\Delta X_t^* = \tilde{\alpha}\tau' X_{t-1}^* + \tilde{\Psi}\Delta\mathbb{X}_{2t}^* + \tilde{\mu} + \varepsilon_t^*$$

or, in companion form,

$$\Delta\mathbb{X}_t^* = \tilde{A}\tilde{B}'\mathbb{X}_{t-1}^* + \tilde{M} + \mathbb{E}_{xt}^*,$$

where $\tilde{M} = (\tilde{\mu}', 0, \dots, 0)'$ and the remaining quantities are as defined in the proof of Proposition 1. Likewise, with $Q_z := (\mathbb{B}^*, \mathbb{A}_\perp^*)$ such that $\mathbb{Z}_t^* := Q_z'\mathbb{X}_t^*$, it holds that

$$\Delta\mathbb{Z}_t^* = \left(a_T b' + \frac{1}{T}c_T b_\perp'\right)\mathbb{Z}_{t-1}^* + m_T + \mathbb{E}_{zt}^*, \tag{D.8}$$

where $m_T := Q_z'\tilde{M}$. Moreover, as $T \to \infty$, $a_T, c_T, m_T = O_p(1)$ with

$$a_T \xrightarrow{p} a := b(\mathbb{B}^{*\prime}\mathbb{A}^*) \quad \text{and} \quad m_T \xrightarrow{p} m := Q_z'(\mu_0^{\prime *}, 0, \dots, 0)'. \tag{D.9}$$

Also with $\gamma = \xi_\perp'\bar{\tau}'C\mu_0$ as defined in the proof of Lemma D.1,

$$c_T \xrightarrow{w} c := Q_Z'\begin{pmatrix} N_\mu(\bar{\tau}\xi_\perp\gamma_\perp, 0)'\beta_{0\perp}^* \\ 0_{(k-1)p\times(p-r^*)}\end{pmatrix}\left(\alpha_{0\perp}^{*\prime}\Gamma_0^*\beta_{0\perp}^*\right)^{-1}, \tag{D.10}$$

where N_μ is defined below in (D.11). To see that c has the limit in (D.10), then as in (C.14), consider $T\tilde{\alpha}\tau'\beta_{0\perp}^*$. With D_T^μ defined in (D.3) and $V_T := \bar{\tau}\xi_\perp(\gamma_\perp/\sqrt{T}, \bar{\gamma}/T)$,

$$\text{(D.11)}\quad \begin{aligned} \operatorname{wlim}\big(T\tilde{\alpha}\tau'\beta^*_{0\perp}\big) &= \operatorname{wlim}\big(TS_{01}\tau D^\mu_T\big(D^{\mu\prime}_T\tau' S_{11}\tau D^\mu_T\big)^{-1}D^{\mu\prime}_T\tau'\beta^*_{0\perp}\big)\\ &= \operatorname{wlim}\left(\sqrt{T}S_{01}V_T\left(\int_0^1 G^c_\mu G^{c\prime}_\mu\,du\right)^{-1}(\bar{\tau}\xi_\perp\gamma_\perp, 0)'\beta^*_{0\perp}\right)\\ &=: N_\mu(\bar{\tau}\xi_\perp\gamma_\perp, 0)'\beta^*_{0\perp}. \end{aligned}$$

Now turn to the error correction process $\mathbb{Z}^*_t$ in (D.8), which, due to the accumulation of m_T together with (C.19), satisfies

$$\text{(D.12)}\quad T^{-1}\mathbb{Z}^*_{\lfloor Tu\rfloor} \overset{w^*}{\to}_p \mu(u) = \int_0^u b_\perp \exp\big(\pi^*(u-s)\big)\,ds b'_\perp m = \varphi u,$$

where φ is the $(p-r^*)$-dimensional vector defined by

$$\varphi := \alpha^{*\prime}_{0\perp}\mu^*_0 = \alpha^{*\prime}_{0\perp}\Gamma^*_0 C\mu_0,$$

and

$$\pi^* := b'_\perp c = \alpha^{*\prime}_{0\perp}N_\mu(\bar{\tau}\xi_\perp\gamma_\perp, 0)'\beta^*_{0\perp}\big(\alpha^{*\prime}_{0\perp}\Gamma^*_0\beta^*_{0\perp}\big)^{-1}.$$

To see this, observe that $b'_\perp m = \alpha^{*\prime}_{0\perp}\mu^*_0 = \alpha^{*\prime}_{0\perp}\Gamma^*_0 C\mu_0$ as

$$\text{(D.13)}\quad \mu^*_0 = \Gamma^*_0 C\mu_0 - \alpha^*_0\beta^{*\prime}_0\eta$$

by Lemma D.1. This implies $\pi^* b'_\perp m = 0$, since $(\bar{\tau}\xi_\perp\gamma_\perp, 0)'C\mu_0 = \gamma'_\perp\gamma = 0$ and

$$\begin{aligned} C^*_z b'_\perp m &= \beta^*_{0\perp}\big(\alpha^{*\prime}_{0\perp}\Gamma^*_0\beta^*_{0\perp}\big)^{-1}\alpha^{*\prime}_{0\perp}\Gamma^*_0 C\mu_0\\ &= \beta^*_{0\perp}\bar{\beta}^{*\prime}_{0\perp}C\mu_0 = C\mu_0. \end{aligned}$$

Note that we may equivalently state (D.12) with $\varsigma_t := \bar{\varphi}' b'_\perp\mathbb{Z}^*_t$ as

$$\text{(D.14)}\quad T^{-1}\varsigma_{\lfloor Tu\rfloor} = T^{-1}\bar{\varphi}' b'_\perp\mathbb{Z}^*_{\lfloor Tu\rfloor} \overset{w^*}{\to}_p u.$$

Next, consider the remaining linear directions of $\mathbb{Z}^*_t$ as given by

$$\mathbb{W}^*_t := Q'_w\mathbb{Z}^*_t = (b, b_\perp\varphi_\perp)'\mathbb{Z}^*_t.$$

Using $(b, b_\perp\bar{\varphi}_\perp)Q'_w + b_\perp\varphi\bar{\varphi}' b'_\perp = I_{pk-1}$, we find

$$\text{(D.15)}\quad \Delta\mathbb{W}^*_t = \left(a^w_T b'_w + \frac{1}{T}c^w_T b'_{w\perp}\right)\mathbb{W}^*_{t-1} + \frac{1}{T^{3/2}}c^z_T\varsigma_{t-1} + m^w_T + \mathbb{E}^*_{wt},$$

where $b'_w = (I_{r^*+p(k-1)}, 0)$, $b'_{w\perp} = (0, I_{p-r^*-1})$, and $\mathbb{E}^*_{wt} = Q'_w \mathbb{E}^*_{zt}$. Moreover,

$$a^w_T = Q'_w a_T \overset{p}{\to} a^w = Q'_w b\big(\mathbb{B}^{*\prime}\mathbb{A}^*\big) = b_w\big(\mathbb{B}^{*\prime}\mathbb{A}^*\big),$$
$$c^w_T = Q'_w c_T \bar{\varphi}_\perp \overset{w}{\to} c^w := Q'_w Q'_Z \begin{pmatrix} N_\mu(\bar{\tau}\xi_\perp\gamma_\perp, 0)'\beta^*_{0\perp} \\ 0_{p(k-1)\times(p-r^*)} \end{pmatrix} \big(\alpha^{*\prime}_{0\perp}\Gamma^*_0\beta^*_{0\perp}\big)^{-1}\bar{\varphi}_\perp, \tag{D.16}$$

while

$$\begin{aligned} c^z_T &= \sqrt{T}Q'_w c_T \varphi \\ &\overset{w}{\to} c^w := Q'_w Q'_z \begin{pmatrix} N_\mu(0, \bar{\tau}\xi_\perp\bar{\gamma})'\beta^*_{0\perp} \\ 0_{p(k-1)\times(p-r^*)} \end{pmatrix} \big(\alpha^{*\prime}_{0\perp}\Gamma^*_0\beta^*_{0\perp}\big)^{-1}\varphi \\ &= Q'_w Q'_z \begin{pmatrix} N_\mu(0, 1)' \\ 0_{p(k-1)\times 1} \end{pmatrix}. \end{aligned}$$

With m^w_T as in (D.15), consider the decomposition

$$m^w_T = Q'_w m_T = Q'_w Q'_z \tilde{M} = b_w d_T + \frac{1}{\sqrt{T}} b_{w\perp} e_T,$$

where $d_T := \mathbb{B}^{*\prime}\tilde{M} \overset{p}{\to} d = \mathbb{B}^{*\prime}(\mu^{*\prime}_0, 0, \dots, 0)'$ and $e_T := \sqrt{T}\varphi'_\perp \mathbb{A}^{*\prime}_\perp \tilde{M} \overset{w}{\to} e$. To show the latter convergence, rewrite e_T as

$$\begin{aligned} e_T &= \sqrt{T}\varphi'_\perp \mathbb{A}^{*\prime}_\perp \tilde{M} = \sqrt{T}\varphi'_\perp \alpha^{*\prime}_{0\perp}\tilde{\mu} = \sqrt{T}\varphi'_\perp \alpha^{*\prime}_{0\perp}\big(\tilde{\mu} - \mu^*_0\big) \\ &= \sqrt{T}\varphi'_\perp\big(\alpha^{*\prime}_{0\perp}M_{0c\cdot 2} - \varphi\big) - \sqrt{T}\varphi'_\perp\alpha^{*\prime}_{0\perp}\tilde{\alpha}\tau' M_{1c\cdot 2} \\ &=: e_{1T} - e_{2T}, \end{aligned} \tag{D.17}$$

with $M_{ic\cdot 2} = M_{ic} - M_{i2}M^{-1}_{22}M_{2c}$, $i = 0, 1$, and where we have used $\varphi = \alpha^{*\prime}_{0\perp}\Gamma^*_0 C\mu_0 = \alpha^{*\prime}_{0\perp}\mu^*_0$. By definition, $\alpha^{*\prime}_{0\perp}E(M_{0c\cdot 2}) = \alpha^{*\prime}_{0\perp}\Gamma^*_0 C\mu_0 = \varphi$ and the first term e_{1T} in (D.17) is of order $O_p(1)$ by standard application of the central limit theorem (CLT) for stationary processes. Next, use D^μ_T and V_T to rewrite e_{2T} as

$$\begin{aligned} e_{2T} = \varphi'_\perp\alpha^{*\prime}_{0\perp}\bigg(N_\mu \int_0^1 G_\mu(s)\,ds + \sqrt{T}\big(S_{01}\beta^*_0\big(\phi'\Sigma_{\beta\beta}\phi\big)^{-1} - \alpha^*_0\big)\eta^*\bigg) \\ + o_p(1), \end{aligned}$$

where $\eta^* := \text{plim}(\beta^{*\prime}_0 M_{1c\cdot 2})$. Again, e_{2T} is $O_p(1)$ by (D.11) and observing that $E(S_{01}\beta^*_0)(\phi'\Sigma_{\beta\beta}\phi)^{-1} = \alpha^*_0$.

By the recursion in (D.15), $\mathbb{W}_t^* = \mathbb{W}_{1t}^* + \mathbb{W}_{2t}^* + \mathbb{W}_{3t}^*$, where

$$\mathbb{W}_{1t}^* := \sum_{j=1}^{t} \Phi_{w,T}^{t-j} \mathbb{E}_{wj}^*,$$

$$\mathbb{W}_{2t}^* := \sum_{j=1}^{t} \Phi_{w,T}^{t-j} \left(b_w d_T + \frac{1}{\sqrt{T}} b_{w\perp} e_T \right),$$

$$\mathbb{W}_{3t}^* := \frac{1}{T^{3/2}} \sum_{j=1}^{t} \Phi_{w,T}^{t-j} c_T^z \mathfrak{s}_{j-1},$$

with $\Phi_{w,T} := I + a_T^w b_w' + \frac{1}{T} c_T^w b_{w\perp}'$. By (C.19) and the weak convergence in (D.16), we find, as in the proof of Proposition 1 of Cavaliere, Nielsen, and Rahbek (2015),

$$\text{(D.18)} \quad T^{-1/2} \mathbb{W}_{1\lfloor Tu \rfloor}^* = T^{-1/2} \sum_{j=1}^{\lfloor Tu \rfloor} \Phi_{w,T}^{\lfloor Tu \rfloor - j} \mathbb{E}_{wj}^* \overset{w^*}{\to}_p b_{w\perp} \left(\int_0^u \exp\big(\pi_w^*(u-s)\big) \, d\varphi_\perp' \alpha_{0\perp}^{*\prime} W_u^* \right),$$

where $\pi_w^* = b_{w\perp}' c$ and we have used that, by definition, $b_{w\perp}' Q_w' Q_Z' = \varphi_\perp' \alpha_{0\perp}^{*\prime}$. Next, using (C.19), Lemma D.3, and the convergence of a_T^w, c_T^w, and e_T^w, we find

$$\text{(D.19)} \quad T^{-1/2} \mathbb{W}_{2\lfloor Tu \rfloor}^* = T^{-1/2} \sum_{t=1}^{\lfloor Tu \rfloor} \Phi_{w,T}^{t} \left(b_w d_T + \frac{1}{\sqrt{T}} b_{w\perp} e_T \right) \overset{w^*}{\to}_p b_{w\perp} \left(\int_0^u \exp\big(\pi_w^*(u-s)\big) \, ds \right) e.$$

Finally, $\mathbb{W}_{3\lfloor T\cdot \rfloor}^* = o_p^*(T^{1/2})$ by (D.14).

Consider next X_t^*, which, similarly to the case of no deterministics in (C.22), one may decompose as

$$\text{(D.20)} \quad X_t^* = C_z^* \big(\bar{\varphi}_\perp \varphi_\perp' + \varphi \bar{\varphi}' \big) b_\perp' \mathbb{Z}_t^* + S_t^* = C_z^* \bar{\varphi}_\perp b_{w\perp}' \mathbb{W}_t^* + C_z^* \varphi \mathfrak{s}_t + S_t^*.$$

Here S_t^* is defined as in (C.22) in terms of $\mathbb{B}^{*\prime} \mathbb{X}_t^*$ and, as argued below, $\max_t \|S_t^*\| = o_p^*(T^{1/2})$. For $\mathfrak{s}_t$, use (D.14) and note that by (D.13), $C_z^* \varphi = \beta_{0\perp}^* (\alpha_{0\perp}^{*\prime} \Gamma_0^* \beta_{0\perp}^*) \alpha_{0\perp}^{*\prime} \mu_0^*$. Finally, use (D.18) and (D.19) to see that the remaining $(p - r^* - 1)$ stochastic trends satisfy $T^{-1/2} b_{w\perp}' \mathbb{W}_{\lfloor T\cdot \rfloor}^* \overset{w^*}{\to}_p Z^w(\cdot)$, where Z^w solves

$$\text{(D.21)} \quad dZ^w(u) = \big(\pi_w^* Z^w(u) + e \big) \, du + \varphi_\perp' \alpha_{0\perp}^{*\prime} \, dW_u^*.$$

Finally, consider S_t^*, which can be decomposed as in (C.23). Let $\mathbb{X}_t^\dagger$ denote the companion form of $X_t^\dagger$, which is as defined in (6) of Cavaliere, Nielsen, and Rahbek (2015), with the intercept μ_0^* added on the right hand side, and define $\rho_i = b_w'(\Phi_{w,T}^i - \Phi_w^i)$, where

$$\Phi_{w,T} := I_{pk-1} + a_T^w b_w' + \frac{1}{T} c_T^w b_{w\perp}' \quad \text{and} \quad \Phi_w := I_{pk-1} + a^w b_w'.$$

By definition, $\mathbb{B}^{*\prime}\mathbb{X}_t^* = b_w'\mathbb{W}_t^*$ and simple substitution gives

$$\mathbb{B}^{*\prime}\mathbb{X}_t^* - \mathbb{B}^{*\prime}\mathbb{X}_t^\dagger = \xi_t + \delta_t,$$

where $\xi_t = \sum_{i=0}^{t-1} \rho_i(\mathbb{E}_{wt-i}^* + b_w d)$ with $\max_t \| \operatorname{Var}^* \xi_t\| = O_p(T^{-1})$ as in the proof of Proposition 1. Moreover, $\delta_t := \delta_{1t} + \delta_{2t} + \delta_{3t}$, where

$$\delta_{1t} := T^{-1/2} \sum_{i=0}^{t-1} \rho_i (b_w f_T + b_{w\perp} e_T),$$

$$\delta_{2t} := T^{-1/2} b_w' \sum_{i=0}^{t-1} \Phi_w^i b_w f_T,$$

$$\delta_{3t} := T^{-3/2} b_w' \sum_{i=0}^{t-1} \Phi_{w,T}^i c_T^z s_{t-1-i},$$

with $f_T := T^{1/2}(d_T - d)$. As $\|\rho_i\| \leq \rho = O_p(T^{-1})$ and f_T, e_T are of order $O_p(1)$, then $\max_t \delta_{1t} = o_p(1)$ since

$$\|\delta_{1t}\| \leq (T\rho)\big(c_T T^{-1/2}\big) = O_p\big(T^{-1/2}\big),$$

where $c_T = O_p(1)$. Then $f_T = O_p(1)$ follows by similar arguments as in (D.17). Next, $\|\delta_{2t}\| = O_p(T^{-1/2})$ as $f_T = O_P(1)$ and $b_w'\Phi_w^i b_w = (I + b_w' a^w)^i$, which is exponentially decreasing. Finally, $\|\delta_{3t}\| = O_p^*(T^{-1/2})$ as $T^{1/2}\delta_{3t} = T^{1/2}\mathbb{W}_{3t}^* = O_p^*(1)$; see also (D.19). *Q.E.D.*

D.2. *Auxiliary Lemmas*

For the co-integrated VAR model with an intercept μ in (11) of Cavaliere, Nielsen, and Rahbek (2015), we show next that the LR test $Q_T(\tau)$ of $\beta = \tau$ is asymptotically χ^2 distributed even when $C\mu_0 = 0$, that is, when no linear trend is present, which extends Johansen (1995), where the case $C\mu_0 \neq 0$ is covered.

LEMMA D.2: *Under Assumptions* 1 *and* 2, *if* $C\mu_0 = 0$, *then as* $T \to \infty$, $Q_T(\tau) \overset{w}{\to} \chi^2_{r_0(p-r_0)}$.

The next technical lemma extends Theorem A.14 in Johansen (1995) so as to deal with the model with intercept under the alternative.

LEMMA D.3: *Assume that a and b are $(n \times m)$ matrices, with $m \leq n$, $a'b$ of full rank m, and $\rho(b'a + I_m) < 1$. Moreover, let $f, g \in \mathbb{R}^n$ such that $a'_\perp f = 0$. Then, as $T \to \infty$,*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tu]} \left(I_n + ab' + \frac{1}{T} D\right)^t \left(f + \frac{1}{\sqrt{T}} g\right) \to \int_0^u \exp(CDs)\, ds Cg$$

in $\mathcal{D}^n$, where $C = b_\perp (a'_\perp b_\perp)^{-1} a'_\perp$.

PROOF OF LEMMA D.2: The model is given by (11) and we prove the results following the arguments outlined in Johansen (1995, proof of Lemma 13.8). Under the hypothesis, $\beta_0 = \tau$, and using the coordinate system $(\beta_0, \beta_{0\perp} T^{-1/2})$, it follows that the standard eigenvalue problem in the limit solves

$$\det\left(\lambda \Sigma_{\beta\beta} - \Sigma_{\beta 0} \Sigma_{00}^{-1} \Sigma_{0\beta}\right) \det\left(\lambda \int_0^1 G_B G'_B\, du\right),$$

with $G_B = B - \int_0^1 B_s\, ds$, $B = T^{-1/2} \beta'_{0\perp} C_g W$, and $W := \text{wlim}(T^{-1/2} \alpha'_{0\perp} \sum_{t=1}^{\lfloor T\cdot \rfloor} \varepsilon_t)$. This establishes $\bar{\beta}'_{0\perp}(\tilde{\beta} - \beta_0) = o_p(T^{-1/2})$. Moreover, standard arguments give

$$T \bar{\beta}'_{0\perp}(\tilde{\beta} - \beta_0) \xrightarrow{w} \left(\int_0^1 G_B G'_B\, du\right)^{-1} \int_0^1 G_B\, dW' \Omega_0^{-1} \alpha_0 \left(\alpha'_0 \Omega_0^{-1} \alpha_0\right)^{-1}.$$

Next, we find similarly

$$Q_T(\tau) \xrightarrow{w} \text{tr}\Bigg\{ \Omega_0^{-1} \alpha_0 \left(\alpha'_0 \Omega_0^{-1} \alpha_0\right)^{-1} \alpha_0 \Omega_0^{-1} \times \int_0^1 dW G_B \left(\int_0^1 G_B G'_B\, du\right)^{-1} \int_0^1 G_B\, dW' \Bigg\},$$

which is $\chi^2_{(r_0(p-r_0))}$ by mixed Gaussianity. *Q.E.D.*

PROOF OF LEMMA D.3: By Theorem A.14 in Johansen (1995),

$$\left(I_n + ab' + \frac{1}{T} D\right)^T \to \exp(CD) C$$

and, hence,

$$T^{-1} \sum_{t=1}^{\lfloor Tu \rfloor} \left(I_n + ab' + \frac{1}{T} D\right)^t \to \int_0^u \exp(CDs)\, ds C.$$

Thus, the result is implied by showing that $T^{-1/2}\sum_{t=1}^{\lfloor Tu\rfloor}(I+ab'+\frac{1}{T}D)^t f\to 0$. Consider first the b direction where, by Johansen (1995, equation (A.22)), with κ a positive (generic) constant,

$$\left\| b'\left(I_n+ab'+\frac{1}{T}D\right)^t - b'\left(I_n+ab'\right)^t\right\| \leq \kappa/T$$

and, hence,

$$\left\| T^{-1/2}\sum_{t=1}^{T} b'\left(I_n+ab'+\frac{1}{T}D\right)^t f\right\| \leq \kappa/T + T^{-1/2}\left\|\sum_{t=1}^{T}\left(I_m+b'a\right)^t b' f\right\| \to 0$$

as $\rho(b'a+I_m)<1$. Next, by Johansen (1995, equation (A.23)),

$$\left\| a_\perp'\left(I_n+ab'+\frac{1}{T}D\right)^t - \left(I_{m-n}+\frac{1}{T}a_\perp' D b_\perp\left(a_\perp' b_\perp\right)^{-1}\right)^t a_\perp'\right\| \leq \kappa/T.$$

Hence, since $a_\perp' f=0$,

$$\left\| T^{-1/2}\sum_{t=1}^{T} a_\perp'\left(I_n+ab'+\frac{1}{T}D\right)^t f\right\| \leq \kappa/T + T^{-1/2}\left\|\sum_{t=1}^{T}\left(I_{n-m}+\frac{1}{T}a_\perp' D b_\perp\left(a_\perp' b_\perp\right)^{-1}\right)^t a_\perp' f\right\| = \kappa/T \to 0,$$

as required. *Q.E.D.*

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