Econometrica Supplementary Material

SUPPLEMENT TO "LINEAR VOTING RULES"

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HANS PETER GRÜNER
Department of Economics, University of Mannheim and CEPR

THOMAS TRÖGER
Department of Economics, University of Mannheim

THIS SUPPLEMENT CONTAINS omitted proofs or sketches of proofs of claims in the main text, analyses of model variants with weights on turnout, separate populations of R-agents and S-agents, and with a committee, and a table of notations.

REMAINING CLAIMS IN THE "UNBIASED CASE" $E[\tilde{v}] = 0$ IN THE PROOF OF PROPOSITION 1

The unbiased case is special because the expected utility (10) is independent of P^A. Using the notation introduced in the proof of Proposition 1, it remains to verify the following two claims.

1. Let $\mu^{A*} = 0$. Then there exist numbers μ^{R*} and μ^{S*} such that setting $M = M^*$ maximizes $\mathcal{L}(M)$ among all rules in $\mathcal{M}_n$; moreover, (34) and (35) hold.

2. Let $\mu^{A*} = 0$, $\mu^{R*} = 1$, and $\mu^{S*} = 0$. Then setting $M = \hat{M}^R$ maximizes $\mathcal{L}(M)$, where the feasible set is restricted to the R-one-sided rules.

We need an auxiliary result, Lemma 5. Consider a relaxed maximization problem in which the equilibrium conditions are replaced by inequalities:

$$\text{(relax)} \quad \max_{(M,\Delta^R,\Delta^S)} \quad \int \max\{v_i\Delta^R - c_i, 0\}\, \mathrm{d}F(v_i, c_i) + \int \max\{-v_i\Delta^S - c_i, 0\}\, \mathrm{d}F(v, c)$$

$$\begin{aligned} \text{s.t.} \quad & \Delta^R - d^R\big(M, t^R(\Delta^R), t^S(\Delta^S)\big) \le 0, \quad (R) \\ & \Delta^S - d^S\big(M, t^R(\Delta^R), t^S(\Delta^S)\big) \le 0, \quad (S) \\ & \Delta^R \ge 0, \Delta^S \ge 0, \\ & M \in \mathcal{M}_n. \end{aligned}$$

Lemma 5 below justifies our focus on the relaxed problem.

LEMMA 5: *Problem* (*relax*) *always has a solution such that both* (R) *and* (S) *are satisfied with equality. In particular, if* $E[\tilde{v}] = 0$, *then any solution to* (*opt*) *also solves problem* (*relax*).

PROOF: Observe first that (relax) always has a solution. This follows from Weierstraß's Maximum-Value Theorem (note that (Δ^R, Δ^S) belongs to the compact set $[0, 1]^2$ because $d^R \le 1$ and $d^S \le 1$).

Consider any solution (M, Δ^R, Δ^S) to (relax). We will construct from it another solution to (relax) such that the constraints (R) and (S) are satisfied with equality. Let $\tau^S = t^S(\Delta^S)$ and $\tau^R = t^R(\Delta^R)$.

Hans Peter Grüner: gruener@uni-mannheim.de
Thomas Tröger: troeger@uni-mannheim.de

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Case $\tau^S > 0$ *and* $\tau^R > 0$ *and* $\tau^R + \tau^S < 1$. We claim that at (M, Δ^R, Δ^S), both constraints (R) and (S) are satisfied with equality.

Because $\Delta^R > 0$ (from $\tau^R > 0$), constraint (R) implies that there exists $(\hat{r}, \hat{s})$ such that $M(\hat{r}+1, \hat{s}) - M(\hat{r}, \hat{s}) > 0$. Thus,

$$M(\hat{r}+1, \hat{s}) > 0 \quad \text{and} \quad M(\hat{r}, \hat{s}) < 1. \tag{78}$$

Suppose that constraint (R) is satisfied with strict inequality. Then locally only constraint (S) is relevant. Thus, (78) implies

$$0 \le \Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}+1, \hat{s}) - \Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}+1, \hat{s}-1), \tag{79}$$

$$0 \ge \Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}, \hat{s}) - \Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}, \hat{s}-1). \tag{80}$$

(If (79) does not hold, then one can slightly decrease $M(\hat{r}+1, \hat{s})$, thus making the left-hand side of (S) strictly smaller than 0, followed by an increase of Δ^R (or Δ^S) that is so small that both constraints remain satisfied; the increase of Δ^R increases the welfare. A similar contradiction is obtained by increasing $M(\hat{r}, \hat{s})$ if (80) does not hold.)

But the expression

$$\Big(\Pr_{\tau^R,\tau^S}^{n-1}(r, s) - \Pr_{\tau^R,\tau^S}^{n-1}(r, s-1)\Big) = \Pr_{\tau^R,\tau^S}^{n}(r, s)\frac{(n-r-s)/\big(1-\tau^R-\tau^S\big) - s/\tau^S}{n}$$

is strictly decreasing in r, a contradiction to (79) and (80). The proof that constraint (S) is satisfied with equality is analogous.

Case $\tau^R + \tau^S = 1$. Then $\tau^R = F^R$ and $\tau^S = F^S$. We claim that at (M, Δ^R, Δ^S), both constraints (R) and (S) are satisfied with equality.

The steps leading to (79) and (80) are as in the previous case. Here, $\hat{s} = n-1-\hat{r}$ in (78) because only the tallies (r, s) with $r+s = n-1$ occur with a positive probability from any agent's point of view. Thus, (79) and (80) simplify to

$$\Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}+1, n-2-\hat{r}) = 0, \qquad \Pr_{\tau^R,\tau^S}^{n-1}(\hat{r}, n-1-\hat{r}) = 0.$$

The first equation implies $\hat{r} = n-1$, a contradiction to the second equation. Thus, the constraints (R) and (S) are satisfied with equality.

Case $\tau^R > 0$ *and* $\tau^S = 0$. (The case $\tau^R = 0$ and $\tau^S > 0$ is analogous.) In the objective of (relax), the right-most integral $= 0$. Thus, another solution to (relax) is given by $(\hat{M}^R, \Delta^R, 0)$ with $\hat{M}^R_{rs} = M_{r0}$ for all (r, s). At $(\hat{M}^R, \Delta^R, 0)$, constraint (S) is satisfied with equality. Also, constraint (R) is satisfied with equality because otherwise one could increase Δ^R (while (S) remains satisfied with equality).

Case $\tau^S = 0$ *and* $\tau^R = 0$. Then the objective of (relax) obtains the value 0 so that another solution to (relax) is given by $(M^{\text{const}}, 0, 0)$ with $M^{\text{const}}_{rs} = 0$ for all (r, s). The solution $(M^{\text{const}}, 0, 0)$ satisfies both constraints (R) and (S) with equality. This completes the proof of Lemma 5. *Q.E.D.*

PROOF OF CLAIM 1: By Lemma 5, $(M^*, \Delta^{R*}, \Delta^{S*})$ solves problem (relax). It is not possible to change M^* to some M such that both constraints (S) and (R) become strict, because if so, then one could increase Δ^{R*} and Δ^{S*} slightly while keeping the constraints satisfied and increasing the objective. In other words, by the separating-hyperplane theorem, there exist μ^{R*} and μ^{S*} such that M^* maximizes $\mathcal{L}$ with $\mu^{A*}=0$, and (34) and (35) hold. *Q.E.D.*

PROOF OF CLAIM 2: We know that $(\hat{M}^R, \Delta^{R*}, 0)$ also solves problem (relax). Changing $\hat{M}^R$ to some other R-one-sided rule cannot make constraint (R) become strict, because otherwise one could increase Δ^{R*} slightly while keeping the constraints satisfied and increasing the objective. Thus $M=\hat{M}^R$ maximizes $\mathcal{L}(M)=d^R(M,\tau^{R*},0)$. From this, Claim 2 is immediate. *Q.E.D.*

MODEL VARIANT WITH WEIGHTS ON TURNOUT

Consider a variant of problem (opt) in which the objective instead is to maximize a weighted average of expected utility, the S-participation rate, and the R-participation rate. Then the conclusions of Proposition 1 still hold.

To see this, fix an optimal mechanism-equilibrium pair $(M^*, \Delta^{R*}, \Delta^{S*})$ with corresponding participation pair (τ^{R*}, τ^{S*}). At any mechanism-equilibrium pair of the form $(M, \Delta^{R*}, \Delta^{S*})$, the welfare is

$$\zeta^W E[\tilde{v}]\rho^A\big(M,\tau^{R*},\tau^{S*}\big) \\ + \underbrace{\zeta^W E\big[\max\big\{\tilde{v}\Delta^{R*}-\tilde{c},0\big\}\mathbf{1}_{\tilde{v}>0}\big] + \zeta^W E\big[\max\big\{-\tilde{v}\Delta^{S*}-\tilde{c},0\big\}\mathbf{1}_{\tilde{v}<0}\big] + \zeta^R\tau^{R*}+\zeta^S\tau^{S*}}_{\text{independent of } M},$$

where some weights $\zeta^W\geq 0$, $\zeta^R\geq 0$, $\zeta^S\geq 0$, with $\zeta^W+\zeta^R+\zeta^S=1$ are given.

Thus, the same arguments as in the proof of Proposition 1 apply (with $\zeta^W E[\tilde{v}]=0$ now being the "unbiased case"). The only additional complication is that we cannot exclude anymore the possibility that the planner wants everybody to show up, that is, $\tau^{R*}+\tau^{S*}=1$ or, equivalently,

$$\tau^{R*}=F^R, \qquad \tau^{S*}=F^S.$$

Consider this case.

We now give a proof of the first claim in Proposition 1. The starting point is that only the tallies with $r+s=n$ occur with positive probability; we consider the set $\hat{\mathcal{M}}$ of voting rules that are equal to M^* on the line $r+s=n$ and also yield the same reform-at-abstention probability as M^*; we show that full participation is an equilibrium under any $M\in\hat{\mathcal{M}}$, and $\hat{\mathcal{M}}$ contains a linear rule.

Here are the details. Note that

$$\underset{\tau^{R*},\tau^{S*}}{\overset{n-1}{\Pr}}(r,s)>0 \iff r+s=n-1; \qquad \underset{\tau^{R*},\tau^{S*}}{\overset{n}{\Pr}}(r,s)>0 \iff r+s=n.$$

For any $M\in\mathcal{M}_n$, the participation pair (F^R, F^S) is an equilibrium if

$$d^R\big(M,F^R,F^S\big)\geq\Delta^{R*} \quad \text{and} \quad d^S\big(M,F^R,F^S\big)\geq\Delta^{S*}. \tag{81}$$

Thus, if $E[\tilde{v}]>0$, then we can set $\mu^{A*}=1$ and there exist Lagrangian multipliers μ^{R*} and μ^{S*} satisfying (34) such that $M=M^*$ maximizes $\mathcal{L}(M)$. The same holds with $\mu^{A*}=-1$ if

$E[\tilde{v}] < 0$. Still the same conclusion holds with $\mu^{A*} = 0$ if $E[\tilde{v}] = 0$, and in this case both (34) and (35) hold (to see this, one follows the same arguments as the section "Remaining claims in the unbiased case in the proof of Proposition 1" above).

Similarly to the computation leading to (38), we find that (36) holds with

$$\mu_{rs} = \begin{cases} \Pr^{n-1}_{F^R,F^S}(r,s)\big(\mu^{A*} + \mu^{S*} - \mu^{R*}\big) & \text{if } r+s = n-1, \\ \Pr^{n}_{F^R,F^S}(r,s)\dfrac{r\mu^{R*}/F^R - s\mu^{S*}/F^S}{n} & \text{if } r+s = n. \end{cases} \tag{82}$$

This implies

$$\mu^{A*} + \mu^{S*} - \mu^{R*} = 0 \tag{83}$$

because otherwise either $M^*_{rs} = 1$ for all $r+s = n-1$, implying $\Delta^{R*} = d^R(M^*, F^R, F^S) \leq 0$, contradicting $\tau^{R*} > 0$, or $M^*_{rs} = 0$ for all $r+s = n-1$, which leads to a similar contradiction using S instead of R.

We conclude that (35) holds even if $E[\tilde{v}] \neq 0$ because otherwise $\mu^{R*} = \mu^{S*} = 0$ by (34), in contradiction with $\mu^{A*} \neq 0$ and (83).

For all (r, s) with $r+s = n$, (82) yields that μ_{rs} has the same sign as $r\mu^{R*}/F^R - s\mu^{S*}/F^S$. This together with (34) and (35) implies that there exists r^* such that, for all $r = 0, \dots, n$,

$$\text{if } r < r^*, \quad \text{then } M^*_{r,n-r} = 0; \qquad \text{if } r > r^*, \quad \text{then } M^*_{r,n-r} = 1. \tag{84}$$

Now define $\rho^{A*} = \rho^A(M^*, F^R, F^S)$ and

$$\hat{\mathcal{M}} = \big\{M \in \mathcal{M}_n \mid \rho^A\big(M, F^R, F^S\big) = \rho^{A*}, M_{r,n-r} = M^*_{r,n-r} \text{ for all } r = 0, \dots, n\big\}.$$

For all $M \in \hat{\mathcal{M}}$,

$$d^R\big(M, F^R, F^S\big) = \sum_{r+s=n-1} \Pr^{n-1}_{F^R,F^S}(r,s) M_{r+1,s} - \rho^{A*} = d^R\big(M^*, F^R, F^S\big) \tag{85}$$

and, similarly,

$$d^S\big(M, F^R, F^S\big) = d^S\big(M^*, F^R, F^S\big). \tag{86}$$

Therefore, (F^R, F^S) is an equilibrium for all $M \in \hat{\mathcal{M}}$. Thus, to complete the proof, it is sufficient to find a linear rule $\hat{M} \in \hat{\mathcal{M}}$.

For all $\hat{r} = 0, \dots, n$, define $\hat{\mathcal{M}}^{\hat{r}} = \{M \in \hat{\mathcal{M}} \mid \forall r < \hat{r} : M_{r,n-1-r} = 0\}$. Let $\hat{r}$ be maximal with the property $\hat{\mathcal{M}}^{\hat{r}} \neq \emptyset$. Note that $\hat{r} < n$ (otherwise we would have $0 \geq d^S(M, F^R, F^S) = \Delta^{S*}$ for all $M \in \hat{\mathcal{M}}$). Choose $\hat{M} \in \hat{\mathcal{M}}^{\hat{r}}$ with minimal $\hat{M}_{\hat{r},n-1-\hat{r}}$. By construction, $\hat{M}_{\hat{r},n-1-\hat{r}} > 0$.

Suppose that there exists $r' > \hat{r}$ such that $\hat{M}_{r',n-1-r'} < 1$. For all small $\varepsilon > 0$, the rule $M^\varepsilon \in \hat{\mathcal{M}}$, where we define

$$\begin{aligned} M^\varepsilon_{\hat{r},n-1-\hat{r}} &= \hat{M}_{\hat{r},n-1-\hat{r}} - \varepsilon \Pr\big(r', n-1-r'\big), \\ M^\varepsilon_{r',n-1-r'} &= \hat{M}_{\hat{r},n-1-\hat{r}} + \varepsilon \Pr(\hat{r}, n-1-\hat{r}), \\ M'_{rs} &= \hat{M}_{rs} \quad \text{otherwise.} \end{aligned}$$

This contradicts the minimality property of $\hat{M}$ because $M^{\varepsilon}_{\hat{r},n-1-\hat{r}} < \hat{M}_{\hat{r},n-1-\hat{r}}$. In summary, $\hat{M}$ has the following property: for all $r = 0, \dots, n-1$,

$$\text{if } r < \hat{r}, \quad \text{then } \hat{M}_{r,n-1-r} = 0; \qquad \text{if } r > \hat{r}, \quad \text{then } \hat{M}_{r,n-1-r} = 1. \tag{87}$$

Moreover, using (84) together with $\hat{M} \in \hat{\mathcal{M}}$,

$$\text{if } r < r^*, \quad \text{then } \hat{M}_{r,n-r} = 0; \qquad \text{if } r > r^*, \quad \text{then } \hat{M}_{r,n-r} = 1. \tag{88}$$

If we had $\hat{r} < r^* - 1$, then (87) and (88) would imply that

$$d^R(\hat{M}, F^R, F^S) = -\Pr_{F^R,F^S}^{n-1}(\hat{r}, n-1-\hat{r})\hat{M}_{\hat{r},n-1-\hat{r}} - \Pr_{F^R,F^S}^{n-1}(r^*-1, n-r^*)(1-\hat{M}_{r^*,n-r^*}) \le 0,$$

contradicting (85). Similarly, if we had $\hat{r} > r^*$, then it would follow that $d^S(\hat{M}, F^R, F^S) \le 0$, contradicting (86). Thus,

$$\hat{r} = r^* - 1 \quad \text{or} \quad \hat{r} = r^*.$$

Hence, one can define $\hat{M}_{rs}$ for all $r + s \le n-2$ such that $\hat{M}$ is linear.

Here is the argument for the second claim in the proof of Proposition 1. Define $\hat{M}$, r^*, and $\hat{r}$ as in the proof of the first claim. Suppose that $\hat{r} = r^* - 1$ (the argument is analogous if $\hat{r} = r^*$). Define the upper linear rules M', M'', and M''' via $M'_{rs} = \mathbf{1}_{s<n-r^*}$, $M''_{rs} = \mathbf{1}_{s<n-r^* \text{or} r \ge r^*}$, and $M'''_{rs} = \mathbf{1}_{s \le n-r^*}$. Because

$$0 < \Delta^{R*} = d^R(\hat{M}, F^R, F^S) = \Pr_{F^R,F^S}^{n-1}(r^*-1, n-r^*)(\hat{M}_{r^*,n-r^*} - \hat{M}_{r^*-1,n-r^*}),$$

we have $\hat{M}_{r^*,n-r^*} - \hat{M}_{r^*-1,n-r^*} > 0$. Hence, defining

$$\kappa' = 1 - \hat{M}_{r^*,n-r^*}, \qquad \kappa'' = \hat{M}_{r^*,n-r^*} - \hat{M}_{r^*-1,n-r^*}, \qquad \kappa''' = \hat{M}_{r^*-1,n-r^*},$$

the rule $\check{M} = \kappa' M' + \kappa'' M'' + \kappa''' M'''$ is a convex combination of M', M'', and M''', and $\check{M}_{rs} = \hat{M}_{rs}$ for all $r + s \ge n-1$. Thus, $(\check{M}, \Delta^{R*}, \Delta^{S*})$ is optimal.

MODEL VARIANT WITH 0-COST AGENTS

The assumptions on the type distribution F are as before, except that now $0 < \Pr[\tilde{c} = 0] < 1$. Accordingly, the random variable $\tilde{v}/\tilde{c}$ (with distribution $\mathcal{F}$) can take the value $+\infty$, with probability $\Pr[\tilde{c} = 0, \tilde{v} > 0]$, or the value $-\infty$, with probability $\Pr[\tilde{c} = 0, \tilde{v} < 0]$.

An equilibrium is still described by a pair (Δ^R, Δ^S), with the additional understanding that any type (v_i, c_i) with $c_i = 0$ and $v_i > 0$ takes action R if $\Delta^R > 0$ and otherwise takes action A; similarly for types with $c_i = 0$ and $v_i < 0$.

Equilibrium conditions are as before, where we define the functions t^R and t^S such that

$$t^R(\Delta^R) = \begin{cases} 1 - \mathcal{F}(1/\Delta^R) & \text{if } \Delta^R > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$t^S(\Delta^S) = \begin{cases} \mathcal{F}(-1/\Delta^S) & \text{if } \Delta^S > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Any equilibrium can alternatively be expressed in terms of the participation pair (τ^R, τ^S), where $\tau^R = 0$ if $\Delta^R = 0$ and $\tau^R \geq \Pr[\tilde{c} = 0, \tilde{v} > 0]$ otherwise; similarly for τ^S.

The conclusion of Proposition 1 still holds without changes; the proof remains as before.

MODEL VARIANT WITH SEPARATE POPULATIONS OF R-AGENTS AND S-AGENTS

Suppose there is a fixed number n_R (resp. n_S) of R-agents (resp., S-agents) with types (v_i, c_i) i.i.d. distributed according to a distribution F_R on $\mathbb{R}_+ \times \mathbb{R}_+$ (resp., F_S on $\mathbb{R}_- \times \mathbb{R}_+$). This captures the setting in Palfrey and Rosenthal (1985) as a special case. Let $(\tilde{v}_R, \tilde{c}_R)$ and $(\tilde{v}_S, \tilde{c}_S)$ denote random variables with distributions F_R and F_S, respectively. Let $\mathcal{F}_R$ denote the cdf for $\tilde{v}_R/\tilde{c}_R$ and $\mathcal{F}_S$ denote the cdf for $\tilde{v}_S/\tilde{c}_S$. We assume that, for $x = R, S$, the distribution $\mathcal{F}_x$ has no atoms, $E[\tilde{c}_x] < \infty$, $E[\tilde{v}_R] < \infty$, and $E[\tilde{v}_S] > -\infty$.

A voting rule is given by a mapping $M : \{0, 1, \dots, n_R\} \times \{0, 1, \dots, n_S\} \to [0, 1]$. The set of voting rules is denoted $\mathcal{M}_{n_R, n_S}$. Linear rules are defined via a representation with parameters (ξ, ξ^R, ξ^S) such that (1) and (2) hold, where we use the new shortcut $n = n_R \cdot n_S$.

We focus on equilibria in which all R-agents use the same strategy, and so do all S-agents. Any strategy is defined via a pair (Δ^R, Δ^S) with $\Delta^R \geq 0$ and $\Delta^S \geq 0$. The strategy of the R-agents (resp., S-agents) is given by (4), restricted to valuations $v_i > 0$ (resp., $v_i < 0$). Let

$$t^R(\Delta^R) = 1 - \mathcal{F}_R\left(\frac{1}{\Delta^R}\right), \qquad t^S(\Delta^S) = \mathcal{F}_S\left(-\frac{1}{\Delta^S}\right).$$

Let $\Pr^l_p(t)$ denote the probability of t successes in a binomial distribution with any parameters (l, p). An agent's anticipated pivotalities under a rule M are given by

$$d^R(M, \tau^R, \tau^S) = \sum_{r \leq n_R - 1} \sum_{s \leq n_S} \Pr^{n_R - 1}_{\tau^R}(r) \Pr^{n_S}_{\tau^S}(s)(M_{r+1,s} - M_{r,s}),$$

$$d^S(M, \tau^R, \tau^S) = \sum_{r \leq n_R} \sum_{s \leq n_S - 1} \Pr^{n_R}_{\tau^R}(r) \Pr^{n_S - 1}_{\tau^S}(s)(M_{r,s} - M_{r,s+1}),$$

where τ^R (resp., τ^S) is the probability that a given R-agent (resp., S-agent) participates. Equilibrium conditions are (6) for the R-agents and (7) for the S-agents. As in the main model, we can represent any equilibrium (Δ^R, Δ^S) alternatively via the participation pair (τ^R, τ^S).

Any mechanism-equilibrium pair $m = (M, \tau^R, \tau^S)$ yields a *reform-at-R-abstention probability*

$$\rho^{\mathrm{AR}}(m) = \sum_{r \leq n_R - 1} \sum_{s \leq n_S} \Pr^{n_R - 1}_{\tau^R}(r) \Pr^{n_S}_{\tau^S}(s) M_{r,s}$$

and a *reform-at-S-abstention probability*

$$\rho^{\mathrm{AS}}(m) = \sum_{r \leq n_R} \sum_{s \leq n_S - 1} \Pr^{n_R}_{\tau^R}(r) \Pr^{n_S - 1}_{\tau^S}(s) M_{r,s}.$$

Given pivotalities Δ^R and Δ^S and a reform-at-R-abstention probability P^{AR}, the interim expected utility of an R-agent of type (v_i, c_i), $v_i > 0$, is

$$v_i P^{\mathrm{AR}} + \max\{v_i \Delta^R - c_i, 0\}.$$

Similarly, the expected utility of an S-agent of type (v_i, c_i), $v_i < 0$, is

$$v_i P^{\mathrm{AS}} + \max\{-v_i \Delta^S - c_i, 0\}.$$

We consider a social planner who is interested in maximizing a weighted average of an R-agent's utility and an S-agent's utility, with weights $\kappa_R > 0$ and $\kappa_S > 0$. For instance, $\kappa_R = n_R$ and $\kappa_S = n_S$. The resulting objective is

$$\begin{aligned} W^{\Delta^R, \Delta^S, P^{\mathrm{AR}}, P^{\mathrm{AS}}} &= \kappa_R E[\tilde{v}_R] P^{\mathrm{AR}} + \kappa_S E[\tilde{v}_S] P^{\mathrm{AS}} \\ &\quad + \kappa_R E\big[\max\{\tilde{v}_R \Delta^R - \tilde{c}_R, 0\}\big] + \kappa_S E\big[\max\{-\tilde{v}_S \Delta^S - \tilde{c}_S, 0\}\big]. \end{aligned}$$

The planner solves

$$\begin{aligned} (\text{optRS}) \qquad \max_{(M, \Delta^R, \Delta^S)} \quad & W^{\Delta^R, \Delta^S, \rho^{\mathrm{AR}}(M, t^R(\Delta^R), t^S(\Delta^S)), \rho^{\mathrm{AS}}(M, t^R(\Delta^R), t^S(\Delta^S))} \\ \text{s.t.} \quad & (6), (7), \\ & \Delta^R \geq 0, \Delta^S \geq 0, \\ & M \in \mathcal{M}_{n_R, n_S}. \end{aligned}$$

PROPOSITION 5: *Consider the model variant with fixed populations of R-agents and S-agents. Then any solution to problem* (*optRS*) *is linear.*

PROOF: Let $W(M, \tau^R, \tau^S)$ denote the welfare from any mechanism-equilibrium pair.

Consider an optimal mechanism-equilibrium pair $m^* = (M^*, \tau^{R*}, \tau^{S*})$ with corresponding pivotalities Δ^{R*} and Δ^{S*}.

First, we show that $\tau^{R*} < 1$ and $\tau^{S*} < 1$. Suppose that $\tau^{R*} = 1$. Then only tallies of the form (n_R, s) can occur with positive probability in equilibrium. Then $(0, \tau^{S*})$ is an equilibrium in the mechanism $\hat{M}$ defined via $\hat{M}_{rs} = M^*_{n_R, s}$. Moreover, $W(\hat{M}, 0, \tau^{S*}) = W(m^*) + \kappa_R E[\tilde{c}_R]$, contradicting the optimality of m^*. The proof that $\tau^{S*} < 1$ is analogous.

In the following, we assume

$$\tau^{R*} > 0 \quad \text{or} \quad \tau^{S*} > 0,$$

because otherwise the linearity claim is trivial. For any rule M, the welfare contribution that is associated with the option to abstain is denoted

$$\rho^{RS*}(M) \stackrel{\text{def}}{=} \kappa_R E[\tilde{v}_R] \rho^{\mathrm{AR}}\big(M, \tau^{R*}, \tau^{S*}\big) + \kappa_S E[\tilde{v}_S] \rho^{\mathrm{AS}}\big(M, \tau^{R*}, \tau^{S*}\big).$$

This function is not identically 0: if $\tau^{S*} > 0$, then

$$\frac{\partial \rho^{RS*}}{\partial M_{0, n_S}} = \kappa_R E[\tilde{v}_R] \overset{n_R - 1}{\underset{\tau^{R*}}{\Pr}}(0) \overset{n_S}{\underset{\tau^{S*}}{\Pr}}(n_S) > 0;$$

similarly, if $\tau^{R*} > 0$, then $\partial \rho^{RS*} / \partial M_{n_R, 0} > 0$.

By optimality, $M = M^*$ maximizes $\rho^{RS*}(M)$ subject to the equilibrium conditions $d^R(M, \tau^{R*}, \tau^{S*}) = \Delta^{R*}$ and $d^S(M, \tau^{R*}, \tau^{S*}) = \Delta^{S*}$.

Thus, there exist numbers μ^{R*} and μ^{S*} such that M^* maximizes the Lagrangian

$$\mathcal{L}^{RS}(M) = \rho^{RS*}(M) + \mu^{R*} d^R\big(M, \tau^{R*}, \tau^{S*}\big) + \mu^{S*} d^S\big(M, \tau^{R*}, \tau^{S*}\big).$$

Hence, if μ_{rs} denotes the weight of M_{rs} in $\mathcal{L}^{RS}(M)$, then (36) holds.

Case $\tau^{R*} > 0$ and $\tau^{S*} > 0$. Then

$$\Pr^{n_R}_{\tau^{R*}}(r) \Pr^{n_S}_{\tau^{S*}}(s) > 0 \quad \text{for all } (r, s). \tag{89}$$

We have

$$\begin{aligned}
\mu_{rs} &= \kappa_R E[\tilde{v}_R] \Pr^{n_R-1}_{\tau^{R*}}(r) \Pr^{n_S}_{\tau^{S*}}(s) + \kappa_S E[\tilde{v}_S] \Pr^{n_R}_{\tau^{R*}}(r) \Pr^{n_S-1}_{\tau^{S*}}(s) \\
&\quad + \mu^{R*}\Big(\Pr^{n_R-1}_{\tau^{R*}}(r-1) - \Pr^{n_R-1}_{\tau^{R*}}(r)\Big) \Pr^{n_S}_{\tau^{S*}}(s) + \mu^{S*} \Pr^{n_R}_{\tau^{R*}}(r) \Big(\Pr^{n_S-1}_{\tau^{S*}}(s) - \Pr^{n_S-1}_{\tau^{S*}}(s-1)\Big) \\
&= \frac{\Pr^{n_R}_{\tau^{R*}}(r) \Pr^{n_S}_{\tau^{S*}}(s)}{n_R n_S} \Bigg(\kappa_R E[\tilde{v}_R] \frac{n_S(n_R - r)}{1 - \tau^{R*}} + \kappa_S E[\tilde{v}_S] \frac{n_R(n_S - s)}{1 - \tau^{S*}} \\
&\quad + \mu^{R*}\left(\frac{r}{\tau^{R*}} - \frac{(n_R - r)}{1 - \tau^{R*}}\right) n_S + \mu^{S*} n_R \left(\frac{s}{\tau^{S*}} - \frac{(n_S - s)}{1 - \tau^{S*}}\right)\Bigg) \\
&= \frac{\Pr^{n_R}_{\tau^{R*}}(r) \Pr^{n_S}_{\tau^{S*}}(s)}{n} \big(r\xi^R - s\xi^S - n\xi\big),
\end{aligned}$$

where we use the shortcuts

$$\xi = \big(\kappa_R E[\tilde{v}_R] - \mu^{R*}\big) \frac{1}{1 - \tau^{R*}} + \big(\kappa_S E[\tilde{v}_S] - \mu^{S*}\big) \frac{1}{1 - \tau^{S*}}, \tag{90}$$

$$\xi^R = \frac{\mu^{R*} n_S}{\tau^{R*}} - \big(\kappa_R E[\tilde{v}_R] - \mu^{R*}\big) \frac{n_S}{1 - \tau^{R*}}, \tag{91}$$

$$\xi^S = \frac{\mu^{S*} n_R}{\tau^{S*}} - \big(\kappa_S E[\tilde{v}_S] - \mu^{S*}\big) \frac{n_R}{1 - \tau^{S*}}. \tag{92}$$

Using (89), we conclude that (2) holds.

To show (1), observe first that $\xi^R \geq 0$ (otherwise (36) implies that $M^*_{r+1,s} \leq M^*_{rs}$ for all (r, s), implying $\Delta^{R*} = 0$ and hence $\tau^{R*} = 0$). Similarly, $\xi^S \geq 0$. Now suppose that $\xi^R = 0$ and $\xi^S = 0$. Then (91) yields

$$\frac{\mu^{R*}/\big(\kappa_R E[\tilde{v}_R]\big)}{1 - \mu^{R*}/\big(\kappa_R E[\tilde{v}_R]\big)} = \frac{\tau^{R*}}{1 - \tau^{R*}},$$

implying $\mu^{R*}/(\kappa_R E[\tilde{v}_R]) = \tau^{R*}$. Similarly, $\mu^{S*}/(\kappa_S E[\tilde{v}_S]) = \tau^{S*}$ by (92). Thus, $\xi = 2 > 0$ by (90). This completes the proof of (1). Hence, M^* is a linear mechanism.

Case $\tau^{R*} > 0$ *and* $\tau^{S*} = 0$. (The case $\tau^{R*} = 0$ and $\tau^{S*} > 0$ is analogous.) Here, $\Pr^{n_S}_{\tau^{S*}}(0) = 1$ and

$$\Pr^{n_R}_{\tau^{R*}}(r) > 0 \quad \text{for all } r.$$

The pair $(\tau^{R*}, 0)$ is an equilibrium in the R-one-sided mechanism $\hat{M}^R$ defined via $\hat{M}^R_{rs} = M^*_{r0}$. It is sufficient to show that $\hat{M}$ is linear.

By optimality, $\hat{M}^R$ maximizes, across all R-one-sided rules M, the objective $\rho^{\mathrm{AR}}(M, \tau^{R*}, \tau^{S*})$ subject to the equilibrium condition $d^R(M, \tau^{R*}, 0) = \Delta^{R*}$. Thus, there exists a number μ^{R*} such that $\hat{M}^R$ maximizes $\mathcal{L}^{RS}$ with $\mu^{S*} = 0$, where the feasible set is given by the R-one-sided rules.

Letting μ_{r0} denote the weight of M_{r0} in $\mathcal{L}^{RS}(M)$,

$$\hat{M}^R_{rs} = \begin{cases} 1 & \text{if } \mu_{r0} > 0, \\ 0 & \text{if } \mu_{r0} < 0. \end{cases}$$

We have

$$\begin{aligned} \mu_{r0} &= \frac{\Pr^{n_R}_{\tau^{R*}}(r)}{n_R n_S} \left(\kappa_R E[\tilde{v}_R] \frac{n_S(n_R - r)}{1 - \tau^{R*}} + \kappa_S E[\tilde{v}_S] \frac{n_R n_S}{1 - \tau^{S*}} + \mu^{R*} \left(\frac{r}{\tau^{R*}} - \frac{(n_R - r)}{1 - \tau^{R*}} \right) n_S \right) \\ &= \frac{\Pr^{n_R}_{\tau^{R*}}(r)}{n} \left(r\xi^R - n\xi \right), \end{aligned}$$

where ξ is given by (90) with $\mu^{S*} = 0$ and ξ^R is given by (91). Thus, (2) holds.

The proof of (1) is also analogous to the case $\tau^{R*} > 0$ and $\tau^{S*} > 0$. Hence, $\hat{M}^R$ is a linear mechanism. *Q.E.D.*

MODEL VARIANT WITH A COMMITTEE

Suppose the mechanism designer randomly chooses a committee of $n' < n$ agents (with equal probability for each possible set of committee members) and allows them to participate in a rule $M \in \mathcal{M}_{n'}$. Redefining the pivotality functions d^R and d^S and the reform-at-abstention probability function ρ^A with n replaced by n', the equilibrium conditions remain as before, and the expected utility (10) is replaced by

$$\begin{aligned} W^{\Delta^R, \Delta^S, P^A, n'} = {} & E[\tilde{v}]P^A + \frac{n'}{n} \left(E\left[\max\{\tilde{v}\Delta^R - \tilde{c}, 0\} \mathbf{1}_{\tilde{v}>0}\right] + E\left[\max\{-\tilde{v}\Delta^S - \tilde{c}, 0\} \mathbf{1}_{\tilde{v}<0}\right] \right) \\ & + \frac{n - n'}{n} E[\tilde{v}] \left(\Delta^R t^R(\Delta^R) - \Delta^S t^S(\Delta^S) \right). \end{aligned}$$

Consider the resulting problem (opt n'). Following the same arguments as the proof of Proposition 1, we have the following:

PROPOSITION 6: *Let* $1 \leq n' < n$. *Any solution to problem* (*opt* n') *is linear.*

PROOFS OF THE CLAIMS IN FOOTNOTE 22

CLAIM: *The committee size n is optimal if the participation cost is small.*

To prove this, make the assumptions of Proposition 4, and let $W^*(c, n')$ denote the welfare obtained at a solution to problem (opt n') at any participation cost $c > 0$ and any committee size $n' \leq n$. Define $W^*(0, n')$ as the optimum welfare in the setting without participation cost, and with a committee size of n'. The claim follows from Step 1 and Step 2 below.

Step 1. For all $n' \leq n$, the function $W^*(c, n')$ is continuous in c at $c = 0$.

To see this, suppose that $n' = n$ (similar arguments apply if $n' < n$). Applying Lemma 4 with any M,

$$\lim_{c\to 0, c>0} \inf W^*(c, n) \geq \lim_{c\to 0, c>0} \inf W\big(c, M, \tilde{\tau}^R(c), \tilde{\tau}^S(c)\big) = W^*(0, n).$$

To complete Step 1, it remains to show that

$$\lim_{c\to 0, c>0} \sup W^*(c, n) \leq W^*(0, n). \tag{93}$$

Consider any sequence of participation costs $(c_l)_{l=1,2,\dots}$, $c_l > 0$, $c_l \to 0$, and any sequence of mechanism-equilibrium pairs $(m_l)_{l=1,2,\dots}$, $m_l = (M_l, \tau_l^R, \tau_l^S)$, such that m_l is optimal at participation cost c_l, and such that the sequence (m_l) has a limit $m_0 = (M_0, \tau_0^R, \tau_0^S)$. By construction,

$$W(c_l, m_l) = W^*(c_l, n) \quad \text{for all } l.$$

Thus, to prove (93), it is sufficient to show (i) that (τ_0^R, τ_0^S) is an equilibrium under M_0 at 0 participation cost, implying $W(0, m_0) \leq W^*(0, n)$, and (ii) that

$$\lim_{l\to\infty} W(c_l, m_l) = W(0, m_0).$$

To see (i), consider the pivotalities

$$\Delta_l^R = d^R(m_l) \quad \text{and} \quad \Delta_l^S = d^S(m_l).$$

At the participation pair (τ_0^R, τ_0^S) under rule M_0, the pivotalities are

$$\Delta_0^R = d^R(m_0) \quad \text{and} \quad \Delta_l^S = d^S(m_0).$$

Thus, using that d^R and d^S are continuous functions,

$$\lim_{l\to\infty} \Delta_l^R = \Delta_0^R \quad \text{and} \quad \lim_{l\to\infty} \Delta_l^S = \Delta_0^S. \tag{94}$$

At 0 participation cost, the equilibrium conditions are simple: $\tau_0^R = F^R$ if $\Delta_0^R > 0$, and $\tau_0^S = F^S$ if $\Delta_0^S > 0$. These conditions follow immediately from (94) if we take limits $l \to \infty$ in

$$\tau_l^R = t^R\big(\Delta_l^R\big) = 1 - \mathcal{H}\big(c_l/\Delta_l^R\big), \qquad \tau_l^S = t^S\big(\Delta_l^S\big) = \mathcal{H}\big(-c_l/\Delta_l^S\big).$$

To see (ii), consider the welfare expression on the right-hand side in (72). Plugging in $(c, m) = (c_l, m_l)$, we obtain an expression for the welfare $W(c_l, m_l)$; plugging in $(c, m) =$

$(0, m_0)$ instead, we obtain an expression for the welfare $W(0, m_0)$. Because the right-hand side in (72) is continuous in c, m, Δ^R, and Δ^S, equation (ii) follows. This completes Step 1.

Step 2. $W^*(0, n') < W^*(0, n)$ *for all* $n' < n$.

At zero participation cost, by the genericity assumption (26), any optimal mechanism has the property that R is implemented if and only if at least t^* agents prefer R (Barberà and Jackson (2006)). This property cannot be achieved with a committee of any size $n' < n$.

CLAIM: *There is an example in which a random dictator yields a higher welfare than the solution to* (*opt*).

The example will be "unbiased," that is, we construct a type distribution such that $E[\tilde{v}] = 0$. Moreover, the example will be such that on one side of the electorate, zero participation is immediate because the participation cost exceeds the valuation. Thus, there exists an optimal rule that is one-sided. Finally, the example will be such that on the participating side, the cost and the valuation are sufficiently close together so that at least some types abstain, implying that almost no welfare is created under the best one-sided rule. A random dictator, however, can achieve a non-vanishing utility for herself and, thus, a non-vanishing welfare.

Consider the following class of two-point valuation-cost distributions:

$$\hat{G} \sim \begin{pmatrix} (-1, \hat{c}) & (\hat{v}, \hat{c}) \\ 1 - \hat{p} & \hat{p} \end{pmatrix},$$

where $1 < \hat{c} < \hat{v}$ and $0 < \hat{p} < 1$ are chosen such that the distribution is unbiased, that is, the expected valuation $\hat{p}\hat{v} - (1 - \hat{p}) = 0$.

Let

$$\overline{\Delta}^R = \max_{r=0,\ldots,n-1} \Pr_{\hat{p},0}^{n-1}(r, 0) < 1$$

denote the probability of the most likely tally of others' votes, from the perspective of an R-agent, assuming that all R-agents participate and all S-agents abstain. Assume that $\hat{c}$ is sufficiently close to $\hat{v}$ so that

$$\hat{v}\overline{\Delta}^R < \hat{c}. \tag{95}$$

Let F be an ε-approximation of $\hat{G}$ with ε so close to zero that

$$1 + \varepsilon < \hat{c}, \tag{96}$$

$$\hat{v} - \varepsilon > \hat{c}, \tag{97}$$

$$2\varepsilon < \frac{\hat{p}(\hat{v} - \hat{c})}{n}. \tag{98}$$

What welfare can be achieved with a random dictator? Consider the rule such that abstention of the dictator leads to the outcome S. Then, due to (97), the dictator will participate and implement R if and only if she supports R. Thus, conditional on being selected as dictator, an agent has the expected utility $\hat{p}(\hat{v} - \hat{c})$. Conditional on not being selected as the dictator, an agent obtains zero because the distribution $\hat{G}$ (and thus F) is unbiased and types are stochastically independent. In summary, the random-dictator welfare equals $\frac{\hat{p}(\hat{v}-\hat{c})}{n}$. Due to (98), it remains to show that

$$\text{the solution to (opt) yields a welfare} \leq 2\varepsilon.$$

To see this, let $m^* = (M^*, \tau^{R*}, \tau^{S*})$ denote a solution to (opt). Let $(\Delta^{R*}, \Delta^{S*})$ denote the corresponding pivotalities.

We have $\tau^{S*} = 0$ because even the most strongly affected S-agent will abstain, given that $\Delta^{S*}(1+\varepsilon) \leq 1+\varepsilon < \hat{c}$ from (96).

Thus, w.l.o.g. we can assume that M^* is an R-one-sided linear mechanism. Let r^* be such that $M_{rs} = 0$ if $r < r^*$, $M_{rs} = 1$ if $r > r^*$, and $M_{rs} = M_{r^*,0}$ if $r = r^*$.

Suppose that there is full participation of R-agents, $\tau^{R*} = \hat{p}$. Then

$$\Delta^{R*} = \Pr_{\hat{p},0}^{n-1}\big(r^*-1, 0\big)M_{r^*,0} + \Pr_{\hat{p},0}^{n-1}\big(r^*, 0\big)(1 - M_{r^*,0}) \leq \overline{\Delta}^R.$$

Using (95), this implies $\hat{v}\Delta^{R*} < \hat{c}$, contradicting the participation of type $\hat{v}$.

Consider the remaining case of partial abstention of R-agents, $\tau^{R*} < \hat{p}$. Here, the R-agent with the lowest valuation finds it optimal to abstain, implying $\Delta^{R*}(\hat{v} - \varepsilon) < \hat{c}$. Thus, the welfare (10) satisfies

$$\begin{aligned} W^{\Delta^{R*},0,\rho^A(m^*)} &\leq E[\tilde{v}]\rho^A\big(m^*\big) + (\hat{v}+\varepsilon)\Delta^{R*} - \hat{c} \\ &\leq 0 + 2\varepsilon\Delta^{R*} \\ &\leq 2\varepsilon. \end{aligned}$$

It is straightforward to extend these arguments towards an example in which the random dictatorship yields a higher welfare than any optimal rule with a committee size of at least two.

PROOF OF LEMMA 3

Additional notation is required. Let

$$\mathcal{D} = \big\{\big(\tau^R, \tau^S\big) \mid \tau^R + \tau^S \leq 1, 0 \leq \tau^R \leq p^R, 0 \leq \tau^S \leq p^S + p^0\big\}$$

denote the set of participation pairs that are feasible if the type distribution approximates $\hat{F}$. Consider any $(\tau^R, \tau^S) \in \mathcal{D}$. Let $(\hat{v}_i, c_i)$ $(i = 1, \ldots, n)$ denote i.i.d. random vectors with distribution $\hat{F}$. Let $\tilde{a}_i$ $(i = 1, \ldots, n)$ denote i.i.d. random variables that describe each agent i's action R, S, or A such that the participation pair (τ^R, τ^S) arises if agent i's type is a realization of $(\hat{v}_i, c_i)$. That is, the joint distribution of $(\tilde{a}_i, \hat{v}_i)$ is given by $\Pr[\tilde{a}_i = R, \hat{v}_i = v^R] = \tau^R$, $\Pr[\tilde{a}_i = S, \hat{v}_i = v^R] = 0$, $\Pr[\tilde{a}_i = R, \hat{v}_i = v^0] = 0$, $\Pr[\tilde{a}_i = S, \hat{v}_i = v^0] = \max\{0, \tau^S - p^S\}$, $\Pr[\tilde{a}_i = R, \hat{v}_i = v^S] = 0$, and $\Pr[\tilde{a}_i = S, \hat{v}_i = v^S] = \min\{\tau^S, p^S\}$.

For any $m = (M, \tau^R, \tau^S) \in \mathcal{M}_n \times \mathcal{D}$, the welfare, ignoring the participation costs, is denoted

$$\hat{W}(m) = \frac{1}{n}E\left[\sum_{i=1}^{n}\hat{v}_i M_{\tilde{r},\tilde{s}}\right], \tag{99}$$

where we use the following random variables for the number of R and S votes, respectively:

$$\tilde{r} = \left|\{j \mid \tilde{a}_j = R\}\right| \quad \text{and} \quad \tilde{s} = \left|\{j \mid \tilde{a}_j = S\}\right|.$$

Elaborating the expectation (99),

$$\hat{W}(m) = \frac{1}{n} \sum_{r+s \le n} \Pr^{n}_{\tau^R, \tau^S}(r, s) \hat{w}_{rs}\left(\tau^R, \tau^S\right) M_{rs},$$

where we use a discrete analogue to (15),

$$\hat{w}_{rs}\left(\tau^R, \tau^S\right) = r\hat{\eta}^R\left(\tau^R\right) + s\hat{\eta}^S\left(\tau^S\right) + (n - r - s)\hat{\eta}^A\left(\tau^R, \tau^S\right)$$

that is defined using the shortcuts

$$\begin{aligned} \hat{\eta}^R\left(\tau^R\right) &= E[\hat{v}_i \mid \tilde{a}_i = R], \\ \hat{\eta}^S\left(\tau^S\right) &= E[\hat{v}_i \mid \tilde{a}_i = S], \\ \hat{\eta}^A\left(\tau^R, \tau^S\right) &= E[\hat{v}_i \mid \tilde{a}_i = A]. \end{aligned}$$

Observe that $\hat{W}$ is continuous in m. Moreover, using the definition (20),

$$\hat{w}_{rs}\left(p^R, p^S\right) = w(r, s). \tag{100}$$

Consider any ε-approximation F of $\hat{F}$. By construction, using the functions defined below (15),

$$\begin{aligned} \left|\eta^R\left(\tau^R\right) - \hat{\eta}^R\left(\tau^R\right)\right| &\le \varepsilon \quad \text{if } \tau^R > 0, \\ \left|\eta^S\left(\tau^S\right) - \hat{\eta}^S\left(\tau^S\right)\right| &\le \varepsilon \quad \text{if } \tau^S > 0, \\ \left|\eta^A\left(\tau^R, \tau^S\right) - \hat{\eta}^A\left(\tau^R, \tau^S\right)\right| &\le \varepsilon \quad \text{if } \tau^R + \tau^S < 1. \end{aligned}$$

Thus, for all (τ^R, τ^S),

$$\left|\omega_{rs}\left(\tau^R, \tau^S\right) - \hat{w}_{rs}\left(\tau^R, \tau^S\right)\right| \le n\varepsilon, \tag{101}$$

where we use the definition (15). This implies

$$\left|W_F(m) - \hat{W}(m)\right| \le n\varepsilon + \max\left\{c^R, c^S\right\} \quad \text{if } \tau^S \le p^S, \tag{102}$$

where we use W as defined in (14) and make its dependence on F explicit with a lower index.

The following lemma determines what would be the first-best in the setting with the discrete type distribution $\hat{F}$, ignoring the participation costs; an important aspect of this result is the fact that the solution is unique.

LEMMA 6: *Consider a three-point distribution $\hat{F}$ such that* (23), (24), *and* (25) *hold. The set of solutions to the problem*

$$(*) \qquad \max_{m \in \mathcal{M}_n \times \mathcal{D}} \hat{W}(m)$$

is given by the singleton $(\overline{M}, p^R, p^S)$.

PROOF: Let (M, τ^R, τ^S) denote a maximizer of $(*)$. Using (99),

$$\hat{W}(M, \tau^R, \tau^S) \leq \frac{1}{n} E\left[\max\left\{0, \sum_{i=1}^{n} \hat{v}_i\right\}\right] = \hat{W}(\overline{M}, p^R, p^S).$$

Because (M, τ^R, τ^S) is optimal, the "$\leq$" is in fact an "$=$". Hence,

$$\Pr\left[\sum_{i=1}^{n} \hat{v}_i > 0 \text{ and } M_{\tilde{r},\tilde{s}} < 1\right] = 0, \tag{103}$$

$$\Pr\left[\sum_{i=1}^{n} \hat{v}_i < 0 \text{ and } M_{\tilde{r},\tilde{s}} > 0\right] = 0. \tag{104}$$

First of all, this implies

$$\tau^R \geq p^R \quad \text{or} \quad \tau^S \geq p^S. \tag{105}$$

Suppose not. Then $\Pr[\hat{v}_1 = \cdots = \hat{v}_n = v^R, \tilde{a}_1 = \cdots = \tilde{a}_n = A] > 0$, implying $M_{0,0} = 1$ by (103). Similarly, $\Pr[\hat{v}_1 = \cdots = \hat{v}_n = v^S, \tilde{a}_1 = \cdots = \tilde{a}_n = A] > 0$, implying $M_{0,0} = 0$ by (104), a contradiction. Next,

$$\text{if } \tau^R < p^R, \quad \text{then } \tau^S = 1 - p^R. \tag{106}$$

Suppose not. Then, using (105),

$$\Pr\left[\hat{v}_1 = v^S, \hat{v}_2 = \cdots = \hat{v}_n = v^R, \tilde{a}_1 = S, \tilde{a}_2 = \cdots = \tilde{a}_n = A\right] > 0,$$

implying $M_{0,1} = 1$ by (24) and (103). On the other hand, using $\tau^S < 1 - p^R$,

$$\Pr\left[\hat{v}_1 = v^S, \hat{v}_2 = \cdots = \hat{v}_n = v^0, \tilde{a}_1 = S, \tilde{a}_2 = \cdots = \tilde{a}_n = A\right] > 0,$$

implying $M_{0,1} = 0$ by (104), a contradiction.

Next,

$$\tau^R = p^R. \tag{107}$$

Suppose not, that is, $\tau^R < p^R$. Thus, using (106) and $\hat{r}$ from (25),

$$\begin{aligned}\Pr\big[&\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^S,\\ &\tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = A, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = S\big] > 0,\end{aligned}$$

implying $M_{0,n-\hat{r}} = 0$ by (104). On the other hand,

$$\begin{aligned}\Pr\big[&\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^0,\\ &\tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = A, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = S\big] > 0,\end{aligned}$$

implying $M_{0,n-\hat{r}} = 1$ by (25) and (103), a contradiction.

Next,

$$\tau^S \geq p^S. \tag{108}$$

Suppose not. Then, using (107),

$$\Pr\big[\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^S, \\ \tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = R, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = A\big] > 0,$$

implying $M_{\hat{r},0} = 0$ by (25) and (104). On the other hand,

$$\Pr\big[\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^0, \\ \tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = R, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = A\big] > 0,$$

implying $M_{\hat{r},0} = 1$ by (25) and (103), a contradiction.

The last step towards finding the optimal participation rates is to show that

$$\tau^S = p^S. \tag{109}$$

Suppose not. Then, using (107) and (108),

$$\Pr\big[\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^S, \\ \tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = R, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = S\big] > 0,$$

implying $M_{\hat{r},n-\hat{r}} = 0$ by (25) and (104). On the other hand,

$$\Pr\big[\hat{v}_1 = \cdots = \hat{v}_{\hat{r}} = v^R, \hat{v}_{\hat{r}+1} = \cdots = \hat{v}_n = v^0, \\ \tilde{a}_1 = \cdots = \tilde{a}_{\hat{r}} = R, \tilde{a}_{\hat{r}+1} = \cdots = \tilde{a}_n = S\big] > 0,$$

implying $M_{\hat{r},n-\hat{r}} = 1$ by (25) and (103), a contradiction.

From (107) and (109), $\tilde{r} = |\{j \mid \hat{v}_j = v^R\}|$ and $\tilde{s} = |\{j \mid \hat{v}_j = v^S\}|$.

Consider any (r, s) and a realization of $(\hat{v}_1, \dots, \hat{v}_n)$ such that $\tilde{r} = r$ and $\tilde{s} = s$. Thus, $\sum_{i=1}^n \hat{v}_i = w(r, s)$. By (23), (i) $\sum_{i=1}^n \hat{v}_i > 0$ or (ii) $\sum_{i=1}^n \hat{v}_i < 0$. In the case (i), $M_{rs} = 1$ by (103); in the case (ii), $M_{rs} = 0$ by (104). Thus, $M = \overline{M}$. This completes the proof of Lemma 6. *Q.E.D.*

By (23), (100), and (101), we can choose $\varepsilon > 0$ so small that for all (r, s),

$$\omega_{rs}\big(p^R, p^S\big) \text{ has the same sign as } w(r, s). \tag{110}$$

Denote

$$\overline{\Delta}^R = d^R\big(\overline{M}, p^R, p^S\big) > 0 \quad \text{and} \quad \overline{\Delta}^S = d^S\big(\overline{M}, p^R, p^S\big).$$

Let

$$\overline{c} = \frac{1}{2}\min\big\{v^R\overline{\Delta}^R, -v^S\overline{\Delta}^S\big\}.$$

By continuity, there exist $\overline{\varepsilon}(c^R, c^S) > 0$ and an open neighborhood $\mathcal{N}$ of the point $(\overline{M}, p^R, p^S)$ such that, for all $c^R < \overline{c}$, $c^S < \overline{c}$, $\varepsilon < \overline{\varepsilon}(c^R, c^S)$ and $m \in \mathcal{N} \cap (\mathcal{M}_n \times \mathcal{D})$,

$$\big(v^R - \varepsilon\big)d^R(m) > c^R, \qquad \big(-v^S - \varepsilon\big)d^S(m) > c^S, \qquad -v^0 + \varepsilon < c^0. \tag{111}$$

In particular, then (p^R, p^S) is the unique equilibrium of M among all participation pairs (τ^R, τ^S) with $(M, \tau^R, \tau^S) \in \mathcal{N} \cap (\mathcal{M}_n \times \mathcal{D})$.

Let W^{**} denote the maximum value of problem $(*)$. We show that there exist $\delta > 0$, $\overline{c}' > 0$, and $\overline{\varepsilon}' > 0$ such that (112) holds for all $c^R < \overline{c}'$, $c^S < \overline{c}'$, $\varepsilon < \overline{\varepsilon}'$, any ε-approximation F of $\hat{F}$, and any $(M, \tau^R, \tau^S) \in \mathcal{M}_n \times \mathcal{D}$ with $\tau^S \leq p^S$:

$$\text{if } W_F\big(M, \tau^R, \tau^S\big) > W^{**} - \delta, \quad \text{then } \big(M, \tau^R, \tau^S\big) \in \mathcal{N}. \tag{112}$$

To see why, suppose (112) fails. Then there exist sequences $\delta_j \to 0$, $c_j^R \to 0$, $c_j^S \to 0$, $\varepsilon_j \to 0$, a sequence F_j, where F_j is an ε_j-approximation of $\hat{F}_j$ (defined like $\hat{F}$ with the replacements $c^R = c_j^R$ and $c^S = c_j^S$), and a sequence $(M_j, \tau_j^R, \tau_j^S) \in \mathcal{M}_n \times \mathcal{D}$ with $\tau_j^S \leq p^S$ such that $W_{F_j}(M_j, \tau_j^R, \tau_j^S) > W^{**} - \delta_j$ and $(M_j, \tau_j^R, \tau_j^S) \notin \mathcal{N}$. By Bolzano–Weierstraß, there exists a limit point $(\hat{M}, \hat{\tau}^R, \hat{\tau}^S) \notin \mathcal{N}$ with $\hat{\tau}^S \leq p^S$ that yields by (102) the limit welfare $\hat{W}(\hat{M}, \hat{\tau}^R, \hat{\tau}^S) \geq W^{**}$. Moreover, $(\hat{M}, \hat{\tau}^R, \hat{\tau}^S) \neq \overline{M}$, contradicting Lemma 6.

We can assume that $n\overline{\varepsilon}' + \overline{c}' < \delta$.

Thus, using (102), for all $c^R < \overline{c}'$, $c^S < \overline{c}'$, $\varepsilon < \overline{\varepsilon}'$, and any ε-approximation F of $\hat{F}$,

$$W_F\big(\overline{M}, p^R, p^S\big) \geq \hat{W}\big(\overline{M}, p^R, p^S\big) - n\varepsilon - \max\big\{c^R, c^S\big\} > W^{**} - \delta. \tag{113}$$

Now consider any $c^R < \min\{\overline{c}, \overline{c}'\}$, $c^S < \min\{\overline{c}, \overline{c}'\}$, $\varepsilon < \min\{\overline{\varepsilon}(c^R, c^S), \overline{\varepsilon}'\}$ and any ε-approximation F of $\hat{F}$. Consider any optimal mechanism-equilibrium pair $(M^*, \tau^{R*}, \tau^{S*})$. Then

$$W_F\big(M^*, \tau^{R*}, \tau^{S*}\big) \geq W_F\big(\overline{M}, p^R, p^S\big) \overset{(113)}{>} W^{**} - \delta.$$

Together with (112), this implies $(M^*, \tau^{R*}, \tau^{S*}) \in \mathcal{N}$.

Hence, $(\tau^{R*}, \tau^{S*}) = (p^R, p^S)$ by (111). Given these participation rates, the welfare is

$$W_F\big(M^*, p^R, p^S\big) = \frac{1}{n} \sum_{r+s \leq n} \Pr_{p^R, p^S}^{n}(r, s) \omega_{rs}\big(p^R, p^S\big) M_{rs}^* - p^R c^R - p^S c^S,$$

implying that the unique best rule is $M^* = \overline{M}$ because of (110). This completes the proof of Lemma 3.

PROOF OF (69) AND (70)

Here we provide omitted computations towards the proof of Lemma 4. We use the notation introduced in the proof of Lemma 4.

As an auxiliary step, we establish a result on the multinomial probabilities (5). We will have to deal with higher-order partial derivatives of functions of τ^R and τ^S. We will use the lower index $(l)\tau^S$ for the lth partial derivative with respect to τ^S, evaluated at $(\tau^R, \tau^S) = (F^R, F^S)$. The lower index $(l)\tau^R$ is analogous.

LEMMA 7: *Let $l = 0, 1, \ldots$. Consider any tally (r, s) with $r + s \leq n - 1$. If $r + s \leq n - 2 - l$, then $\Pr_{(l)\tau^R}^{n-1}(r, s) = 0$ and $\Pr_{(l)\tau^S}^{n-1}(r, s) = 0$. Suppose that $r + s \geq n - 1 - l$. Then*

$$\Pr_{(l)\tau^R}^{n-1}(r, s) = \frac{(n-1)!}{(n-1-s-l)!s!} \big(F^R\big)^{n-1-s-l} \big(F^S\big)^s \binom{l}{n-1-r-s} (-1)^{n-1-r-s}$$

and

$$\Pr_{(l)\tau^S}^{n-1}(r,s) \quad = \frac{(n-1)!}{r!(n-1-r-l)!}\left(F^R\right)^r\left(F^S\right)^{n-1-r-l}\binom{l}{n-1-r-s}(-1)^{n-1-r-s}.$$

PROOF: This is a straightforward computation. The only non-vanishing derivative of $(1-\tau^R-\tau^S)^{n-1-r-s}$ is the $(n-1-r-s)$th derivative. Thus, both derivatives vanish if $r+s\leq n-2-l$. In case $r+s\geq n-1-l$, using the Leibniz rule we find

$$\Pr_{(l)\tau^R}^{n-1}(r,s) = \frac{(n-1)!(-1)^{n-1-r-s}}{r!s!(n-1-r-s)!}\frac{r!\left(F^R\right)^{r-l+n-1-r-s}\left(F^S\right)^s}{(r-l+n-1-r-s)!}(n-1-r-s)!\binom{l}{n-1-r-s}.$$

Similarly,

$$\Pr_{(l)\tau^S}^{n-1}(r,s) = \frac{(n-1)!(-1)^{n-1-r-s}}{r!s!(n-1-r-s)!}\frac{s!\left(F^R\right)^{r}\left(F^S\right)^{s-l+n-1-r-s}}{(s-l+n-1-r-s)!}(n-1-r-s)!\binom{l}{n-1-r-s}.$$

Cancelling terms yields the expressions in the lemma. *Q.E.D.*

We will use the shortcut

$$\begin{aligned} x &= \Pr_{(k-1)\tau^R}^{n-1}\left(t^*-1,q\right) = \Pr_{(k-1)\tau^S}^{n-1}\left(t^*-1,q\right) \\ &= \frac{(n-1)!}{\left(t^*-1\right)!q!}\left(F^R\right)^{t^*-1}\left(F^S\right)^q(-1)^{k-1}. \end{aligned}$$

In order to compute the relevant higher-order derivatives of $(\check{c},\check{\tau}^R)$, we derive expressions for some higher-order derivatives of d^R, d^S, and ρ^A with respect to τ^R and/or τ^S, evaluated at $(\tau^R,\tau^S)=(F^R,F^S)$. Note that

$$\begin{aligned} d^S\left(M,\tau^R,\tau^S\right) = &\sum_{r+s\leq n-k-2}(M_{r,s}-M_{r,s+1})\Pr_{\tau^R,\tau^S}^{n-1}(r,s) \\ &+\Pr_{\tau^R,\tau^S}^{n-1}\left(t^*-1,q\right)+\mathbf{1}_{t^*>1}\cdot M_{t^*-2,q}\Pr_{\tau^R,\tau^S}^{n-1}\left(t^*-2,q\right). \end{aligned} \tag{114}$$

From Lemma 7, all terms in $(\sum\dots)$ vanish if we take the lth derivative ($l\leq k$) w.r.t. τ^S or τ^R at (F^R,F^S). Moreover, using the definitions of k and x,

$$\begin{aligned} \Pr_{(k)\tau^S}^{n-1}\left(t^*-1,q-1\right) &= -x\frac{q}{F^S}, \\ \Pr_{(k)\tau^S}^{n-1}\left(t^*-2,q\right) &= -x\frac{t^*-1}{F^R}, \\ \Pr_{(k)\tau^S}^{n-1}\left(t^*-1,q\right) &= x\frac{q}{F^S}k. \end{aligned}$$

Thus, for all $l \geq 0$,

$$d^S_{(l)\tau^S} = x \cdot \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ 1 & \text{if } l = k-1, \\ \dfrac{q}{F^S}k - M_{t^*-2,q}\dfrac{t^*-1}{F^R} & \text{if } l = k. \end{cases} \tag{115}$$

Next,

$$d^S_{(1)\tau^R} = \mathbf{1}_{k=2} \Pr^{n-1}_{(1)\tau^R}\big(t^*-1, q\big) = \mathbf{1}_{k=2}x. \tag{116}$$

The following formulas (117), (118), (119), and (120) are proved below.

For all $l \geq 0$,

$$d^R_{(l)\tau^S} = x \cdot \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ -1 & \text{if } l = k-1, \\ -(1-M_{t^*-2,q})\dfrac{t^*-1}{F^R} - \dfrac{q}{F^S}(k-1) & \text{if } l = k. \end{cases} \tag{117}$$

Moreover,

$$d^R_{(1)\tau^R} = d^*\frac{t^*-1}{F^R} - \mathbf{1}_{k>2} \cdot d^*\frac{n-t^*}{F^S}. \tag{118}$$

For all $l \geq 0$,

$$\rho^A_{(l)\tau^S} = x \cdot \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ 1 & \text{if } l = k-1, \\ -M_{t^*-2,q} \cdot \dfrac{t^*-1}{F^R} + \dfrac{q(k-1)}{F^S} & \text{if } l = k. \end{cases} \tag{119}$$

Finally,

$$\rho^A_{(1)\tau^R} = \mathbf{1}_{k>2} \cdot d^*\frac{n-t^*}{F^S}. \tag{120}$$

PROOF OF (117): Note that

$$\begin{aligned} d^R\big(M, \tau^R, \tau^S\big) = & \sum_{r+s \leq n-k-2} (M_{r+1,s} - M_{r,s}) \Pr^{n-1}_{\tau^R,\tau^S}(r, s) \\ & + \mathbf{1}_{t^*>1} \cdot (1 - M_{t^*-2,q}) \Pr^{n-1}_{\tau^R,\tau^S}\big(t^*-2, q\big) \\ & + \sum_{\hat{s}=0}^{k-2} \Pr^{n-1}_{\tau^R,\tau^S}\big(t^*-1, n-t^*-\hat{s}\big), \end{aligned} \tag{121}$$

where we have used the parameter $\hat{s} = n - t^* - s$ instead of s to write the last sum (observe that $k-1 \leq n-t^*$ by definition of k).

Taking the lth ($l \leq k$) derivative of (121) and evaluating at $1-\tau^R-\tau^S=0$, all terms in the first row vanish because $(1-\tau^R-\tau^S)$ occurs in all terms with an exponent $> k$. Similarly, the second row vanishes unless $l=k$.

We begin by showing (117) for $1 \leq l \leq k-2$. Consider the lth derivative of the third row in (121), evaluated at $1-\tau^R-\tau^S=0$. Within the lth derivative expression as represented according to the general Leibniz product rule, only the term resulting from taking the $\hat{s}$th derivative of $(1-\tau^R-\tau^S)^{\hat{s}}$ (and taking the $(l-\hat{s})$th derivative of $(\tau^S)^{n-t^*-\hat{s}}$) does not vanish. Thus,

$$\begin{aligned}
d^R_{(l)\tau^S} &= \sum_{\hat{s}=0}^{l} \binom{n-1}{t^*-1 \quad \hat{s}} \left(F^R\right)^{t^*-1} \frac{(n-t^*-\hat{s})!}{\left(n-t^*-\hat{s}-(l-\hat{s})\right)!} \left(F^S\right)^{n-t^*-\hat{s}-(l-\hat{s})} \cdot \hat{s}!(-1)^{\hat{s}} \binom{l}{\hat{s}} \\
&= \left(F^R\right)^{t^*-1} \left(F^S\right)^{n-t^*-l} \sum_{\check{s}=0}^{l} \frac{(n-1)!}{(t^*-1)!(n-t^*-\check{s})!\check{s}!} \frac{(n-t^*-\check{s})!}{(n-t^*-l)!} \cdot (\check{s})!(-1)^{\check{s}} \binom{l}{\check{s}} \\
&= \left(F^R\right)^{t^*-1} \left(F^S\right)^{n-t^*-l} \sum_{\check{s}=0}^{l} \frac{(n-1)!}{(t^*-1)!} \frac{1}{(n-t^*-l)!} \cdot (-1)^{\check{s}} \binom{l}{\check{s}} \\
&= \left(F^R\right)^{t^*-1} \left(F^S\right)^{n-t^*-l} \frac{(n-1)!}{(t^*-1)!} \frac{1}{(n-t^*-l)!} \sum_{\check{s}=0}^{l} (-1)^{\check{s}} \binom{l}{\check{s}} \\
&= 0.
\end{aligned}$$

To show (117) for $l=k-1$, we use (121) and the general Leibniz product rule,

$$\begin{aligned}
d^R_{(k-1)\tau^S} &= \sum_{\hat{s}=0}^{k-2} \binom{n-1}{t^*-1 \quad \hat{s}} \left(F^R\right)^{t^*-1} \left(F^S\right)^{\overbrace{n-t^*-\hat{s}-(k-1-\hat{s})}^{=n-t^*-(k-1)=q}} \\
&\quad \cdot \frac{(n-t^*-\hat{s})!}{\left(n-t^*-(k-1)\right)!} \hat{s}!(-1)^{\hat{s}} \binom{k-1}{\hat{s}} \\
&= \binom{n-1}{t^*-1 \quad q} (k-1)! \left(F^R\right)^{t^*-1} \left(F^S\right)^{q} \cdot \sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \binom{k-1}{\hat{s}} \\
&= -x,
\end{aligned}$$

where we have used the identity $\sum_{\hat{s}=0}^{k-1}(-1)^{\hat{s}}\binom{k-1}{\hat{s}}=0$.

To show (117) for $l=k$, note that

$$\begin{aligned}
d^R_{(k)\tau^S} &= \mathbf{1}_{t^*>1, M_{t^*-2,q}=0} \binom{n-1}{t^*-2 \quad q} \left(F^R\right)^{t^*-2} \left(F^S\right)^{q} k!(-1)^k \\
&\quad + \mathbf{1}_{q>0} \sum_{\hat{s}=0}^{k-2} \binom{n-1}{t^*-1 \quad \hat{s}} \left(F^R\right)^{t^*-1} \left(F^S\right)^{\overbrace{n-t^*-\hat{s}-(k-\hat{s})}^{=n-t^*-k=q-1}} \\
&\quad \cdot \frac{(n-t^*-\hat{s})!}{(n-t^*-k)!} \hat{s}!(-1)^{\hat{s}} \binom{k}{\hat{s}}
\end{aligned}$$

$$= -\mathbf{1}_{t^*>1, M_{t^*-2,q}=0} \cdot x \frac{t^*-1}{F^R}$$
$$+ \mathbf{1}_{q>0} \begin{pmatrix} & n-1 & \\ t^*-1 & & q-1 \end{pmatrix} k! \left(F^R\right)^{t^*-1} \left(F^S\right)^{q-1} \cdot \sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \binom{k}{\hat{s}}$$
$$= -\mathbf{1}_{t^*>1, M_{t^*-2,q}=0} \cdot x \frac{t^*-1}{F^R} - \mathbf{1}_{q>0} \cdot x \frac{q}{F^S} (k-1),$$

where we have used the identity

$$\sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \binom{k}{\hat{s}} = -\sum_{\hat{s}=k-1}^{k} (-1)^{\hat{s}} \binom{k}{\hat{s}} = -(-1)^{k-1} k - (-1)^k = (-1)^k (k-1).$$

This completes the proof of (117). *Q.E.D.*

PROOF OF (118): Using (121),

$$d^R_{(1)\tau^R} = \mathbf{1}_{t^*>1} \begin{pmatrix} & n-1 & \\ t^*-1 & & 0 \end{pmatrix} (t^*-1) \left(F^R\right)^{t^*-2} \left(F^S\right)^{n-t^*} - \mathbf{1}_{k>2} \begin{pmatrix} & n-1 & \\ t^*-1 & & 1 \end{pmatrix} \left(F^R\right)^{t^*-1} \left(F^S\right)^{n-t^*-1}$$
$$= d^* \frac{t^*-1}{F^R} - \mathbf{1}_{k>2} \cdot d^* \frac{n-t^*}{F^S}.$$
Q.E.D.

PROOF OF (119): Note that

$$\begin{aligned}
\rho^A\left(M, \tau^R, \tau^S\right) = & \sum_{r+s \le n-k-2} M_{r,s} \Pr^{n-1}_{\tau^R, \tau^S}(r, s) \\
& + \mathbf{1}_{q>0} \begin{pmatrix} & n-1 & \\ t^*-1 & & q-1 \end{pmatrix} \left(\tau^R\right)^{t^*-1} \left(\tau^S\right)^{q-1} \left(1-\tau^R-\tau^S\right)^k \\
& + \mathbf{1}_{t^*>1, M_{t^*-2,q}=1} \begin{pmatrix} & n-1 & \\ t^*-2 & & q \end{pmatrix} \left(\tau^R\right)^{t^*-2} \left(\tau^S\right)^q \left(1-\tau^R-\tau^S\right)^k \\
& + \begin{pmatrix} & n-1 & \\ t^*-1 & & q \end{pmatrix} \left(\tau^R\right)^{t^*-1} \left(\tau^S\right)^q \left(1-\tau^R-\tau^S\right)^{k-1} \\
& + \sum_{r+s \ge n-1-k, r \ge t^*} \begin{pmatrix} n-1 \\ r \quad s \end{pmatrix} \left(\tau^R\right)^r \left(\tau^S\right)^s \left(1-\tau^R-\tau^S\right)^{\overbrace{n-1-r-s}^{\le k}}.
\end{aligned} \tag{122}$$

Taking the lth derivative ($1 \le l \le k-2$), only terms in the last sum can be non-vanishing because $l < k-1$. In the last sum, any term with $n-1-r-s > l$ vanishes, and any term with $s + (n-1-r-s) < l$ vanishes. Thus, using the general Leibniz product rule,

$$\rho^A_{(l)\tau^S} = \sum_{n-1-l \le r+s \le n-1, r \ge t^*, n-1-r \ge l} \begin{pmatrix} n-1 \\ r \quad s \end{pmatrix} \left(F^R\right)^r \binom{l}{n-1-r-s}$$
$$\cdot \frac{s!}{(n-1-r-l)!} \left(F^S\right)^{\overbrace{s-(l-(n-1-r-s))}^{=n-1-r-l}} (n-1-r-s)! (-1)^{n-1-r-s}$$

$$= \sum_{r=t^*}^{n-1-l} \left(F^R\right)^r \frac{(n-1)!}{r!(n-1-r-l)!} \left(F^S\right)^{n-1-r-l}$$

$$\cdot \sum_{n-1-l-r \le s \le n-1-r} \binom{l}{n-1-r-s} (-1)^{n-1-r-s}.$$

The last sum equals 0, as can be seen by using the variable $\check{s} = n-1-r-s$ instead of s. This shows (119) for $1 \le l \le k-2$.

The above computation also works if $l = k-1$ or $l = k$, showing that the fifth row on the right-hand side of (122) can be ignored.

Consider $l = k-1$. The $(k-1)$th derivative of the fourth row on the right-hand side of (122) equals x, while the $(k-1)$th derivatives of the second and third rows vanish.

Consider $l = k$. The kth derivative of the second and third rows on the right-hand side of (122) are obtained by taking the kth derivative of $(1-\tau^R-\tau^S)^k$, yielding the terms

$$\begin{aligned}
&\mathbf{1}_{q>0} \binom{n-1}{t^*-1 \quad q-1} \left(F^R\right)^{t^*-1} \left(F^S\right)^{q-1} k!(-1)^k \\
&\quad = \mathbf{1}_{q>0} \frac{(n-1)!}{(t^*-1)!(q-1)!} \left(F^R\right)^{t^*-1} \left(F^S\right)^{q-1} (-1)^k \\
&\quad = -\mathbf{1}_{q>0} \cdot x \frac{q}{F^S},
\end{aligned} \tag{123}$$

and

$$\begin{aligned}
&\mathbf{1}_{t^*>1, M_{t^*-2,q}=1} \binom{n-1}{t^*-2 \quad q} \left(F^R\right)^{t^*-2} \left(F^S\right)^{q} k!(-1)^k \\
&\quad = \mathbf{1}_{t^*>1, M_{t^*-2,q}=1} \cdot \frac{(n-1)!}{(t^*-2)!q!} \left(F^R\right)^{t^*-2} \left(F^S\right)^{q} (-1)^k \\
&\quad = -\mathbf{1}_{t^*>1, M_{t^*-2,q}=1} \cdot x \frac{t^*-1}{F^R}.
\end{aligned}$$

The last remaining term is obtained by taking the kth derivative of the fourth row on the right-hand side of (122). Using the Leibniz product rule, we take the $(k-1)$th derivative of $(1-\tau^R-\tau^S)^{k-1}$ and the first derivative of $(\tau^S)^q$ and multiply with $\binom{k}{1} = k$, yielding the term

$$\begin{aligned}
&\mathbf{1}_{q>0} \binom{n-1}{t^*-1 \quad q} \left(F^R\right)^{t^*-1} q \left(F^S\right)^{q-1} (k-1)!(-1)^{k-1} k \\
&\quad = \mathbf{1}_{q>0} \frac{(n-1)!}{(t^*-1)!(q-1)!} \left(F^R\right)^{t^*-1} \left(F^S\right)^{q-1} (-1)^{k-1} k \\
&\quad = \mathbf{1}_{q>0} \cdot x \frac{q}{F^S} k.
\end{aligned}$$

Adding this to (123), we obtain the last term in (119). This completes the proof of (119).

Q.E.D.

PROOF OF (120):

$$\rho^A_{(1)\tau^R} = \mathbf{1}_{k=2}\begin{pmatrix} & n-1 & \\ t^*-1 & & q\end{pmatrix}\left(F^R\right)^{t^*-1}\left(F^S\right)^q(-1)$$

$$+\sum_{r+s=n-2, r\geq t^*}\begin{pmatrix} n-1 \\ r \quad s\end{pmatrix}\left(F^R\right)^r\left(F^S\right)^s(-1)$$

$$+\underbrace{\sum_{r+s=n-1, r\geq t^*, r\geq 1}\begin{pmatrix} n-1 \\ r \quad s\end{pmatrix} r\left(F^R\right)^{r-1}\left(F^S\right)^s}_{=\sum_{\hat r+s=n-2, \hat r\geq t^*-1}\binom{n-1}{\hat r \; s}(F^R)^{\hat r}(F^S)^s,\text{ where } \hat r = r-1}$$

$$= -\mathbf{1}_{k=2}\begin{pmatrix} & n-1 & \\ t^*-1 & & q\end{pmatrix}\left(F^R\right)^{t^*-1}\left(F^S\right)^q + \begin{pmatrix} & n-1 & \\ t^*-1 & & n-1-t^*\end{pmatrix}\left(F^R\right)^{t^*-1}\left(F^S\right)^{n-1-t^*}$$

$$= -\mathbf{1}_{k=2}\cdot d^*\frac{n-t^*}{F^S} + d^*\frac{n-t^*}{F^S}$$

$$= \mathbf{1}_{k>2}\cdot d^*\frac{n-t^*}{F^S}.$$

Using (67) and (63), $\phi_{(1)\tau^S} = (0,0)^T$. Thus, (68) implies (69) for $l=1$. *Q.E.D.*

We proceed by induction over l to show (69). Suppose the formula in (69) holds for some l and we want to show it for $l+1$, where $l+1<k$. Applying the chain rule and general Leibniz product rule to (68), it is sufficient to show $\phi_{(l')\tau^S} = (0,0)^T$ for all $l'\leq l+1$. Consider the first factor, $\mathcal{H}^{-1}(1-\check\tau^R(\tau^S))$, of the first component of $\phi_{\partial\tau^S}$. By the chain rule and the induction hypothesis, the first l derivatives of this factor vanish at $\tau^S = F^S$. Hence, the first l derivatives of the first component of $\phi_{\partial\tau^S}$ vanish at m^*. Of the second component of $\phi_{\partial\tau^S}$, the term $\mathcal{H}^{-1}(\tau^S)$ vanishes at $\tau^S=F^S$, and, because $l<k-1$, by (115), the first l derivatives of $d^S(M,\check\tau^R(\tau^S),\tau^S)$ also vanish at $\tau^S=F^S$. Hence, the first l derivatives of the second component of $\phi_{\partial\tau^S}$ vanish at m^*. This completes the induction.

From (68),

$$\begin{pmatrix} \mathrm{d}^k\check c/\mathrm{d}\left(\tau^S\right)^k \\ \mathrm{d}^k\check\tau^R/\mathrm{d}\left(\tau^S\right)^k\end{pmatrix} = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}\left(\tau^S\right)^{k-1}}\left(\phi^{-1}_{\partial c,\partial\tau^R}\cdot\phi_{\partial\tau^S}\right).$$

Because $\phi_{(l')\tau^S}(c,m^*) = (0,0)^T$ for all $l'\leq k-1$ from the induction above,

$$\begin{pmatrix} \left.\dfrac{\mathrm{d}^k\check c}{\mathrm{d}\left(\tau^S\right)^k}\right|_{\tau^S=F^S} \\ \left.\dfrac{\mathrm{d}^k\check\tau^R}{\mathrm{d}\left(\tau^S\right)^k}\right|_{\tau^S=F^S}\end{pmatrix} = -\phi^{-1}_{\partial c,\partial\tau^R}\big|_{m^*}\cdot\phi_{(k)\tau^S}$$

$$= -\frac{h^*}{d^*}\begin{pmatrix} 0 & \dfrac{d^*}{h^*} \\ -1 & -1\end{pmatrix}$$

$$\cdot\frac{\mathrm{d}^{k-1}}{\mathrm{d}\left(\tau^S\right)^{k-1}}\left.\begin{pmatrix}\mathcal{H}^{-1}\left(1-\check\tau^R\left(\tau^S\right)\right)d^R_{\tau^S} \\ \mathcal{H}^{-1}\left(\tau^S\right)d^S_{\tau^S} + \left(\mathcal{H}^{-1}\right)'\left(\tau^S\right)d^S\end{pmatrix}\right|_{\tau^S=F^S}. \tag{124}$$

By (69) and the chain rule,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}}\big(\mathcal{H}^{-1}\big(1-\check{\tau}^R(\tau^S)\big)d^R_{\tau^S}\big)\Big|_{\tau^S=F^S}=0.$$

Moreover, using the general Leibniz product rule and (115),

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}}\big(\mathcal{H}^{-1}(\tau^S)d^S_{\tau^S}\big)\Big|_{\tau^S=F^S}=(k-1)\frac{1}{h^*}d^S_{(k-1)\tau^S}\overset{(115)}{=}\frac{k-1}{h^*}x.$$

Similarly,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}}\big(\big(\mathcal{H}^{-1}\big)'(\tau^S)d^S\big)\Big|_{\tau^S=F^S}=\frac{1}{h^*}x.$$

Thus, (124) implies

$$\begin{pmatrix}\dfrac{\mathrm{d}^k\check{c}}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S=F^S}\\ \dfrac{\mathrm{d}^k\check{\tau}^R}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S=F^S}\end{pmatrix}=-\frac{h^*}{d^*}\begin{pmatrix}0 & \dfrac{d^*}{h^*}\\ -1 & -1\end{pmatrix}\circ\begin{pmatrix}0\\ \dfrac{k}{h^*}x\end{pmatrix},$$

yielding (70).

PROOF OF (76) AND (77)

Here, we provide omitted computations for the proof of Lemma 4. We use the notation introduced in the proof of Lemma 4. Consider

$$\big(\check{\rho}^A\big)'(\tau^S)=\frac{\partial\rho^A}{\partial\tau^R}\cdot\big(\check{\tau}^R\big)'(\tau^S)+\frac{\partial\rho^A}{\partial\tau^S},\tag{125}$$

$$\big(\check{d}^R\big)'(\tau^S)=\frac{\partial d^R}{\partial\tau^R}\cdot\big(\check{\tau}^R\big)'(\tau^S)+\frac{\partial d^R}{\partial\tau^S},\tag{126}$$

$$\big(\check{d}^S\big)'(\tau^S)=\frac{\partial d^S}{\partial\tau^R}\cdot\big(\check{\tau}^R\big)'(\tau^S)+\frac{\partial d^S}{\partial\tau^S}.\tag{127}$$

Using (125), (119), and (69),

$$\big(\check{\rho}^A\big)^{(l)}=0\quad\text{for all }1\le l\le k-2.\tag{128}$$

Using (126), (117), and (69),

$$\big(\check{d}^R\big)^{(l)}=0\quad\text{for all }1\le l\le k-2.\tag{129}$$

Similarly, (127), (115), and (69),

$$\big(\check{d}^S\big)^{(l)}=0\quad\text{for all }1\le l\le k-2.\tag{130}$$

Applying the general Leibniz product rule to the second term on the right-hand side in equation (75), noting that the first derivative of the second factor in this term vanishes, and using (129), the only non-vanishing term in the $l-1$th derivative comes from taking the $l-1$th derivative of the first factor. Analogous reasoning applies to the third term on the right-hand side in (75). Applying the general Leibniz rule to the fourth and fifth terms and using (69), the $l-1$th derivative of the sum of these terms converges to the lth derivative of c. In summary,

$$\begin{aligned}\check{W}^{(l)} &:= \left.\frac{\mathrm{d}^l \check{W}}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=F^S} \\ &= (\check{\rho}^A)^{(l)}(E^R - E^S) + (\check{d}^R)^{(l)}E^R + (\check{d}^S)^{(l)}E^S - \check{c}^{(l)} \quad \text{for all } l = 1, \dots, k, \end{aligned} \tag{131}$$

where we can use the shortcuts

$$E^R = E[\tilde{v}\mathbf{1}_{\tilde{v}>0}] > 0, \qquad E^S = E[-\tilde{v}\mathbf{1}_{\tilde{v}<0}] > 0, \tag{132}$$

because

$$\begin{aligned}\int_{\overline{v}>0} \overline{v} g(\overline{v})\,\mathrm{d}\mathcal{H}(\overline{v}) &= \int_{\overline{v}>0} \overline{v} \int z\,\mathrm{d}Z_{\overline{v}}(z)\,\mathrm{d}\mathcal{H}(\overline{v}) \\ &= \int_{v>0} v\,\mathrm{d}H(v, z) = E[\tilde{v}\mathbf{1}_{\tilde{v}>0}] = E^R\end{aligned}$$

and a similar computation applies to E^S.

Using (131) together with (128), (129), and (130), one obtains (76) for all $l < k-1$.

Using (125), (126), (127), and (69),

$$\begin{aligned}(\check{\rho}^A)^{(k-1)} &= \rho^A_{(k-1)\tau^S} \overset{(119)}{=} d^S_{(k-1)\tau^S}, \\ (\check{d}^R)^{(k-1)} &= d^R_{(k-1)\tau^S} \overset{(117)}{=} -d^S_{(k-1)\tau^S}, \\ (\check{d}^S)^{(k-1)} &= d^S_{(k-1)\tau^S}.\end{aligned}$$

Thus, (131) yields (76) for $l = k-1$. In order to find $\check{W}^{(k)}$, we have to evaluate the right-hand side of (131) at $l = k$. Note that

$$\begin{aligned}(\check{\rho}^A)^{(k)} &= (\check{\tau}^R)^{(k)} \cdot \rho^A_{(1)\tau^R} + \rho^A_{(k)\tau^S} \\ &= x\mathbf{1}_{k>2}\frac{(n-t^*)k}{F^S} - xM_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + x\frac{q(k-1)}{F^S},\end{aligned} \tag{133}$$

where the first equation follows from the chain rule, the general Leibniz rule, and (69), and the second equation follows from (119), (120), and (70). Similarly,

$$\begin{aligned}(\check{d}^S)^{(k)} &= (\check{\tau}^R)^{(k)} \cdot d^S_{(1)\tau^R} + d^S_{(k)\tau^S} \\ &= -x\mathbf{1}_{k=2}\frac{(n-t^*)k}{F^S} - xM_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + x\frac{kq}{F^S},\end{aligned} \tag{134}$$

where the derivatives that occur on the right-hand side of (134) have been computed in (115), (116), and (70). Similarly,

$$\begin{aligned}\left(\check{d}^R\right)^{(k)} &= \left(\check{\tau}^R\right)^{(k)} \cdot d^R_{(1)\tau^R} + d^R_{(k)\tau^S} \\ &= x \cdot \frac{\left(t^*-1\right)k}{F^R} - x\mathbf{1}_{k>2} \cdot \frac{\left(n-t^*\right)k}{F^S} \\ &\quad - x(1-M_{t^*-2,q}) \cdot \frac{t^*-1}{F^R} - x\frac{q(k-1)}{F^S},\end{aligned} \tag{135}$$

where the derivatives that occur on the right-hand side of (135) have been computed in (117), (118), and (70).

Plugging (133), (134), (135), and (70) into (131) at $l = k$, the variable x cancels out and we find

$$\begin{aligned}\frac{\check{W}^{(k)}}{\check{c}^{(k)}} = &-\frac{h^*}{k}\left(\mathbf{1}_{k>2}\frac{\left(n-t^*\right)k}{F^S} - M_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + \frac{q(k-1)}{F^S}\right)\left(E^R - E^S\right) \\ &- \frac{h^*}{k}\left(\frac{\left(t^*-1\right)k}{F^R} - \mathbf{1}_{k>2} \cdot \frac{\left(n-t^*\right)k}{F^S}\right. \\ &\left.- (1-M_{t^*-2,q})\frac{t^*-1}{F^R} - \frac{q(k-1)}{F^S}\right)E^R \\ &- \frac{h^*}{k}\left(-\mathbf{1}_{k=2}\frac{\left(n-t^*\right)k}{F^S} - M_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + \frac{kq}{F^S}\right)E^S \\ &- 1.\end{aligned}$$

The terms with $\mathbf{1}_{k>2}$ and $\mathbf{1}_{k=2}$ can be summarized into a single term. Thus

$$\begin{aligned}\frac{\check{W}^{(k)}}{\check{c}^{(k)}} = &-\frac{h^*}{k}\left(-M_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + \frac{q(k-1)}{F^S}\right)\left(E^R - E^S\right) \\ &- \frac{h^*}{k}\left(\frac{\left(t^*-1\right)k}{F^R} - (1-M_{t^*-2,q})\frac{t^*-1}{F^R} - \frac{q(k-1)}{F^S}\right)E^R \\ &- \frac{h^*}{k}\left(-\frac{\left(n-t^*\right)k}{F^S} - M_{t^*-2,q} \cdot \frac{t^*-1}{F^R} + \frac{kq}{F^S}\right)E^S \\ &- 1.\end{aligned}$$

Similarly, the terms with $1 - M_{t^*-2,q}$ and with $M_{t^*-2,q}$ can be summarized into a single term. Thus,

$$\begin{aligned}\frac{\check{W}^{(k)}}{\check{c}^{(k)}} = &-\frac{h^*}{k}\frac{q(k-1)}{F^S}\left(E^R - E^S\right) \\ &- \frac{h^*}{k}\left(\frac{\left(t^*-1\right)k}{F^R} - \frac{t^*-1}{F^R} - \frac{q(k-1)}{F^S}\right)E^R\end{aligned}$$

$$
\begin{aligned}
&-\frac{h^*}{k}\left(-\frac{(n-t^*)k}{F^S}+\frac{kq}{F^S}\right)E^S \\
&-1 \\
&=-\frac{h^*}{k}\left(\frac{(t^*-1)k}{F^R}-\frac{t^*-1}{F^R}\right)E^R-\frac{h^*}{k}\left(-\frac{(n-t^*)k}{F^S}+\frac{q}{F^S}\right)E^S \\
&-1 \\
&=h^*\left(-\frac{t^*-1}{F^R}+\frac{t^*-1}{kF^R}\right)E^R+h^*\left(\frac{n-t^*}{F^S}-\frac{n-t^*-k+1}{kF^S}\right)E^S \\
&-1 \\
&=h^*\left(-\frac{t^*-1}{F^R}+\frac{t^*-1}{kF^R}\right)E^R+h^*\left(\frac{n-t^*+1}{F^S}-\frac{n-t^*+1}{kF^S}\right)E^S \\
&-1 \\
&=-h^*\left(1-\frac{1}{k}\right)\left((t^*-1)\frac{E^R}{F^R}-(n-t^*+1)\frac{E^S}{F^S}\right)-1,
\end{aligned}
$$

implying (77).

CLAIMS IN FOOTNOTE 32

In Lemma 8, we consider the R-one-sided rule M^{R*} that, together with full R-participation (or compulsory participation), is welfare-maximizing at zero participation cost. That is, at zero participation cost, M^{R*} yields the same welfare, defined as W^* in (30), as the best compulsory rule. We show that, at small participation costs, the rule M^{R*} has an equilibrium with almost full R-participation. We provide a formula (137) for the first-order welfare effect of introducing a participation cost. Lemma 9 is analogous for the S-one-sided rule M^{S*} that, together with full S-participation (or compulsory participation), is welfare-maximizing at zero participation cost.

The first claim in Footnote 32 amounts to showing that at least one of the first-order welfare effects (137) and (138) is > -1, where -1 is the first-order welfare effect of introducing a participation cost given a compulsory rule. To obtain the required inequality, it suffices to consider the terms due to types around 0 abstaining; these are the terms in (137) and (138) that are proportional to $\mathcal{H}'(0) > 0$. The sum of the proportionality factors is

$$\frac{E^R}{F^R}-V(t^*)+\frac{E^S}{F^S}+V(t^*-1)=0.$$

Thus, one of the proportionality factors is nonnegative, implying that either the right-hand side in (137) is $\geq F^R > -1$ or the right-hand side in (138) is $\geq F^S > -1$.

To see the second claim in Footnote 32, concerning the possibility that compulsory voting dominates voluntary voting, consider any environment such that $\frac{E^R}{F^R} - V(t^*) < 0$ and $\mathcal{H}'(0)$ is sufficiently large. In such an environment, the right-hand side in (137) is < -1. This implies that, at small participation costs, the equilibrium established in Lemma 8 yields a lower welfare than what can be achieved in the optimal rule with compulsory participation. The rule M^{R*} may have other equilibria; however, by the implicit-function theorem, the equilibrium in Lemma 8 is unique among the equilibria in a neighborhood of full R-participation; at small participation costs, any equilibrium outside this neighborhood yields a lower welfare than the equilibrium established in Lemma 8 because the welfare in any other equilibrium does not converge to W^* as the participation cost vanishes.

The lemmas below refer to equilibria in one-sided voting rules. A pair $(\tau^R, 0)$ with $\tau^R > 0$ is an equilibrium under an R-one-sided rule M if and only if the types (v_i, cz_i) with $v_i/z_i = \mathcal{H}^{-1}(1-\tau^R)$ are indifferent between participating and abstaining, that is,

$$\mathcal{H}^{-1}\big(1-\tau^R\big)d^R\big(M, \tau^R, 0\big) - c = 0. \tag{136}$$

In such equilibria, only tallies of the form $(r, 0)$ occur with positive probability. S-one-sided mechanisms are treated analogously.

Lemma 8 establishes existence and properties of equilibria with almost full R-participation in the R-one-sided linear rule $\mathbf{1}_{r\geq t^*}$. Lemma 9 is analogous for the S-one-sided case. We use the shortcuts introduced in (132).

LEMMA 8: *Make the assumptions of Proposition* 4. *Then, for all c sufficiently close to* 0, *there exists an equilibrium* $(\tilde{\tau}^R(c), 0)$ $(\to (F^R, 0)$ *as* $c \to 0)$ *in the mechanism* $M^{R*} = \mathbf{1}_{r\geq t^*}$ *that yields a welfare such that*

$$\lim_{c\to 0} W\big(c, M^{R*}, \tilde{\tau}^R(c), 0\big) = W^*.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}c} W\big(c, M^{R*}, \tilde{\tau}^R(c), 0\big)\bigg|_{c=0} = \mathcal{H}'(0)\bigg(\frac{E^R}{F^R} - V\big(t^*\big)\bigg) - F^R. \tag{137}$$

To prove Lemma 8, one applies the implicit-function theorem to the equilibrium condition (136) in order to describe the equilibrium $(\tau^R, 0)$ as a function of c. The details are omitted.

To prove the following analogous result for the S-one sided rule $\mathbf{1}_{s<n-t^*}$, one replaces $F(v) \to 1 - F(-v)$ and $t^* \to n - t^* + 1$.

LEMMA 9: *Make the assumptions of Proposition* 4. *Then, for all c sufficiently close to* 0, *there exists an equilibrium* $(0, \tilde{\tau}^S(c))$ $(\to (0, F^S)$ *as* $c \to 0)$ *in* $M^{S*} = \mathbf{1}_{s<n-t^*}$ *that yields a welfare such that*

$$\lim_{c\to 0} W\big(c, M^{S*}, 0, \tilde{\tau}^S(c)\big) = W^*.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}c} W\big(c, M^{S*}, 0, \tilde{\tau}^S(c)\big)\bigg|_{c=0} = \mathcal{H}'(0)\bigg(\frac{E^S}{F^S} + V\big(t^* - 1\big)\bigg) - F^S. \tag{138}$$

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