Econometrica Supplementary Material

SUPPLEMENT TO "MECHANISM DESIGN WITH LIMITED COMMITMENT"

(*Econometrica*, Vol. 90, No. 4, July 2022, 1499–1536)

LAURA DOVAL
Economics Division, Graduate School of Business, Columbia University

VASILIKI SKRETA
Department of Economics, University of Texas at Austin and Department of Economics, University College London

NOTATIONAL CONVENTIONS.

Measurable sets. If X is a Polish space, then $\mathcal{B}_X$ denotes its Borel σ-algebra and $\widetilde{X}$ denotes a measurable subset of X, that is, $\widetilde{X}$ is an element of $\mathcal{B}_X$.

Product sets and measures on product sets. For any two measurable spaces, X and Y, and a Borel measure ν on $X \times Y$, ν_X and ν_Y denote the marginals of ν on X and Y, respectively. Given a product space $X \times Y$, proj_Y denotes the projection of $X \times Y$ onto Y.

Throughout the Appendix, we define different distributions that arise in the game. Because we endow product spaces with their product topology and their product Borel σ-algebra, it is enough to define these new measures on the measurable rectangles and we follow this strategy throughout.

Transition probabilities and composition. Given two Polish spaces, X and Y, a transition probability from X to Y is a measurable map $\kappa : X \mapsto \Delta(Y)$. If κ is a transition probability from X to Y, then we denote by $\kappa(\cdot|x)$ the measure on Y induced by κ evaluated at x. If κ is a transition probability from X to Y and κ' is a transition probability from Y to Z, then their composition $\kappa \otimes \kappa'$ is the transition probability from X to $Y \times Z$ such that

$$\left(\kappa \otimes \kappa'\right)(\widetilde{Y} \times \widetilde{Z}|x) = \int_{\widetilde{Y}} \kappa'(\widetilde{Z}|y)\kappa(dy|x).$$

In particular, a measure ν on X can be seen as a transition probability from $\{\emptyset\}$ to X, so that $\nu \otimes \kappa'$ defines a measure on $Y \times Z$.

Mixing and public randomization. The set $\Omega = [0, 1]$ appears frequently; it is endowed with its Borel σ-algebra. We denote by l the Lebesgue measure on Ω. The set Ω is used in two ways: to define the public randomization device and to define the principal's mixed strategies in Lemma D.1. We reserve the notation $\omega \in \Omega$ to indicate a realization of the public randomization device.

Disintegration. The proof of Theorem 1 makes frequent use of the notion of disintegration Pollard (2002, Appendix F). Let X and Z denote two Polish spaces and let ν denote a measure on Z. Let $f : Z \mapsto X$ denote a measurable mapping. The family

---

Laura Doval: laura.doval@columbia.edu
Vasiliki Skreta: vskreta@gmail.com

$\{\lambda_x : x \in X\}$ is a disintegration of ν according to f if every λ_x is a probability measure concentrated on $f^{-1}(\{x\})$ and for every bounded measurable function $h : Z \mapsto \mathbb{R}$ the map $x \mapsto \int_Z h\, d\lambda_x$ is Borel measurable and $\int_Z h\, d\nu = \int_X \int_Z h\, d\lambda_x\, d(\nu \circ f^{-1})(dx)$.

In the special case when $Z = X \times Y$ and $f = \mathsf{proj}_X$, we say that $\{\lambda_x : x \in X\}$ is the proj_X-disintegration of ν. While technically λ_x is a measure on $X \times Y$ concentrated on $\{x\} \times Y$, it is usually identified as a measure on Y (cf., Kallenberg, 2017) and we do the same. That is, we define the proj_X-disintegration of ν as the map $\lambda : X \times \mathcal{B}_Y \mapsto [0, 1]$ satisfying:

(i) For every $\widetilde{Y} \in \mathcal{B}_Y$, $x \mapsto \lambda_x(\widetilde{Y})$ is $\mathcal{B}_X$-measurable,

(ii) For ν_X-almost every $x \in X$, $\widetilde{Y} \mapsto \lambda_x(\widetilde{Y})$ is a probability measure,

(iii) and for every bounded measurable function $h : X \times Y \mapsto \mathbb{R}$, $\int_{X\times Y} h(x, y)\nu(d(x, y)) = \int_X \int_Y h(x, y)\lambda_x(dy)\nu_X(dx)$,

where $\nu_X = \nu \circ \mathsf{proj}_X^{-1}$. Theorem 1.23 in Kallenberg (2017) ensures that $\{\lambda_x : x \in X\}$ exists and is unique ν_X-almost everywhere.

APPENDIX D: THE MECHANISM-SELECTION GAME

Histories and Strategies. We formally define the histories of the game (and hence the extensive form) together with the principal and the agent's strategies.

Let $\mathcal{M}_{i,j}$ denote the set of transition probabilities from M_i to $S_j \times A$, since M_i is at most countable, $\mathcal{M}_{i,j}$ is Polish. With this notation, $\mathcal{M}_{\mathcal{I}} = \cup_{i,j\in\mathcal{I}}\mathcal{M}_{i,j}$. Recall that to simplify notation, we follow the convention that when the agent does not participate the input message is $\emptyset$, the output message is $\emptyset$, and the allocation is a^*. Thus, when the principal offers a mechanism in $\mathcal{M}_{i,j}$, the possible private outcomes are $M_iS_jA_\emptyset \equiv (M_i \times S_j \times A) \cup \{(\emptyset, \emptyset, a^*)\}$, while the public outcomes are $S_jA_\emptyset \equiv (S_j \times A) \cup \{(\emptyset, a^*)\}$. We endow $M_iS_jA_\emptyset$ and $S_jA_\emptyset$ with the disjoint topology and we note that they are Polish sets under that topology.[1]

With the above notation, an outcome at the end of period t is an element of $\mathsf{Z}_A = \cup_{i,j\in\mathcal{I}}(\mathcal{M}_{i,j} \times M_iS_jA_\emptyset) \times \Omega$; the public component of the outcome in period t is an element of $\mathsf{Z} = \cup_{i,j\in\mathcal{I}}(\mathcal{M}_{i,j} \times S_jA_\emptyset) \times \Omega$. Since $\mathcal{I}$ is at most countable, Z_A, Z are Polish when endowed with the disjoint topology. For $t \geq 1$, public histories at the beginning of period t are $H^t = \Omega \times \mathsf{Z}^{t-1}$, while private histories are $\Theta \times H_A^t \equiv \Theta \times \Omega \times \mathsf{Z}_A^{t-1}$, with the understanding that $\mathsf{Z}^0 = \mathsf{Z}_A^0 = \{\emptyset\}$ and $\emptyset$ denotes the empty history. The information sets of the principal can be described by a measurable function $\zeta_{Pt} : \Theta \times H_A^t \mapsto H^t$ where ζ_{Pt} is the projection of $\Theta \times H_A^t$ onto $\Omega \times \mathsf{Z}^t$.

The principal's behavioral strategy is a collection $(\sigma_{Pt})_{t=1}^T$ where $\sigma_{Pt} : H^t \mapsto \Delta(\mathcal{M}_{\mathcal{I}})$ is a measurable function. The agent's behavioral strategy $(\sigma_{At})_{t=1}^T$ is a collection $\sigma_{At} \equiv (\pi_t, r_t)_{t=1}^T$ such that $\pi_t : \Theta \times H_A^t \times \mathcal{M}_{\mathcal{I}} \mapsto \Delta(\{0, 1\})$ and $r_t : \Theta \times H_A^t \times \mathcal{M}_{\mathcal{I}} \mapsto \Delta(\cup_{i\in\mathcal{I}}M_i)$ are measurable and $r_t(\theta, h_A^t, \mathbf{M}_t)(M^{\mathbf{M}_t}) = 1$.

Induced Distributions and Payoffs. Given the strategy profile $\sigma = (\sigma_P, \sigma_A)$ and a node (θ, h_A^t), we define transition probabilities from $\Theta \times H_A^t$ to $\mathcal{M}_{\mathcal{I}}$, from $\Theta \times H_A^t \times \mathcal{M}_{\mathcal{I}}$ to

[1] Since $M_i \times S_j \times A$ is disjoint from $\{(\emptyset, \emptyset, a^*)\}$, the disjoint topology is constructed as follows. Take a subset $B \subseteq M_iS_jA_\emptyset$ to be open if $B \cap (M_i \times S_j \times A)$ and $B \cap \{(\emptyset, \emptyset, a^*)\}$ are open in the respective topologies. This topology is the finest topology that makes the canonical injections of $M_i \times S_j \times A$ and $\{(\emptyset, \emptyset, a^*)\}$ into $M_iS_jA_\emptyset$ continuous (cf. Fremlin, 2010, Vol. 2, 214L).

$\cup_{i,j\in\mathcal{I}}M_iS_jA_\emptyset$ and from $\Theta\times H_A^t\times\mathcal{M}_\mathcal{I}\times\cup_{i,j\in\mathcal{I}}M_iS_jA_\emptyset$ to Ω as follows:

$$\begin{aligned}
\kappa_t^{\sigma_P}\big(\cup_{i,j\in\mathcal{I}}\widetilde{\mathcal{M}}_{i,j}|\theta,h_A^t\big) &= \sum_{i,j\in\mathcal{I}}\sigma_{P_t}\big(\widetilde{\mathcal{M}}_{i,j}|h^t\big),\\
\kappa_t^{\sigma_A}\big(\widetilde{M_iS_jA_\emptyset}|\theta,h_A^t,\mathbf{M}_t\big) &= \big(1-\pi_t\big(\theta,h_A^t,\mathbf{M}_t\big)\big)\mathbb{1}\big[(\emptyset,\emptyset,a^*)\in\widetilde{M_iS_jA_\emptyset}\big]\\
&\quad+\pi_t\big(\theta,h_A^t,\mathbf{M}_t\big)\\
&\quad\times\int_{\widetilde{M_iS_jA_\emptyset}}r_t\big(\theta,h_A^t,\mathbf{M}_t\big)\otimes\varphi^{\mathbf{M}_t}\big(d(m_t,s_t,a_t)\big),\\
\kappa_{t+1}^{\omega}\big(\widetilde{\Omega}|\theta,h_A^t,\mathbf{M}_t,m_t,s_t,a_t\big) &= \int_{\widetilde{\Omega}}l(d\omega_{t+1}),
\end{aligned}\tag{D.1}$$

where h^t denotes the projection of (θ,h_A^t) onto $\Omega\times\mathsf{Z}^{t-1}$ and the notation presumes that $\mathbf{M}_t\in\mathcal{M}_{i,j}$. Note that $\kappa_t^\sigma\equiv\kappa_t^{\sigma_P}\otimes\kappa_t^{\sigma_A}\otimes\kappa_{t+1}^\omega$ defines a transition probability from $\Theta\times H_A^t$ to Z_A.

Let μ_1 denote the initial distribution on Θ. The Ionescu–Tulcea extension theorem (Pollard, 2002) guarantees the existence of a sequence of probability measures $P_t^\sigma=\mu_1\otimes\kappa_1^\omega\otimes\bigotimes_{\tau=1}^{t-1}\kappa_\tau^\sigma$ defined on the product sets $(\Theta\times H_A^t)_{t=1}^{T+1}$ and a probability measure P^σ on $(\Theta\times H_A^{T+1},\mathcal{B}_\Theta\otimes\mathcal{B}_{H_A^{T+1}})$ such that for each $t\geq 1$, the marginal of P^σ on $\Theta\times H_A^t$ is P_t^σ.

Note that P^σ determines a distribution over $\Theta\times H_A^{T+1}$. The principal and the agent's payoffs, however, are defined over $\Theta\times A^T$. We record for future reference the definition of the distribution on $\Theta\times A^T$ induced by P^σ.

DEFINITION D.1: Fix an assessment (σ_P,σ_A,μ). The distribution $\eta^\sigma\in\Delta(\Theta\times A^T)$ induced by the assessment is defined as follows:

$$\eta^\sigma\big(\widetilde{\Theta}\times\widetilde{A^T}\big)=\int_{\Theta\times H_A^{T+1}}\mathbb{1}\big[\mathrm{proj}_{\Theta\times A^T}\big(\theta,h_A^{T+1}\big)\in\widetilde{\Theta}\times\widetilde{A^T}\big]P^\sigma\big(d\big(\theta,h_A^{T+1}\big)\big).$$

Thus, the principal's payoff under assessment (σ_P,σ_A,μ), $W(\sigma,\mu)$, is given by

$$\int_{\Theta\times H_A^{T+1}}W\big(\mathrm{proj}_{\Theta\times A^T}\big(\theta,h_A^{T+1}\big)\big)P^\sigma\big(d\big(\theta,h_A^{T+1}\big)\big)=\int_{\Theta\times A^T}W\big(a^T,\theta\big)\eta^\sigma\big(d\big(\theta,a^T\big)\big),\tag{D.2}$$

while the agent's payoff when her type is θ, $U(\sigma,\mu,\theta)$, is given by

$$\int_{\Theta\times H_A^{T+1}}U\big(\mathrm{proj}_{\Theta\times A^T}\big(\theta',h_A^{T+1}\big)\big)P^{\sigma|\theta}\big(d\big(\theta',h_A^{T+1}\big)\big)=\int_{\Theta\times A^T}U\big(a^T,\theta'\big)\eta_\theta^\sigma\big(d\big(\theta',a^T\big)\big),\tag{D.3}$$

where (i) $P^{\sigma|\theta}$ is the induced probability over $\Theta\times H_A^{T+1}$ determined by $\delta_\theta\otimes\kappa_1^\omega\otimes\bigotimes_{t=1}^T\kappa_t^\sigma$, where δ_θ is the Dirac measure on θ, and (ii) η_θ^σ is the proj_Θ-disintegration of η^σ.

Belief System, Conditional Distributions, and Payoffs. We now introduce the necessary notation to define the principal's beliefs along the game and the principal and the agent's payoffs conditional on the history of the game. The belief system is a collection $(\mu_t)_{t\geq 1}$ such that for all $t \geq 1$, $\mu_t : H^t \mapsto \Delta(\Theta \times H_A^t)$ is a transition probability.

In a slight abuse of notation, let $P^{\sigma|(\theta,h_A^t)}$ denote the distribution on $\Theta \times H_A^{T+1}$ induced by $\delta_{(\theta,h_A^t)} \otimes \bigotimes_{\tau\geq t} \kappa_\tau^\sigma$, which exists by the Ionescu–Tulcea theorem. More generally, suppose that in period t the principal's beliefs are given by the transition probability $\mu_t : H^t \mapsto \Delta(\Theta \times H_A^t)$.[2] Let $P^{\sigma|\mu_t(h^t)}$ denote the distribution on $\Theta \times H_A^{T+1}$ induced by $\mu_t(h^t) \otimes \bigotimes_{\tau\geq t} \kappa_\tau^\sigma$. This induces a *conditional* distribution over $\Theta \times A^T$, which we record for future reference.

DEFINITION D.2: Fix an assessment $(\sigma_P, \sigma_A, \mu)$. The conditional outcome distribution $\eta^{\sigma|h^t} \in \Delta(\Theta \times H_A^{T+1})$ induced by the assessment is defined as follows: $\eta^{\sigma|h^t} = P^{\sigma|\mu_t(h^t)} \circ \mathsf{proj}_{\Theta\times A^T}^{-1}$.

Then we can write the principal's payoffs at h^t as follows:

$$\begin{aligned} W(\sigma, \mu|h^t) &= \mathbb{E}_{\eta^{\sigma|h^t}}\big[W(\cdot)\big] \\ &= \int_{\Theta\times H_A^t} \mathbb{E}^{P^{\sigma|(\theta,h_A^t)}}\big[W(a^{t-1}, \cdot, \theta)\big]\mu_t\big(d(\theta, h_A^t)|h^t\big) \\ &= \int_{\Theta\times H_A^t}\int_{\mathcal{M}_\mathcal{I}} \mathbb{E}^{P^{\sigma|(\theta,h_A^t,\mathbf{M}_t)}}\big[W(a^{t-1}, \cdot, \theta)\big]\sigma_{Pt}(d\mathbf{M}_t|h^t)\mu_t\big(d(\theta, h_A^t)|h^t\big) \\ &= \int_{\mathcal{M}_\mathcal{I}} W(\sigma, \mu|h^t, \mathbf{M}_t)\sigma_{Pt}(d\mathbf{M}_t|h^t), \end{aligned} \tag{D.4}$$

where $W(\sigma, \mu|h^t, \mathbf{M}_t)$ is the principal's payoff at history h^t, when he offers mechanism $\mathbf{M}_t$.

Let $H_{A-}^{t+1} = H_A^t \times \cup_{i,j}(\mathcal{M}_{i,j} \times M_i S_j A_\emptyset)$, and similarly let $H_-^{t+1} = H^t \times \cup_{i,j}(\mathcal{M}_{i,j} \times S_j A_\emptyset)$. Conditional on offering $\mathbf{M}_t \in \mathcal{M}_{i,j}$ at h^t, the principal's beliefs, the mechanism, and the agent's strategy induce a distribution over $\Theta \times H_{A-}^{t+1}$ as follows:

$$\begin{aligned} &\mathbb{P}_{t+1}^-\big(\widetilde{\Theta} \times \widetilde{H}_A^t \times \{\mathbf{M}_t\} \times \widetilde{M_i S_j A_\emptyset}|\mu_t(h^t), \mathbf{M}_t\big) \\ &\quad = \int_{\widetilde{\Theta}\times\widetilde{H}_A^t} \kappa_t^{\sigma_A}\big(\widetilde{M_i S_j A_\emptyset}|\theta, h_A^t, \mathbf{M}_t\big)\mu_t\big(d(\theta, h_A^t)|h^t\big). \end{aligned} \tag{D.5}$$

Note that $\mathbb{P}_{t+1}^-$ defines a transition probability from $\Delta(\Theta \times H_A^t) \times \mathcal{M}_\mathcal{I}$ to $\Theta \times H_{A-}^{t+1}$. Furthermore, we can use $\mathbb{P}_{t+1}^-(\cdot|\mu_t(h^t), \mathbf{M}_t)$ to define a joint distribution over H_-^{t+1} as follows:

$$\begin{aligned} &\nu_{t+1}\big(\widetilde{H_-^{t+1}}|\mu_t(h^t), \mathbf{M}_t\big) \\ &\quad = \int_{\Theta\times H_{A-}^{t+1}} \mathbb{1}\big[(\theta, h_{A-}^{t+1}) \in \mathsf{proj}_{H_-^{t+1}} \widetilde{H_-^{t+1}}\big]\mathbb{P}_{t+1}^-\big(d(\theta, h_{A-}^{t+1})|\mu_t(h^t), \mathbf{M}_t\big). \end{aligned} \tag{D.6}$$

[2]The construction that follows formally yields $\mu_{t+1} : H^{t+1} \mapsto \Delta(\Theta \times H_A^{t+1})$ as a transition probability, thereby justifying why we can take μ_t to be a transition probability as well.

The notation ν_{t+1} signifies that this is the analogue of the distribution in equation (4) in Appendix A.

The joint probability $\mathbb{P}^{-}_{t+1}$ is the one that the principal uses to update his beliefs about the agent's type conditional on offering $\mathbf{M}_t$ at h^t. Formally, let $\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}} \equiv \widetilde{\Theta} \times \widetilde{H}^t_A \times \{\mathbf{M}_t\} \times \widetilde{M_i S_j A_\emptyset}$. Then

$$\begin{aligned} &\mathbb{P}^{-}_{t+1}\big(\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}}|\mu_t\big(h^t\big), \mathbf{M}_t\big) \\ &\quad = \int_{\text{proj}_{H^{t+1}_{-}} \widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}}} \mu_{t+1}\big(\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}}|h^{t+1}_{-}\big)\nu_{t+1}\big(dh^{t+1}_{-}|\mu_t\big(h^t\big), \mathbf{M}_t\big), \end{aligned} \tag{D.7}$$

where the equality uses that $\nu_{t+1}(\cdot|\mu_t(h^t), \mathbf{M}_t) = \mathbb{P}^{-}_{t+1}(\mu_t(h^t), \mathbf{M}_t) \circ \text{proj}^{-1}_{H^{t+1}_{-}}$. The right-hand side defines the principal's updated beliefs conditional on h^{t+1}_{-} as the disintegration of $\mathbb{P}^{-}_{t+1}$ according to $\text{proj}_{H^{t+1}_{-}}$. Pollard (2002, Appendix F, Theorem 6) implies that μ_{t+1} is a measurable function from H^{t+1}_{-} to $\Delta(\Theta \times H^{t+1}_{A-})$.

Equations (6) and (7) imply that we can write the principal's payoff at h^t when he offers mechanism $\mathbf{M}_t$ as follows:

$$\begin{aligned} &W\big(\sigma, \mu|h^t, \mathbf{M}_t\big) \\ &\quad = \int_{\Theta \times H^{t+1}_{A-}} \mathbb{E}^{P^{\sigma|(\theta, h^{t+1}_{A-})}}\big[W\big(a^{t-1}, a_t, \cdot, \theta\big)\big]\mathbb{P}^{-}_{t+1}\big(d\big(\theta, h^{t+1}_{A-}\big)|\mu_t\big(h^t\big), \mathbf{M}_t\big) \\ &\quad = \int_{H^{t+1}_{-}} \int_{\Theta \times H^{t+1}_{A-}} \mathbb{E}^{P^{\sigma|(\theta, h^{t+1}_{A-})}}\big[W\big(a^{t-1}, a_t, \cdot, \theta\big)\big] \\ &\qquad \times \mu_{t+1}\big(d\big(\theta, h^{t+1}_{A-}\big)|h^{t+1}_{-}\big)\nu_{t+1}\big(dh^{t+1}_{-}|\mu_t\big(h^t\big), \mathbf{M}_t\big). \end{aligned} \tag{D.8}$$

Note that this is the analogue of equation (5) in Appendix A. We can similarly define the agent's payoffs at h^t when the principal offers $\mathbf{M}_t$ as follows:

$$\begin{aligned} &U\big(\sigma|\theta, h^t_A, \mathbf{M}_t\big) \\ &\quad = \int_{M_i S_j A_\emptyset} \mathbb{E}^{P^{\sigma|(\theta, h^t_A, \mathbf{M}_t, m_t, s_t, a_t)}}\big[U\big(a^{t-1}, a_t, \cdot, \theta\big)\big]\kappa^{\sigma_A}_t\big(d(m_t, s_t, a_t)|\theta, h^t_A, \mathbf{M}_t\big). \end{aligned} \tag{D.9}$$

To complete the definition of the principal's period- $t+1$ beliefs as a function of H^{t+1}, consider the joint probability on $\Theta \times H^{t+1}_A$ induced by the principal's belief, the mechanism, the agent's strategy, and the public randomization device:

$$\begin{aligned} &\mathbb{P}_{t+1}\big(\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}} \times \widetilde{\Omega}|\mu_t\big(h^t\big), \mathbf{M}_t\big) \\ &\quad = \int_{\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}}} \kappa^{\omega}_{t+1}\big(\widetilde{\Omega}|\theta, h^{t+1}_{A-}\big)\mathbb{P}^{-}_{t+1}\big(d\big(\theta, h^{t+1}_{A-}\big)|\mu_t\big(h^t\big), \mathbf{M}_t\big) \\ &\quad = \mathbb{P}^{-}_{t+1}\big(\widetilde{\Theta} \times \widetilde{H^{t+1}_{A-}}|\mu_t\big(h^t\big), \mathbf{M}_t\big)l(\widetilde{\Omega}). \end{aligned} \tag{D.10}$$

Therefore, we have

$$\begin{aligned}&\mathbb{P}_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|\mu_t(h^t),\mathbf{M}_t\big)\\&\quad=l(\widetilde{\Omega})\int_{\mathrm{proj}_{H^{t+1}_{-}}\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}}\mu_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}|h^{t+1}_{-}\big)\nu_{t+1}\big(dh^{t+1}_{-}|\mu_t(h^t),\mathbf{M}_t\big),\end{aligned}$$

but also

$$\begin{aligned}&\mathbb{P}_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|\cdot\big)\\&=\int_{\mathrm{proj}_{H^{t+1}}(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega})}\mu'_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|h^{t+1}\big)\big(\mathbb{P}_{t+1}\big(\mu_t(h^t),\mathbf{M}_t\big)\circ\mathrm{proj}^{-1}_{H^{t+1}}\big)\big(dh^{t+1}\big)\\&=\int_{(\mathrm{proj}_{H^{t+1}_{-}}\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}})\times\widetilde{\Omega}}\mu'_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|h^{t+1}\big)\big(\mathbb{P}_{t+1}\big(\mu_t(h^t),\mathbf{M}_t\big)\circ\mathrm{proj}^{-1}_{H^{t+1}}\big)\big(dh^{t+1}\big)\\&=\int_{\mathrm{proj}_{H^{t+1}_{-}}\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}}\int_{\widetilde{\Omega}}\mu'_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|h^{t+1}_{-},\omega'\big)\\&\quad\times l\big(d\omega'\big)\big(\mathbb{P}^{-}_{t+1}\big(\mu_t(h^t),\mathbf{M}_t\big)\circ\mathrm{proj}^{-1}_{H^{t+1}_{-}}\big)\big(dh^{t+1}_{-}\big)\\&=\int_{\mathrm{proj}_{H^{t+1}_{-}}\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}}\int_{\widetilde{\Omega}}\mu'_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|h^{t+1}_{-},\omega'\big)l\big(d\omega'\big)\nu_{t+1}\big(\mu_t\big(dh^{t+1}_{-}|\mu_t(h^t),\mathbf{M}_t\big),\end{aligned}$$

where the notation μ'_{t+1} signifies that a priori this is not the distribution μ_{t+1}. Thus,

$$\int_{\mathrm{proj}_{H^{t+1}_{-}}\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}}\int_{\widetilde{\Omega}}\begin{bmatrix}\mu'_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}\times\widetilde{\Omega}|h^{t+1}_{-},\omega'\big)-\\\mu_{t+1}\big(\widetilde{\Theta}\times\widetilde{H^{t+1}_{A-}}|h^{t+1}_{-}\big)\end{bmatrix}l\big(d\omega'\big)\nu_{t+1}\big(dh^{t+1}_{-}|\mu_t(h^t),\mathbf{M}_t\big)=0,$$

which shows that $\nu_{t+1}(\mu_t(h^t),\mathbf{M}_t)$-almost surely the principal's beliefs about the agent's private history do not depend on the realization of the public randomization device. Thus, we can define the principal's beliefs as a measurable function from H^{t+1} to $\Delta(\Theta\times H^{t+1}_A)$.

Perfect Bayesian Equilibrium.

DEFINITION D.3: An assessment (σ_P,σ_A,μ) is sequentially rational if for all t and public histories h^t:

1. If $\mathbf{M}_t\in\operatorname{supp}\sigma_{Pt}(h^t)$, $W(\sigma,\mu|h^t,\mathbf{M}_t)\geq W(\sigma,\mu|h^t,\mathbf{M}'_t)$ for all $\mathbf{M}'_t\in\mathcal{M}_{\mathcal{I}}$,
2. For all $\mathbf{M}_t\in\mathcal{M}_{\mathcal{I}}$, $U(\sigma|\theta,h^t_A,\mathbf{M}_t)\geq U(\sigma_P,\sigma'_A|\theta,h^t_A,\mathbf{M}_t)$ for all $\theta\in\Theta$, $h^t_A\in H^t_A(h^t)$, σ'_A.

DEFINITION D.4: The system of beliefs $(\mu_t)_{t\geq1}$ satisfies Bayes' rule where possible if for all t, all public histories h^t, and mechanisms $\mathbf{M}_t$, it satisfies equation (7) $\nu_{t+1}(\mu_t(h^t),\mathbf{M}_t)$-almost surely.

DEFINITION D.5: An assessment (σ_P,σ_A,μ) is a *perfect Bayesian equilibrium* if it is sequentially rational and satisfies Bayes' rule where possible.

D.1. *Proof of Theorem 1*

D.1.1. *Proof of Proposition 3*

Fix a PBE assessment $(\sigma_P, \sigma_A, \mu)$ and suppose there exists h^t and an agent type θ, such that $\sigma_{At}(\theta, h^t_A) \neq \sigma_{At}(\theta, \overline{h}^t_A)$ for some $h^t_A, \overline{h}^t_A \in H^t_A(h^t)$.

We begin by arguing that for each mechanism $\mathbf{M}_t$, it must be the case that $U(\sigma|\theta, h^t_A, \mathbf{M}_t) = U(\sigma|\theta, \overline{h}^t_A, \mathbf{M}_t)$. First, the strategy of (θ, h^t_A) is feasible for $(\theta, \overline{h}^t_A)$ and vice versa. Second, since $h^t_A, \overline{h}^t_A \in H^t_A(h^t)$, the allocation through period $t-1$, a^{t-1}, is the same for (θ, h^t_A) and $(\theta, \overline{h}^t_A)$, so that at both nodes the agent evaluates continuation payoffs using the same payoff function, $U(a^{t-1}, \cdot, \theta)$. It follows that the agent at (θ, h^t_A) is indifferent between $\sigma_{At}(\theta, h^t_A)$, $\sigma_{At}(\theta, \overline{h}^t_A)$, and any randomization over $\sigma_{At}(\theta, h^t_A)$ and $\sigma_{At}(\theta, \overline{h}^t_A)$.

Moreover, the same reasons imply that the same indifference holds for any public history h^τ that succeeds h^t. That is, for any $\tau \geq t$ and h^τ that succeeds h^t, and any $h^\tau_A, \overline{h}^\tau_A$ that succeed h^t_A and $\overline{h}^t_A$, respectively, the agent is indifferent over $\sigma_{A\tau}(\theta, h^\tau_A)$ and $\sigma_{A\tau}(\theta, \overline{h}^\tau_A)$.

We use this indifference to construct a new strategy for the agent, σ'_A, and a new PBE assessment $(\sigma_P, \sigma'_A, \mu')$ that implements the same distribution over outcomes, $\Theta \times A^T$ starting from h^t.

To define the agent's new strategy, σ'_A, we use the following construction to simplify notation. Given a history h^t and a mechanism $\mathbf{M}_t \in \mathcal{M}_{i,j}$, we can extend the mechanism and the agent's strategy so as to subsume the agent's participation decision into her reporting strategy and the mechanism as follows (see the discussion after the statement of Theorem 1). Let $M_{i\emptyset} = M_i \cup \{\emptyset\}$. Let $\varphi^{\mathbf{M}_t}_\pi : M_{i\emptyset} \mapsto \Delta(S_j A_\emptyset)$ be such that for $m \in M_i$, $\varphi^{\mathbf{M}_t}_\pi(\cdot|m) = \varphi^{\mathbf{M}_t}(\cdot|m)$, whereas $\varphi^{\mathbf{M}_t}_\pi(\cdot|\emptyset) = \delta_{a^*}$. Finally, let

$$\begin{aligned} &r_{\pi t}(\theta, h^t_A, \mathbf{M}_t)(\widetilde{M}_{i\emptyset}) \\ &\quad = (1 - \pi_t(\theta, h^t_A, \mathbf{M}_t))\mathbb{1}[\emptyset \in \widetilde{M}_{i\emptyset}] + \pi_t(\theta, h^t_A, \mathbf{M}_t) r_t(\theta, h^t_A, \mathbf{M}_t)(\widetilde{M}_{i\emptyset}), \end{aligned}$$

denote the agent's extended reporting strategy. By construction, $r_{\pi,t}$ is a transition probability from $\Theta \times H^t_A \times \mathcal{M}_\mathcal{I}$ to $\cup_{i\in\mathcal{I}} M_{i\emptyset}$ such that $r_{\pi,t}(\theta, h^t_A, \mathbf{M}_t)(M^{\mathbf{M}_t}_\emptyset) = 1$.

Fix h^t. For each $\tau \geq t$ and each h^τ that follows h^t, if $\mathbf{M}_\tau \in \mathcal{M}_{i,j}$ let

$$\mathbb{R}_\tau(\widetilde{\Theta} \times \widetilde{M}_{i\emptyset}|h^\tau, \mathbf{M}_\tau) = \int_{\widetilde{\Theta}\times H^\tau_A} r_{\pi\tau}(\theta, h^\tau_A, \mathbf{M}_\tau)(\widetilde{M}_{i\emptyset})\mu_\tau(d(\theta, h^\tau_A)|h^\tau) \tag{D.11}$$

denote the joint probability over $\Theta \times M_{i\emptyset}$ induced by the principal's belief at h^τ together with the agent's (extended) reporting strategy. By construction, $\mathbb{R}_\tau$ is a transition probability from $H^\tau \times \mathcal{M}_\mathcal{I}$ to $\Theta \times \cup_{i\in\mathcal{I}} M_{i\emptyset}$. Proposition 7.27 in Bertsekas and Shreve (1978) implies that a transition probability $r'_{\pi,\tau}(\cdot, h^\tau, \mathbf{M}_\tau) : \Theta \times H^\tau \times \mathcal{M}_\mathcal{I} \mapsto \Delta(M_{i\emptyset})$ exists such that

$$\mathbb{R}_\tau(\widetilde{\Theta} \times \widetilde{M}_{i\emptyset}|h^\tau, \mathbf{M}_\tau) = \int_{\widetilde{\Theta}} r'_{\pi\tau}(\theta, h^\tau, \mathbf{M}_\tau)(\widetilde{M}_{i\emptyset})\mu_{\tau\Theta}(d\theta|h^\tau) \tag{D.12}$$

where $\mu_{\tau\Theta}(h^\tau)$ denotes the marginal on Θ of $\mu_\tau(\cdot|h^\tau)$. The above construction does not necessarily define (i) $(\pi_\tau(\theta, h^\tau_A, \mathbf{M}_\tau), r_\tau(\theta, h^\tau_A, \mathbf{M}_\tau))$ for types θ not in the support of $\mu_{\tau\Theta}(\cdot|h^\tau)$, (ii) r_τ for types in the support of $\mu_{\tau\Theta}(\cdot|h^\tau)$ that do not participate in the mechanism. For types not in the support of $\mu_\tau(h^\tau)$, define their strategy at h^τ by choosing an arbitrary (and agent-history independent) randomization over $\{(\pi_\tau(\theta, h^\tau_A, \mathbf{M}_\tau), r_\tau(\theta, h^\tau_A, \mathbf{M}_\tau))|h^\tau_A \in H^\tau_A(h^\tau)\}$. Similarly, for types in the support of $\mu_{\tau\Theta}(\cdot|h^\tau)$ that do not participate in the mechanism define their reporting strategy at h^τ by choosing an arbitrary (and agent-history independent) randomization over $\{r_\tau(\theta, h^\tau_A, \mathbf{M}_\tau)|h^\tau_A \in H^\tau_A(h^\tau)\}$.

We now verify that two properties hold. First, that if we were to change the agent's strategy from h^t onwards by $r'_{\pi,\cdot}$, then the principal's beliefs over Θ would remain the same. Second, that the the principal's "prior" $\mu_t(h^t)$ and the strategy profile (σ_P,σ'_A) induce the same probability over the terminal outcomes $(\theta, a^{t-1}, a_{\geq t})$ as $\mu_t(h^t)$ and the strategy profile (σ_P,σ_A). It follows that σ_P continues to be a sequentially rational strategy for the principal.

To verify the first property, denote by $(\sigma_P,\sigma'_A,\mu')$ the assessment in which the agent follows the aforementioned strategy starting from h^t onwards. We inductively show that for $\tau \geq t$, if $\mu'_{\tau\Theta}(\cdot|h^\tau) = \mu_{\tau\Theta}(\cdot|h^\tau)$, then $\mu'_{\tau+1\Theta}(\cdot|h^\tau, \mathbf{M}_\tau, \cdot) = \mu_{\tau+1\Theta}(\cdot|h^\tau, \mathbf{M}_\tau, \cdot)$. Since by construction $\mu'_{t\Theta}(\cdot|h^t) = \mu_{t\Theta}(\cdot|h^t)$, this closes the inductive argument.

Fix h^τ,$\mathbf{M}_\tau$, for some h^τ that can be reached from h^t under the agent's strategy σ_A.[3] Associated with the new assessment there is also a new kernel for the agent's strategy, $\kappa^{\sigma'_A}_\tau$, and therefore, a new version of the transition probability defined in equation (5), which we denote by $\mathbb{P}^{-\prime}_{\tau+1}$. Evaluating $\mathbb{P}^{-\prime}_{\tau+1}$ at $(\mu'_\tau(\cdot|h^\tau), \mathbf{M}_\tau)$, we have

$$\begin{aligned}
&\mathbb{P}^{-\prime}_{\tau+1}\big(\widetilde{\Theta} \times H^\tau_A \times \{\mathbf{M}_\tau\} \times \widetilde{M_i S_j A_\emptyset}|\mu'_\tau\big(h^\tau\big), \mathbf{M}_\tau\big) \\
&= \int_{\widetilde{\Theta}\times H^\tau_A} \kappa^{\sigma'_A}_\tau\big(\widetilde{M_i S_j A_\emptyset}|\theta, h^\tau_A, \mathbf{M}_\tau\big)\mu'_\tau\big(d\big(\theta, h^\tau_A\big)|h^\tau\big) \\
&= \int_{\widetilde{\Theta}} \kappa^{\sigma'_A}\big(\widetilde{M_i S_j A_\emptyset}|\theta, h^\tau_A, \mathbf{M}_\tau\big)\mu'_{\tau\Theta}\big(d\theta|h^\tau\big) \\
&= \int_{\widetilde{\Theta}} \kappa^{\sigma'_A}_\tau\big(\widetilde{M_i S_j A_\emptyset}|\theta, h^\tau_A, \mathbf{M}_\tau\big)\mu_{\tau\Theta}\big(d\theta|h^\tau\big) \\
&= \int_{\widetilde{\Theta}} \int_{\mathrm{proj}_{M_{i\emptyset}} \widetilde{M_i S_j A_\emptyset}} \varphi^{\mathbf{M}_t}_\pi(\mathrm{proj}_{S_j A_\emptyset} \widetilde{M_i S_j A_\emptyset}|m) r'_{\pi\tau}\big(dm|\theta, h^\tau, \mathbf{M}_\tau\big)\mu_{\tau\Theta}\big(d\theta|h^\tau\big) \\
&= \int_{\widetilde{\Theta}\times H^\tau_A} \int_{\mathrm{proj}_{M_{i\emptyset}} \widetilde{M_i S_j A_\emptyset}} \varphi^{\mathbf{M}_t}_\pi(\mathrm{proj}_{S_j A_\emptyset} \widetilde{M_i S_j A_\emptyset}|m) r_{\pi\tau}\big(dm|\theta, h^\tau_A, \mathbf{M}_\tau\big)\mu_\tau\big(d\big(\theta, h^\tau_A\big)|h^\tau\big) \\
&= \mathbb{P}^{-}_{\tau+1}\big(\widetilde{\Theta} \times H^\tau_A \times \{\mathbf{M}_\tau\} \times \widetilde{M_i S_j A_\emptyset}|\mu_\tau\big(h^\tau\big), \mathbf{M}_\tau\big),
\end{aligned} \tag{D.13}$$

where the second equality uses that $\kappa^{\sigma'_A}_\tau(\cdot|\theta, h^\tau_A, \mathbf{M}_\tau)$ is measurable in the public history, the third equality uses that the marginal on Θ of the principal's beliefs $\mu'_\tau(\cdot|h^\tau)$ coincides with those in the original assessment, $\mu_\tau(\cdot|h^\tau)$, the fourth equality uses the definition of the new strategy, the fifth equality follows from equations (11) and (12), and the last from the definition of $\mathbb{P}^{-}_{\tau+1}$ under the old strategy profile. Equation (7) then implies that the marginal of $\mu'_{\tau+1}(\cdot|h^\tau, \mathbf{M}_\tau, \cdot)$ on Θ coincides with that of $\mu_{\tau+1}(\cdot|h^\tau, \mathbf{M}_\tau, \cdot)$.

[3]While we change the agent's strategy at all histories, which succeed $(h^t, \mathbf{M}_t)$, Bayes' rule where possible ties the beliefs at h^t and the beliefs at h^τ only at those histories h^τ that are on the path of the (agent's) strategy. This is why when we check that the principal's beliefs over Θ have not changed we do so along the path of the agent's strategy profile starting at h^t.

To verify the second property, let $\kappa_\tau^{\sigma'} \equiv \kappa_\tau^{\sigma_P} \otimes \kappa_\tau^{\sigma'_A} \otimes \kappa_{\tau+1}^{\omega}$. The Ionescu–Tulcea theorem guarantees the existence of a sequence of probability measures $Q_\tau^{\sigma'} = \mu_t(h^t) \otimes \delta_{a^{t-1}} \otimes \prod_{n=t}^{\tau-1} \kappa_n^{\sigma'}$ defined on the product sets $(\Theta \times H_A^t \times A^{t-1} \times \prod_{n=t}^{\tau-1} \mathsf{Z}_A)_{\tau=t}^T$ and a probability measure $Q^{\sigma'}$ on $\Theta \times H_A^t \times A^{t-1} \times \prod_{\tau=t}^{T} \mathsf{Z}_A$ such that for all $\tau \geq t$, $Q_\tau^{\sigma'}$ coincides with the marginal of $Q^{\sigma'}$ on $\Theta \times H_A^t \times A^{t-1} \times \prod_{n=t}^{\tau-1} \mathsf{Z}_A$. Analogously, one can define Q^σ, Q_τ^σ for the original assessment. Note that Q^σ is the distribution under which the principal's payoffs are computed at h^t, that is, $Q^\sigma = P^{\sigma|\mu_t(h^t)}$ (see equation (4)).[4]

Lemma 10.4 in Bertsekas and Shreve (1978) implies that along the path of play starting from h^t and for each $\tau \geq t$ the principal's updated beliefs at $h^{\tau+1}$, $\mu_{\tau+1}(h^{\tau+1})$, correspond to the distribution $Q_{\tau+1}^\sigma$ conditional on the principal's information set, $h^{\tau+1}$. That is,

$$Q_{\tau+1}^\sigma\left(\left\{(\theta, h_A^{\tau+1}) \in \widetilde{\Theta} \times \widetilde{H}_A^{\tau+1} : h_A^{\tau+1} \in \bigcup_{h^{\tau+1} \in \widetilde{H}^{\tau+1}} H_A^{\tau+1}(h^{\tau+1})\right\}\right)$$
$$= \int_{\widetilde{H}^{\tau+1}} \mu_{\tau+1}(\widetilde{\Theta} \times \widetilde{H}_A^{\tau+1} | h^{\tau+1}) Q_{\tau+1}^\sigma(d(\theta, h_A^{\tau+1})), \tag{D.14}$$

and similarly for $Q_{\tau+1}^{\sigma'}$ and $\mu'_{\tau+1}$.

The definition of $Q_{\tau+1}^{\sigma'}$ together with equation equation (14) imply that for $\tau \geq t$ we have

$$Q_{\tau+1}^{\sigma'}\left(\left\{(\theta, h_A^{\tau+1}) : \theta \in \widetilde{\Theta}, h_A^{\tau+1} \in \bigcup_{h^\tau \in \widetilde{H}^\tau} \widetilde{H}_A^\tau(h^t) \times \widetilde{\mathsf{Z}_A}\right\}\right)$$
$$= \int_{\widetilde{\Theta} \times \cup_{h^\tau \in \widetilde{H}^\tau} \widetilde{H}_A^\tau(h^\tau)} \kappa_\tau^{\sigma'}(\widetilde{\mathsf{Z}_A} | \theta, h_A^\tau) Q_\tau^{\sigma'}(d(\theta, h_A^\tau))$$
$$= \int_{\widetilde{H}^\tau} \int_{\widetilde{\Theta} \times \widetilde{H}_A^\tau(h^\tau)} \kappa_\tau^{\sigma'}(\widetilde{\mathsf{Z}_A} | \theta, h_A^\tau) \mu'_\tau(d(\theta, h_A^\tau) | h^\tau) Q_\tau^{\sigma'}(d(\theta, h_A^\tau)), \tag{D.15}$$

where the second equality uses equation (14). Evaluating equation (15) at $\widetilde{H}_A^\tau(h^\tau) = H_A^\tau(h^\tau)$ (i.e., taking the marginal over the agent-histories, h_A^τ), and applying Fubini's theorem, we have

$$Q_{\tau+1}^{\sigma'}\left(\left\{(\theta, h_A^{\tau+1}) : \theta \in \widetilde{\Theta}, h_A^{\tau+1} \in \bigcup_{h^\tau \in \widetilde{H}^\tau} H_A^\tau(h^t) \times \widetilde{\mathsf{Z}_A}\right\}\right)$$
$$= \int_{\widetilde{H}^\tau} \int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \kappa_\tau^{\sigma'}(\widetilde{\mathsf{Z}_A} | \theta, h_A^\tau) \mu'_\tau(d(\theta, h_A^\tau) | h^\tau) Q_\tau^{\sigma'}(d(\theta, h_A^\tau))$$
$$= \int_{\widetilde{H}^\tau} \int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \kappa_\tau^{\sigma}(\widetilde{\mathsf{Z}_A} | \theta, h_A^\tau) \mu_\tau(d(\theta, h_A^\tau) | h^\tau) Q_\tau^{\sigma'}(d(\theta, h_A^\tau)), \tag{D.16}$$

[4]We introduce the Q^σ notation to define the collection $(Q_\tau^\sigma)_{\tau \geq t}$ and avoid confusing it with P_τ^σ defined on page 3.

where the second equality uses equations (11) and (12), as we did in equation (13).[5] Equation (16) implies that if we could change $Q_\tau^{\sigma'}$ for Q_τ^σ, then we would conclude that

$$Q_{\tau+1}^{\sigma'}\left(\left\{(\theta, h_A^{\tau+1}) : \theta \in \widetilde{\Theta}, h_A^{\tau+1} \in \bigcup_{h^\tau \in \widetilde{H}^\tau} H_A^\tau(h^t) \times \widetilde{Z_A}\right\}\right)$$

$$= Q_{\tau+1}^{\sigma}\left(\left\{(\theta, h_A^{\tau+1}) : \theta \in \widetilde{\Theta}, h_A^{\tau+1} \in \bigcup_{h^\tau \in \widetilde{H}^\tau} H_A^\tau(h^t) \times \widetilde{Z_A}\right\}\right),$$

which inductively implies that we can perform the change since for $\tau = t$ we have $Q_t^\sigma = Q_t^{\sigma'} = \mu_t(h^t)$. This implies that for each finite τ $Q_{\tau+1}^{\sigma'}$ and $Q_{\tau+1}^\sigma$ generate the same distribution over $\Theta \times A^{\tau+1}$, and hence so do $Q^{\sigma'}$ and Q^σ. Thus, the distribution over outcomes from h^t onwards is unchanged.

These two steps also imply that $(\sigma_P, \sigma_A', \mu')$ is a PBE assessment.

D.1.2. *Proof of Proposition 4*

PROOF OF PROPOSITION 4: Let $(\sigma_P, \sigma_A, \mu)$ be as in the statement of Proposition 4. Let h^t be a public history and let $\mathbf{M}_t$ denote the mechanism that the principal offers at h^t under σ_{Pt}. Let Θ^+ denote the support of the principal's beliefs at h^t, $\mu_t(h^t)$.

We begin by performing some auxiliary changes to the distribution $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$ defined in equation (10). First, we extend it to the product set $\Theta \times H_A^t \times \{\mathbf{M}_t\} \times M^{\mathbf{M}_t} \cup \{\emptyset\} \times S^{\mathbf{M}_t} \cup \{\emptyset\} \times A \times \Omega$, so as to be able to invoke disintegration results in Kallenberg (2017), which are stated for product spaces.[6] Second, in a slight abuse of notation, we denote by $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$ the marginal over $H_A^t \times M^{\mathbf{M}_t} \cup \{\emptyset\}$ of $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$. That is, in what follows, $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$ is a distribution over $\Theta \times S^{\mathbf{M}_t} \cup \{\emptyset\} \times A \times \Omega$. The reason is that the agent's strategy is measurable with respect to the public history.

The first step of the proof is to use the distribution $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$ over period-t allocations and continuation histories to define a measure over period-t allocations, continuation beliefs, and potentially continuation equilibria. To this end, de-

[5]To verify the second equality, note the following. Letting $\widetilde{Z_A} = \cup_{i,j}(\widetilde{\mathcal{M}}_{ij} \times \widetilde{M_i S_j A_\emptyset}) \times \widetilde{\Omega}$,

$$\int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \kappa_\tau^{\sigma'}(\widetilde{Z_A}|\theta, h_A^\tau)\mu_\tau'(d(\theta, h_A^\tau)|h^\tau)$$

$$= \int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \sum_{i,j\in\mathcal{I}}\left(\int_{\widetilde{\mathcal{M}}_{ij}} \int_{\widetilde{\Omega}} \kappa_\tau^{\sigma_A'}(\widetilde{M_i S_j A_\emptyset}|\theta, h_A^t, \mathbf{M}_\tau) l(d\omega_{\tau+1})\sigma_{P\tau}(d\mathbf{M}_\tau|h^\tau)\right)\mu_\tau'(d(\theta, h_A^\tau)|h^\tau)$$

$$= \sum_{i,j\in\mathcal{I}} \int_{\widetilde{\mathcal{M}}_{i,j}} \int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \int_{\widetilde{\Omega}} \kappa_\tau^{\sigma_A'}(\widetilde{M_i S_j A_\emptyset}|\theta, h_A^\tau, \mathbf{M}_\tau) l(d\omega_{\tau+1})\mu_\tau'(d(\theta, h_A^\tau)|h^\tau)\sigma_{P\tau}(d\mathbf{M}_\tau)$$

$$= \sum_{i,j\in\mathcal{I}} \int_{\widetilde{\mathcal{M}}_{i,j}} \int_{\widetilde{\Theta} \times H_A^\tau(h^\tau)} \int_{\widetilde{\Omega}} \kappa_\tau^{\sigma_A}(\widetilde{M_i S_j A_\emptyset}|\theta, h_A^\tau, \mathbf{M}_\tau) l(d\omega_{\tau+1})\mu_\tau(d(\theta, h_A^\tau)|h^\tau)\sigma_{P\tau}(d\mathbf{M}_\tau),$$

where the first equality follows by definition, the second from Fubini's theorem, and the third from equations (11) and (12) once we verify that the beliefs in the new assessment have the same marginal over Θ as the beliefs in the old assessment.

[6]Formally, for any set $B = \widetilde{\Theta} \times \widetilde{H}_A^t \times \{\mathbf{M}_t\} \times \widetilde{M} \times \widetilde{S} \times \widetilde{A} \times \widetilde{\Omega} \in \Theta \times H_A^t \times M^{\mathbf{M}_t} \cup \{\emptyset\} \times S^{\mathbf{M}_t} \cup \{\emptyset\} \times A \times \Omega$,

$$\mathbb{P}_{t+1}(B|\mu_t(h^t), \mathbf{M}_t) = \int_{\Theta \times H_A^{t+1}} \mathbb{1}[(\theta, h_A^{t+1}) \in B]\mathbb{P}_{t+1}(d(\theta, h_A^{t+1})|\mu_t(h^t), \mathbf{M}_t).$$

fine the measurable map T that maps tuples formed by period-t output messages and allocations together with period-$t+1$ realizations of the public randomization device, $(s_t, a_t, \omega_{t+1}) \in S^{\mathbf{M}_t} A_\emptyset \times \Omega$, to a belief-allocation-public randomization device tuple, $(\mu_{t+1}(\cdot|h^t, z_{(s_t,a_t)}(\mathbf{M}_t), \omega_{t+1}), a_t, \omega'(s_t, \omega_{t+1}))$, where $\omega' : S^{\mathbf{M}_t} \cup \{\emptyset\} \times \Omega \mapsto [0,1]$ is a measurable bijection, which exists by Kuratowski's theorem (see Srivastava, 2008) and $z_{(s_t,a_t)}(\mathbf{M}_t)$ is shorthand notation for $(\mathbf{M}_t, s_t, a_t)$. Second, define the measure $\mathbb{P}'$ over $\Theta \times \Delta(\Theta) \times A \times \Omega$ as follows:

$$\mathbb{P}'(\widetilde{\Theta} \times \widetilde{U} \times \widetilde{A} \times \widetilde{\Omega}') = \mathbb{P}_{t+1}(\widetilde{\Theta} \times T^{-1}(\widetilde{U} \times \widetilde{A} \times \widetilde{\Omega}')|\mu_t(h^t), \mathbf{M}_t). \tag{D.17}$$

By definition, we have that the principal's payoff is the same under $\mathbb{P}'$ and $\mathbb{P}_{t+1}(\mu_t(h^t), \mathbf{M}_t)$:

$$\begin{aligned}
&\int_\Theta \int_{M^{\mathbf{M}_t} S^{\mathbf{M}_t} A_\emptyset} \int_\Omega \mathbb{E}^{P^{\sigma|(\theta,h^t,z_{(s_t,a_t)}(\mathbf{M}_t),\omega_{t+1})}}\left[W\left(a^{t-1}, a_t, \cdot, \theta\right)\right] \\
&\quad \times \mathbb{P}_{t+1}\left(d\left(\theta, h^{t+1}\right)|\mu_t\left(h^t\right), \mathbf{M}_t\right) \\
&= \int_\Theta \int_{\Delta(\Theta)\times A\times\Omega'} \mathbb{E}^{P^{\sigma|(\theta,h^t,\mathbf{M}_t,T^{-1}(\mu,a_t,\omega'))}}\left[W\left(a^{t-1}, a_t, \cdot, \theta\right)\right] \\
&\quad \times \mathbb{P}'\left(d\left(\theta, \mu, a_t, \omega'\right)\right).
\end{aligned} \tag{D.18}$$

Similarly, whenever θ is in the support of $\mu_t(h^t)$, the agent's payoff remains the same under the distribution $\mathbb{P}'$ *conditional on* θ. To see this, let $\{\lambda_\theta : \theta \in \Theta\}$ denote the proj_Θ-disintegration of $\mathbb{P}'$. Noting that $\mathbb{P}'_\Theta = \mu_t(h^t)$, Theorem 1.23 in Kallenberg (2017) implies that $\lambda_\theta(\cdot) = (\kappa_t^{\sigma_A}(\theta, h^t, \mathbf{M}_t) \otimes \kappa_{t+1}^\omega) \circ T^{-1}$ $\mu_t(h^t)$-almost surely. Thus, we have that

$$\begin{aligned}
&\int_{M^{\mathbf{M}_t} S^{\mathbf{M}_t} A_\emptyset} \int_\Omega \mathbb{E}^{P^{\sigma|(\theta,h^t,z_{(s_t,a_t)}(\mathbf{M}_t),\omega_{t+1})}}\left[U\left(a^{t-1}, a_t, \cdot, \theta\right)\right] \\
&\quad \times \left(\kappa_t^{\sigma_A} \otimes \kappa_{t+1}^\omega\right)\left(d(m_t, s_t, a_t, \omega_{t+1})|\theta, h^t, \mathbf{M}_t\right) \\
&= \int_{\Delta(\Theta)\times A\times\Omega'} \mathbb{E}^{P^{\sigma|(\theta,h^t,\mathbf{M}_t,T^{-1}(\mu,a_t,\omega'))}}\left[U\left(a^{t-1}, a_t, \cdot, \theta\right)\right]\lambda_\theta\left(d\left(\mu, a_t, \omega'\right)\right).
\end{aligned} \tag{D.19}$$

Note that equations (18) and (19) correspond to equations (9) and (12) in the main Appendix.

The second step verifies two properties of $\mathbb{P}'$: (i) the principal's beliefs over Θ coincide with μ whenever μ is the *output message* and (ii) conditional on μ, the allocation and the selection of continuation equilibria (as indexed by ω') are independent of the agent's type θ. To verify (i), note that equation (7) implies that $\mathbb{P}'$ satisfies the following:

$$\begin{aligned}
&\mathbb{P}'(\widetilde{\Theta} \times \widetilde{U} \times \widetilde{A} \times \widetilde{\Omega}') \\
&\quad = \int_{T^{-1}(\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}')} \mu_{t+1}\left(\widetilde{\Theta}|h^t, z_{(s_t,a_t)}(\mathbf{M}_t), \omega_{t+1}\right) \\
&\quad \times l(d\omega_{t+1})\nu_{t+1}\left(d\left(h^t, z_{(s_t,a_t)}(\mathbf{M}_t)\right)|\mu_t\left(h^t\right), \mathbf{M}_t\right) \\
&\quad = \int_{T^{-1}(\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}')} T_{\Delta(\Theta)}(s_t, a_t, \omega_{t+1})(\widetilde{\Theta})l(d\omega_{t+1})\nu_{t+1}\left(d\left(h^t, z_{(s_t,a_t)}(\mathbf{M}_t)\right)|\mu_t\left(h^t\right), \mathbf{M}_t\right)
\end{aligned}$$

$$= \int_{\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'} \mu(\widetilde{\Theta})\big(\big(\nu_{t+1}\big(\mu_t\big(h^t\big),\mathbf{M}_t\big)\otimes\kappa^{\omega}_{t+1}\big)\circ T^{-1}\big)\big(d\big(\mu,a_t,\omega'\big)\big)$$

$$= \int_{\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'} \mu(\widetilde{\Theta})\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}\big(d\big(\mu,a_t,\omega'\big)\big), \tag{D.20}$$

where the first equality follows from equation (7), $T_{\Delta(\Theta)}$ in the third term denotes the first coordinate of T, the third equality follows from the change in integration variables, and the last from the definition of $\mathbb{P}'$. Equation (20) implies that when the principal "observes (μ, a_t, ω')," the principal's beliefs update to μ. Formally, consider the $\mathsf{proj}_{\Delta(\Theta)\times A\times\Omega}$-disintegration of $\mathbb{P}'$. Then

$$\mathbb{P}'\big(\widetilde{\Theta}\times\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'\big) = \int_{\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'} \lambda_{(\mu,a_t,\omega')}(\widetilde{\Theta})\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}\big(d\big(\mu,a_t,\omega'\big)\big). \tag{D.21}$$

Equations (20) and (21) together with Theorem 1.23 in Kallenberg (2017) implies that $\lambda_{(\mu,a_t,\omega')}(\cdot) = \mu(\cdot)$-$\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}$ almost surely.

To verify (ii) and complete the assertion that conditional on μ, the principal's beliefs update to μ, consider the $\mathsf{proj}_{\Delta(\Theta)}$-disintegration of $\mathbb{P}'$, $\{\lambda_\mu : \mu \in \Delta(\Theta)\}$. Formally,

$$\begin{aligned}
\mathbb{P}'\big(\widetilde{\Theta}\times\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'\big) &= \int_{\widetilde{U}} \lambda_\mu\big(\widetilde{\Theta}\times\widetilde{A}\times\widetilde{\Omega}'\big)\mathbb{P}'_{\Delta(\Theta)}(d\mu) \\
&= \int_{\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}'} \mu(\widetilde{\Theta})\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}\big(d\big(\mu,a_t,\omega'\big)\big) \\
&= \int_{\widetilde{U}\times A\times\Omega'} \mu(\widetilde{\Theta})\mathbb{1}\big[(a_t,\omega)\in\widetilde{A}\times\widetilde{\Omega}'\big]\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}\big(d\big(\mu,a_t,\omega'\big)\big) \\
&= \int_{\widetilde{U}}\int_{A\times\Omega'} \mu(\widetilde{\Theta})\lambda'_\mu\big(d\big(a_t,\omega'\big)\big)\mathbb{P}'_{\Delta(\Theta)}(d\mu) \\
&= \int_{\widetilde{U}} \mu(\widetilde{\Theta})\int_{\widetilde{A}\times\widetilde{\Omega}'} \lambda'_\mu\big(d\big(a_t,\omega'\big)\big)\mathbb{P}'_{\Delta(\Theta)}(d\mu) \\
&= \int_{\widetilde{U}} \mu(\widetilde{\Theta})\lambda'_\mu\big(\widetilde{A}\times\widetilde{\Omega}'\big)\mathbb{P}'_{\Delta(\Theta)}(d\mu),
\end{aligned} \tag{D.22}$$

where the first equality follows from the definition of disintegration, the second equality follows from equation (20), the third is a rewriting of the integral, the fourth uses the $\mathsf{proj}_{\Delta(\Theta)}$-disintegration of $\mathbb{P}'_{\Delta(\Theta)\times A\times\Omega'}$, $\{\lambda'_\mu : \mu \in \Delta(\Theta)\}$, and the fifth the property that "conditional on μ," $\mu(\widetilde{\Theta})$ is constant. It follows from this that $\Theta \perp (A, \Omega')|\Delta(\Theta)$.

The third step uses the above properties to construct a canonical mechanism, initially defined for those types in the support of $\mu_t(h^t)$, which we then extend to all types. With this in mind, consider again the proj_Θ-disintegration of $\mathbb{P}'$, $\{\lambda_\theta : \theta \in \Theta\}$. Theorem 1.25 in Kallenberg (2017) implies that three transition probabilities, $\beta : \Theta \mapsto \Delta(\Delta(\Theta))$, $\alpha : \Theta \times \Delta(\Theta) \mapsto \Delta(A)$, and $\gamma : \Theta \times \Delta(\Theta) \times A \mapsto \Delta(\Omega')$, exist such that $\lambda_\theta = \beta \otimes \alpha \otimes \gamma$. Furthermore, Equation (D.22) and Theorem 1.27 in Kallenberg (2017) together imply that α and γ do not depend on the agent's type.[7] Thus, we can write

$$\begin{aligned}\mathbb{P}'(\widetilde{\Theta}\times\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}') &= \int_{\widetilde{\Theta}}\lambda_\theta(\widetilde{U}\times\widetilde{A}\times\widetilde{\Omega}')\mu_t(d\theta|h^t) \\ &= \int_{\widetilde{\Theta}}\int_{\widetilde{U}}\int_{\widetilde{A}}\gamma(\widetilde{\Omega}'|\theta,\mu,a_t)\alpha(da_t|\theta,\mu)\beta(d\mu|\theta)\mu_t(d\theta|h^t) \\ &= \int_{\widetilde{\Theta}}\int_{\widetilde{U}}\int_{\widetilde{A}}\gamma(\widetilde{\Omega}'|\mu,a_t)\alpha(da_t|\mu)\beta(d\mu|\theta)\mu_t(d\theta|h^t). \qquad \text{(D.23)}\end{aligned}$$

We now define the canonical mechanism $\varphi^{\mathbf{M}_t^C}$. For $\theta\in\Theta^+$, define $\varphi^{\mathbf{M}_t^C}(\cdot|\theta)=\beta\otimes\alpha$. For $\theta\notin\Theta^+$, let $R^*(\theta)$ denote the set of solutions to

$$\begin{aligned}\max_{r\in\Delta(\Theta^+)}\int_{\Theta^+}\biggl[\int_{\Delta(\Theta)\times A\times\Omega'}&\mathbb{E}^{P^{\sigma}|(\theta,h^t,\mathbf{M}_t,T^{-1}(\mu,a_t,\omega'))}\bigl[U(a^{t-1},a_t,\cdot,\theta)\bigr] \\ &\times\gamma(d\omega'|\mu,a_t)\alpha(da_t|\mu)\beta(d\mu|\theta')\biggr]r(d\theta'), \qquad \text{(D.24)}\end{aligned}$$

The objective function in equation (24) corresponds to the payoff from reporting (possibly at random) a type in Θ^+ and then conditional on (μ,a_t,ω'), play proceeding as in the original strategy profile. The objective is continuous in r[8] and $\Delta(\Theta^+)$ is compact since Θ^+ is compact (Theorem 15.11 in Aliprantis and Border, 2006). Then the maximization is well-defined. Theorem 18.19 in Aliprantis and Border (2006) implies that a measurable selector $r^*(\theta)\in R^*(\theta)$ exists. Use this to define $\varphi^{\mathbf{M}_t^C}$ for $\theta\notin\Theta^+$ as follows:

$$\varphi^{\mathbf{M}_t^C}(\widetilde{U}\times\widetilde{A}|\theta)=\int_{\Theta_+}\varphi^{\mathbf{M}_t^C}(\widetilde{U}\times\widetilde{A}|\theta')r^*(\theta)(d\theta'),$$

which is measurable by composition of measurable functions. Note that this defines $\varphi^{\mathbf{M}_t^C}$ as a transition probability from Θ to $\Delta(\Theta)\times A$.[9] Let $\mathbf{M}_t^C=(\Theta,\Delta(\Theta),\varphi^{\mathbf{M}_t^C})$.

Continuation strategies are modified so that when the outcome of the mechanism is (μ,a_t), we draw $\omega'\in[0,1]$ according to $\gamma(\mu,a_t)$ and play proceeds as it did after $(h^t,z_{(s_t,a_t)}(\mathbf{M}_t),\omega_{t+1})$ where $T^{-1}(\mu,a_t,\omega')=(s_t,a_t,\omega_{t+1})$.[10] That is, $(\sigma_P,\sigma_A,$

[7]To facilitate checking the application of Theorem 1.27 in Kallenberg (2017) to our setting, we now use his notation. For any sets Y, X, and Z, and joint measure ν on $Y\times X\times Z$, let $\lambda_{YX|Z}$ denote the $(\nu_Z,\mathrm{proj}_{Y\times X})$–disintegration of ν. Then equation (22) shows that $\lambda_{\Theta A[0,1]|\Delta(\Theta)}=\lambda_{\Theta|\Delta(\Theta)}\lambda_{A[0,1]|\Delta(\Theta)}$, $\mathbb{P}_{\Delta(\Theta)}$-almost everywhere. By Theorem 1.25 in Kallenberg (2017), $\lambda_{\Theta A[0,1]|\Delta(\Theta)}=\lambda_{\Theta|\Delta(\Theta)}\otimes\lambda_{A[0,1]|\Theta|\Delta(\Theta)}$, which means that $\lambda_{A[0,1]|\Theta|\Delta(\Theta)}=\lambda_{A[0,1]|\Delta(\Theta)}$ $\mathbb{P}_{\Delta(\Theta)}$-almost everywhere. Theorem 1.27 in Kallenberg (2017) shows that $\lambda_{A[0,1]|\Delta(\Theta)|\Theta}=\lambda_{A[0,1]|\Theta|\Delta(\Theta)}$ $\mathbb{P}_{\Theta\Delta(\Theta)}$-almost everywhere. Together with the observation that $\lambda_{\Delta(\Theta)A[0,1]|\Theta}=\lambda_{\Delta(\Theta)|\Theta}\lambda_{A[0,1]|\Delta(\Theta)|\Theta}$, completes the claim.

[8]The term in brackets is bounded above by the payoff the agent of type θ obtains in equilibrium. Serfozo (1982, Theorem 3.5) then implies continuity of the objective in r.

[9]To see this, fix a measurable subset C of $\Delta(\Delta(\Theta)\times A\times\Omega)$ and let B denote a measurable subset of $[0,1]$. Then the set $\{\theta\in\Theta:\varphi^{\mathbf{M}_t^C}(C|\theta)\in B\}=\{\theta\in\Theta_+:\varphi^{\mathbf{M}_t^C}(C|\theta)\in B\}\cup\{\theta\in\Theta\setminus\Theta_+:\varphi^{\mathbf{M}_t^C}(C|\theta)\in B\}$. Each set is in $\mathcal{B}_\Theta$ by construction and, therefore, their union is in $\mathcal{B}_\Theta$.

[10]At the risk of introducing more notation, one could use the probability integral transform and make the distribution on $[0,1]$ be the uniform distribution. Now, the probability integral transform requires that the distribution be continuous. This can always be guaranteed by applying the result in Lehmann (2012), which shows that for any (real-valued) random variable X, one can always construct an information-equivalent random variable X^*, the distribution of which is continuous.

$\mu)|_{(h^t, z_{(\mu,a_t)}(\mathbf{M}_t^C), \omega')} = (\sigma_P, \sigma_A, \mu)|_{(h^t, \mathbf{M}_t, T^{-1}(\mu, a_t, \omega'))}$. Instead, conditional on not participating in the mechanism, $\mathbf{M}_t^C$, set $(\sigma_P, \sigma_A, \mu)|_{(h^t, z_{(\emptyset,a^*)}(\mathbf{M}_t^C), \omega_{t+1})} = (\sigma_P, \sigma_A, \mu)|_{(h^t, \mathbf{M}_t, z_{(\emptyset,a^*)}(\mathbf{M}_t), \omega_{t+1})}$. Modify the agent's strategy so that when the principal offers $\mathbf{M}_t^C$ and the agent participates, then the agent truthfully reports her type. Modify the agent's participation strategy as follows. For types in Θ^+, the agent participates with probability 1. For types not in Θ^+, set $\pi_t(\theta, h^t, \mathbf{M}_t^C) = 1$ only if the value of the problem in equation (24) is larger than the utility the agent obtains by not participating. It follows that the new strategy profile remains a PBE. *Q.E.D.*

D.2. *Pure Strategies for the Principal Are Without Loss of Generality*

LEMMA D.1: *For every PBE assessment $(\sigma_P, \sigma_A, \mu)$ of the mechanism-selection game $G_{\mathcal{I}}$, an outcome-equivalent PBE assessment $(\sigma_P', \sigma_A', \mu')$ of $G_{\mathcal{I}}$ exists such that the principal plays a pure strategy.*

PROOF: Let $(\sigma_P, \sigma_A, \mu)$ denote a PBE assessment of the $G_{\mathcal{I}}$. The proof proceeds as follows: We construct a sequence of assessments $((\sigma_P^n, \sigma_A^n, \mu^n))_{n\in\mathbb{N}_0}$ such that for $n = 0$,$(\sigma_P^n, \sigma_A^n, \mu^n) = (\sigma_P, \sigma_A, \mu)$, and for $n \geq 1$, $(\sigma_P^n, \sigma_A^n, \mu^n) \equiv (\sigma_{Pt}^n, \sigma_{At}^n, \mu_t^n)_{t=1}^T$ is such that for $t \leq n-1$, $(\sigma_{Pt}^n, \sigma_{At}^n, \mu_t^n) = (\sigma_{Pt}^{n-1}, \sigma_{At}^{n-1}, \mu_t^{n-1})$. Furthermore, in σ^n the principal's strategy is pure through period n. The assessment $(\sigma_P', \sigma_A', \mu')$ in the statement of Lemma D.1 is then obtained as $(\sigma_P', \sigma_A', \mu') = \lim_{n\to\infty}(\sigma_P^n, \sigma_A^n, \mu^n)$. The proof uses the representation of behavioral strategies in Aumann (1964, Lemma F). Given a public history h^t, recall h_-^t denotes h^t up to, but not including the realization of the public randomization device. Then the principal's behavioral strategy at $h^t = (h_-^t, \omega_t)$ can be represented as a measurable function $\sigma_{Pt}(h_-^t, \omega_t, \cdot) : [0,1] \mapsto \mathcal{M}_{\mathcal{I}}$, where $[0,1]$ is endowed with the Lebesgue measure, l.

Fix $t \geq 1$, and suppose we have defined $(\sigma_P^n, \sigma_A^n, \mu^n)$ for $n \leq t-1$. We now define it for t. By Kuratowski's theorem, a bijection $b : [0,1] \mapsto [0,1]^2$ exists such that b and b^{-1} are measurable. For any measurable $\widetilde{Y}$ in $[0,1]$, define $L_b(\widetilde{Y}) = \int_0^1 \int_0^1 \mathbb{1}[(\omega, x) \in b(\widetilde{Y})] l(dx) l(d\omega)$. Define $(\sigma_P^t, \sigma_A^t, \mu^t)$ as follows. For $n \leq t-1$, $(\sigma_{Pn}^t, \sigma_{An}^t, \mu_n^t)$ coincide with $(\sigma_{Pn}^{t-1}, \sigma_{An}^{t-1}, \mu_n^{t-1})$. For $n = t$, endow the public randomization device with the measure L_b and let $\sigma_{Pt}^t(h_-^t, \omega_t, x) = \sigma_{Pt}^{t-1}(h_-^t, b(\omega_t))$, for all $x \in X$.[11] That is, at h_-^t, when the public randomization device coincides with ω_t, the principal plays *with probability* 1 the mechanism that he would have played under $(\sigma_P^{t-1}, \sigma_A^{t-1}, \mu^{t-1})$ when the public randomization device equals $b_1(\omega_t)$ and the principal's randomization device equals $b_2(\omega_t)$. Similarly, let $\sigma_{At}^t(h_-^t, \omega_t, \mathbf{M}_t) = \sigma_{At}^{t-1}(h_-^t, b_1(\omega_t), \mathbf{M}_t)$. That is, at h_-^t, when the public randomization device coincides with ω_t and the principal plays $\mathbf{M}_t$, the agent follows the strategy that she would have followed under $(\sigma_P^{t-1}, \sigma_A^{t-1}, \mu^{t-1})$ when the public randomization device equals $b_1(\omega_t)$ and the principal plays $\mathbf{M}_t$. For $n \geq t+1$, let $\sigma_{Pn}^t(h_-^t, \omega_t, \cdot) = \sigma_{Pn}^{t-1}(h_-^t, b_1(\omega_t), \cdot)$ and $\sigma_{An}^t(h_-^t, \omega_t, \cdot) = \sigma_{An}^{t-1}(h_-^t, b_1(\omega_t), \cdot)$. Because $(\sigma_P, \sigma_A, \mu)$ is a PBE assessment, it follows immediately that $(\sigma_P^n, \sigma_A^n, \mu^n)$ is an outcome-equivalent PBE assessment for all $n \geq 1$. Furthermore, the principal plays a pure strategy in each period under $\lim_{n\to\infty}(\sigma_P^n, \sigma_A^n, \mu^n)$. This concludes the proof. *Q.E.D.*

[11] As in footnote 10, one can redefine the strategies so that the public randomization device has the uniform distribution.

APPENDIX E: CONTINUUM TYPE SPACES

E.1. *The Agent-Extensive Form*

In this section, we introduce the notation to formally define the extensive form game $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$ for some $t \geq 1$, $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$ and a dynamic mechanism given a^{t-1}, $(\varphi_\tau)_{\tau\geq t}$ as in Definition 3. As it will become clear, the notation is essentially the same as in Appendix D except that we no longer condition on the mechanism chosen by the principal.

Histories and agent strategies. As in the mechanism-selection game, we subsume the participation decision into the input–output message notation and let $MSA_\emptyset$ denote the set $M \times S \times A \cup \{(\emptyset, \emptyset, a^*)\}$. Thus, an outcome in period $\tau \geq t$ is an element of $MSA_\emptyset$. In a slight abuse of notation, we let $\mathsf{Z}_A = MSA_\emptyset \times \Omega$ and $\mathsf{Z} = SA_\emptyset \times \Omega$. Then, for $\tau \geq t$, agent histories are given by $\Theta \times \{a^{t-1}\} \times H^\tau_{At} = \Theta \times \{a^{t-1}\} \times \mathsf{Z}_A^{\tau-t}$ and public histories are $H^\tau_t = \{a^{t-1}\} \times \mathsf{Z}^{\tau-t}$.[12]

The agent's behavioral strategy is a collection $\sigma_A \equiv (\pi_\tau, r_\tau)_{\tau=t}^T$ such that $\pi_\tau : \Theta \times H^\tau_{At} \mapsto \Delta(\{0, 1\})$ and $r_\tau : \Theta \times H^\tau_{At} \mapsto \Delta(M)$ are transition probabilities.

Induced distributions and payoffs. Given a dynamic mechanism $(\varphi_\tau)_{\tau\geq t}$, an agent-strategy σ_A, and a node $(\theta, a^{t-1}, h^\tau_A)$, we define transition probabilities from $\Theta \times \{a^{t-1}\} \times H^\tau_{At}$ to $MSA_\emptyset$ and from $\Theta \times \{a^{t-1}\} \times H^\tau_{At} \times MSA_\emptyset$ to Ω as follows:

$$\kappa^{\sigma_A}_\tau\big(\widetilde{MSA_\emptyset}|\theta, a^{t-1}, h^\tau_{At}\big) = \big(1 - \pi_\tau\big(\theta, h^\tau_{At}\big)\big)\mathbb{1}\big[(\emptyset, \emptyset, a^*) \in \widetilde{MSA_\emptyset}\big] + \pi_\tau\big(\theta, h^\tau_{At}\big) \int_{\widetilde{MSA_\emptyset}} r_\tau\big(\theta, h^\tau_{At}\big) \otimes \varphi_\tau\big(h^\tau_t\big)\big(d(m_\tau, s_\tau, a_\tau)\big),$$

$$\kappa^\omega_{\tau+1}\big(\widetilde{\Omega}|\theta, a^{t-1}, h^\tau_{At}, m_\tau, s_\tau, a_\tau\big) = \int_{\widetilde{\Omega}} l(d\omega_{\tau+1}),$$

where h^τ_t denotes the projection of $(\theta, a^{t-1}, h^\tau_{At})$ onto $(SA_\emptyset \times \Omega)^{\tau-t}$. Note that $\kappa^\sigma_\tau \equiv \kappa^{\sigma_A}_\tau \otimes \kappa^\omega_{\tau+1}$ defines a transition probability from $\Theta \times \{a^{t-1}\} \times H^\tau_{At}$ to Z_A.

Recall that $\mu_t \times \delta_{a^{t-1}}$ denotes the initial distribution on $\Theta \times A^{t-1}$. Like in the mechanism-selection game, the Ionescu–Tulcea extension theorem (Pollard (2002)) guarantees the existence of a sequence of probability measures $P^\sigma_{t,\tau} = \mu_t \otimes \delta_{a^{t-1}} \otimes \bigotimes_{s=t}^{\tau-1} \kappa^\sigma_s$ defined on the product sets $(\Theta \times A^{t-1} \times H^\tau_{At})^T_{\tau=t}$ and a probability measure P^σ_t on $(\Theta \times A^{t-1} \times H^T_{At}, \mathcal{B}_\Theta \otimes \mathcal{B}_{A^{t-1}} \otimes \bigotimes_{\tau=t}^T \mathcal{B}_{\mathsf{Z}_A})$ such that for each $\tau \geq t$, the marginal of P^σ_t on $\Theta \times H^\tau_{At}$ is $P^\sigma_{t,\tau}$.[13]

Given P^σ_t, we can define the outcome distribution induced by P^σ_t, $\eta^{(\varphi_\tau)_{\tau\geq t}, \sigma_A}$ as in Definition D.1, and hence, the principal and the agent's payoffs as in equations (2) and (3).

Belief system, conditional distributions, and payoffs. The belief system is a collection $(\mu_\tau)_{\tau\geq t}$ such that for all $\tau \geq t$, $\mu_\tau : H^\tau_t \mapsto \Delta(\Theta \times H^\tau_{At})$ is a transition probability. Fix a history h^τ_t. Using the belief system at h^τ_t together with the kernels, we can define a conditional outcome distribution $\eta^{\sigma|h^\tau_t}$ as in Definition D.2.

[12] We are adding a^{t-1} so as to define all outcome distributions of the game as distributions over $\Delta(\Theta \times A^T)$.

[13] Note that the distribution P^σ_t corresponds to the distribution P^σ in Appendix D: it is the distribution over the terminal nodes starting from the root of the extensive-form game.

Furthermore, the belief at h_t^τ, $\mu_\tau(h_t^\tau)$, together with the agent's strategy induce a distribution over $\Theta \times H_{At}^\tau \times MSA_\emptyset \equiv H_{At-}^{\tau+1}$ as follows:

$$\mathbb{P}_{t,\tau+1}^{-}\big(\widetilde{\Theta} \times \widetilde{H}_{At}^\tau \times \widetilde{MSA_\emptyset} | \mu_\tau(h_t^\tau)\big) = \int_{\widetilde{\Theta} \times \widetilde{H}_A^\tau} \kappa_\tau^{\sigma_A}\big(\widetilde{MSA_\emptyset} | \theta, h_{At}^\tau\big)\mu_\tau\big(d\big(\theta, h_{At}^\tau\big)|h_t^\tau\big). \tag{E.1}$$

This is the distribution that the principal uses to update his beliefs about the agent's type at history h_t^τ. Formally,

$$\begin{aligned}&\int_{\widetilde{\Theta}\times\widetilde{H_{At-}^{\tau+1}}} \mathbb{1}\big[\mathsf{proj}_{H_{t-}^{\tau+1}}\big(\theta, h_{At-}^{\tau+1}\big) \in \widetilde{H_{t-}^{\tau+1}}\big]\mathbb{P}_{t,\tau+1}^{-}\big(d\big(\theta, h_{At-}^{\tau+1}\big)|\mu_\tau\big(h_t^\tau\big)\big)\\ &\quad= \int_{\widetilde{H_{t-}^{\tau+1}}} \mu_{\tau+1}\big(\widetilde{\Theta} \times \widetilde{H_{At-}^{\tau+1}} | h_{t-}^{\tau+1}\big)\nu_{t,\tau+1}\big(dh_{t-}^{\tau+1}|\mu_\tau\big(h_t^\tau\big)\big),\end{aligned} \tag{E.2}$$

where again we are defining $\nu_{t,\tau+1}(\mu_\tau(h_t^\tau)) = \mathbb{P}_{t,\tau+1}^{-}(\mu_\tau(h_t^\tau)) \circ \mathsf{proj}_{H_{t-}^{\tau+1}}^{-1}$. This allows us to obtain the analogues of equations (8) and (9) in this setting:

$$\begin{aligned}W\big(\sigma, \mu | h_t^\tau\big) &= \int_{H_{t-}^{\tau+1}} \int_{\Theta\times H_{At-}^{\tau+1}} \mathbb{E}^{P_t^{\sigma|(\theta,h_{At-}^{\tau+1})}}\big[W\big(a^\tau, \cdot, \theta\big)\big]\\ &\quad\times \mu_{\tau+1}\big(d\big(\theta, h_{At-}^{\tau+1}\big)|h_{t-}^{\tau+1}\big)\nu_{t,\tau+1}\big(dh_{t-}^{\tau+1}|\mu_\tau\big(h_t^\tau\big)\big),\\ U\big(\sigma|\theta, h_{At}^\tau\big) &= \int_{MSA_\emptyset} \mathbb{E}^{P^{\sigma|(\theta,h_{At-}^{\tau+1})}}\big[U\big(a^\tau, \cdot, \theta\big)\big]\kappa_\tau^{\sigma_A}\big(d(m_\tau, s_\tau, a_\tau)|\theta, h_{At}^\tau\big)\end{aligned}$$

Finally, as in Appendix D we define the joint probability over $\Theta \times H_A^{\tau+1}$ given $\mu_\tau(h^\tau)$:

$$\mathbb{P}_{t,\tau+1}\big(\widetilde{\Theta} \times \widetilde{H_{At-}^{\tau+1}} \times \widetilde{\Omega} | \mu_\tau\big(h_t^\tau\big)\big) = \int_{\widetilde{\Omega}} \kappa_{\tau+1}^\omega\big(\widetilde{\Omega}|\theta, h_{At-}^{\tau+1}\big)\mathbb{P}_{t,\tau+1}^{-}\big(d\big(\theta, h_{At-}^{\tau+1}\big)|\mu_\tau\big(h_t^\tau\big)\big), \tag{E.3}$$

and we recall that we can define Bayes' rule also on the basis of $\mathbb{P}_{\tau+1}$. Namely,

$$\begin{aligned}&\mathbb{P}_{t,\tau+1}\big(\widetilde{\Theta} \times \widetilde{H_{At-}^{\tau+1}} \times \widetilde{\Omega} | \mu_\tau\big(h_t^\tau\big)\big)\\ &\quad= \int_{\mathsf{proj}_{H_t^{\tau+1}}\widetilde{\Theta}\times\widetilde{H_{At-}^{\tau+1}}\times\widetilde{\Omega}} \mu_{\tau+1}\big(\widetilde{\Theta} \times \widetilde{H_{At-}^{\tau+1}} \times \widetilde{\Omega}|h_{t-}^{\tau+1}, \omega\big)\big(\mathbb{P}_{t,\tau+1}\big(\mu_\tau\big(h_t^\tau\big)\big) \circ \mathsf{proj}_{H_t^{\tau+1}}^{-1}\big)\big(dh_t^{\tau+1}\big)\\ &\quad= \int_{\mathsf{proj}_{H_-^{\tau+1}}\widetilde{\Theta}\times\widetilde{H_{At-}^{\tau+1}}} \int_{\widetilde{\Omega}} \mu_{\tau+1}\big(\widetilde{\Theta} \times \widetilde{H_{At-}^{\tau+1}}|h_{t-}^{\tau+1}, \omega\big)l(d\omega)\nu_{t,\tau+1}\big(dh_{t-}^{\tau+1}|\mu_\tau\big(h_t^\tau\big)\big).\end{aligned} \tag{E.4}$$

Agent PBE. An agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$ is an assessment (σ_A,μ) such that:

1. σ_A is sequentially rational (cf., part (ii) of Definition D.3),
2. μ satisfies Bayes' rule where possible. That is for all $\tau \geq t$, $\mu_{\tau+1}$ satisfies equation (26) $\nu_{t,\tau+1}(\mu_\tau(h_t^\tau))$-almost surely.

E.2. *Canonical PBE-Feasible Outcomes*

We define the correspondence $\mathcal{O}^C$ of canonical PBE-feasible outcomes (Definition E.3). Denote by $\mathcal{M}_{\Theta,\Delta(\Theta)}$ the set $\{\varphi : \Theta \mapsto \Delta(\Delta(\Theta) \times A) : \varphi \text{ is measurable}\}$. Throughout Section E.2, all mechanisms belong to $\mathcal{M}_{\Theta,\Delta(\Theta)}$.

To define the correspondence $\mathcal{O}^C$, we first define *canonical* dynamic mechanisms, *canonical* agent-PBE, and the set of outcomes the principal anticipates upon a deviation.

DEFINITION E.1—Canonical dynamic mechanisms: For $t \geq 1$ and $a^{t-1} \in A^{t-1}$, a dynamic mechanism given a^{t-1}, $(\varphi_\tau^C)_{\tau\geq t}$ is *canonical* if for all $\tau \geq t$ and $(s^{\tau-t}, a^{\tau-t}, \omega^{\tau-t}) \in (\Delta(\Theta)A_\emptyset \times \Omega)^{\tau-t}$, $\varphi_\tau^C(s^{\tau-t}, a^{\tau-t}, \omega^{\tau-t})$ is a canonical mechanism.

Fix $t \geq 1$, a tuple $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$, and a canonical dynamic mechanism given a^{t-1}, $(\varphi_\tau^C)_{\tau\geq t}$, and we can define a *canonical* agent-assessment of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$ similar to Definition 2 in the main text, by just eliminating item 1.

Finally, we define the set of outcomes the principal expects he will face if in some period $t \geq 1$, when his belief is μ_t, and the allocation through period $t-1$ is a^{t-1}, he deviates to $\varphi_t' : \Theta \mapsto \Delta(\Delta(\Theta) \times A)$, $\mathsf{D}_{\mathcal{O}^C}(\mu_t, a^{t-1}, \varphi_t')$, as follows:

$$\left\{\begin{array}{l} \eta' \in \Delta(\Theta \times A^T) : \eta' = \eta^{(\varphi_\tau')_{\tau\geq t}, \sigma_A'} \text{ where} (\varphi_\tau')_{\tau\geq t} = (\varphi_t', (\varphi_\tau')_{\tau\geq t+1}) \text{ and} \\ \qquad \text{(i) } (\varphi_\tau')_{\tau\geq t} \text{ is a dynamic mechanism given } a^{t-1} \\ \qquad \text{(ii) } (\sigma_A', \mu') \text{ is an agent-PBE of } \Gamma(\mu_t, a^{t-1}, (\varphi_\tau')_{\tau\geq t}) \\ \qquad \text{(iii) } (\forall (\mu', a', \omega') \in \Delta(\Theta)A_\emptyset \times \Omega) \\ \qquad \eta^{(\varphi_\tau')_{\tau\geq t}, \sigma_A' | \mu', a', \omega'} \in \mathcal{O}^C(\mu_{t+1}(\mu', a', \omega'), a^{t-1}, a') \end{array}\right\}. \tag{E.5}$$

Using equation (29), we can extend the definition of sequential rationality.

DEFINITION E.2—Sequential rationality: Fix $t \geq 1$, (μ_t, a^{t-1}), a canonical dynamic mechanism given a^{t-1}, $(\varphi_\tau^C)_{\tau\geq t}$, and an agent-PBE (σ_A, μ) of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$. $(\varphi_\tau^C)_{\tau\geq t}$ is *sequentially rational* given (σ_A, μ) if the following hold:

1. For all $\varphi_t' : \Theta \mapsto \Delta(\Delta(\Theta) \times A)$, a distribution $\eta' \in \mathsf{D}_{\mathcal{O}^C}(\cdot, \varphi_t')$ exists such that the principal prefers $\eta^{(\varphi_\tau)_{\tau\geq t}, \sigma_A}$ to η';
2. For all $h_t^{t+1} = (\mu', a', \omega') \in \Delta(\Theta)A_\emptyset \times \Omega$, $\eta^{(\varphi_\tau)_{\tau\geq t}, \sigma_A | h_t^{t+1}} \in \mathcal{O}^C(\mu_{t+1}(h^{t+1}), a^{t-1}, a')$.

DEFINITION E.3—Canonical PBE-feasible outcomes: Fix $t \geq 1$, $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$. The distribution $\eta \in \Delta(\Theta \times A^T)$ is a canonical PBE-feasible outcome at (μ_t, a^{t-1}) if a canonical dynamic mechanism given a^{t-1}, $(\varphi_\tau^C)_{\tau\geq t}$, and a canonical agent-PBE (σ_A, μ) of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$ exist such that:

1. η is the outcome distribution induced by (σ_A, μ) in $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$, and
2. $(\varphi_\tau^C)_{\tau\geq t}$ is sequentially rational given (σ_A, μ).

$\mathcal{O}^C(\mu_t, a^{t-1})$ denotes the set of canonical PBE-feasible outcomes at (μ_t, a^{t-1}).

REMARK E.1: Note that equation (29) requires neither that the continuation dynamic mechanism is canonical nor that the agent-PBE is canonical from $t+1$ onwards. Adding these requirements would lead to the same definition, at the cost of more complicated notation. The reason is that requirement (iii) in equation (29) implies that the continuation distributions must be induced by some canonical agent-PBE of some canonical dynamic mechanism.

E.3. *Preliminary Steps and Results for the Proof of Theorem 2*

Proving Theorem 2 requires we show that for all $t \geq 1$, and tuples $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$, we have that $\mathcal{O}_{\mathcal{I}}^*(\mu_t, a^{t-1}) = \mathcal{O}^C(\mu_t, a^{t-1})$. There are two challenges relative to the proof of Theorem 1. First, outcome distributions in $\mathcal{O}_{\mathcal{I}}^*(\mu_t, a^{t-1})$ are defined relative to

$\mathcal{O}^*_{\mathcal{I}}(\mu_{t+1}, a^{t-1}, a_t)$, so that to show that $\mathcal{O}^*_{\mathcal{I}}(\mu_t, a^{t-1}) = \mathcal{O}^C(\mu_t, a^{t-1})$. We also need to know how $\mathcal{O}^*_{\mathcal{I}}(\mu_{t+1}, a^{t-1}, a_t)$ relates to $\mathcal{O}^C(\mu_{t+1}, a^{t-1}, a_t)$. Second, the proof of Theorem 1 used the behavioral strategies of the principal and the agent and proceeded forwards, modifying at each step the assessment in period t while leaving the continuation assessment unchanged. Instead, $\mathcal{O}^*_{\mathcal{I}}$ and $\mathcal{O}^C$ are defined relative to the principal and the agent's "full strategy" from period t onwards.

To overcome these difficulties, we proceed as follows. To deal with the first, we use a trick from dynamic games and define two operators, $\mathcal{T}$ and $\mathcal{T}^C$, which describe the set of (canonical) PBE-feasible outcomes relative to some set of feasible continuation distributions, and whose fixed points correspond to the correspondences $\mathcal{O}^*_{\mathcal{I}}$ and $\mathcal{O}^C$. This allows us to fix the set of continuation outcome distributions for both solution concepts making it easier to show that they implement the same set of outcome distributions. To deal with the second challenge, we show below that the set $\mathcal{M}_{\Theta,\Delta(\Theta)}$ is in bijection with the set of mechanisms with input and output messages $\mathcal{I} = \{(M, S)\}$. This implies we can always translate a dynamic mechanism $(\varphi_\tau)_{\tau\geq t}$ with input and output messages (M, S) and an agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$ to a dynamic mechanism $(\varphi'_\tau)_{\tau\geq t}$ with input and output messages $(\Theta, \Delta(\Theta))$ and an agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi'_\tau)_{\tau\geq t})$.

E.3.1. *The Operators $\mathcal{T}$ and $\mathcal{T}^C$*

Feasible continuation distributions. Throughout this section, we consider correspondences $\psi : \cup_{t=1}^T (\Delta(\Theta) \times A^{t-1}) \rightrightarrows \Delta(\Theta \times A^T)$, which describe the sets from which we draw continuation distributions. Each ψ satisfies the following feasibility condition:

$$\begin{aligned}&(\forall t \geq 1)\\ &\quad \big(\forall (\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}\big)\eta \in \psi\big(\mu_t, a^{t-1}\big)\\ &\qquad \Rightarrow \quad \begin{cases}\operatorname{supp}\eta \subseteq \big\{\big(\theta, \widetilde{a}^T\big) : \widetilde{a}^{t-1} = a^{t-1}\big\},\\ \eta_\Theta = \mu_t.\end{cases}\end{aligned} \tag{F}$$

We denote the set of all such correspondences by Ψ.

$\mathcal{T}$ and $\mathcal{T}^C$: We now define two operators, $\mathcal{T}, \mathcal{T}^C : \Psi \mapsto \Psi$. The first operator $\mathcal{T}$ takes a correspondence ψ in Ψ and determines for each $t \geq 1$ and each $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$, the set of outcome distributions that are feasible when the principal selects mechanisms with message sets (M, S) and continuation distributions must be drawn from ψ. With this in mind, define the set $\mathsf{D}^{M,S}_\psi(\mu_t, a^{t-1}, \varphi'_t)$ as follows:

$$\left\{\begin{array}{l}\eta' \in \Delta\big(\Theta \times A^T\big) : \eta' = \eta^{(\varphi'_\tau)_{\tau\geq t}, \sigma'_A} \text{ where} \big(\varphi'_\tau\big)_{\tau\geq t} = \big(\varphi'_t, \big(\varphi'_\tau\big)_{\tau\geq t+1}\big) \text{ and}\\ \qquad\qquad \text{(i) } \big(\varphi'_\tau\big)_{\tau\geq t} \text{ is a dynamic mechanism given } a^{t-1}\\ \qquad\qquad \text{(ii) } \big(\sigma'_A, \mu'\big) \text{ is an agent-PBE of } \Gamma\big(\mu_t, a^{t-1}, \big(\varphi'_\tau\big)_{\tau\geq t}\big)\\ \qquad\qquad \text{(iii) } \big(\forall \big(s', a', \omega'\big) \in SA_\emptyset \times \Omega\big)\\ \qquad\qquad \eta^{(\varphi'_\tau)_{\tau\geq t}, \sigma'_A | s', a', \omega'} \in \psi\big(\mu_{t+1}\big(s', a', \omega'\big), a^{t-1}, a'\big)\end{array}\right\}. \tag{E.6}$$

This definition coincides with that in the main text except that in item (iii) we require that the continuation outcome distribution is an element of $\psi(\cdot)$. The superscript in $\mathsf{D}^{M,S}_\psi$ allows us to keep track of the message sets used by the dynamic mechanisms. We now extend the definition of sequential rationality (Definition 4) to account for the correspondence ψ.

DEFINITION E.4—Sequential rationality: Fix $t \geq 1$, (μ_t, a^{t-1}), a dynamic mechanism $(\varphi_\tau)_{\tau\geq t}$ given a^{t-1}, and an agent-PBE (σ_A, μ) of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$. $(\varphi_\tau)_{\tau\geq t}$ is *sequentially rational* given (σ_A, μ) and ψ if the following hold:

1. For all $\varphi_t' : M \mapsto \Delta(S \times A)$, a distribution $\eta' \in \mathsf{D}_\psi^{M,S}(\cdot, \varphi_t')$ exists such that the principal prefers $\eta^{(\varphi_\tau)_{\tau\geq t},\sigma_A}$ to η', and
2. For all $h_t^{t+1} = (s', a', \omega') \in SA_\emptyset \times \Omega$, $\eta^{(\varphi_\tau)_{\tau\geq t},\sigma_A|h_t^{t+1}} \in \psi(\mu_{t+1}(h_t^{t+1}), a^{t-1}, a')$.

DEFINITION E.5—$\mathcal{T}$: The outcome distribution $\eta \in \Delta(\Theta \times A^T)$ is *PBE-feasible at (μ_t, a^{t-1}) given ψ* if a dynamic mechanism given a^{t-1}, $(\varphi_\tau)_{\tau\geq t}$, and an agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$, (σ_A, μ), exist such that:

1. η is the outcome distribution induced by (σ_A, μ) on $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$, and
2. $(\varphi_\tau)_{\tau\geq t}$ is sequentially rational given (σ_A, μ) and ψ.

For each $t \geq 1$ and $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$, define $\mathcal{T}(\psi)(\mu_t, a^{t-1})$ to be the set of PBE-feasible outcomes at (μ_t, a^{t-1}) given ψ.

Thus, $\mathcal{T}$ takes in a correspondence ψ and produces an alternative correspondence $\mathcal{T}(\psi) \in \Psi$. Note that $\mathcal{O}_\mathcal{I}^*$ is a fixed point of $\mathcal{T}$ and we take it to be the largest (in set inclusion order) fixed point of $\mathcal{T}$.

The operator $\mathcal{T}^C$ performs a similar operation in terms of canonical mechanisms and canonical agent-PBE.

DEFINITION E.6—$\mathcal{T}^C$: The outcome distribution $\eta \in \Delta(\Theta \times A^T)$ is *canonical PBE-feasible at (μ_t, a^{t-1}) given ψ* if a canonical dynamic mechanism $(\varphi_\tau^C)_{\tau\geq t}$ and a canonical agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$, (σ_A, μ), exist such that the following hold:

1. η is the outcome distribution induced by (σ_A, μ) on $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau^C)_{\tau\geq t})$,
2. $(\varphi_\tau^C)_{\tau\geq t}$ is sequentially rational given (σ_A, μ) and ψ (using $D_\psi^{\Theta,\Delta(\Theta)}$).

For each $t \geq 1$ and $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$, we define $\mathcal{T}^C(\psi)(\mu_t, a^{t-1})$ to be the set of canonical PBE-feasible outcomes at (μ_t, a^{t-1}) given ψ.

E.3.2. *Bijection Between $\mathcal{M}_{\Theta,\Delta(\Theta)}$ and $\mathcal{M}_{i,j}$*

PROPOSITION E.1—Bijection: *Fix $\mathcal{I}$ and $i, j \in \mathcal{I}$. Then a bijection exists between $\mathcal{M}_{i,j}$ and $\mathcal{M}_{\Theta,\Delta(\Theta)}$.*

PROOF: Fix $i, j \in \mathcal{I}$. By Kuratowski's theorem, two measurable bijections exist, t, b, where $t : M_i \mapsto \Theta$ and $b : S_j \mapsto \Delta(\Theta)$. For each $\varphi \in \mathcal{M}_{i,j}$, define $\varphi'(\varphi) \in \mathcal{M}_{\Theta,\Delta(\Theta)}$ as follows: for all $m \in M_i$, $\varphi'(\varphi)(b(\widetilde{S}) \times \widetilde{A}|t(m)) = \varphi(\widetilde{S} \times \widetilde{A}|m)$. We now verify that $\varphi'(\cdot)$ is an injection. Let φ_1, φ_2 be such that $\varphi'(\varphi_1) = \varphi'(\varphi_2)$. Because b is a bijection, for all $\widetilde{S} \in S_j$ there exists a unique $\widetilde{U} \in \Delta(\Theta)$ such that $\widetilde{S} = b^{-1}(\widetilde{U})$. Similarly, because t is a bijection, for all $m \in M_i$, there exists a unique θ such that $m = t^{-1}(\theta)$. Then we have

$$\varphi_1(\widetilde{S} \times \widetilde{A}|m) = \varphi_1\big(b^{-1}(\widetilde{U}) \times \widetilde{A}|t^{-1}(m)\big) = \varphi_2\big(b^{-1}(\widetilde{U}) \times \widetilde{A}|t^{-1}(m)\big) = \varphi_2(\widetilde{S} \times \widetilde{A}|m),$$

where the second equality follows from the assumption that $\varphi'(\varphi_1) = \varphi'(\varphi_2)$. This implies that $\varphi_1 \equiv \varphi_2$. One can similarly construct an injection from $\mathcal{M}_{\Theta,\Delta(\Theta)}$ to $\mathcal{M}_{i,j}$. Theorem 1.2 in Aliprantis and Border (2006) then implies that a bijection exists between these sets. *Q.E.D.*

E.4. *Proof of Theorem 2*

The proof of Theorem 2 follows from Proposition E.2.

PROPOSITION E.2: *For all* $\psi \in \Psi$, $\mathcal{T}(\psi) = \mathcal{T}^C(\psi)$.

PROOF OF PROPOSITION E.2: Fix $\psi \in \Psi$, $t \geq 1$, and $(\mu_t, a^{t-1}) \in \Delta(\Theta) \times A^{t-1}$. We show that (i) $\mathcal{T}(\psi) \subseteq \mathcal{T}^C(\psi)$ and (ii) $\mathcal{T}^C(\psi) \subseteq \mathcal{T}(\psi)$.

To show that (i) holds, let $\eta \in \mathcal{T}(\psi)$ and let $(\varphi_\tau)_{\tau\geq t}$, (σ_A, μ) denote the dynamic mechanism and the agent-PBE of $\Gamma(\mu_t, a^{t-1}, (\varphi_\tau)_{\tau\geq t})$ that induce η. That $\eta \in \mathcal{T}^C(\psi)(\mu_t, a_t)$ follows from the following observations.

First, following the same steps as in the proof of Proposition 4 in Section D.1, one can construct a dynamic canonical mechanism $(\varphi^C_\tau)_{\tau\geq t}$ and a canonical agent-PBE of $\Gamma(\mu_t, a_t, (\varphi^C_\tau)_{\tau\geq t})$ that implement η, thus satisfying item (1) in Definition E.6. Furthermore, because all continuation distributions over outcomes are preserved, item (2) in Definition E.6 also holds.

Second, let Φ denote the bijection between $\mathcal{M}_{\Theta,\Delta(\Theta)}$ and $\mathcal{M}_{M,S}$, which exists by Proposition E.1. Thus, for every deviation to a mechanism φ'_t in $\mathcal{M}_{\Theta,\Delta(\Theta)}$, we can find an equivalent $\Phi(\varphi'_t) \in \mathcal{M}_{M,S}$ and $\eta' \in \mathsf{D}^{M,S}_\psi(\mu_1, a^{t-1}, \Phi(\varphi'_t))$ that deters the deviation to $\Phi(\varphi'_t)$ and by successive application of Proposition E.1 can be shown to belong in $\mathsf{D}^{\Theta,\Delta(\Theta)}_\psi(\mu_t, a^{t-1}, \varphi'_t)$. It follows that $\mathcal{T}(\psi)(\mu_t, a^{t-1}) \subseteq \mathcal{T}^C(\psi)(\mu_t, a^{t-1})$.

The proof that $\mathcal{T}^C(\psi)(\mu_t, a^{t-1}) \subseteq \mathcal{T}(\psi)(\mu_t, a^{t-1})$ follows immediately from Proposition E.1, so we omit it. *Q.E.D.*

REFERENCES

ALIPRANTIS, CHARALAMBOS, AND KIM BORDER (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag Berlin and Heidelberg GmbH & Company KG. [13,19]

AUMANN, ROBERT J. (1964): "Mixed and Behavior Strategies in Infinite Extensive Games," Technical Report, Princeton University. [14]

BERTSEKAS, DIMITRI P., AND STEVEN E. SHREVE (1978): *Stochastic Optimal Control,* Mathematics in Science and Engineering Vol. 139. New York–London: Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]. [7,9]

FREMLIN, DAVID HEAVER (2010): *Measure Theory*, Vol. 2. Torres Fremlin. [2]

KALLENBERG, OLAV (2017): *Random Measures, Theory and Applications*. Springer. [2,10-13]

LEHMANN, ERICH LEO (2012): "Comparing Location Experiments," in *Selected Works of EL Lehmann*. Springer, 779–791. [13]

POLLARD, DAVID (2002): *A User's Guide to Measure Theoretic Probability*. Cambridge University Press. [1,3,5,15]

SERFOZO, RICHARD (1982): "Convergence of Lebesgue Integrals With Varying Measures," Sankhyā: The Indian Journal of Statistics, Series A, 380–402. [13]

SRIVASTAVA, SASHI MOHAN (2008): *A Course on Borel Sets*, Vol. 180. Springer Science & Business Media. [11]

Co-editor Alessandro Lizzeri handled this manuscript.

Manuscript received 20 November, 2018; final version accepted 27 January, 2022; available online 23 February, 2022.