Econometrica Supplementary Material

SUPPLEMENT TO "THE FARSIGHTED STABLE SET"

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O.1. FARSIGHTED STABLE SETS: EXISTENCE AND SOME CONNECTIONS

THE EXISTENCE QUESTION WAS one that von Neumann and Morgenstern (1944, Section 4.6.3) regarded as vitally important:

> There can, of course, be no concessions as regards existence. If it should turn out that our requirements concerning a solution... are, in any special case, unfulfillable,—this would certainly necessitate a fundamental change in the theory. Thus a general proof of the existence of solutions... for all particular cases[1] is most desirable. It will appear from our subsequent investigations that this proof has not yet been carried out in full generality but that in all cases considered so far solutions were found.

One presumes that von Neumann and Morgenstern were referring to transferable-utility games, because it is quite easy to show that vNM stable sets do not exist over the entire domain of all characteristic functions. Stearns (1964) constructed such an example, and we do something analogous for the farsighted stable set in Example O.2 below. But the TU case proved to be much harder to resolve, and the question was not settled until Lucas (1968) provided a 10-person example of a TU game without a vNM stable set.

O.1.1. *Two Examples*

We reproduce the celebrated Lucas example here, as it will provide a noteworthy instance of the role played by separability.

EXAMPLE O.1: Given a 10-player TU game (Lucas (1968)): $v(N) = 5$, $v(1,3,5,7,9) = 4$, $v(3,5,7,9) = v(1,5,7,9) = v(1,3,7,9) = 3$, $v(3,5,7) = v(1,5,7) = v(1,3,7) = v(3,5,9) = v(1,5,9) = v(1,3,9) = v(1,4,7,9) = v(3,6,7,9) = v(2,5,7,9) = 2$, $v(1,2) = v(3,4) = v(5,6) = v(6,7) = v(7,8) = v(9,10) = 1$, and $v(S) = 0$ for all other coalitions S.

This is a game with a nonempty core but without a vNM stable set. Yet a farsighted stable set exists.

Although this game is not superadditive, it has the property that any efficient payoff can be achieved through the grand coalition.[2] A payoff allocation u belongs to the core of this game if and only if it satisfies the two conditions

$$u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = u_7 + u_8 = u_9 + u_{10} = 1 \tag{O.1}$$

[1]In the terminology of games: for all numbers of participants and for all possible rules of the game.

[2]Indeed, as Lucas (1968) points out, the superadditive cover of this game also does not possess a stable set.

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and

$$\text{(O.2)} \qquad u_1 + u_3 + u_5 + u_7 + u_9 \geq 4.$$

As Lucas observed, this game admits allocations that solve (O.1) and (O.2), so the core is nonempty. However, because $v(S) = 1$ for S consisting of adjacent pairs beginning with odd indices (e.g., $\{1, 2\}$ or $\{5, 6\}$), it follows from (O.1) that the interior of the core is empty. At the same time, there are core allocations that are separable. Consider any u in the core such that every odd player receives $u_i > 0.8$. For instance, consider u^*, where

$$u_i^* = \begin{cases} 0.9 & \text{if } i \text{ is odd,} \\ 0.1 & \text{if } i \text{ is even.} \end{cases}$$

It can be shown that the only coalitions that can achieve u^* are coalitions that consist of pairs of adjacent players starting with odd indices. Thus, if u^* is feasible for a strict subpartition, it must be one in which each coalition consists of such adjacent pairs. But then, by the fact that the subpartition is strict, the complement must contain at least one more such adjacent pair. For this pair u^* is feasible. Therefore, u^* is separable. By Theorem 2, $[u^*]$ is a farsighted stable set.[3]

Yet we know from Lucas (1968) that this game admits no vNM stable set.

Separability plays a key role here in establishing the existence of a single-payoff farsighted stable set, even when there is no vNM stable set. As we have seen in Section 5, farsighted stable sets do exist in a large class of games that have no separable allocation or even a core allocation. How general, then, is the existence property? Certainly, one cannot hope that the existence of a farsighted stable set is invariably guaranteed.

EXAMPLE O.2: Given a three-player NTU "roommate game" (Lucas (1992), Banerjee, Konishi, and Sönmez (2001), Diamantoudi and Xue (2003): $V(S) = \{v \leq a^S\}$ for all $S \subset N$, where

$$a^{\{12\}} = (3, 2), \quad a^{\{23\}} = (3, 2), \quad a^{\{13\}} = (2, 3)$$

and $V(N)$ is the comprehensive hull of $\{a^{\{12\}}, a^{\{23\}}, a^{\{13\}}\}$.

[3] Harsanyi stable sets exist as well, though these are very different from the farsighted stable set; more on this contrast in Section O.4. Define $\bar{u}$ by $\bar{u}_i = 0.8$ if i is odd and by $\bar{u}_i = 0.2$ if i is even. The allocation $\bar{u}$ is part of a Harsanyi stable set as it satisfies the conditions of Béal, Durieu, and Solal (2008); see also Diamantoudi and Xue (2005). Notice also that $\bar{u}$ is feasible for the five-player coalition consisting of all the odd numbered players but not for any coalition in the complement. The state $x = (\bar{u}, N)$. It is, therefore, not separable, and so by Theorem 2, it cannot be a farsighted stable set for any effectivity correspondence satisfying Conditions (i) and (ii).

It is easy to see that this example has an empty core and does not possess either a vNM stable set or a farsighted stable set.[4]

But this example is not entirely definitive for the following reason. This very example is well known as an easy counterexample to the existence of the traditional vNM set for general NTU games. But that did not prevent a search for a counterexample in the TU context, a question that was finally settled after many years by Lucas (Example O.1 above). For the same reason, we would like to know if there are corresponding counterexamples to the existence of the farsighted stable set in TU games. Or does every TU game necessarily admit at least one such set? So far, it appears from our analysis (see especially Section 5) that the answer is in the affirmative. But the question remains open. In the remainder of this section we consider some special cases in which the existence issue can be approached by drawing a connection with the vNM stable set.

O.1.2. *Main Simple Solutions and Farsighted Stability*

Consider Example 3, which is a symmetric majority game, and suppose that there are just three players. It is not difficult to verify that there is a farsighted stable set that consists of the three imputations $(0.5, 0.5, 0)$, $(0.5, 0, 0.5)$, and $(0, 0.5, 0.5)$ (and the accompanying winning coalitions). It also happens to be a vNM stable set. In fact, this configuration is an example of what von Neumann and Morgenstern (1944) called a *main simple solution*. Suppose there is $\mathbf{a} \in \Re^N_+$ such that $\sum_{i\in S} a_i = 1$ for every minimal winning coalition S. For every such S, define u^S to be the imputation where $u^S_i = a_i$ for $i \in S$ and $a^S_i = 0$ for $i \notin S$. Then the set of all such imputations is a (finite) vNM stable set and is called a main simple solution. von Neumann and Morgenstern (1944) showed that such solutions exist for a class of simple games, which includes a subclass of weighted majority games. (There are, however, games in which there is no such solution, such as Example 2.)

It can be shown that any main simple solution U with an associated vector $\mathbf{a} \gg 0$ corresponds to a farsighted stable set. Apart from the possible connection via discriminatory sets (also made via Example 3), this is a second link between vNM stable sets and farsighted stable sets. More precisely, suppose $U \in \triangle$ is a main simple solution in the sense of von Neumann and Morgenstern with vector $\mathbf{a} \gg 0$. Consider the corresponding set of states: $F^* = \bigcup_{u\in U}[u]$. We claim that F^* is a farsighted stable set. To see that it satisfies external stabil-

[4]It follows from Bhattacharya and Brosi (2011) that there do exist Harsanyi farsighted stable sets in this example if imputations are defined as individually rational payoffs on the boundary of $V(N)$. For example, $(3, 2, 0)$ is a singleton Harsanyi stable set. The farsighted objection from $(0, 3, 2)$ to this imputation involves player 1 leaving the grand coalition and assigning 0 to player 2 (moving to the imputation $(0, 0, 2)$), which leads 1 and 2 to move to $(3, 2, 0)$. This argument obviously does not work if the departure of 1 from the grand coalition results in $(0, 3, 2)$.

ity, consider a state x such that $u(x) \notin U$. Then there exists a minimal winning coalition T such that $a_i > u_i(x)$ for all $i \in T$ and $\sum_{i \in T} a_i = 1$. Let y be the state with T as the winning coalition and $\mathbf{a}_T$ as the payoff to T. Clearly, there is an objection (in one step), via coalition T, from x to y. To prove internal stability, suppose $x, y \in F^*$ and there is a farsighted objection leading from x to y. Since $u(x) \in U$, there is a minimal winning coalition T such that $u_i(x) = a_i > 0$ for all $i \in T$ and $u_i(x) = 0$ for $i \notin T$. Since $u_i(x) = a_i > 0$ for all $i \in T$, $T \subseteq W(x)$. None of the players in T can gain by moving to y: they will either receive 0 or a_i at y. By Lemma 2, there must be a coalition $S \subseteq W(x)$ such that $S \cap T = \emptyset$ and $W(y) - S$ is losing. But that is impossible since $T \subseteq W(x) - S$ is a winning coalition. This establishes the internal stability of F^*.

O.1.3. *The Demand Bargaining Set and Farsighted Stability*

For games that possess a main simple solution, there is also an interesting connection with the *demand bargaining set* of Morelli and Montero (2003). One of the important differences between this solution concept and the traditional bargaining set is that there is a given vector of demands for each player and a counterobjecting coalition is required to meet these demands for each of its players. These demands are suggestive of a connection with the vector of payoffs associated with a main simple solution. Indeed, Morelli and Montero (2003) show that in all constant sum simple games that admit a main simple solution, the unique imputation corresponding to the demand bargaining set is precisely the main simple solution. It follows from our earlier discussion that it also corresponds to a farsighted stable set. There is one noteworthy difference, however. While the (unique) demand bargaining set coincides with the main simple solution of such games, there may well be other vNM stable sets and farsighted stable sets, as in Example 3.

O.1.4. *Pillage Games and Farsighted Stability*

Pillage games, introduced in Jordan (2006), provide another interesting case in which farsightedness does not modify the myopic notion of a stable set. These are games in which the ability of a coalition to change a given allocation depends on its power, and power itself depends on the current allocation. Given an allocation $u \in \triangle$, the *power* of coalition S is denoted $\pi(S, u)$, where $\pi(S, u') \geq \pi(S, u)$ if $u'_S \geq u_S$ and $\pi(S, u') > \pi(S, u)$ if $u'_S \gg u_S$. It is also assumed that expanding a coalition does not reduce its power. A coalition can change an allocation through pillage if it has more power than those who stand to lose. Formally, in our framework, let $\triangle$ be the set of states and define the effectivity correspondence to be such that $S \in E(u, u')$ if and only if $\pi(S, u) > \pi(L, u)$, where $L = \{i \in N \mid u'_i < u_i\}$. Applying the standard notion of domination, say that an allocation u' *dominates* u if there is a coalition S such that $u'_S \gg u_S$ and $S \in E(u, u')$.

Note that S can be taken, without loss of generality, to be $W = \{i \in N \mid u'_i > u_i\}$. The implicit assumption is that those who have nothing to gain or lose remain neutral. The core and the vNM stable set can be defined as in Section 2.2. This is so even though a pillage game is not specified in terms of a characteristic function. While a coalition that pillages is reminiscent of a winning coalition in a simple game, in that it can expropriate the resources of the less powerful, its power to do so depends on the current allocation.

Jordan (2006) shows that the core of a pillage game is often empty, or nonempty in uninteresting ways. He then characterizes the stable set for a variety of different power functions. At first sight, it appears that the stable set in this context is also subject to the Harsanyi critique. However, as Jordan (2006) demonstrates, properly accounting for the behavior of otherwise neutral players, through a consistent expectations function, preserves the stability of the stable set even in a farsighted sense. This feature of the vNM stable set in pillage games can be shown directly in our framework as follows.

In defining farsighted dominance, we should assume that *all* players are farsighted, including those who see no change in their payoff in a single step of a farsighted move. With this in mind, say that u' farsightedly dominates u if there is a collection of allocations $u^0, u^1, \dots, u^m$ (with $u^0 = u$ and $u^m = u'$) and a corresponding collection of coalitions $S^1, \dots, S^m$, such that for all $k = 1, \dots, m$,

$$\pi(S^k, u^{k-1}) > \pi(L', u), \quad \text{where} \quad L' = \{i \in N \mid u'_i < u_i\}$$

and

$$u'_{S^k} \gg u^{k-1}_{S^k}.$$

Observe that S^1, which initiates the farsighted move, has more power than those who eventually lose. This implies that S^1 could have moved directly from u to u', which can also be expressed more succinctly as

$$\pi(W', u) > \pi(L', u) \quad \text{where} \quad W' = \{i \in N \mid u'_i > u_i\}.$$

Thus, u' farsightedly dominates u if and only if it (myopically) dominates u.

O.2. A REMARK ON HEDONIC GAMES

In a hedonic game, the payoff to every individual is well defined once we know the particular coalition to which she belongs. That is, a hedonic game is fully described by a collection of vectors $\{a^S\}$, one for every coalition S, such that $V(S) = \{v \in \mathbb{R}^S \mid v \leq a^S\}$ and $\bar{V}(S) = \{a^S\}$. As we have already observed in the main text, hedonic games are particularly well suited for studying farsighted stability, since such games are free of any ambiguity about how the formation of a coalition affects outsiders. Banerjee, Konishi, and Sönmez (2001) and Bogomolnaia and Jackson (2002) provide several conditions under which the core partition (or the coalition structure core) of a hedonic game is nonempty. Diamantoudi and Xue (2003) show that if all players have strict preferences across coalitions, then every core partition yields a single-payoff farsighted stable set.

This result can be obtained as a corollary of Theorem 2 by observing that every core allocation of a hedonic game with strict preferences is separable.[5] To see this, consider a hedonic game described by the collection $\{a^S\}$. Suppose further that players have strict preferences across coalitions, so that if $i \in S \cap T$ and $S \neq T$, then $a_i^S \neq a_i^T$. Suppose $y = (u, \pi) \in C(N, V, E)$. Let $S(i)$ denote the coalition in π that contains i; then $u = (a_i^{S(i)})_{i \in N}$. Suppose $u_T \in V(T)$ for every T in some strict subpartition $\mathcal{T}$. If every $T \in \mathcal{T}$ is in π, then u trivially satisfies separability. Otherwise, there is a coalition $T \in \mathcal{T}$ such that $T \notin \pi$, which means that $T \neq S(j)$ for every $j \in T$. Because $u_T \in V(T)$, $a_j^T \geq a_j^{S(j)}$ for all $j \in T$. By the assumption of strict preferences, all these inequalities must be strict. But this contradicts the hypothesis that $y \in C(N, V, E)$.

One condition that has been shown to be sufficient for the nonemptiness of the core partition in a hedonic game is the *top coalition property.* For a subset Q of N, coalition S is said to be a *top coalition of* Q if for every $i \in S$ and $T \subseteq Q$ such that $i \in T$, $a_i^S \geq a_i^T$. A game (N, V) has the top coalition property if every $Q \subseteq N$ has a top coalition. Banerjee, Konishi, and Sönmez (2001) show that if preferences are strict, this property implies the existence of a *unique* core partition. Diamantoudi and Xue (2003) prove that in this case, the unique core partition is also the unique farsighted stable set. That yields a complete characterization of farsighted stable sets in hedonic games with the top coalition property. Another instance of hedonic games in which the only farsighted stable sets are single-payoff core/separable allocations is provided by Mauleon, Vannetelbosch, and Vergote (2011), who prove this for a matching game with strict preferences. We do not yet know if such a characterization holds more generally for hedonic games with a nonempty core.

O.3. A FURTHER RESTRICTION ON THE EFFECTIVITY CORRESPONDENCE

We introduce a mild additional restriction on the effectivity correspondence that will be used in the analysis below. Recall that Conditions (i) and (ii) in the main text dealt with coalitional sovereignty for untouched and deviating coalitions. Condition (iii), introduced for simple games, went a step further to consider "affected" coalitions. It stated that if a subcoalition of a winning coalition were to break away, leaving behind a residual that is *still* winning, then the residual stays intact and no individual in that residual can be made immediately worse off by the deviation. That condition made perfect sense, as

[5] If preferences are not strict, it is possible that a core allocation in a hedonic game does not satisfy separability (see Example 3 in Diamantoudi and Xue (2003)) and is, therefore, not a single-payoff farsighted stable set.

there would be the same surplus left for a smaller coalition, but of course it only holds for simple games. What follows is an extension to a more general class.

Let T move. Consider a residual coalition W; that is, $W = S - T$ for some $S \in \pi(x)$, where $S \cap T \neq \emptyset$. We assume that W remains as an unbroken coalition. We also want to assign a "default payoff" to W immediately following T's departure (as emphasized earlier, this can change in subsequent stages). The idea we wish to formalize is that when a coalition moves, it cannot control the payoffs accruing to the residual players; there is an exogenously specified, unique default payoff to the residuals. Formally, a *default function* f^W maps payoffs v in R^W to fresh payoffs $f^W(v)$ on the coalitional frontier $\bar{V}(W)$. The interpretation is that W is a residual after some members from a larger coalition have left: W was enjoying a payoff of v just before that move (i.e., $v = u_W(x)$, where x was the earlier state), and $f^W(v)$ is the payoff vector that it receives just after the move.[6]

A default function f^W is *monotonic* if for every $v \in R^W$, either $f^W(v)_i > v_i$ for all $i \in W$ or $0 < f^W(x)_i < v_i$ for all $i \in W$ with $v_i > 0$, or $f^W(x)_i = v_i$ for all $i \in W$. In words, immediately following the move, the payoffs to every residual member either uniformly go up or go down relative to what they were getting before, or they do not change at all.

Now we can formally state our additional restriction on the effectivity correspondence.

Condition (iv). There exists a monotonic, continuous default function f^W defined for every subcoalition W such that whenever W is a residual in a move from x to y, $W \in \pi(y)$ and $u(y)_W = f^W(u(x)_W)$.

It is easy to check that Condition (iv) implies Condition (iii) for the special class of simple games studied in the main text.

O.4. PROOF OF THE ASSERTION PERTAINING TO EXAMPLE 1

The stable sets of Theorem 2 are essentially singletons, as in the case of the Harsanyi sets. But the two collections could not be more different. Our stable sets are entirely "compatible" with the core, in the sense that every farsighted stable set of Theorem 2 contains a payoff allocation that belongs to the coalition structure core. (Some nonseparable core allocations are excluded, but these are boundary exceptions: every *interior* core allocation is separable and, therefore, compatible with farsighted stability.) In contrast, a Harsanyi set *cannot contain any allocation in the interior of the core*.

Example 1 is designed to emphasize the difference between the two sets. We reproduce that example here:

[6]The default could, in general, depend both on the pre-move state as well as the identity of the moving coalition. It simplifies the exposition to assume that the default payoff depends only on the pre-move payoff profile to W.

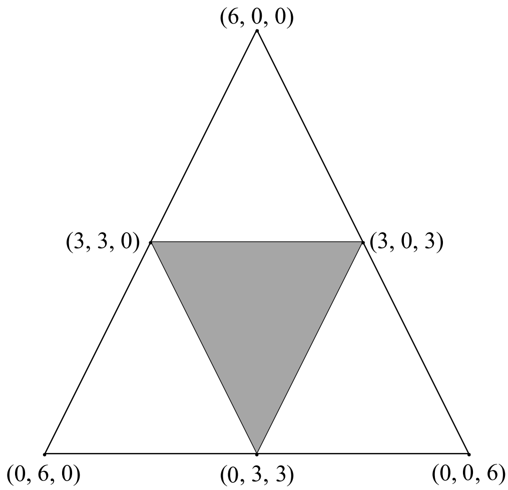

FIGURE O.1.—Comparison of stable sets.

EXAMPLE O.3: Given a three-player TU convex game: $v(S) = 3$ for S such that $|S| = 2$, and $v(N) = 6$. The set of efficient allocations is depicted in Figure O.1. The core is the convex hull of $(3, 3, 0)$, $(0, 3, 3)$, and $(3, 0, 3)$, shown as the inverted central triangle in Figure O.1.

In the main text, we assert that "under a weak additional condition on the effectivity correspondence, there are no other farsighted stable sets in this example, single-payoff or otherwise." The following observation formalizes this assertion:

OBSERVATION 1: *Suppose that the effectivity correspondence satisfies Conditions* (i), (ii), *and* (iv). *Then F is a farsighted stable set of the game in Example* O.3 *if and only if $F = \{(N, u)\}$, where u is an imputation in the interior of the core*.

To prove this observation we shall rely on a lemma.

LEMMA O.1: *Suppose (N, v) is a three-person, superadditive TU game with $v(N) > v(S)$ for all $S \subset N$. If default functions are continuous and monotonic, then any state in a farsighted stable set must consist of the grand coalition and an imputation. Moreover, there is at least one such state in which all players receive a strictly positive payoff*.

PROOF: We begin by showing that if F is a farsighted stable set, then it cannot include a state x' where $u(x') = 0$. Suppose not. This implies that no state

x such that $u(x) \gg 0$ can belong to F. Any such x must, therefore, be farsightedly dominated by some $y \in F$, with $u(y)_i = 0$ for some $i \in N$. Note that every coalition that forms in the move form x to y must provide a strictly positive payoff to all its players.[7] The last coalition to form must, therefore, be a doubleton, that is, $\pi(y) = (\{i, j\}, \{k\})$ with $u(y)_i > 0$ and $u(y)_j > 0$. But then y dominates x', contradicting the supposition that $x' \in F$.

Next, we show that no state with an intermediate coalition structure can be in F. Suppose, to the contrary, that there is a state $x^i = (\pi^i, u^i) \in F$, where $\pi^i = (\{j, k\}, \{i\})$. The case in which $v(\{j, k\}) = 0$ has already been covered, so there are two cases that remain:

(i) The case $(u^i_j, u^i_k) \gg 0$. Consider $x = (N, u)$, where $u \gg u^i$ and $f_{\{j,k\}}(x) = (u^i_j, u^i_k)$. In other words, if from state x player i were to form a singleton coalition, the resulting payoffs are precisely u^i. This is possible because the default function is monotonic and continuous.[8] Because $x^i \in F$, it follows from internal stability that $x \notin F$. So there exists a farsighted objection, initiated by some coalition S from x to x^1, leading eventually to $y \in F$. If $S = \{i\}$, then, since $f_{\{j,k\}}(x) = (u^i_j, u^i_k)$, $x^1 = x^i \in F$ and there cannot be any further change. But this cannot be a farsighted objection since S receives 0 at x^i. Thus $S \neq \{i\}$ and it must include either j or k. Without loss of generality, suppose $j \in S$. Let S' be the coalition in $\pi(y)$ that contains j. Since $u_j(y) > u_j(x) > 0$, S' must be either a doubleton or the grand coalition. Moreover, all players in S' must receive positive payoffs at y. Suppose j were to stand alone at state x^i, precipitating the coalition structure of singletons, resulting in a 0 payoff to each player. From this state, S' can now form and implement y. Since $u_j(y) > u_j(x) > u_j(x^i)$, we have constructed a farsighted objection leading from x^i to $y \in F$, which contradicts the hypothesis that $x^i \in F$.

(ii) The case $x^i = (\pi^i, u^i) \in F$, where $\pi = (\{j, k\}, i)$, $u^i_j > 0$, $u^i_k = u^i_i = 0$. Note that k can induce the finest coalition structure. As we have already argued, this state cannot be stable; there must be a farsighted objection leading to a state in F. If k is part of any coalition in such a move, it must end up with more than 0. But then k could precipitate such a farsighted move from x^i by first inducing the finest coalition structure, which contradicts the hypothesis that $x^i \in F$. Thus, the only farsighted move from the finest coalition structure must involve players i and j, both of whom must end up with strictly positive amounts in the end. But this is not possible as we have already ruled out, in the previous paragraph, the possibility of there being a state in F consisting of a doubleton coalition with strictly positive payoffs.

[7] After all, each player must find it strictly profitable to participate in the final move.

[8] The argument is the following. Let u' be an imputation where $u'_j > u^i_j > 0$ and $u'_k = u^i_k > 0$. Since $u'_j + u'_k > v(\{j, k\})$ and $(u'_j, u'_k) \gg 0$, by monotonicity, $f_{\{j,k\}}(u')_k < u^i_k$. Similarly, for an imputation u'' with $u''_j = u^i_j$ and $u''_k > u^i_k$, monotonicity implies that $f_{\{j,k\}}(u'')_j < u^i_j$. It now follows from the continuity of the default function that there is some imputation, u, on the line segment between u' and u'' such that $f_{\{j,k\}}(u) = u^i_{\{j,k\}}$.

Since every stable state involves the grand coalition, dominance of states outside the farsighted stable set can only come from (stable) states in which all players receive a strictly positive payoff. And there must be at least one such state. Otherwise it will be impossible for a farsighted stable state to block any state, in particular one with a strictly positive payoff allocation.[9] *Q.E.D.*

PROOF OF OBSERVATION 1: It follows from Theorem 2 that $\{(N, u)\}$ is a farsighted stable set for every imputation $u \in \mathring{C}(N, v)$. Suppose there is a farsighted stable set F that does not fit this description. Lemma O.1 tells us that every state in F must consist of the grand coalition and an imputation. Since no farsighted stable set can contain another, F must consist only of imputations that are *not* in $\mathring{C}(N, v)$. By Lemma O.1, there is at least one such state $x = (N, u) \in F$ such that $u \gg 0$, and because $u \notin \mathring{C}(N, v)$, $u_i \geq 3$ for some $i \in N$. Without loss of generality, suppose $u_1 \geq 3$. Consider a state y, where $\pi(y) = (\{2, 3\}, \{1\})$ and $(u_2(y), u_3(y)) \geq (u_2, u_3)$. Such a state exists because $u_2 + u_3 \leq 3$. Of course, by Lemma O.1, $y \notin F$. Neither 2 nor 3 has any interest in moving from y to x, which implies that x does not farsightedly dominate y. So there must be some other $z \in F$ that does, where $\pi(z) = N$ and $u(z) \gg 0$. Either player 2 or player 3 has to be part of the first coalition that moves from y to z. Without loss of generality, suppose this coalition includes player 2 and, therefore, $u_2(z) > u_2(y) \geq u_2$. Now there are two possibilities:

(i) The case $u_1(z) \geq 3$. Since $u \gg 0$ and $u_1 \geq 3$, we know that $u_1 + u_3 > 3$. By monotonicity, this implies that if player 2 were to stand alone at x, player 1 would be left with a payoff strictly less than player 3 (recall that $v(\{1, 3\}) = 3$). Then player 1 can do better by standing alone, precipitating 0 for everyone, and finally having the grand coalition move to z. We have, therefore, constructed a farsighted objection leading from x to z, which contradicts internal stability.

(ii) The case $u_1(z) < 3$. If $u_3(x) \geq u_3(z)$, then player 1 can initiate a farsighted move from z to x. The argument is similar to that used above: player 1 stands alone, counting on player 3 to do the same, followed by a final move by N to x. If $u_3(z) > u_3(x)$, then it is easy to see that players 2 and 3 can leave sequentially to make a move from x to z. In either case, we have a contradiction to the supposition that $x, z \in F$.

Thus, we have shown that there is no farsighted stable set F that does not conform to the description in Theorem 2. *Q.E.D.*

O.5. ANOTHER PERTURBATION OF A SIMPLE GAME WITH VETO

We consider a three-person example of a simple game in which player 1 is a veto player, but needs at least one other player to win. This is exactly Example 2, which is reproduced here as follows.

[9]This is analogous to the fact that in a simple game a farsighted stable set must include a regular state.

EXAMPLE O.4: We have $N = \{1, 2, 3\}$, $v(N) = v(\{1, 2\}) = v(\{1, 3\}) = 1$, and $v(S) = 0$ for all other S. Player 1 can be viewed as the veto player. The coalition structure core has the single payoff $(1, 0, 0)$.

This is a nonoligarchic simple game, with $S^* = \{1\}$ and with $N - S^* = \{23\}$ as the minimal veto coalition. So Corollary 1 applies and we have a full characterization of farsighted stable sets. Each such set must provide a fixed payoff to player 1, strictly smaller than 1, with the remaining surplus split in all divisions between players 2 and 3.

Theorem 3 implies that if this game were to be changed to one that is oligarchic, so that we switch $v(\{1\})$ to equal 1, then there is just one farsighted stable set, consisting of the single payoff $(1, 0, 0)$.[10] The purpose of this section is to exhibit an even smaller perturbation to produce a game with separable allocations, and to show that all the farsighted stable sets described above disappear as soon as the perturbation is in place; the only farsighted stable sets that remain are the single-payoff sets corresponding to each of the separable allocations.

EXAMPLE O.5: Change Example O.4 by setting $v(N) = 1 + \delta$ for any $\delta > 0$. (Think of δ as small.) Now the interior of the core is nonempty; see the shaded area in Figure O.2.

This is not a simple game any more, but the perturbation is "finer" because δ can be made arbitrarily small. Moreover, because every interior core allocation

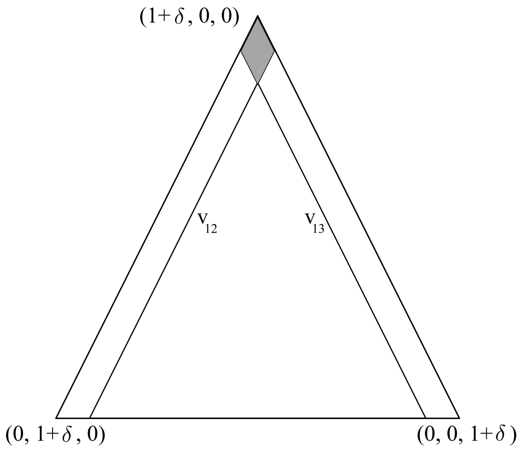

FIGURE O.2.—Perturbed veto game.

[10]Strictly speaking, Theorem 3 cannot be applied directly because this modification violates zero normalization. However, it is easy to see that the proof still applies to this particular example.

is separable, Theorem 2 is back in force again. So every interior core allocation, partnered with the grand coalition, is a farsighted stable set. At the same time, the following observation holds.

OBSERVATION 2: *Under Conditions* (i), (ii), *and* (iv) *on the effectivity correspondence, the only farsighted stable sets in Example* O.5 *are the single-payoff sets described by Theorem* 2. *In particular, the discriminatory stable sets of Example* O.4 *described in Corollary* 1 *must entirely disappear*.

If there is a farsighted stable set that is not a single-payoff state in the interior of the core, then as we saw in the proof of Observation 1, it must be disjoint from $\mathring{C}(N, v)$ and must include a strictly positive payoff state with the grand coalition. In an argument that parallels the proof of Lemma 1, it can be shown, that the only such candidates for farsighted stability are collections of states that all generate the same payoff for individual 1. However, now these sets cannot stretch from one end of the payoff simplex to the other, as they did for the pure veto game. In particular, neither player 2 nor 3 can be given a payoff lower than their marginal contribution to the grand coalition, which is δ. See Figure O.3 for a depiction of the possible sets that remain as potential candidates.

The argument for this claim will be familiar from the proof of Observation 1. Suppose that player 3 is pushed below δ in some farsighted stable state y containing all three players and unilaterally leaves the arrangement. The overall payoff available to players 1 and 2 is now strictly less than $u_1(y) + u_2(y)$. By monotonicity, the resulting default payoff must be lower for both 1 and 2. Next, suppose 1 leaves {12}, precipitating the singletons with zero payoff. In the final step, the grand coalition can make an improvement by moving to a point on the

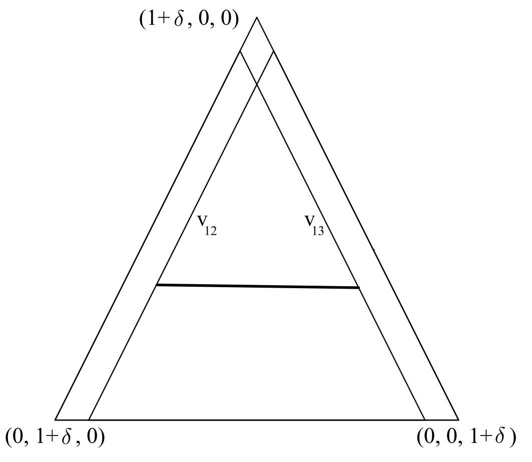

FIGURE O.3.—Possible candidate for farsighted stable set.

horizontal line segment shown in Figure O.3. We have constructed a farsighted objection to y, starting with a departure by player 3, followed by player 1, and then the grand coalition.

But such "truncated" segments cannot be powerful enough to meet the external stability requirement. Consider the leftmost extremity of the line segment in Figure O.3, which is a payoff allocation of the form $p = (c, 1 - c, \delta)$. Let z be a state such that $\pi(z) = (\{1, 2\}, \{3\})$ and $u(z) = (c, 1 - c, 0)$. By Lemma O.1, z cannot be stable. However, farsighted dominance of z must come from some imputation on the truncated line segment. But that is not possible since 1 and 2 will not participate in any such move, and player 3 cannot unilaterally make such a change.

PROOF OF OBSERVATION 2: Suppose there is a farsighted stable set F that does not conform to the description in Theorem 2. Then, as noted in the proof of Observation 1, F must consist only of imputations that are *not* in $\mathring{C}(N, v)$. Moreover, there is at least one such state $x = (N, u) \in F$ such that $u \gg 0$. For the remainder of the proof, fix a stable set F and $x = (N, u) \in F$ such that $u \gg 0$.

CLAIM 1: *For any other $z \in F$ and $u_1(z) \geq u_1(x)$, and if $(u_2(z), u_3(z)) \gg 0$, $u_1(z) = u_1(x)$.*

Because player 1 must be part of every coalition with a positive worth, this follows from the same argument that we used in proving Lemma 1. Note that by Lemma O.1, $\pi(z) = N$ for every $z \in F$, which implies that if z is a stable state that farsightedly dominates some other state, then all players must participate in such a move and $u(z) \gg 0$. For such z, therefore, $u_1(z) = u_1$.[11]

CLAIM 2: *If $u_3 < \delta$ (or $u_2 < \delta$), then x is the unique state in F that provides each player a strictly positive payoff.*

Suppose $u_3 < \delta$ and there is $y \in F$ such that $y \neq x$ and $u(y) \gg 0$. By Claim 1, $u_1(y) = u_1$. This implies that either $u_3(y) < u_3$ or $u_3(y) > u_3$. In the former case, we can construct a farsighted objection from y to x as follows. Suppose player 3 leaves the grand coalition at state y. Because $u_3(y) < u_3 < \delta$, then $u_1(y) + u_2(y) > 1$. Since $u \gg 0$, this implies, by monotonicity, that player 3's departure causes player 1's payoff to fall below u_1. Next, suppose player 1 leaves $\{1, 2\}$, precipitating zero payoffs. The grand coalition can then make a final move to x, restoring player 1's payoff to u_1. Thus x farsightedly dominates y, a contradiction. If $u_3(y) > u_3$, player 3 can trigger a similar farsighted objection, this time leading from x to y. (The case $u_2 < \delta$ is similar.)

[11]We have not ruled out the possibility that there may be $z \in F$, where some players receive 0, but this will not be relevant for the remainder of the proof.

CLAIM 3: *We have* $\min\{u_2, u_3\} \geq \delta$. (*The supposition in Claim* 2 *does not hold*.)

Suppose $u_3 < \delta$, which means that $u_1 + u_2 > 1$. Since $u \notin \mathring{C}(N, v)$, this implies that $u_1 + u_3 \leq 1$. Thus, there exists a state $z = (\{1, 3\}, \{2\}, w)$, where $(w_1, w_3) \geq (u_1, u_3)$ (and $w_2 = 0$). By Lemma O.1, $z \notin F$ and there must be a farsighted objection that leads from z to a stable state in which all players receive a strictly positive payoff. By Claim 2, the only such state is x. However, x cannot farsightedly dominate z. Players 1 or 3 cannot gain in having z replaced with x, so they cannot be part of any coalition that initiates a move from z. And since player 2 is in a singleton, she cannot on her own change the state. This proves that $u_3 \geq \delta$. A similar argument shows that $u_2 \geq \delta$.

CLAIM 4: *Every state* y *with* $\pi(y) = N$, $u_1(y) = u_1$, *and* $\min\{u_2(y), y_3(y)\} \geq \delta$ *belongs to* F.

Suppose not. Then there is y satisfying the stated properties but farsightedly dominated by $z \in F$. By Claim 2, $u_1(z) = u_1(y) = u_1$. Thus, the interests of players 2 and 3 are opposed in the move from y to z; both cannot gain. So $\{23\}$ cannot be the *joint* first mover. Therefore, it is either 2 or 3 who must be the first mover. Say it is 2; then $u_2(z) > u_2(y)$. But then $u_3(z) < u_3(y)$. Moreover, because $u_2(y) \geq \delta$ to begin with, the departure of player 2 does not decrease the aggregate payoff available to $\{13\}$. By monotonicity, both players 1 and 3 must then enjoy an intermediate payoff no smaller than what was available under y. So neither 1 nor 3 will participate in further moves. That eliminates this possibility.

We have shown that if there is farsighted stable set that is not as described in Theorem 2, it must consist of a truncated line segment depicted in Figure O.3 (and possibly some points on the edges of the simplex where player 1 gets more than u_1). The leftmost extremity of this line segment is a payoff allocation of the form $p = (c, 1 - c, \delta)$. Now, consider the state z where $\pi(z) = (\{1, 2\}, \{3\})$ and $u(z) = (c, 1 - c, 0)$. If there is $y \in F$ that farsightedly dominates z, then $u(y) \gg 0$ and y must be on the line segment. Neither player 1 nor player 2 can gain from such a move, so they cannot be part of any coalition that initiates the move from z. But player 3 cannot unilaterally make a change, which leads us to conclude that there cannot be a farsighted domination of z from any state in F, that is, $z \in F$. But this contradicts Lemma O.1. *Q.E.D.*

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