Econometrica Supplementary Material

SUPPLEMENT TO "RANDOM CHOICE AND PRIVATE INFORMATION"

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THIS SUPPLEMENT USES the notation and definitions established in the main paper. Theorems, propositions, and lemmas are numbered S.1, S.2, etc. in this supplement. Numbers without the prefix S refer to those in the main paper.

S.1. REPRESENTATION THEOREMS

In this section, we provide an axiomatic treatment of our model. We first introduce a more general representation. Let $\mathbb{R}^X$ be the space of affine utility functions $u : \Delta X \to \mathbb{R}$ and π be a measure on $\Delta S \times \mathbb{R}^X$. Interpret π as the joint distribution over beliefs and tastes. Assume that u is non-constant π-a.s. The corresponding regularity condition on π is as follows.

DEFINITION: π is *regular* if $q \cdot (u \circ f) = q \cdot (u \circ g)$ with π-measure 0 or 1 for any f, g.

We now define a *random subjective expected utility (RSEU)* representation as follows.

DEFINITION—RSEU Representation: ρ is *represented by* a regular π if, for $f \in F \in \mathcal{K}$,

$$\rho_F(f) = \pi\big\{(q, u) \in \Delta S \times \mathbb{R}^X | q \cdot (u \circ f) \geq q \cdot (u \circ g)\ \forall g \in F\big\}.$$

This is a RUM model where both beliefs and tastes are random. In the individual interpretation, this describes an agent who receives unobservable shocks to both beliefs and tastes. In the group interpretation, this describes a group with unobserved heterogeneity in both beliefs and utilities. Note that in the special case where the marginal distribution of π on utilities is degenerate, this reduces to an information representation where the signal distribution μ is the marginal distribution of π on beliefs.

We now introduce the axioms. The first four are familiar restrictions on RCRs. Let ext F denote the set of extreme acts of $F \in \mathcal{K}$.[1]

AXIOM S.1—Monotonicity: *ρ is monotone*.

[1]Formally, $f \in F$ is an extreme act if it cannot be expressed as $ag + (1 - a)h$ for any g or h in F and $a \in (0, 1)$.

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AXIOM S.2—Linearity: *ρ is linear.*

AXIOM S.3—Extremeness: $\rho_F(\operatorname{ext} F) = 1$.

AXIOM S.4—Continuity: *ρ is continuous*.

Monotonicity follows from the fact that when menus are enlarged by adding new acts, the probability of choosing old acts can only decrease. Linearity and extremeness follow from the fact that the random utilities in our model are linear (i.e., agents are subjective expected utility maximizers). In fact, linearity is *the* version of the independence axiom tested in many experimental settings (see Kahneman and Tversky (1979), for example). Since linear utilities are used for evaluation, extremeness means that mixtures of acts in a menu are never chosen (aside from indifferences). Continuity is standard other than the adjustment for indifferences. Note that if $\mathcal{H}$ is the Borel σ-algebra, then $\mathcal{K}_0 = \mathcal{K}$ and our continuity axiom condenses to the usual continuity.[2] These first four axioms are necessary and sufficient for random expected utility (Gul and Pesendorfer (2006)).

We now present the new axioms. Call a state $s \in S$ *null* if the RCR treats all acts that differ only in that state the same, that is, $\rho_{F\cup f}(f) = \rho_{F\cup g}(g)$ whenever $f(s') = g(s')$ for all $s' \neq s$. Given an act f and state $s \in S$, let $f(s)$ also denote the constant act that yields the payoff $f(s)$ in every state. Call a menu *constant* if it contains only constant acts. Given a menu F and state $s \in S$, let $F(s) = \bigcup_{f\in F} f(s)$ denote the constant menu consisting of $f(s)$ for all $f \in F$. We now present the random choice analog of state-independence.

AXIOM S.5—S-independence: *Suppose* $f(s_1) = f(s_2)$, $F_1(s_1) = F_2(s_2)$, *and* $F_i(s) = f(s)$ *for all* $s \neq s_i$. *If* s_1 *is not null, then* $\rho_{F_1}(f) = \rho_{F_1\cup F_2}(f)$.

S-independence states that if an act f yields the same payoff in states s_1 and s_2, payoffs of menu F_1 in s_1 are the same as those of menu F_2 in s_2, and acts in F_i only differ in s_i, then the probability of choosing f in F_1 is the same as choosing f in $F_1 \cup F_2$. This is because only payoffs in state s_i matter in menu F_i, so state-independent utilities imply that f is optimal in F_1 if and only if it is also optimal in $F_1 \cup F_2$. This is the random choice version of the state-independence axiom.[3] Finally, non-degeneracy rules out the trivial case of universal indifference.

AXIOM S.6—Non-degeneracy: $\rho_F(f) < 1$ *for some* F *and* $f \in F$.

[2]In general, though, the RCR is not continuous over all menus and is in fact discontinuous at precisely those menus that contain indifferences. Nevertheless, every menu is arbitrarily (Hausdorff) close to some menu in $\mathcal{K}_0$, so continuity is preserved over almost all menus.

[3]Under deterministic choice, Theorem S.1 implies that S-independence is equivalent to the standard state-independence axiom in the presence of the other axioms.

We now present the first representation theorem. Axioms S.1–S.6 are necessary and sufficient for a RSEU representation.

THEOREM S.1: *ρ satisfies Axioms S.1–S.6 if and only if it has a RSEU representation.*

Theorem S.1 shows that the characterization of random expected utility can be comfortably extended to the realm of Anscombe–Aumann acts and the axioms of subjective expected utility yield intuitive random choice analogs. For an information representation, we need one additional restriction.

AXIOM S.7—C-determinism: $\rho_F(f) \in \{0, 1\}$ *for constant* F.

C-determinism states that the RCR is deterministic over menus consisting only of constant acts. Under an information representation, choice is stochastic only as a result of varying beliefs. Since beliefs are irrelevant for constant acts, choice must be deterministic. Thus, adding C-determinism results in an information representation.

THEOREM S.2: *ρ satisfies Axioms S.1–S.7 if and only if it has an information representation.*

Note that if we allow the utility u to be constant, then the non-degeneracy axiom can be dropped without loss of generality. However, the uniqueness of μ in the representation would obviously fail in Theorem 1.

We also provide an alternate axiomatization of an information representation. Consider the following condition.

AXIOM S.5′—S-monotonicity: *If* $\rho_{F(s)}(f(s)) = 1$ *for all* $s \in S$, *then* $\rho_F(f) = 1$.

S-monotonicity states that if an act is the best regardless of which state occurs, then it must be chosen for sure. Similar to S-independence, it is the random choice analog of the standard state-monotonicity condition from deterministic choice. As in the Anscombe–Aumann model, replacing S-independence with S-monotonicity results in an alternate axiomatization for an information representation.

THEOREM S.3: *ρ satisfies Axioms S.1–S.4, S.5′, S.6–S.7 if and only if it has an information representation.*

Our treatment so far assumes beliefs are completely subjective. Finally, as in Section 7, we present an axiomatization that includes as a primitive the observed frequency of states $r \in \Delta S$ (assume r has full support as before). A calibrated information representation is an information representation where the signal distribution agrees with the objective r.

DEFINITION—Calibrated Information Representation: ρ is represented by (μ, u) and

$$r = \int_{\Delta S} q\mu(dq).$$

We now introduce a consistency axiom that relates r with RCR. Recall that the conditional worst act $\underline{f}^s$ coincides with the worst act if s occurs and with the best act otherwise.

AXIOM S.8—Consistency: *The mean of* $\underline{f}^s_\rho$ *is* r_s *for all* $s \in S$.

Axioms S.1–S.8 are necessary and sufficient for a calibrated information representation. In other words, calculating means of test functions allows an analyst to check if the RCR is consistent with the objective frequency r.

THEOREM S.4: *ρ satisfies Axioms S.1–S.8 if and only if it has a calibrated information representation.*

PROOF: Let ρ satisfy Axioms S.1–S.7 so by Theorem S.2, ρ is represented by some (μ, u). Without loss of generality, normalize u such that $u(\overline{f}) = 1$ and $u(\underline{f}) = 0$. From Lemma A.6, we know that the mean of $\underline{f}^s_\rho$ is given by

$$\int_{[0,1]} a\, d\underline{f}^s_\rho(a) = 1 - \int_{\Delta S} q \cdot \left(u \circ \underline{f}^s\right)\mu(dq)$$
$$= 1 - \int_{\Delta S} (1 - q_s)\mu(dq) = \int_{\Delta S} q_s\mu(dq).$$

Thus, ρ satisfies Axiom S.8 iff $r = \int_{\Delta S} q_s\mu(dq)$, as desired. *Q.E.D.*

S.1.1. *Proof of Theorem S.1*

Before proving Theorem S.1, we first present a few useful lemmas. The first shows that without loss of generality, we can consider random choice on a subspace of acts without ties. First, note that we can associate each act $f \in H$ with its corresponding vector $f \in \mathbb{R}^{S\times X}$. Consider a collection of acts $f_1, f_2, \dots, f_k$ that are tied with $g_1, g_2, \dots, g_k$, respectively, where

$$z_i := \frac{f_i - g_i}{\|f_i - g_i\|} \neq 0,$$

and $z_i \cdot z_j = 0$ for all $i \neq j$. Let $Z := \text{lin}\{z_1, \dots, z_k\}$ be the linear space spanned by all z_i with $Z = 0$ if no such z_i exists. Let k be maximal in that, for any two acts f and g that are tied, $f - g \in Z$. Lemmas A.3 and A.4 ensure that k is well-defined, and note that the vectors z_i form an orthonormal basis for the space of tied acts.

We now define a projection $\varphi : H \to \mathbb{R}^{S\times X}$ that maps acts to a subspace orthogonal to Z and hence contains no ties. Formally, define

$$\varphi(f) := f - \sum_i (f \cdot z_i) z_i,$$

and let $W := \text{lin}(\varphi(H))$ be the linear space containing the image of φ with orthonormal basis $\{w_1, \dots, w_m\}$. Note that $\{z_1, \dots, z_k, w_1, \dots, w_m\}$ form an orthonormal basis for the space of all acts.

LEMMA S.1: *Suppose ρ is monotonic and linear.*
(1) $\varphi(f) = \varphi(g)$ *if and only if f and g are tied.*
(2) $w \cdot \varphi(f) = w \cdot f$ *for all $w \in W$.*

PROOF: We first prove (1). Suppose f and g are tied so $f - g \in Z$ by the definition of Z. Thus,

$$f = g + \sum_i \alpha_i z_i$$

for some coefficients α_i. This implies that

$$\begin{aligned}\varphi(f) &= g + \sum_i \alpha_i z_i - \sum_i \left[\left(g + \sum_j \alpha_j z_j\right) \cdot z_i\right] z_i \\ &= g - \sum_i (g \cdot z_i) z_i = \varphi(g),\end{aligned}$$

as desired. For the converse, suppose $\varphi(f) = \varphi(g)$ so

$$\begin{aligned}f - \sum_i (f \cdot z_i) z_i &= g - \sum_i (g \cdot z_i) z_i, \\ f - g &= \sum_i \big((f - g) \cdot z_i\big) z_i \in Z.\end{aligned}$$

Thus, f and g are tied as desired. This proves (1).

We now prove (2). Note that $\varphi(f) \cdot z_i = 0$ for all $f \in H$. Since $W = \text{lin}(\varphi(H))$ and φ is a linear mapping, $w \cdot z_i = 0$ for all $w \in W$. Thus, for all $w \in W$,

$$w \cdot \varphi(f) = w \cdot \left(f - \sum_i (f \cdot z_i) z_i\right) = w \cdot f,$$

proving (2). *Q.E.D.*

The next lemma shows that we can apply the Gul and Pesendorfer (2006) random expected utility representation theorem to obtain a random utility representation.

LEMMA S.2: *If ρ satisfies Axioms S.1–S.4, then there exists a measure ν on W such that*

$$\rho_F(f) = \nu\{w \in W | w \cdot f \geq w \cdot g \ \forall g \in F\}.$$

PROOF: Recall that W has basis $\{w_1, \dots, w_m\}$. Let Δ be the m-dimensional probability simplex and define the mapping $T : H \to \Delta$, where

$$[T(f)]_i = \lambda\left[\varphi(f) \cdot \left(w_i - \sum_j w_j\right)\right] + \frac{1}{m}.$$

Note that since H is bounded, we can always find some small enough λ such that $[T(f)]_i \geq 0$ for all i and $\sum_i [T(f)]_i = 1$ so $T(H) \subset \Delta$. Now, for each finite set of lotteries $D \subset \Delta$, we can find a $p^* \in \Delta$ and $a \in (0, 1)$ such that $Dap^* \subset T(H)$. Thus, we can define an RCR τ on Δ such that

$$\tau_D(p) := \rho_F(f),$$

where $T(F) = Dap^*$ and $T(f) = pap^*$. Linearity and Lemma S.1 ensure that τ is well-defined.

Since the mappings φ and T are both affine, Axioms S.1–S.4 correspond exactly to the axioms of Gul and Pesendorfer (2006) on Δ. Thus, by their Theorem 3, there exists a measure $\tilde{\nu}$ on Δ such that, for any menu F that contain no ties,

$$\begin{aligned}\rho_F(f) &= \tau_{T(F)}\big(T(f)\big) \\ &= \tilde{\nu}\big\{v \in \Delta | v \cdot \big(T(f)\big) \geq v \cdot \big(T(g)\big) \ \forall g \in F\big\}.\end{aligned}$$

Now, note that

$$\begin{aligned}v \cdot \big(T(f)\big) &= \sum_i v_i\left(\lambda\left[\varphi(f) \cdot \left(w_i - \sum_j w_j\right)\right] + \frac{1}{m}\right) \\ &= \lambda\varphi(f) \cdot \sum_i (v_i - 1)w_i + \frac{1}{m} \\ &= \lambda\varphi(f) \cdot \zeta(v) + \frac{1}{m},\end{aligned}$$

where $\zeta(v) := \sum_i (v_i - 1)w_i \in W$. Hence, if we let $\nu := \tilde{\nu} \circ \zeta^{-1}$ be the measure on W induced by $\tilde{\nu}$, then

$$\begin{aligned}\rho_F(f) &= \tilde{\nu}\{v \in \Delta | \varphi(f) \cdot \zeta(v) \geq \varphi(g) \cdot \zeta(v)\ \forall g \in F\} \\ &= \nu\{w \in W | \varphi(f) \cdot w \geq \varphi(f) \cdot w\ \forall g \in F\} \\ &= \nu\{w \in W | w \cdot f \geq w \cdot g\ \forall g \in F\},\end{aligned}$$

where the last equality follows from Lemma S.1.

Now, for any menu $F \in \mathcal{K}$, let $F^* \subset F$ denote the submenu that does not contain ties. By Lemma S.1, if f and f^* are tied, then

$$w \cdot f = w \cdot \varphi(f) = w \cdot \varphi(f^*) = w \cdot f^*.$$

Thus, by Lemma A.3, for any $f \in F$,

$$\begin{aligned}\rho_F(f) = \rho_{F^*}(f^*) &= \nu\{w \in W | w \cdot f^* \geq w \cdot g^*\ \forall g^* \in F^*\} \\ &= \nu\{w \in W | w \cdot f \geq w \cdot g\ \forall g \in F\},\end{aligned}$$

as desired. *Q.E.D.*

The next lemma shows that non-degeneracy ensures that there is at least one state that is not null. We will use the shorthand notation $\rho(F, G) := \rho_{F \cup G}(F)$.

LEMMA S.3: *If ρ is non-degenerate, then there exists a non-null state.*

PROOF: Suppose ρ is non-degenerate but all states are null and consider any two acts f and g. Order the states $S = \{s_1, \ldots, s_n\}$ and define a sequence of acts $f^i \in H$ for $1 \leq i \leq n$ such that

$$f^i(s_j) = \begin{cases} g(s_j) & \text{if } j \leq i, \\ f(s_j) & \text{if } j > i. \end{cases}$$

In other words, f^i coincides with g on states s_1 to s_i and with f on states s_{i+1} to s_n. Since f^i and f^{i+1} differ only on one state and every state is null,

$$\rho(f^i, f^{i+1}) = 1 = \rho(f^{i+1}, f^i).$$

Thus, f^i and f^{i+1} are tied for all i, so by Lemma A.2, f and g are tied. This implies $\rho_F(f) = 1$ for any $f \in F$, contradicting non-degeneracy, as desired. *Q.E.D.*

We are now ready to prove Theorem S.1. We wish to show the following are equivalent:

(1) ρ satisfies Axioms S.1–S.6,
(2) ρ is represented by some regular π.

Suppose (1) is true. By Lemma S.2, we know there is a measure ν on W such that

$$\rho_F(f) = \nu\left\{w \in W \,\Big|\, \sum_s w_s\big(f(s)\big) \geq \sum_s w_s\big(g(s)\big)\ \forall g \in F\right\},$$

where $w_s : \Delta X \to \mathbb{R}$ denotes the linear utility corresponding to $w \in W$ in state $s \in S$. Note that if a state $s_0 \in S$ is null, then for any two acts f and g where $f(s) = g(s)$ for all $s \neq s_0$,

$$1 = \rho(f, g) = \nu\big\{w \in W | w_{s_0}\big(f(s)\big) \geq w_{s_0}\big(g(s)\big)\ \forall g \in F\big\}.$$

Since this is true for all acts f and g, this means that w_{s_0} is constant ν-a.s. Hence, without loss of generality, we can set $w_s = 0$ for any null state $s \in S$.

Let $S^* \subset S$ be the set of non-null states, which is non-empty by Lemma S.3. Now, consider two non-null states $s_1 \in S^*$ and $s_2 \in S^*$. For any $p \in \Delta X$, $q \in \Delta X$, and $i \in \{1, 2\}$, define the following sets:

$$H_i(p, q) := \big\{w \in W | w_{s_i}(p) > w_{s_i}(q)\big\},$$
$$I_i(p, q) := \big\{w \in W | w_{s_i}(p) = w_{s_i}(q)\big\}.$$

Also, define

$$R(p, q) := \bigcap_i H_i(p, q) \cup \bigcap_i I_i(p, q) \cup \bigcap_i H_i(q, p).$$

We will show that for any two lotteries p and q, $\nu(R(p, q)) = 1$, that is, $w_{s_1}(p) \geq w_{s_1}(q)$ iff $w_{s_2}(p) \geq w_{s_2}(q)$ ν-a.s.

Fix two lotteries p and q and consider acts f, g, and h where

$$p = f(s_1) = f(s_2) = g(s_2) = h(s_1),$$
$$q = g(s_1) = h(s_2),$$

and $f(s) = g(s) = h(s)$ for all $s \notin \{s_1, s_2\}$. Since s_1 is not null, S-independence implies that

$$\begin{aligned}
&\nu\big\{w \in W | w_{s_1}(p) \geq w_{s_1}(q)\big\} \\
&\quad = \rho(f, g) = \rho(f, g \cup h) \\
&\quad = \nu\big\{w \in W | w_{s_1}(p) \geq w_{s_1}(q) \text{ and } w_{s_2}(p) \geq w_{s_2}(q)\big\}.
\end{aligned}$$

This implies that

$$\nu\big(\big(H_1(p, q) \cup I_1(p, q)\big) \cap H_2(q, p)\big) = 0.$$

Since s_2 is not null, S-independence also implies that $\rho(f, h) = \rho(f, g \cup h)$, so

$$\nu\big(\big(H_2(p, q) \cup I_2(p, q)\big) \cap H_1(q, p)\big) = 0.$$

By symmetric argument, we also have

$$\nu\big(\big(H_1(q, p) \cup I_1(p, q)\big) \cap H_2(p, q)\big) = 0,$$
$$\nu\big(\big(H_2(q, p) \cup I_2(p, q)\big) \cap H_1(p, q)\big) = 0.$$

This all implies that

$$\nu\big(R(p, q)\big) = \nu\big(H_1(p, q) \cup I_1(p, q) \cup H_1(q, p)\big) = 1,$$

as desired.

We now show that ν-a.s. $w_{s_1}(p) \geq w_{s_1}(q)$ iff $w_{s_2}(p) \geq w_{s_2}(q)$ for all p and q. In other words, we will show that $\nu(R) = 1$, where

$$R := \big\{w \in W | w_{s_1}(p) \geq w_{s_1}(q) \text{ iff } w_{s_2}(p) \geq w_{s_2}(q) \\ \text{for all } \{p, q\} \subset \Delta X\big\}.$$

Let $C \subset \Delta X$ be a countable dense subset of ΔX and note that

$$R \subset R^* := \bigcap_{(p,q) \in C \times C} R(p, q).$$

We will show that $R = R^*$. Suppose otherwise, so there is some $w \in R^* \setminus R$. Without loss of generality, we can find two lotteries p and q where $w_{s_1}(p) \geq w_{s_1}(q)$ but $w_{s_1}(p) < w_{s_1}(q)$. Since s_1 is not null, we can find sequences of lotteries $p_k \in C$ and $q_k \in C$ where $p_k \to p$, $q_k \to q$, and

$$w_{s_1}(p_k) \geq w_{s_1}(p) \geq w_{s_1}(q) \geq w_{s_1}(q_k).$$

Since $w \in R^*$, this means that $w_{s_2}(p_k) \geq w_{s_2}(q_k)$ for all k, so by continuity, $w_{s_2}(p) \geq w_{s_2}(q)$, yielding a contradiction. Thus, $R = R^*$, so

$$\nu(R) = \nu\big(R^*\big) = 1,$$

as $\nu(R(p, q)) = 1$ for all $(p, q) \in C \times C$.

Fix some non-null state $s^* \in S^*$ so, for any other $s \in S^*$, we know that ν-a.s. $w_{s^*}(p) \geq w_{s^*}(q)$ iff $w_s(p) \geq w_s(q)$ for all p and q. In other words, w_{s^*} and w_s represent the same linear order over lotteries so ν-a.s.

$$w_s = a_s w_{s^*} + b_s.$$

Define the mapping $\phi_1 : W \to \Delta S$ where

$$\big[\phi_1(w)\big](s) = \frac{a_s}{\sum_{s \in S^*} a_s}$$

if $s \in S^*$ and $[\phi_1(w)](s) = 0$ otherwise. Also let $\phi_2 : W \to \mathbb{R}^X$ be such that $\phi_2(w) = w_{s^*}$ and $\phi = (\phi_1, \phi_2)$. Thus,

$$\begin{aligned}\rho_F(f) = \nu\Big\{w \in W \Big| & \\ & \sum_{s \in S^*}\big(a_s w_s\big(f(s)\big) + b_s\big) \geq \sum_{s \in S^*}\big(a_s w_s\big(g(s)\big) + b_s\big)\ \forall g \in F\Big\} \\ &= \nu\big\{w \in W | \phi_1(w) \cdot \big(\phi_2(w) \circ f\big) \geq \phi_1(w) \cdot \big(\phi_2(w) \circ g\big)\ \forall g \in F\big\} \\ &= \pi\big\{(q, u) \in \Delta S \times \mathbb{R}^X | q \cdot (u \circ f) \geq q \cdot (u \circ g)\ \forall g \in F\big\},\end{aligned}$$

where $\pi := \nu \circ \phi^{-1}$ is the measure on $\Delta S \times \mathbb{R}^X$ induced by ν. Finally, we show that π is regular. Consider any two acts f and g and note that if they are tied, then $q \cdot (u \circ f) = q \cdot (u \circ g)$ π-a.s. On the other hand, if they are not tied, then

$$\begin{aligned}&\pi\big\{(q, u) \in \Delta S \times \mathbb{R}^X | q \cdot (u \circ f) = q \cdot (u \circ g)\big\} \\ &\quad = \rho(f, g) - \big(1 - \rho(g, f)\big) = 0.\end{aligned}$$

Thus π is regular and ρ is represented by π, proving (2).

Now, suppose (2) is true. Monotonicity, linearity, and extremeness all follow trivially from the representation. To show non-degeneracy, suppose ρ is degenerate. Thus, for any two constant acts f and g,

$$1 = \rho(f, g) = \rho(g, f) = \pi\big\{(q, u) \in \Delta S \times \mathbb{R}^X | u \circ f = u \circ g\big\},$$

so u is constant, yielding a contradiction. Thus, non-degeneracy is satisfied. To show S-independence, suppose $p = f(s_1) = f(s_2)$, $D = F_1(s_1) = F_2(s_2)$, and $F_i(s) = f(s)$ for all $s \neq s_i$. Since u is non-constant, every state is not null and

$$\rho_{F_i}(f) = \pi\big\{(q, u) \in \Delta S \times \mathbb{R}^X | u(p) \geq u(q) \text{ for all } q \in D\big\} = \rho_{F_1 \cup F_2}(f).$$

This proves S-independence.

Finally, we show continuity. Let $F_k \to F$ where $\{F_k, F\} \subset \mathcal{K}_0$. Note that for any $\{f, g\} \subset F_k$, f and g are not tied. Since π is regular, this implies that $q \cdot (u \circ f) = q \cdot (u \circ g)$ with π-measure zero. Now, define

$$\mathcal{I} := \bigcup_{\{f,g\} \subset F_k \cup F} \big\{(q, u) \in \Delta S \times \mathbb{R}^X | q \cdot (u \circ f) = q \cdot (u \circ g)\big\}$$

as the set of all beliefs and utilities that rank some $\{f, g\} \subset F_k \cup F$ the same. Note that $\pi(\mathcal{I}) = 0$, so if we let $\mathcal{Q} := \Delta S \times \mathbb{R}^X \setminus \mathcal{I}$, then $\mu(\mathcal{Q}) = 1$. Let π^* be the restriction of π on $\mathcal{Q}$. We will now define random variables $\xi_k : \mathcal{Q} \to H$ and $\xi : \mathcal{Q} \to H$ that have distributions ρ_{F_k} and ρ_F, respectively. For each F_k, let $\xi_k : \mathcal{Q} \to H$ be such that

$$\xi_k(q, u) := \arg\max_{f \in F_k} q \cdot (u \circ f),$$

and define ξ similarly for F. Note that these are well-defined because there exists a unique maximizer for every $(q, u) \in \mathcal{Q}$. Now, for any measurable set $E \subset H$,

$$\begin{aligned} \xi_k^{-1}(E) &= \big\{(q, u) \in \mathcal{Q} | \xi_k(q \cdot u) \in E \cap F_k\big\} \\ &= \bigcup_{f \in E \cap F_k} \big\{(q, u) \in \mathcal{Q} | q \cdot (u \circ f) > q \cdot (u \circ g)\ \forall g \in F_k\big\}, \end{aligned}$$

which is measurable. Hence, ξ_k and ξ are random variables. Note that

$$\begin{aligned} \pi^* \circ \xi_k^{-1}(E) &= \sum_{f \in E \cap F_k} \pi^*\big\{(q, u) \in \mathcal{Q} | q \cdot (u \circ f) > q \cdot (u \circ g)\ \forall g \in F_k\big\} \\ &= \sum_{f \in E \cap F_k} \pi\big\{(q, u) \in \mathcal{Q} | q \cdot (u \circ f) \geq q \cdot (u \circ g)\ \forall g \in F_k\big\} \\ &= \rho_{F_k}(E \cap F_k) \\ &= \rho_{F_k}(E), \end{aligned}$$

so ρ_{F_k} and ρ_F are the distributions of ξ_k and ξ, respectively. Note that for any $(q, u) \in \mathcal{Q} \subset \Delta S \times \mathbb{R}^X$, $q \cdot (u \circ f)$ is bounded and thus continuous in f. Hence, by the maximum theorem, $\xi_k(q, u) = \arg\max_{f \in F_k} q \cdot (u \circ f)$ is continuous in F_k. Since $F_k \to F$, $\xi_k \to \xi$ π^*-a.s. and since a.s. convergence implies convergence in distribution, $\rho_{F_k} \to \rho_F$ as desired.

S.1.2. *Proof of Theorem S.2*

Before proving Theorem 5, we first show a useful lemma. Let τ be an RCR on ΔX. We say τ is *deterministic* if $\tau_D(p) \in \{0, 1\}$ for all sets of lotteries D. We show that if τ has a random expected utility representation and is deterministic, then it reduces to a standard expected utility representation.

LEMMA S.4: *Suppose τ is deterministic and there is a regular measure ν on $\mathbb{R}^X$ such that*

$$\tau_D(p) = \nu\big\{u \in \mathbb{R}^X | u(p) \geq u(q)\ \forall q \in D\big\}.$$

Then ν has a degenerate distribution on some $u^* \in \mathbb{R}^X$.

PROOF: Note that if τ is degenerate, that is $\tau_D(p) = 1$ for all D so we can just set $u^* = 0$. Thus, assume τ is non-degenerate and note that, for any two p and q,

$$\tau(p, q) = \nu\big\{u \in \mathbb{R}^X | u(p) \geq u(q)\big\} \in \{0, 1\}.$$

Since X is finite, we can find two degenerate lotteries y and z such that $\tau(y, x) = \tau(x, z) = 1$ for all $x \in X$. Note that $u(y) > u(z)$ ν-a.s. by non-degeneracy. By linearity, this implies that $\tau(y, p) = \tau(p, z) = 1$ for all $p \in \Delta X$. By determinism and continuity, for any $p \in \Delta X$, we can find an $a_p \in [0, 1]$ such that

$$\tau(y a_p z, p) = \tau(p, y a_p z) = 1.$$

This implies that

$$1 = \nu\big\{u \in \mathbb{R}^X | u(p) = a_p u(y) + (1 - a_p) u(z)\big\}.$$

Since we can always normalize u such that $u(y) = 1$ and $u(z) = 0$, this means that $u(p) = a_p$ ν-a.s. so ν has a degenerate distribution on some $u^* \in \mathbb{R}^X$, as desired. *Q.E.D.*

We are now ready to prove Theorem S.2. We wish to show the following are equivalent:

(1) ρ satisfies Axioms S.1–S.7,

(2) ρ is represented by some (μ, u).

Suppose (1) is true, so by Theorem S.1, ρ is represented by some regular π. Let π_1 and π_2 be the marginal distributions of π on ΔS and $\mathbb{R}^X$, respectively. Define the RCR τ on ΔX such that for every constant menu $F = D$,

$$\tau_D(p) = \rho_D(p) = \pi_2\big\{u \in \mathbb{R}^X | u(p) \geq u(q)\ \forall q \in D\big\}.$$

Since C-determinism implies τ is deterministic, Lemma S.4 implies that π_2 has a degenerate distribution on some u. If we let $\mu = \pi_1$, then μ is regular and ρ is represented by (μ, u), proving (2).

Now, suppose (2) is true, so by Theorem S.1, ρ satisfies Axioms S.1–S.6. To show that ρ satisfies C-determinism, note that for any constant menu F,

$$\begin{aligned}\rho_F(f) &= \mu\big\{q \in \Delta S | u(f) \geq u(g)\ \forall g \in F\big\} \\ &= \begin{cases} 1 & \text{if } u(f) \geq u(g)\ \forall g \in F, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This proves (1), as desired.

S.1.3. *Proof of Theorem S.3*

Let ρ satisfy Axioms S.1–S.4 and S.6–S.7. We wish to show the following are equivalent:

(1) ρ satisfies S-independence,
(2) ρ satisfies S-monotonicity.

Suppose (1) is true, so by Theorem S.2, ρ is represented by some (μ, u). To show that ρ satisfies S-monotonicity, suppose $\rho_{F(s)}(f(s)) = 1$ for all $s \in S$. This implies that $u(f(s)) \geq u(g(s))$ for all $g \in F$ and $s \in S$ so

$$\rho_F(f) = \mu\{q \in \Delta S | q \cdot (u \circ f) \geq q \cdot (u \circ g)\ \forall g \in F\} = 1,$$

proving (2), as desired.

Now, suppose (2) is true. From Lemma S.2, define the RCR τ on ΔX such that

$$\tau_D(p) = \rho_D(p) = \nu\left\{w \in W \Big| \sum_s w_s(p) \geq \sum_s w_s(q)\ \forall q \in D\right\}.$$

C-determinism implies that τ is deterministic, so by Lemma S.4, there is some $u^* \in \mathbb{R}^X$ such that $\tau(p, q) = 1$ iff $u^*(p) \geq u^*(q)$. Now, for every state $s \in S$, define the RCR τ^s on ΔX such that

$$\tau^s_D(p) = \rho_F(f) = \nu\{w \in W | w_s(p) \geq w_s(q)\ \forall q \in D\},$$

where $f(s) = p$, $F(s) = D$, and $F(s') = f(s')$ for all $s' \neq s$.

We now show that τ^s is deterministic for every $s \in S$. Suppose otherwise, so we can find some $p \in D$ and $q \in D$ such that $\tau^s_D(p) \in (0, 1)$ and $\tau^s_D(q) \in (0, 1)$. By monotonicity, this implies that $\tau^s(p, q) < 1$ and $\tau^s(q, p) < 1$. By the contrapositive of S-monotonicity, it must be that $\tau(p, q) < 1$ and $\tau(q, p) < 1$, contradicting C-determinism. Thus, τ^s is deterministic, so by Lemma S.4, there is some $u_s \in \mathbb{R}^X$ such that $\tau^s(p, q) = 1$ iff $u_s(p) \geq u_s(q)$.

Next, we show that if s_1 is not null, then $u_{s_1}(p) \geq u_{s_1}(q)$ implies $u_{s_2}(p) \geq u_{s_2}(q)$. Suppose otherwise, so $u_{s_2}(p) < u_{s_2}(q)$. Let $f(s_1) = f(s_2) = p$, $f_i(s_i) = q$, and $f_i(s) = f(s)$ for all $s \neq s_i$. Note that

$$\rho(f, f_1) = \nu\{w \in W | u_{s_1}(p) \geq u_{s_1}(q)\} = 1,$$
$$\rho(f, f_2) = \nu\{w \in W | u_{s_2}(p) \geq u_{s_2}(q)\} = 0.$$

By the contrapositive of S-monotonicity and C-determinism, $\rho(q, p) = 0$. This implies that $u^*(q) > u^*(p)$. By S-monotonicity again, $\rho(f_1, f) = 1$ so $u_{s_1}(p) = u_{s_1}(q)$. Suppose there exists some r where $u_{s_1}(p) > u_{s_1}(r)$. Let g be the act where $g(s_1) = r$ and $g(s) = f(s)$ for all $s \neq s_1$ so $\rho(g, f) = 0$. By the contrapositive of S-monotonicity again, $\rho(r, p) = 0$ so $u^*(q) > u^*(p) > u^*(r)$. Now, we can find some $a \in (0,1)$ such that $u^*(p) = u^*(qar)$. By S-monotonicity, this means that

$$u_{s_1}(p) = u_{s_1}(qar) = au_{s_1}(q) + (1-a)u_{s_1}(r).$$

However, $u_{s_1}(p) = u_{s_1}(q) > u_{s_1}(r)$, yielding a contradiction. Hence, there can be no such r where $u_{s_1}(p) = u_{s_1}(q) > u_{s_1}(r)$. By symmetric argument, there can be no such r where $u_{s_1}(r) > u_{s_1}(p) = u_{s_1}(q)$. Since the same argument applies for any p and q, this means that s_1 must be null, yielding a contradiction. Thus, $u_{s_1}(p) \geq u_{s_1}(q)$ implies $u_{s_2}(p) \geq u_{s_2}(q)$ whenever s_1 is not null.

Finally, we show that ρ satisfies S-independence. Let s_1 be not null, $p = f(s_1) = f(s_2)$, $D = F_1(s_1) = F_2(s_2)$, and $F_i(s) = f(s)$ for all $s \neq s_i$. Now,

$$\rho_{F_1}(f) = \nu\{w \in W | u_{s_1}(p) \geq u_{s_1}(q)\ \forall q \in D\},$$

$$\rho_{F_1 \cup F_2}(f) = \nu\{w \in W | u_{s_1}(p) \geq u_{s_1}(q) \text{ and } u_{s_2}(p) \geq u_{s_2}(q)\ \forall q \in D\}.$$

Note that if s_2 is null, then $\rho_{F_1}(f) = \rho_{F_1 \cup F_2}(f)$ trivially. Thus assume s_2 is not null, so from above, $u_{s_1}(p) \geq u_{s_1}(q)$ iff $u_{s_2}(p) \geq u_{s_2}(q)$. This implies that $\rho_{F_1}(f) = \rho_{F_1 \cup F_2}(f)$, proving (2), as desired.

S.2. RELATION TO AHN AND SARVER (2013)

In this section, we relate our results to those of Ahn and Sarver (2013). They introduced a condition called consequentialism to link choice behavior from the two time periods.[4] Consequentialism translates into the following in our setting.

AXIOM S.9—Consequentialism: *If* $\rho_F = \rho_G$, *then* $F \sim G$.

However, consequentialism fails as a sufficient condition for linking the two choice behaviors in our setup. This is demonstrated in the following.

EXAMPLE S.1: Let $S = \{s_1, s_2\}$, $X = \{x, y\}$, and $u(ax + (1-a)y) = a$. Associate each belief $q \in \Delta S$ with $t \in [0,1]$, where $t = q_{s_1}$ is the belief in state s_1. Let μ have the uniform distribution and ν have density $6t(1-t)$. Thus, μ is more informative than ν. Let $\succeq$ be represented by (μ, u) and ρ be represented by (ν, u). We show that $(\succeq, \rho)$ satisfies consequentialism. Consider two menus F and G, where $\rho_F = \rho_G$ with support $F^+ \subset F \cap G$. Since $f \in F \setminus F^+$ implies it is dominated by acts in F^+ μ-a.s., it is also dominated by acts in F^+ ν-a.s. Thus, $F \sim F^+$. A symmetric analysis for G yields $G \sim F^+$, so $F \sim G$, proving consequentialism. However, μ and ν are clearly different distributions.

[4]Their second axiom deals with indifferences which we resolve using non-measurability.

The reason for why consequentialism fails in the Anscombe–Aumann setup is that the representation of DLR is more permissive than that of DLST. In the lottery setup, if consequentialism is satisfied, then this extra freedom allows us to construct an ex ante representation that is completely consistent with that of ex post random choice. On the other hand, information is uniquely identified in the representation of DLST, so this lack of flexibility prevents us from performing this construction even when consequentialism is satisfied. A stronger condition is needed to perfectly equate choice behavior from the two time periods.

AXIOM S.10—Strong Consequentialism: *If F_ρ and G_ρ share the same mean, then $F \sim G$.*

The following demonstrates why this is a strengthening of consequentialism.

LEMMA S.5: *For ρ monotonic, $\rho_F = \rho_G$ implies $F_\rho = G_\rho$.*

PROOF: Let ρ be monotonic and let F^+ be the support of ρ_F. We first show that $F^+_\rho = F_\rho$. Let $F^0 := F \setminus F^+$, and for $a \in [0, 1]$, monotonicity yields

$$0 = \rho_F(F^0) \geq \rho_{F \cup f^a}(F^0).$$

Note that by Lemma A.2, both F^0 and F^+ are $\mathcal{H}_F$-measurable. First, suppose f^a is tied with nothing in F. Hence,

$$\rho_{F^+ \cup f^a}(F^+) + \rho_{F^+ \cup f^a}(f^a) = 1 = \rho_{F \cup f^a}(F^+) + \rho_{F \cup f^a}(f^a).$$

By monotonicity, $\rho_{F^+ \cup f^a}(F^+) \geq \rho_{F \cup f^a}(F^+)$ and $\rho_{F^+ \cup f^a}(f^a) \geq \rho_{F \cup f^a}(f^a)$, so

$$F^+_\rho(a) = \rho_{F^+ \cup f^a}(F^+) = \rho_{F \cup f^a}(F^+) = \rho_{F \cup f^a}(F) = F_\rho(a).$$

Now, if f^a is tied with some act in F, then by Lemma A.3 and monotonicity,

$$1 = \rho_F(F^+) = \rho_{F \cup f^a}(F^+) \leq \rho_{F^+ \cup f^a}(F^+).$$

Thus, $F^+_\rho(a) = 1 = F_\rho(a)$ so $F^+_\rho = F_\rho$.

Now, suppose $\rho_F = \rho_G$ for two menus F and G. Since $\rho_F(f) > 0$ iff $\rho_G(f) > 0$, $F^+ = G^+$. We thus have

$$F_\rho = F^+_\rho = G^+_\rho = G_\rho. \qquad Q.E.D.$$

Thus, if strong consequentialism is satisfied, then consequentialism must also be satisfied as $\rho_F = \rho_G$ implies $F_\rho = G_\rho$, which implies that F_ρ and G_ρ must have the same mean. Strong consequentialism delivers the corresponding connection between ex ante and ex post choice behaviors that consequentialism delivered in the lottery setup.

PROPOSITION S.1: *Let $\succeq$ and ρ be represented by (μ, u) and (ν, v), respectively. Then the following are equivalent*:

(1) $(\succeq, \rho)$ *satisfies strong consequentialism*,
(2) $F \succeq G$ *if and only if* $F \succeq_\rho G$,
(3) $(\mu, u) = (\nu, \alpha v + \beta)$ *for* $\alpha > 0$.

PROOF: Note that the equivalence of (2) and (3) follows from Theorem 2 and the uniqueness properties of the subjective learning representation (see Theorem 1 of DLST). That (2) implies (1) is immediate, so we only need to prove that (1) implies (2).

Suppose (1) is true. Since $\succeq_\rho$ is represented by (ν, v), we have $F \sim_\rho G$ implies $F \sim G$. Without loss of generality, we assume both u and v are normalized. First, consider only constant acts and let $\underline{f}$ and $\overline{f}$ be the worst and best acts under v. Now, for any constant act f, we can find $a \in [0, 1]$ such that $\underline{f} a \overline{f} \sim_\rho f$ which implies $\underline{f} a \overline{f} \sim f$. Thus

$$v(f) = v(\underline{f} a \overline{f}) = 1 - a$$

and

$$\begin{aligned} u(f) &= au(\underline{f}) + (1-a)u(\overline{f}) = \big(1 - v(f)\big)u(\underline{f}) + v(f)u(\overline{f}) \\ &= \big(u(\overline{f}) - u(\underline{f})\big)v(f) + u(\underline{f}) \end{aligned}$$

for all constant f. Thus, $u = \alpha v + \beta$ where $\alpha := u(\overline{f}) - u(\underline{f})$ and $\beta := u(\underline{f})$. Since $\underline{f} \cup \overline{f} \sim_\rho \overline{f}$ implies $\underline{f} \cup \overline{f} \sim \overline{f}$, we have $u(\overline{f}) \geq u(\underline{f})$ so $\alpha \geq 0$. If $\alpha = 0$, then $u = \beta$, contradicting the fact that u is non-constant. Thus, $\alpha > 0$.

We can now assume without loss of generality that $\succeq_\rho$ is represented by (ν, u). Now, given any $F \in \mathcal{K}$, we can find some constant act f such that $F \sim_\rho f$, which implies $F \sim g$. Thus,

$$\int_{\Delta S} \sup_{f \in F} q \cdot (u \circ f)\nu(dq) = u(g) = \int_{\Delta S} \sup_{f \in F} q \cdot (u \circ f)\mu(dq),$$

so $\succeq_\rho$ and $\succeq$ represent the same preference, which implies (2). Thus, (1), (2), and (3) are all equivalent. *Q.E.D.*

S.3. RELATION TO CAPLIN AND MARTIN (2015)

In this section, we relate Theorem 5 to Caplin and Martin (2015) who characterized state-dependent random choice with a restriction called *No Improving Action Switches (NIAS)*. In our setting, ρ *satisfies NIAS* if, for all $g \in F \in \mathcal{K}$,

$$\sum_{s \in S} r_s \rho_{s,F}(f) u\big(f(s)\big) \geq \sum_{s \in S} r_s \rho_{s,F}(f) u\big(g(s)\big).$$

Proposition S.2 below shows that under state-dependent information representations, using test functions to calibrate beliefs is equivalent to testing for NIAS. While both approaches use random choice data, checking means of test functions involves restrictions on ex ante (i.e., pre-signal) values of menus, while checking NIAS involves restrictions on ex post (i.e., post-signal) values of acts.

There is also a close relationship between the NIAS expressions and test functions. Recall from Theorem 2 that the ex ante valuation of menus is given by the integral of test functions:

$$V(F)=\int_{[0,1]} F_{\overline{\rho}}(a)\,da.$$

Proposition S.2 also shows that the marginal ex ante value with respect to an act is exactly the left-hand expression of the NIAS inequality. Note that this is a generalization of Theorem 3. This condition linking test functions with NIAS is necessary and sufficient for well-calibrating beliefs. This is all summarized as follows.

PROPOSITION S.2: *Let ρ be represented by (μ, u). Then the following are all equivalent.*

(1) *μ is well-calibrated.*

(2) *For all $F\in\mathcal{K}_s$ and $s\in S$,*

$$\int_{[0,1]} F^s_{\rho}(a)\,da=V(F).$$

(3) *ρ satisfies NIAS.*

(4) *For all $F\in\mathcal{K}$ and $f_a:=af+(1-a)\underline{f}$,*

$$\sum_{s\in S} r_s\rho_{s,F\cup f_a}(f_a)u\big(f(s)\big)=\frac{dV(F\cup f_a)}{da}.$$

PROOF: Let ρ be represented by (μ, u). Note that the equivalence of (1) and (2) follows immediately from Theorem 5 and Lemma B.1. Employ the notation $F_a:=F\cup f_a$. We first show that

$$\frac{dV(F_a)}{da}=\int_{Q^{f_a}_{F_a}} q\cdot(u\circ f)\overline{\mu}(dq).$$

Without loss of generality, assume $u(\underline{f})=0$ and $u(\overline{f})=1$. For $f\in F\in\mathcal{K}$, define

$$Q^f_F:=\big\{q\in\Delta S \mid q\cdot(u\circ f)\geq q\cdot(u\circ g)\ \forall g\in F\big\}.$$

Consider $c := a + \varepsilon$ for $\varepsilon > 0$ and note that we can partition

$$Q_{F_c}^{f_c} = Q_{F_a}^{f_a} \cup \bigcup_{g \in F} \left(Q_{F_a}^{g} \setminus Q_{F_c}^{g}\right).$$

Along with Lemma B.1, this implies that

$$\begin{aligned} V(F_c) - V(F_a) &= \int_{[0,1]} \left((F_c)_{\overline{\rho}}(a') - (F_a)_{\overline{\rho}}(a')\right) da' \\ &= \int_{\Delta S} \left(\sup_{g \in F_c} q \cdot (u \circ g) - \sup_{g \in F_a} q \cdot (u \circ g)\right) \overline{\mu}(dq) \\ &= \int_{Q_{F_a}^{f_a}} q \cdot \left(u \circ (f_c - f_a)\right) \overline{\mu}(dq) \\ &\quad + \sum_{g \in F} \int_{Q_{F_a}^{g} \setminus Q_{F_c}^{g}} q \cdot \left(u \circ (f_c - g)\right) \overline{\mu}(dq). \end{aligned}$$

Note that $u \circ (f_c - f_a) = \varepsilon(u \circ f)$ and for all $q \in Q_{F_a}^{g} \setminus Q_{F_c}^{g} \subset Q_{F_c}^{f_c}$,

$$q \cdot u \circ f_c \geq q \cdot u \circ g \geq q \cdot u \circ f_a.$$

This second inequality is true iff

$$q \cdot \left(u \circ (f_c - g)\right) \leq q \cdot \left(u \circ (f_c - f_a)\right) = \varepsilon q \cdot (u \circ f).$$

Hence, for all $g \in F$,

$$0 \leq \int_{Q_{F_a}^{g} \setminus Q_{F_c}^{g}} q \cdot \left(u \circ (f_c - g)\right) \overline{\mu}(dq) \leq \varepsilon \int_{Q_{F_a}^{g} \setminus Q_{F_c}^{g}} q \cdot (u \circ f) \overline{\mu}(dq).$$

Since $\overline{\mu}(Q_{F_a}^{g} \setminus Q_{F_c}^{g}) \to 0$ as $\varepsilon \to 0$, this implies that

$$\frac{dV(F_a)}{da} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(V(F_{a+\varepsilon}) - V(F_a)\right) = \int_{Q_{F_a}^{f_a}} q \cdot (u \circ f) \overline{\mu}(dq),$$

as desired.

We now show that (1) implies (4) implies (3) implies (1). First, suppose μ is well-calibrated. Note that

$$\begin{aligned} &\sum_{s} r_s \rho_{s,F_a}(f_a) u\left(f(s)\right) \\ &\quad = \sum_{s \in S} r_s \mu_s\left(Q_{F_a}^{f_a}\right) u\left(f(s)\right) = \sum_{s} \int_{Q_{F_a}^{f_a}} q_s u\left(f(s)\right) \frac{r_s}{q_s} \mu_s(dq) \end{aligned}$$

$$= \sum_s \int_{Q_{F_a}^{f_a}} q_s u\big(f(s)\big)\overline{\mu}(dq) = \int_{Q_{F_a}^{f_a}} q \cdot (u \circ f)\overline{\mu}(dq)$$
$$= \frac{dV(F_a)}{da},$$

as desired. Hence, (4) is true.

Now, suppose (4) is true. Note that by considering $a = 0$, we have

$$\sum_{s \in S} r_s \rho_{s,F}(f) u\big(f(s)\big) = \int_{Q_F^f} q \cdot (u \circ f)\overline{\mu}(dq)$$

for all $f \in F \in \mathcal{K}$. Now, given any $\{f, g\} \subset F$, we can always find some G such that $Q_F^f = Q_G^g$. Hence, we have

$$\begin{aligned}\sum_s r_s \mu_s\big(Q_F^f\big) u\big(f(s)\big) &= \int_{Q_F^f} q \cdot (u \circ f)\overline{\mu}(dq) \geq \int_{Q_F^f} q \cdot (u \circ g)\overline{\mu}(dq) \\ &\geq \int_{Q_G^g} q \cdot (u \circ g)\overline{\mu}(dq) = \sum_s r_s \mu_s\big(Q_G^g\big) u\big(g(s)\big) \\ &= \sum_s r_s \mu_s\big(Q_F^f\big) u\big(g(s)\big),\end{aligned}$$

proving (3).

Finally, suppose (3) is true, so for all $g \in F \in \mathcal{K}$ and $\overline{\rho}_F(f) > 0$,

$$\sum_s r_s \rho_{s,F}(f) u\big(f(s)\big) \geq \sum_s r_s \rho_{s,F}(f) u\big(g(s)\big),$$
$$\sum_s \frac{r_s \mu_s\big(Q_F^f\big)}{\overline{\mu}\big(Q_F^f\big)} u\big(f(s)\big) \geq \sum_s \frac{r_s \mu_s\big(Q_F^f\big)}{\overline{\mu}\big(Q_F^f\big)} u\big(g(s)\big),$$
$$q_F(f) \cdot (u \circ f) \geq q_F(f) \cdot (u \circ g),$$

where $q_F : F \to \Delta S$ is such that $q_F(f)(s) := \frac{r_s \mu_s(Q_F^f)}{\overline{\mu}(Q_F^f)}$. Hence, $q_F(f) \in Q_F^f$ for all $f \in F$. For each $s \in S$, define the measure $\nu_s(Q) := \int_Q \frac{q_s}{r_s}\overline{\mu}(dq)$ and note that $\sum_s r_s \nu_s(Q) = \overline{\mu}(Q) = \sum_s r_s \mu_s(Q)$. Now, we also have, for all $g \in F$,

$$\int_{Q_F^f} q \cdot (u \circ f)\overline{\mu}(dq) \geq \int_{Q_F^f} q \cdot (u \circ g)\overline{\mu}(dq),$$
$$\sum_s r_s \int_{Q_F^f} \frac{q_s}{r_s} u\big(f(s)\big)\overline{\mu}(dq) \geq \sum_s r_s \int_{Q_F^f} \frac{q_s}{r_s} u\big(g(s)\big)\overline{\mu}(dq),$$

$$\sum_s \frac{r_s \nu_s(Q_F^f)}{\overline{\mu}(Q_F^f)} u(f(s)) \geq \sum_s \frac{r_s \nu_s(Q_F^f)}{\overline{\mu}(Q_F^f)} u(g(s)),$$
$$p_F(f) \cdot (u \circ f) \geq p_F(f) \cdot (u \circ g),$$

where $p_F : F \to \Delta S$ is such that $p_F(f)(s) := \frac{r_s \nu_s(Q_F^f)}{\overline{\mu}(Q_F^f)}$. Hence, $p_F(f) \in Q_F^f$ for all $f \in F$.

Consider a partition $\mathcal{P}^n$ of ΔS such that, for every $P_i^n \in \mathcal{P}^n$, $P_i^n = Q_F^f$ for some $f \in F$ and $\sup_{\{p,q\} \subset P_i} |p - q| \leq \frac{1}{n}$ for every $i \in \{1, \dots, n\}$. Since both $q_F(f)$ and $p_F(f)$ are in P_i^n,

$$\left| \frac{r_s \mu_s(P_i^n)}{\overline{\mu}(P_i^n)} - \frac{r_s \nu_s(P_i^n)}{\overline{\mu}(P_i^n)} \right| \leq \frac{1}{n},$$
$$\left| \mu_s(P_i^n) - \nu_s(P_i^n) \right| \leq \overline{\mu}(P_i^n) \frac{1}{n r_s}.$$

Now, for any $\psi^n : \Delta S \to \mathbb{R}$ that is $\mathcal{P}^n$-measurable, we have

$$\sum_i \nu_s(P_i^n)\psi_i^n - \frac{1}{n} \sum_i \frac{1}{r_s} \overline{\mu}(P_i^n)\psi_i^n$$
$$\leq \sum_i \mu_s(P_i^n)\psi_i^n \leq \sum_i \nu_s(P_i^n)\psi_i^n + \frac{1}{n} \sum_i \frac{1}{r_s} \overline{\mu}(P_i^n)\psi_i^n.$$

For any measurable $\psi : \Delta S \to \mathbb{R}$, we can find a sequence of $\mathcal{P}^n$-measurable functions such that $\psi^n \to \psi$. Hence by dominated convergence,

$$\lim_{n\to\infty} \left[\int_{\Delta S} \psi^n(q)\nu_s(dq) - \frac{1}{n} \int_{\Delta S} \frac{1}{r_s} \psi^n(q)\overline{\mu}(dq) \right]$$
$$\leq \lim_{n\to\infty} \int_{\Delta S} \psi^n(q)\mu_s(dq)$$
$$\leq \lim_{n\to\infty} \left[\int_{\Delta S} \psi^n(q)\nu_s(dq) + \frac{1}{n} \int_{\Delta S} \frac{1}{r_s} \psi^n(q)\overline{\mu}(dq) \right],$$
$$\int_{\Delta S} \psi(q)\mu_s(dq) = \int_{\Delta S} \psi(q)\nu_s(dq).$$

Hence, $\mu_s = \nu_s$ so μ is well-calibrated and (1) is true. This concludes the proof. *Q.E.D.*

REFERENCES

AHN, D., AND T. SARVER (2013): "Preference for Flexibility and Random Choice," *Econometrica*, 81 (1), 341–361. [14]

CAPLIN, A., AND D. MARTIN (2015): "A Testable Theory of Imperfect Perception," *Economic Journal*, 125 (582), 184–202. [16]

GUL, F., AND W. PESENDORFER (2006): "Random Expected Utility," *Econometrica*, 74, 121–146. [2,6]

KAHNEMAN, D., AND A. TVERSKY (1979): "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, 47, 263–291. [2]

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