Econometrica Supplementary Material

SUPPLEMENT TO "PRODUCTIVITY DISPERSION, BETWEEN-FIRM COMPETITION, AND THE LABOR SHARE"

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APPENDIX A: THEORY

A.1. *Proof of Proposition 1*

FIRST, I FOCUS ON THE PROBLEM of unemployed workers. Their reservation wage $\underline{w}$ is obtained as the solution to $W(\underline{w}) = U$. Equating (3) and (4), I obtain

$$\begin{aligned} b + \mu \int e(z)\big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}\Gamma_0(z) + \lambda(1-u)\int \big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}P(z) \\ = \underline{w} + \chi \int \big(1 - x(z)\big)\big(W\big(w(z)\big) - U\big)\,\mathrm{d}\Gamma_0(z) \\ + \lambda(1-u)\int \big|W\big(w(z)\big) - U\big|_+ \,\mathrm{d}P(z). \end{aligned}$$

Using the fact that an employed worker can quit to unemployment, we have $W(w) \geq U$. I remove the max operators and obtain

$$b + \mu \int e(z)\big(W\big(w(z)\big) - U\big)\,\mathrm{d}\Gamma_0(z) = \underline{w} + \chi \int \big(1 - x(z)\big)\big(W\big(w(z)\big) - U\big)\,\mathrm{d}\Gamma_0(z).$$

Rearranging, we obtain

$$\underline{w} = b + \int \big(\mu e(z) - \chi\big(1 - x(z)\big)\big)\big(W\big(w(z)\big) - U\big)\,\mathrm{d}\Gamma_0(z).$$

Second, I focus on the problem of employed workers. I now prove that employed workers accept a job offer w' whenever $w' > w$. To show that $w' > w \implies W(w') > W(w)$, I proceed by contradiction. Set $w' > w$ and assume that $W(w') \leq W(w)$. Using (4), I obtain

$$\begin{aligned} & r\big(W\big(w'\big) - W(w)\big) \\ & = w' - w - \left(\chi \int \big(1 - x(z)\big)\,\mathrm{d}\Gamma_0(z) + \chi \int x(z)\,\mathrm{d}\Gamma_0(z) + \delta\right)\big(W\big(w'\big) - W(w)\big) \\ & \quad + \lambda(1-u)\int \big[\big|W\big(w(z)\big) - W\big(w'\big)\big|_+ - \big|W\big(w(z)\big) - W(w)\big|_+\big]\,\mathrm{d}P(z). \end{aligned}$$

Using $W(w') \leq W(w)$, we have that

$$\big|W\big(w(z)\big) - W\big(w'\big)\big|_+ \geq \big|W\big(w(z)\big) - W(w)\big|_+,$$

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which implies that

$$r\big(W(w') - W(w)\big)$$
$$\geq w' - w - \left(\chi \int (1 - x(z))\,\mathrm{d}\Gamma_0(z) + \chi \int x(z)\,\mathrm{d}\Gamma_0(z) + \delta\right)\big(W(w') - W(w)\big).$$

Rearranging, we obtain

$$(r + \chi + \delta)\big(W(w') - W(w)\big) \geq w' - w,$$

which is a contradiction. Therefore, it must be that $w' > w \implies W(w') > W(w)$.

A.2. *Proof of Lemma 1*

I now provide a derivation of the employment growth function over the range $[\underline{w}, +\infty)$:

$$\widetilde{g}(w) \equiv \underbrace{\lambda u + \lambda(1-u)\widetilde{P}(w)}_{\text{hiring rate}} - \underbrace{\lambda(1-u)\big(1 - \widetilde{P}(w)\big) - \delta}_{\text{separation rate}}.$$

The hiring rate is determined by the fraction of job offers that are accepted. A fraction u of meetings are with unemployed workers (who accept all job offers above $\underline{w}$), so the hiring rate out of unemployment is λu. The remainder of meetings are with employed workers who require a wage increase to quit. The measure of employed workers working at firms paying less than w is equal to $\widetilde{P}(w)$, so the hiring rate from employment is given by $\lambda(1-u)\widetilde{P}(w)$.

Separations occur due to quits as well as exogenous job destruction. At rate $\lambda(1-u)$, a worker receives a competing job offer. Since firms meet workers proportionally to their size, the distribution of offered wages is precisely the wage distribution $\widetilde{P}(w)$. The rate at which a firm paying w loses a worker to a competitor is thus given by $\lambda(1-u)(1-\widetilde{P}(w))$. In addition, firms lose workers exogenously at rate δ.

Since the wage distribution $\widetilde{P}$ is a CDF, we have that $\widetilde{P}(w)$ is weakly increasing in w and $\lim_{w\to\infty} \widetilde{P}(w) = 1$. Since $\lambda(1-u) > 0$, it follows that $\widetilde{g}(w)$ is weakly increasing and bounded from above by $\lambda - \delta$.

A.3. *Proof of Proposition 2*

The derivation of $k(z)$ and $y(z)$ follow directly from the first-order condition for capital, so I focus on the derivation of $e(z)$, $x(z)$, and $w(z)$.

Entry and Exit Functions

Firms enter whenever $v(z) > 0$ and exit whenever $v(z) = 0$. I now show that there exists a unique $\underline{z} \geq 0$ such that

$$z < \underline{z} \implies v(z) = 0, \qquad z > \underline{z} \implies v(z) > 0.$$

Define $\mathcal{Z}_+ \equiv \{z : v(z) > 0\}$ and $\mathcal{Z}_- \equiv \{z : v(z) = 0\}$. Notice that $\mathcal{Z}_- \cup \mathcal{Z}_+ = [0, +\infty)$. First, I show that $z \in \mathcal{Z}_+ \implies v'(z) > 0$. From (12), we have that

$$v(z) > 0$$
$$\implies \quad rv(z) = \max_{w,k}\big\{zk^\alpha - w - Rk + v(z)\widetilde{g}(w)\big\} + \chi\left(\int v(x)\Gamma_0(dx) - v(z)\right).$$

Using the envelope theorem, we have that when $v(z) > 0$,

$$rv'(z) = k^\alpha(z) + v'(z)g(z) - \chi v(z) \quad \Longrightarrow \quad v'(z) = \frac{k(z)^\alpha}{r + \chi - g(z)}.$$

Assumption 2 combined with Lemma 1 implies that $r + \chi - g(z) > 0$ for all $r > 0$, so we obtain the desired result that $v'(z) > 0$. Firms operate whenever $z \in \mathcal{Z}_+$ and $v(z)$ is continuous over $\mathcal{Z}_+$ (differentiability implies continuity), so we have that for all $z \in \inf \mathcal{Z}_+$, $v(z) = 0$. Therefore, the entry/exit threshold is $\underline{z} \equiv \inf \mathcal{Z}_+$.

Wage Function

As is standard in such proofs, there are two cases to consider: a continuous wage distribution and a wage distribution with a point mass. When the wage distribution is continuous, the function $\widetilde{g}(w)$ is differentiable. From the first-order condition (equation (14)), we have that

$$1 = v(z)\widetilde{g}'\big(w(z)\big).$$

Using the definition $g(z) \equiv \widetilde{g}(w(z))$, it follows from the chain rule that $g'(z) = \widetilde{g}'(w(z))w'(z)$. Plugging back in the first-order condition, we obtain

$$w'(z) = v(z)g'(z).$$

To solve for $w(z)$, I use the fact that the marginal firm (i.e., $z = \underline{z}$) is constrained by the worker's reservation wage $\underline{w}$. Without the constraint $w \geq \underline{w}$, firms would never exit as they would be able to achieve zero flow profit by setting $w = 0$ and $k = 0$. I use this insight to obtain the boundary condition $w(\underline{z}) = \underline{w}$. Solving the ODE forward, I obtain the desired result,

$$w(z) = \underline{w} + \int_{\underline{z}}^{z} v(x)g'(x)\,\mathrm{d}x.$$

I now verify that the second-order condition $v(z)\widetilde{g}''(w(z)) < 0$ holds for all $z > \underline{z}$. Since $v(z) > 0$, we only need to verify that $\widetilde{g}''(w(z)) \leq 0$. From the definition of $g(z)$, we have that

$$g'(z) = \widetilde{g}'\big(w(z)\big)w'(z) \quad \Longrightarrow \quad g''(z) = \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)w''(z).$$

From the first-order condition, we have that $w'(z) = g'(z)v(z) \implies w''(z) = g''(z)v(z) + g'(z)v'(z)$. Putting together,

$$g''(z) = \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)\big(g''(z)v(z) + g'(z)v'(z)\big)$$
$$\Longrightarrow \quad g''(z)\big(1 - \widetilde{g}'\big(w(z)\big)v(z)\big) = \widetilde{g}''\big(w(z)\big)\big(w'(z)\big)^2 + \widetilde{g}'\big(w(z)\big)g'(z)v'(z).$$

But from the first-order condition, we have that $1 - \widetilde{g}'(w(z))v(z) = 0$, so

$$\widetilde{g}''\big(w(z)\big) = -\frac{\widetilde{g}'\big(w(z)\big)g'(z)v'(z)}{\big(w'(z)\big)^2} = -\frac{\big(g'(z)\big)^2 v'(z)}{w'(z)}.$$

Since $v'(z) > 0$ for active firms, we have that $\widetilde{g}''(w(z)) > 0 \iff w'(z) > 0$. The case $w'(z) < 0$ can be ruled out as it would imply that $w(z) < \underline{w}$ for all $z > \underline{z}$ so it must be that $w'(z) \geq 0$. Since I am now considering the case with a continuous wage distribution, $w'(z) = 0$ is not possible as it would imply a point-mass wage distribution. Therefore, we have that the second-order condition is satisfied.

Finally, I use a standard argument (see, e.g., Burdett and Mortensen (1998); Coles and Mortensen (2016)) to rule out the possibility of a point mass. Imagine that all firms offer $w(z) = \underline{w}$ so that the wage distribution $\widetilde{P}(w)$ is given by a point mass. A firm could thus deviate by offering $\underline{w} + \epsilon$ (where $\epsilon > 0$), and increase its growth rate discontinuously from $\widetilde{g}(\underline{w}) = \lambda u - \delta$ to $\widetilde{g}(\underline{w}) = \lambda - \delta$ (see equation (8)). An infinitesimal change in the wage offered would then lead to an increase in the continuation value of a firm (see equation (9)) and the equilibrium would unravel. Therefore, the point mass wage distribution cannot be sustained in equilibrium

One-Shot Deviations

In order for the equilbrium to be Markov perfect, there needs to be no profitable one-shot deviation. What prevents a firm from deviating from $w = w(z)$ to $w = \underline{w}$ over a short time interval $[t, t + \mathrm{d}t)$ and then reverting to $w = w(z)$? Workers do not observe productivity directly, so when a firm lowers its wage, workers interpret it as arising due to a negative productivity shock, and expect the wage change to be persistent (productivity shocks are persistent). As soon as the firm would lower the wage, the growth rate of employment would decrease. The wage of the deviating firm over $[t, t + \mathrm{d}t)$ would no longer solve the first-order condition (14) and, therefore, would lead to a lower expected discounted profits.

A.4. *Derivation of Equation (22)*

I now provide a derivation of each term of the *Kolmogorov forward equation* (22) for the employment-weighted productivity distribution $P(z)$:

$$\dot{P}(z) = \underbrace{(1-u)\lambda P(z)\big(P(z)-1\big)}_{\text{Job-to-job flows}} + \underbrace{\frac{u}{1-u}\mu_e \Gamma(z) + u\lambda P(z)}_{\text{Employment inflows}} - \underbrace{(\chi_x + \delta)P(z)}_{\substack{\text{Employment} \\ \text{outflows}}} + \underbrace{\chi_s\big(\Gamma(z) - P(z)\big)}_{\text{Productivity shocks}}.$$

At Poisson rate $\lambda(1-u)$, an employed worker receives a competing job offer. The distribution of productivity (CDF) of such workers, after they have made their decision whether or not to accept the competing job offer, is $P(z)^2$. The reason is that the distribution of z in the population of workers is precisely the employment-weighted productivity distribution $P(z)$ while the distribution of job offers z' is also $P(z)$, since firms meet worker proportionally to their size. The CDF of $\max\{z, z'\}$ is therefore $P(z)^2$.[22] Using the KFE formula for jump processes, the term that accounts for job-to-job flows is therefore

$$\lambda(1-u)\big(P^2(z) - P(z)\big) = \lambda(1-u)P(z)\big(P(z)-1\big).$$

At Poisson μ_e, an unemployed worker meets a potential firm and enters the workforce with productivity distributed according to $\Gamma(z)$. At rate $\lambda(1-u)$, an unemployed worker

[22]If X and Y are two independent random variables with CDFs F, G, then $\mathbb{P}(\max\{X, Y\} \leq z) = F(z)G(z)$.

meets a firm and enters the workforce with productivity distributed according to $P(z)$. Since the ratio of unemployment to employment is $\frac{u}{1-u}$, the term that accounts for unemployment inflows is

$$\frac{u}{1-u}\big(\mu_e\Gamma(z)+\lambda(1-u)P(z)\big)=\frac{u}{1-u}\mu_e\Gamma(z)+u\lambda P(z).$$

At Poisson $\chi_x+\delta$, a worker with productivity distributed according to $P(z)$ is sent to unemployment due to firm exit or exogenous job destruction so that the term which accounts for employment outflows is $-(\chi_x+\delta)P(z)$. At Poisson rate χ_s, an employed worker's productivity resets to a draw from $\Gamma(z)$, so that the term which accounts for productivity shocks is $\chi_s(\Gamma(z)-P(z))$.

A.5. *Proof of Proposition 3*

I break down the proof into three lemmas. All derivations take $\underline{z}$ as given.

LEMMA 2: *There exists a unique real solution* $u\in[0,1)$ *that solves equation* (21) *with* $\dot{u}=0$.

PROOF: Substituting $\dot{u}=0$ in (21), we have

$$0=(1-u)(\chi_x+\delta)-u\big(\mu_e+\lambda(1-u)\big),$$

which is a quadratic equation of the form $au^2+bu+c=0$ with

$$a=\lambda,\qquad b=-(\lambda+\delta+\chi_x+\mu_e),\qquad c=\chi_x+\delta.$$

First, I verify that the discriminant $\Delta\equiv b^2-4ac$ is positive:

$$\begin{aligned}\Delta&=(\lambda+\delta+\chi_x+\mu_e)^2-4\lambda(\chi_x+\delta)\\&\geq(\lambda+\delta+\mu_e+\chi_x)^2-4\lambda(\chi+\delta)\\&\geq\big(\lambda+\delta+\min\{\chi,\mu\}\big)^2-4\lambda(\chi+\delta).\end{aligned}$$

When $\chi\leq\mu$,

$$\Delta\geq(\lambda+\delta+\chi)^2-4\lambda(\chi+\delta)=(\chi+\delta-\lambda)^2.$$

When $\chi>\mu$,

$$\Delta\geq(\lambda+\delta+\mu)^2-4\lambda(\mu+\delta)=(\mu+\delta-\lambda)^2.$$

Putting together, we have that

$$\Delta\geq\big(\min\{\mu,\chi\}+\delta-\lambda\big)^2>0,$$

where the last inequality follows from Assumption 2. There are therefore two real solutions to (21), which I denote by $u^{(\pm)}$.

$$u^{\pm}=\frac{\lambda+\delta+\chi_x+\mu_e\pm\sqrt{(\lambda+\delta+\chi_x+\mu_e)^2-4\lambda(\chi_x+\delta)}}{2\lambda}.$$

I now use the fact that $\sqrt{\Delta} > \min\{\mu, \chi\} + \delta - \lambda$ to show that $u^+ > 1$:

$$u^+ \geq \frac{\lambda + \delta + \min\{\chi, \mu\} + \sqrt{\Delta}}{2\lambda} > \frac{2\lambda}{2\lambda} = 1.$$

Therefore, u^+ is not the solution (the unemployment rate cannot be greater than one). I now show that u^- is the solution (i.e., it satisfies $0 \leq u^- < 1$). First, we have that

$$u^- \leq \frac{\lambda + \delta + \max\{\mu, \chi\}}{2\lambda} < 1,$$

where the last equality follows from

$$\lambda + \delta + \max\{\mu, \chi\} < 2\lambda \quad \Longleftrightarrow \quad \delta + \max\{\mu, \chi\} - \lambda < 0,$$

which is true under Assumption 2. Finally, we have that $u^{(-)} \geq 0$, since

$$b < \sqrt{b^2 - 4ac} \quad \Longleftrightarrow \quad 4ac \geq 0$$

and $4ac = \lambda(\delta + \chi_x)\delta \geq 0$. *Q.E.D.*

LEMMA 3: *The stationary unemployment rate satisfies* $u < \frac{\chi+\delta}{\lambda}$.

PROOF: Using the fact the discriminant is positive, we have that

$$\Delta > 0 \quad \Longrightarrow \quad u < \frac{\lambda + \min\{\mu, \chi\} + \delta}{2\lambda}.$$

Under Assumption 2, we have that $\lambda < \min\{\mu, \chi\} + \delta$, so that

$$u < \frac{\min\{\mu, \chi\} + \delta}{\lambda} \leq \frac{\chi + \delta}{\lambda}.$$ *Q.E.D.*

LEMMA 4: *There exists a unique function P that is a valid CDF and solves* (22) *with* $\dot{P}(z) = 0$.

PROOF: Setting $\dot{P}(z) = 0$ in (22), I obtain a quadratic equation in $P(z)$ of the form $aP^2(z) + bP(z) + c(z) = 0$, with coefficients given by

$$\begin{aligned} a &= (1-u)\lambda, \\ b &= -\big((1-u)\lambda + \chi_x + \chi_s + \delta - u\lambda\big), \\ c(z) &= \left(\frac{u}{1-u}\mu_e + \chi_s\right)\Gamma(z), \end{aligned}$$

Notice that, when $\dot{u} = 0$, we have from (22) that

$$\frac{u}{1-u}\mu_e = \chi_x + \delta - u\lambda.$$

Moreover, $\chi_x + \chi_s = \chi$, so the coefficients can be simplified to

$$a = (1-u)\lambda, \qquad b = -\big((1-u)\lambda + \chi + \delta - u\lambda\big), \qquad c(z) = (\chi + \delta - u\lambda)\Gamma(z).$$

I now show that the determinant

$$\Delta(z) \equiv \big(\lambda(1-u) + \chi + \delta - \lambda u\big)^2 - 4(1-u)\lambda(\chi + \delta - u\lambda)\Gamma(z),$$

is positive for all z. First, I show that $\Delta'(z) < 0$. Since $\Gamma'(z) > 0$, the statement is implied by $\chi + \delta - u\lambda > 0$, which is true (see Lemma 3). Since $\Delta(z)$ is decreasing, it sufficient to show that it is positive in the limit:

$$\lim_{z\to\infty} \Delta(z) = \big(\lambda(1-u) + \chi + \delta - \lambda u\big)^2 - 4(1-u)\lambda(\chi + \delta - u\lambda) = (\chi + \delta - \lambda)^2 > 0,$$

where the last inequality follows from Assumption 2. We therefore have two real solutions given by

$$\begin{aligned} P^{\pm}(z) = \Big(&\lambda(1-u) + \chi + \delta - u\lambda \\ &\pm \sqrt{\big(\lambda(1-u) + \chi + \delta - \lambda u\big)^2 - 4\lambda(1-u)(\chi + \delta - u\lambda)\Gamma(z)}\Big) \\ &/\big(2\lambda(1-u)\big). \end{aligned}$$

First, I show that $P^+(z) > 1$ for all $z > \underline{z}$, which implies that P^+ is not a valid CDF. Using the fact that $\sqrt{\Delta(z)} \geq \lambda(1-u) + \chi + \delta - u\lambda$, we have that

$$P^+(z) \geq \frac{\lambda(1-u) + \chi + \delta - u\lambda}{\lambda(1-u)} > 1 \quad \Longleftrightarrow \quad \chi + \delta - u\lambda > 0,$$

and the last inequality follows from Lemma 3. I now show that P^- is a valid CDF, meaning that (i) $(P^-)'(z) \geq 0$, (ii) $P^-(\underline{z}) = 0$, and (iii) $\lim_{z\to\infty} P^-(z) = 1$. First,

$$\big(P^-\big)'(z) = -\frac{\Delta'(z)}{\sqrt{\Delta(z)}4a} > 0$$

The last equality follows from $\Delta'(z) < 0$ (already shown). Second,

$$P^-(\underline{z}) = \frac{-b - |b|}{2a} = 0,$$

where the last equality follows from $b < 0$. Finally, I have already shown that $\lim \Delta(z) = (\chi + \delta - \lambda)^2$, which implies, under Assumption 2, that $\lim \sqrt{\Delta(z)} = \chi + \delta - \lambda$. We therefore have that

$$\lim_{z\to\infty} P^-(z) = \frac{\lambda(1-u) + \chi + \delta - u\lambda - (\chi + \delta - \lambda)}{2\lambda(1-u)} = \frac{2\lambda(1-u)}{2\lambda(1-u)} = 1,$$

which concludes the proof. *Q.E.D.*

A.6. *Proof of Proposition 4*

The fact that $\mathrm{LS}(\underline{z}) > 1 - \alpha$ follows directly from (28):

$$\underline{w} = (1-\alpha)y(\underline{z}) + \chi \int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x) \quad \Longrightarrow \quad \mathrm{LS}(\underline{z}) = 1 - \alpha + \frac{\chi \int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x)}{y(\underline{z})}.$$

Since $v(\underline{z})=0$ and $v'(z)>0$, we have that $\int_0^\infty v(x)\,\mathrm{d}\Gamma_0(x)>0$. By continuity of $\mathrm{LS}(z)$, we have that there exist a z' such that for all $z<z'$ we have that $\mathrm{LS}(z)>1-\alpha$. To show that there exists a z'' such that for all $z>z''$, $\mathrm{LS}'(z)<0$, first notice that

$$\mathrm{LS}'(z)<0 \quad \Longleftrightarrow \quad \frac{w'(z)}{w(z)}<\frac{y'(z)}{y(z)} \quad \Longleftrightarrow \quad w'(z)z<\frac{1}{1-\alpha}w(z).$$

To show that there exists a z'' such that this inequality holds for all $z>z''$, I first prove three lemmas.

LEMMA 5: *The value function is bounded from above by*

$$\overline{v}(z)=-\frac{1-\alpha}{r+\chi+\delta-\lambda}y(\underline{z})+\frac{1-\alpha}{r+\chi+\delta-\lambda}y(z).$$

PROOF: Since $w(z)>\underline{w}$ and $g(z)<\lambda-\delta$ (Lemma 1), then we can construct that upper bound $\overline{v}(z)$ by plugging $w(z)=\underline{w}$ (see equation (28) for equilibrium expression for $\underline{w}$) and $g(z)=\lambda-\delta$ in the HJB (9).

$$r\overline{v}(z)=(1-\alpha)\big(y(z)-y(\underline{z})\big)+(\lambda-\delta-\chi)\overline{v}(z).$$

Solving for $\overline{v}(z)$, we obtain the desired result. Under Assumption 2, we have that $r+\chi+\delta-\lambda>0$. *Q.E.D.*

LEMMA 6: *The employment weighted productivity distribution is asymptotically equivalent to* $\Gamma(z)$,

$$P'(z)\sim\frac{\chi+\delta-u\lambda}{\chi+\delta-\lambda}\Gamma'(z)$$

PROOF: From Lemma 8 below, we have that $P'(z)=\frac{\chi+\delta-u\lambda}{\chi-g(z)}\Gamma'(z)$. From Lemma 1, we have that $g(z)\to\lambda-\delta$. *Q.E.D.*

LEMMA 7: *The wage function* $w(z)$ *is bounded from above by* $\overline{w}<\infty$.

PROOF: Let $\overline{w}\equiv\lim_{z\to\infty}w(z)=\underline{w}+\int_{\underline{z}}^\infty v(z)g'(z)\,\mathrm{d}z$. Using the upper bound for $\overline{v}(z)$, we have that $\overline{w}\leq\underline{w}+(1-\alpha)\frac{2\lambda(1-u)}{r+\chi+\delta-\lambda}\int_{\underline{z}}^\infty y(z)\,\mathrm{d}P(z)<\infty$. Notice that the last term is aggregate output, which is finite under Assumption 1. Moreover, Assumption 2 implies that $r+\chi+\delta-\lambda>0$. *Q.E.D.*

Equipped with these results, I now show that there exists a z'' such that for all $z>z''$,

$$w'(z)z<\frac{1}{1-\alpha}w(z).$$

By continuity, I only need to show that the right-hand size converges to a positive number and that the left-hand side converges to zero. The right-hand side converges to $\frac{1}{1-\alpha}\overline{w}>0$. The left-hand side is bounded from below by 0 (since $w'(z)>0$) and is bounded from above by

$$2\lambda(1-u)\overline{v}(z)P'(z)\sim\frac{2\lambda(1-u)}{r+\chi+\delta-\lambda}z^{1+\frac{1}{1-\alpha}}\Gamma'(z),$$

where the asymptotic equivalence relationship uses the two lemmas combined with $w'(z) = 2\lambda(1-u)P'(z)$. Under Assumption 1, we have that $z^{1+\frac{1}{1-\alpha}}\Gamma'(z) \to 0$, which concludes the proof.

A.7. *Proof of Proposition 5*

The aggregate labor share is the ratio of aggregate wages to aggregate output. Rearranging, I obtain

$$\mathrm{LS} = \frac{(1-u)\int_{\underline{z}}^{\infty} w(z)\,\mathrm{d}P(z)}{(1-u)\int_{\underline{z}}^{\infty} y(z)\,\mathrm{d}P(z)} = \frac{\int_{\underline{z}}^{\infty} w(z)\,\mathrm{d}P(z)}{Y} = \int_{\underline{z}}^{\infty} \mathrm{LS}(z)\frac{y(z)}{Y}\,\mathrm{d}P(z).$$

To obtain equation (29), I first prove the following lemma

LEMMA 8: *The relationship between the employment-weighted productivity distribution* $P(z)$ *and the productivity distribution* $\Gamma(z)$ *is given by*

$$\mathrm{d}P(z) = \frac{\chi+\delta-\lambda u}{\chi-g(z)}\,\mathrm{d}\Gamma(z). \tag{44}$$

PROOF: Using (24), I now obtain an expression for $P'(z)$,

$$\begin{aligned} P'(z) &= \frac{\chi+\delta-u\lambda}{\sqrt{\left(\lambda(1-u)+\chi+\delta-\lambda u\right)^2 - 4\lambda(1-u)(\chi+\delta-u\lambda)\Gamma(z)}}\Gamma'(z), \\ &= \frac{\chi+\delta-u\lambda}{\lambda(1-u)+\chi+\delta-u\lambda-2\lambda(1-u)P(z)}\Gamma'(z), \\ &= \frac{\chi+\delta-u\lambda}{\chi-g(z)}\Gamma'(z), \end{aligned}$$

where the third equation uses the definition of $g(z)$ (equation (18)). Rearranging, I obtain the desired result:

$$\frac{\mathrm{d}P(z)}{\mathrm{d}\Gamma(z)} = \frac{\chi+\delta-u\lambda}{\chi-g(z)}.$$

Q.E.D.

Notice that the term $\lambda u - \delta$ is equal to the average growth rate of existing firms $\overline{g} \equiv \int_{\underline{z}}^{\infty} g(z)\,\mathrm{d}P(z)$:

$$\begin{aligned} \int_{\underline{z}}^{\infty} g(z)\,\mathrm{d}P(z) &= \lambda u - \delta + \int_{\underline{z}}^{\infty} \left(\lambda(1-u)P(z) - \lambda(1-u)\left(1-P(z)\right)\right)\mathrm{d}P(z), \\ &= \lambda u - \delta + \lambda(1-u)\frac{1}{2} - \lambda(1-u)\frac{1}{2}, \\ &= \lambda u - \delta. \end{aligned}$$

Substituting $\mathrm{d}P(z) = \frac{\chi - \overline{g}}{\chi - g(z)} \,\mathrm{d}\Gamma(z)$ in the expression for the labor share, I obtain the desired result:

$$\mathrm{LS} = \int_{\underline{z}}^{\infty} \mathrm{LS}(z) \times \frac{\chi - \overline{g}}{\chi - g(z)} \frac{y(z)}{Y} \times \mathrm{d}\Gamma(z).$$

A.8. *Firm Size Distribution*

The law of motion for the measure of firms F is

$$\dot{F} = u\mu_e - \chi_x F.$$

In a stationary equilibrium, we have that $\dot{F} = 0$, which implies that the firm entry rate $\frac{u\mu_e}{F}$ must equal the firm exit rate χ_x. The Kolmogorov forward equation for the joint distribution of productivity and size $\varphi(z, N) : [\underline{z}, +\infty] \times \mathbb{R}_+ \to \mathbb{R}_+$ across firms is given by

$$\dot{\varphi}(z, N) = \underbrace{\chi_s\left(\gamma(z) \int_{\underline{z}}^{\infty} \varphi(x, N)\,\mathrm{d}x - \varphi(z, N)\right)}_{\text{productivity shocks}} - \underbrace{g(z)\frac{\partial}{\partial N}\big(\varphi(z, N)N\big)}_{\text{firm growth}} \tag{45}$$

$$+ \underbrace{\mu_e u\gamma(z)\psi(N-1) - \chi_x\varphi(z, N)}_{\text{firm turnover}}, \tag{46}$$

where $\psi(N)$ is the Dirac delta and $\gamma(z) = \Gamma'(z)$. I now characterize the upper tail of the firm-size distribution.

PROPOSITION 1: *The upper tail of the firm size distribution obeys a power law*

$$\mathbb{P}(N > n) \sim Cn^{-\zeta}, \tag{47}$$

for some constant $C > 0$ *and Pareto exponent* $\zeta > 1$.

PROOF: Suppose that the solution (i.e., $\dot{\varphi}(z, N) = 0$) satisfies $\varphi(z, N) \sim \overline{\varphi}(z)N^{-(1+\zeta)}$, for some positive function $\overline{\varphi}$. Substituting the asymptotic conjecture in KFE, I obtain

$$\begin{aligned} 0 &= \chi_s\left(\gamma(z) \int \overline{\varphi}(z')\,\mathrm{d}z' - \overline{\varphi}(z)\right)N^{-(1+\zeta)} + \zeta g(z)\overline{\varphi}(z)N^{-(1+\zeta)} - \chi_x\overline{\varphi}(z)N^{-(1+\zeta)} \\ &= \chi_s\gamma(z) \int \overline{\varphi}(z') + \zeta g(z)\overline{\varphi}(z) - \chi\overline{\varphi}(z) \\ &= \underbrace{(\chi_s\mathcal{D}_\gamma\mathcal{K} + \zeta\mathcal{D}_g - \chi\mathcal{I})}_{\equiv\mathcal{A}(\zeta)}\overline{\varphi}(z). \end{aligned}$$

For any functions (f, g), I define the operators $\mathcal{D}_\cdot, \mathcal{K}, \mathcal{I}$ by the following actions $\mathcal{D}_f g(z) \equiv f(z)g(z)$, $\mathcal{K}f(z) \equiv \int f(z')\,\mathrm{d}z'$, and $\mathcal{I}f(z) \equiv f(z)$. The Pareto exponent ζ and function $\overline{\varphi}(z)$ are thus the solution to the following equation:

$$\mathcal{A}(\zeta)\overline{\varphi}(z) = 0.$$

See Beare and Toda (2022) and Beare, Seo, and Toda (2021) for a formal treatment. *Q.E.D.*

To compute ζ, I discretize the operator $\mathcal{A}(\zeta)$ (for a guess of ζ) and compute the largest eigenvalue. The solution ζ is such that the largest eigenvalue is zero.

A.9. *Proof of Proposition 6*

I now prove three lemmas that I will use for the proof of Proposition 6.

LEMMA 9: *For all $z > \underline{z}$, the derivative of the firm value function is given by*

$$v'(z) = \frac{\frac{1}{z} y(z)}{r + \chi - g(z)}.$$

PROOF: Using the HJB for continuing firms (9) combined with the envelope theorem, I obtain

$$rv'(z) = (1-\alpha) y'(z) + g(z) v'(z) - \chi v'(z).$$

Solving for $v'(z)$, I obtain the desired result

$$v'(z) = \frac{(1-\alpha) y'(z)}{r + \chi - g(z)} = \frac{1}{z} \frac{y(z)}{r + \chi - g(z)},$$

where the last equality uses the definition of labor productivity $y(z)$. *Q.E.D.*

LEMMA 10: *Suppose that the assumptions of Proposition* 6 *are satisfied. For all $z > \underline{z}$, the CDF of the productivity distribution of active firms satisfies*

$$\sigma \, \mathrm{d}\Gamma(z) = \frac{1}{z} \big(1 - \Gamma(z)\big) \, \mathrm{d}z.$$

PROOF:

$$\mathrm{d}\Gamma(z) = \sigma z^{-(1+\frac{1}{\sigma})} \, \mathrm{d}z \quad \Longrightarrow \quad \sigma z \, \mathrm{d}\Gamma(z) = z^{-\frac{1}{\sigma}} \, \mathrm{d}z = \big(1 - \Gamma(z)\big) \, \mathrm{d}z. \qquad \textit{Q.E.D.}$$

LEMMA 11: *Suppose that the assumptions of Proposition* 6 *are satisfied. The expected value of a new firm is given by*

$$\int v(x) \, \mathrm{d}\Gamma(x) = \underline{v} + \frac{\sigma}{\chi - \overline{g}} Y,$$

where Y is aggregate output.

PROOF:

$$\begin{aligned} \int v(x) \, \mathrm{d}\Gamma(x) &= \underline{v} + \int_{\underline{z}}^{\infty} \int_{\underline{z}}^{x} v'(s) \, \mathrm{d}s \, \mathrm{d}\Gamma(z) \\ &= \underline{v} + \int_{\underline{z}}^{\infty} \left(\int_{\underline{z}}^{x} \frac{1}{s} \frac{y(s)}{\chi - g(s)} \, \mathrm{d}s \right) \mathrm{d}\Gamma(x) \\ &= \underline{v} + \int_{\underline{z}}^{\infty} \frac{1}{s} \frac{y(s)}{\chi - g(s)} \left(\int_{s}^{\infty} \mathrm{d}\Gamma(x) \right) \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \underline{v} + \int_{\underline{z}}^{\infty} \frac{y(s)}{\chi - g(s)} \frac{1}{s} \big(1 - \Gamma(s)\big)\, \mathrm{d}s \\
&= \underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s) \frac{\chi - \overline{g}}{\chi - g(s)}\, \mathrm{d}\Gamma(s) \\
&= \underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s)\, \mathrm{d}P(s).
\end{aligned}
$$

The second equality uses Lemma 9 and the fifth one uses Lemma 10. *Q.E.D.*

I now derive an expression for aggregate profits $\int \pi(z)\, \mathrm{d}P(z)$, where $\pi(z) \equiv (1 - \alpha)y(z) - w(z)$. Using the HJB (9) combined with $r = 0$, we have

$$
\pi(z) = \big(\chi - g(z)\big) v(z) - \chi_s \int v(x)\, \mathrm{d}\Gamma(x).
$$

Integrating against the density $\mathrm{d}P(z)$, we obtain

$$
\begin{aligned}
\int_{\underline{z}}^{\infty} \pi(z)\, \mathrm{d}P(z) &= \int_{\underline{z}}^{\infty} v(x)\big(\chi - g(x)\big)\, \mathrm{d}P(x)\, \mathrm{d}x - \chi_s \int v(x)\, \mathrm{d}\Gamma(z) \\
&= (\chi - \overline{g}) \int_{\underline{z}}^{\infty} v(x)\, \mathrm{d}\Gamma(z) - \chi_s \int v(x)\, \mathrm{d}\Gamma(z) \\
&= (\chi_x - \overline{g}) \int_{\underline{z}}^{\infty} v(x)\, \mathrm{d}\Gamma(z) \\
&= (\chi_x - \overline{g}) \left(\underline{v} + \frac{\sigma}{\chi - \overline{g}} \int_{\underline{z}}^{\infty} y(s)\, \mathrm{d}P(s) \right).
\end{aligned}
$$

The second equation uses Lemma 8, the third one uses the fact that $\chi = \chi_s + \chi_x$, and the fourth uses Lemma 11. Denoting aggregate output and profits as $Y \equiv \int_{\underline{z}}^{\infty} y(s)\, \mathrm{d}P(s)$ and $\Pi \equiv \int_{\underline{z}}^{\infty} \pi(s)\, \mathrm{d}P(s)$, we have that

$$
\Pi = (\chi_x - \overline{g}) \left(\underline{v} + \frac{\sigma}{\chi - \overline{g}} Y \right). \tag{48}
$$

Using the fact that $\Pi / Y = 1 - \alpha - \mathrm{LS}$ combined with $\underline{v} = 0$, we obtain

$$
\mathrm{LS} = 1 - \alpha - \frac{\chi_x - \overline{g}}{\chi - \overline{g}} \sigma,
$$

which is the main result. The last thing to prove is that $0 < \frac{\chi_x - \overline{g}}{\chi - \overline{g}} < 1$. Using the law of motion for employment in steady state (i.e., equation (21)), we have that $\chi_x - \overline{g} = \frac{u}{1-u} \mu_e > 0$, which implies that $\frac{\chi_x - \overline{g}}{\chi - \overline{g}} > 0$. Moreover, $\chi_x < \chi$ implies that $\frac{\chi_x - \overline{g}}{\chi - \overline{g}} < 1$.

APPENDIX B: BENCHMARK MODELS AND EXTENSIONS

B.1. *Burdett and Mortensen (1998) With Capital*

I now present a derivation of the model which I refer to as "Burdett–Mortensen" in the main text. It corresponds to an extension of the model in Burdett and Mortensen

(1998) with capital as a factor of production. Unless stated otherwise, functional forms and notations are exactly as in the baseline model.

Main Equations

The laws of motion for the wage distribution $P(w)$ and firm-level employment $N(w)$ are respectively given by

$$\dot{N}(w) = \underbrace{\lambda P(w)}_{\text{Hires}} - \underbrace{\delta N(w)}_{\text{Layoffs}} - \underbrace{\lambda \overline{F}(w) N(w)}_{\text{Quits}},$$

$$\dot{P}(w) = \underbrace{\delta\big(1\{w \geq 0\} - P(w)\big)}_{\text{Job destruction}} + \underbrace{\lambda\big(F(w)P(w) - P(w)\big)}_{\text{Job creation + Churn}},$$

where $F(w)$ denotes the (endogenously-determined) distribution of wage offers and $\overline{F}(w) \equiv 1 - F(w)$. The value $P(0)$ is by convention the unemployment rate (i.e., the measure of workers working at firms that pay $w = 0$). Setting $\dot{N}(w) = 0$ and $\dot{P}(w) = 0$, I obtain

$$N(w) = \frac{\lambda\delta}{\big(\delta + \lambda\overline{F}(w)\big)^2}, \qquad P(w) = \frac{\delta}{\delta + \lambda\overline{F}(w)}.$$

Let $b > 0$ be the flow value of unemployment. As in Burdett and Mortensen (1998), I focus on a long-run steady-state where firms choose a constant wage policy $w \geq b$ and capital stock per worker $k \geq 0$ as to maximize long-run flow profits defined as

$$\begin{aligned} v(z) &= \max_{w,k} \lambda P(w) \frac{zk^\alpha - Rk - w}{r + \delta + \lambda\overline{F}(w)} \\ &= \max_{w} \delta\lambda \frac{(1-\alpha)y(z) - w}{\big(r + \delta + \lambda\overline{F}(w)\big)\big(\delta + \lambda\overline{F}(w)\big)}. \end{aligned}$$

Flow profits are given by the measure of hires $\lambda P(w)$ times the present-value of a hire. In the second expression, I use the steady-state expression for $P(w)$ and the optimal capital stock is $k(z) = (\frac{z\alpha}{R})^{\frac{1}{1-\alpha}}$ (i.e., same formula as in the baseline model; see Proposition 2).

Denoting $\widetilde{\phi}(w) \equiv \frac{\delta\lambda}{(r+\delta+\lambda\overline{F}(w))(\delta+\lambda\overline{F}(w))}$, the first-order condition for wages is given by

$$\widetilde{\phi}'\big(w(z)\big)\big((1-\alpha)y(z) - w(z)\big) - \widetilde{\phi}\big(w(z)\big) = 0.$$

Defining $\phi(z) \equiv \widetilde{\phi}(w(z))$, I obtain

$$\begin{aligned} &\phi'(z)\big((1-\alpha)y(z) - w(z)\big) - \phi(z)w'(z) = 0 \\ \implies \quad &(1-\alpha)y(z)\phi'(z) = \big(w(z)\phi(z)\big)'. \end{aligned}$$

This is an ODE. Combined with the initial condition $w(\underline{z}) = b$, I obtain the following solution:

$$w(z)\phi(z) - b\phi(\underline{z}) = (1-\alpha)\phi(\underline{z})y(\underline{z}) - \int_{\underline{z}}^{z} y'(x)\phi(x)\,\mathrm{d}x$$

$$\implies \quad w(z) = (1-\alpha)y(z) + \big(b - (1-\alpha)y(\underline{z})\big)\frac{\phi(\underline{z})}{\phi(z)} - (1-\alpha)\int_{\underline{z}}^{z} y'(x)\frac{\phi(x)}{\phi(z)}\,\mathrm{d}x.$$

The entry threshold $\underline{z}$ satisfies $(1-\alpha)y(\underline{z}) = b$, which ensures that the marginal firm makes zero profit. The formula simplifies to

$$w(z) = (1-\alpha)\left[y(z) - \int_{\underline{z}}^{z} y'(x)\frac{\phi(x)}{\phi(z)}\,\mathrm{d}x\right]. \tag{49}$$

Let Γ_0 be the distribution of productivity of potential entrants and $\Gamma(z) = \frac{\Gamma_0(z)-\Gamma_0(\underline{z})}{1-\Gamma_0(\underline{z})}$ be the truncated distribution where the threshold $\underline{z}$ is given by $\underline{z} = \frac{b^{1-\alpha}R^{\alpha}}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}$. Using the fact that the wage schedule is increasing in z, we have that the equilibrium is characterized by

$$\begin{aligned} N(z) &= \frac{\lambda\delta}{\left(\delta+\lambda\overline{\Gamma}(z)\right)^2}, \qquad P(z) = \frac{\delta}{\delta+\lambda\overline{\Gamma}(z)}, \\ \phi(z) &= \frac{\delta\lambda}{\left(r+\delta+\lambda\overline{\Gamma}(z)\right)\left(\delta+\lambda\overline{\Gamma}(z)\right)}, \end{aligned} \tag{50}$$

where $P(z) \equiv P(w(z))$.

Pareto Special Case

I now provide a closed-form solution for the aggregate labor share in the Pareto special case with no discounting (i.e., same assumptions as in Proposition 6). First, note that $r = 0$ implies $\phi(z) = N(z)$. Integrating equation (49) against $N(z)\,\mathrm{d}\Gamma(z)$, I obtain

$$\underbrace{\int_b^{\infty} w(x)N(x)\,\mathrm{d}\Gamma(x)}_{wN} = (1-\alpha)\underbrace{\int_b^{\infty} xN(x)\,\mathrm{d}\Gamma(x)}_{Y} - (1-\alpha)\underbrace{\int_b^{\infty}\int_b^{z} y'(s)N(s)\,\mathrm{d}s\,\mathrm{d}\Gamma(x)}_{\Pi},$$

where wN is aggregate worker compensation, Y is aggregate output, and Π is aggregate profits. Using the fact that $(1-\alpha)y'(s) = \frac{1}{s}y(s)$, I obtain

$$\begin{aligned} \Pi &= \int_b^{\infty} \frac{y(s)}{s}N(s)\left(\int_b^{\infty} 1\{s \le x\}\,d\Gamma(x)\right)dy \\ &= \int_b^{\infty} \frac{y(s)}{s}N(s)\Gamma(s)\,ds. \end{aligned}$$

Finally, using the fact that $1-\Gamma(s) = \sigma s\frac{d\Gamma(s)}{ds}$, I obtain

$$\Pi = \sigma\int_b^{\infty} y(s)N(y)\Gamma(y)\,dy = \sigma Y.$$

The labor share $\mathrm{LS} \equiv wN/Y$ is therefore given by

$$\mathrm{LS} = 1-\alpha-\sigma. \tag{51}$$

Calibration

The free parameters of the model are $(\alpha, R, \lambda, \delta, b, \eta)$. First, I calibrate (α, R, η)—which govern the capital share, discount rate, and productivity dispersion—exactly as in the baseline model (see Section 3).

The remaining parameters are (λ, δ). In the baseline model, those parameters are identified using job reallocation rates across firms as well as the unemployment rate. In Burdett–Mortensen's long-run steady state, there is no job reallocation. Instead, I use the unemployment rate and the layoff rate as empirical targets.

I set $\delta = 1 - e^{-12\times 0.004} = 0.0469$ to match a monthly employment-to-unemployment probability of 0.004 as estimated in Nakamura, Nakamura, Phong, and Steinsson (2019) using Canadian microdata. Then I choose λ to match the unemployment target of 7.13% as in the baseline model.

$$u \equiv P(0) = \frac{\delta}{\delta + \lambda} \quad \Longrightarrow \quad \lambda = \delta \times \frac{1-u}{u} = 0.6109.$$

Finally, b is chosen such that the rank of the entry threshold $\Gamma_0(\underline{z})$ in the Burdett–Mortensen model coincides with the rank of the entry threshold in the full model. This last step ensures that the distribution of productivity amongst active firms (i.e., $\Gamma(z)$) is exactly the same in the baseline model and the Burdett–Mortensen model.

B.2. *Coles and Mortensen (2016) With Balanced-Matching and Capital*

I now present a derivation of the model which I refer to as "Coles–Mortensen" in the main text. It corresponds to a version of the model in Coles and Mortensen (2016) with balanced matching and capital as a factor of production. Unless stated otherwise, functional forms and notation are exactly as in the baseline model.

In the Coles–Mortensen model, the minimum productivity level $\underline{z}$ is exogenous, which implies that the firm exit rate χ_x and productivity shock rate χ_s are exogenous as well (see equation (20)). Coles and Mortensen (2016) assume that flow profits are nonnegative at every date and state, which implies that

$$\underline{w} \leq (1-\alpha) y(\underline{z}).$$

I consider the most conservative case (i.e., where flow profits of the lowest productivity firms are zero, as in the Burdett–Mortensen). Assumption 3 in the baseline model is thus replaced by

$$\underline{w} = (1-\alpha) y(\underline{z}).$$

Main Equations

Given an exogenous $\underline{z}$ (which pins down χ_x and χ_s), the equilibrium unemployment rate u, employment-weighted productivity distribution $P(z)$ are exactly as in the baseline model (see equations (23) and (24)). This implies that $g(z) = \lambda u + \lambda(1-u)P(z) - \lambda(1-u)P(z) - \delta$, $v'(z) = \frac{1}{z}\frac{y(z)}{r+\chi-g(z)}$, and $v(z) - \underline{v} = \int_{\underline{z}}^{z} v'(x)\,\mathrm{d}x$ are also exactly as in the standard model.

I first derive an expression for $v(\underline{z})$, which I then use to derive an expression for $v(z)$ and $w(z)$. Using the HJB for active firms (9), we have

$$rv(z) = \pi(z) + g(z)v(z) + \chi_s \int v(x)\,\mathrm{d}\Gamma(x) - \chi v(z).$$

Evaluating at $z = \underline{z}$ and using the fact that $\underline{w} = (1-\alpha)y(\underline{z})$, I obtain

$$\begin{aligned}(r + \chi - \underline{g})\underline{v} &= \chi_s \int v(x)\,\mathrm{d}\Gamma(x) \\ &= \chi_s \underline{v} + \chi_s \int \big(v(x) - \underline{v}\big)\,\mathrm{d}\Gamma(x),\end{aligned}$$

where $\underline{g} \equiv g(\underline{z})$. Using the fact that $\chi = \chi_s + \chi_x$, I obtain

$$\underline{v} = \frac{\chi_s}{r + \chi_x - \underline{g}} \int \big(v(x) - \underline{v}\big)\,\mathrm{d}\Gamma(x). \tag{52}$$

From the first-order condition for wages, we have that

$$w'(z) = g(z)v(z) \quad \Longrightarrow \quad w(z) = \underline{w} + \int_{\underline{z}}^{z} g'(x)v(x).$$

Using the fact that $\underline{w} = (1-\alpha)y(\underline{z})$, we have that

$$\begin{aligned} w(z) &= (1-\alpha)y(\underline{z}) + \int_{\underline{z}}^{z} g'(x)v(x)\,\mathrm{d}x \\ &= (1-\alpha)y(\underline{z}) + \big(g(z) - \underline{g}\big)\underline{v} + \int_{\underline{z}}^{z} g'(x)\big(v(x) - \underline{v}\big)\,\mathrm{d}x. \end{aligned}$$

Combining with (52), I obtain

$$\begin{aligned} w(z) = (1-\alpha)y(\underline{z}) &+ \big(g(z) - \underline{g}\big)\frac{\chi_s}{r + \chi_x - \underline{g}} \int_{\underline{z}}^{\infty} \big(v(x) - \underline{v}\big)\,\mathrm{d}\Gamma(x) \\ &+ \int_{\underline{z}}^{z} g'(x)\big(v(x) - \underline{v}\big)\,\mathrm{d}x. \end{aligned} \tag{53}$$

This equation is useful because it expresses the wage schedule in the Coles–Mortensen model in terms of objects that depend only on the exogenous threshold $\underline{z}$: $(1-\alpha)y(\underline{z})$, $g(z) - \underline{g}$, $\frac{\chi_s}{r+\chi_x-\underline{g}}$, $g'(z)$, and $v(x) - \underline{v}$. Therefore, the Coles–Mortensen model can be solved entirely in closed form.

Note that, if the baseline model and Coles–Mortensen models are calibrated to imply the same productivity threshold $\underline{z}$, then they will imply the same allocation of workers across firms $P(z)$, and only differ in the equilibrium breakdown between wages and profits $w(z)$.

Calibration

The model parameters are $(\underline{z}, \lambda, \mu, \delta, \chi, \eta)$. First, I choose $\underline{z}$ so that the rank of the entry threshold $\Gamma_0(\underline{z})$ in the Coles–Mortensen model coincides with the rank of the entry threshold in the baseline model. This step ensures that the distribution of productivity among active firms $\Gamma(z)$ is exactly the same in the baseline model and the Coles–Mortensen model. Then I set $(\lambda, \mu, \delta, \chi, \eta)$ to the same values as in the baseline model (see Section 3.1). This calibration strategy implies that the Coles–Mortensen model implies the same targeted moments as the baseline model (see Section 3.1). The only difference between the baseline model and Coles–Mortensen is the wage function, which pins down the split between profits and wages.

Pareto Special Case

I now provide a closed-form solution for the aggregate labor share in the Pareto special case with no discounting (i.e., same assumptions as in Proposition 6). Combining Proposition 11 and equation (52), I obtain the following expression:

$$\underline{v} = \frac{\chi_s}{(\chi_x - \underline{g})(\chi - \overline{g})}\sigma Y.$$

Plugging in equation (48) (which was derived in the baseline model, but the exact same derivation applies in the Coles–Mortensen model), I obtain

$$\Pi = \left(\frac{\chi_x - \overline{g}}{\chi - \overline{g}} + \frac{\chi_x - \overline{g}}{\chi_x - \underline{g}}\frac{\chi_s}{\chi - \overline{g}}\right)\sigma Y.$$

Hence, the aggregate labor share is

$$\mathrm{LS} = 1 - \alpha - \left(\frac{\chi_x - \overline{g}}{\chi - \overline{g}} + \frac{\chi_x - \overline{g}}{\chi_x - \underline{g}}\frac{\chi_s}{\chi - \overline{g}}\right)\sigma.$$

B.3. *Endogenous Vacancy Posting Extension*

I now describe an extension of the baseline model with endogenous vacancy posting (endogenous-λ model). The derivations are brief but detailed. Unless stated otherwise, the environment and notation is exactly as in the baseline model.

Matching

In the baseline model, $\lambda > 0$ is an exogenous parameter that determines the rate at which firms meet workers. I will now interpret λ as a firm-level vacancy rate (i.e., the measure of vacancies per employee). Suppose that firms must pay a cost $c(\lambda, z)$ to post a measure λ of vacancies per employee, which potentially depends on their productivity z. Note that the cost function is allowed to depend directly on the firm type z.

Strategies and Beliefs

As in the baseline model, I restrict attention to Markov strategies that depend only on productivity. The wage schedule, vacancy rate, entry decision, and exist decision are respectively given by $w(z)$, $\lambda(z)$, $e(z)$, and $x(z)$.

Distributions

Define $\widetilde{P}(w)$ to be the wage distribution and $\widetilde{\lambda}(w)$ to be the vacancy rate of firms who pays a wage w. The vacancy-weighted wage distribution $\widetilde{Q}$ and aggregate vacancy rate Λ

are defined as

$$\widetilde{Q}(w) \equiv \frac{1}{\Lambda} \int_0^w \widetilde{\lambda}(w)\, \mathrm{d}\widetilde{P}(w), \tag{54}$$

$$\Lambda \equiv \int_0^\infty \widetilde{\lambda}(w)\, \mathrm{d}\widetilde{P}(w). \tag{55}$$

I denote the employment-weighted and vacancy-weighted productivity distribution as $P(z)$ and $Q(z)$, respectively.

Worker Problem

Denote by U and $W(w)$ the value of being unemployed and the value of being employed at a firm currently paying w, respectively. Equations (3) and (4) in the baseline model become

$$\begin{aligned} rU = b &+ \underbrace{\mu \int e(z) \big| W\big(w(z)\big) - U \big|_+ \, \mathrm{d}\Gamma_0(z)}_{\text{job offers by entering firm}} \\ &+ \underbrace{\Lambda(1-u) \int \big| W\big(w(z)\big) - U \big|_+ \, \mathrm{d}Q(z)}_{\text{job offers by continuing firm}}, \end{aligned} \tag{56}$$

$$\begin{aligned} rW(w) = w &+ \underbrace{\chi \int \big(1 - x(z)\big)\big(W\big(w(z')\big) - W(w)\big)\, \mathrm{d}\Gamma_0(z')}_{\text{wage changes due to productivity shocks}} \\ &+ \underbrace{\Lambda(1-u) \int \big| W\big(w(z')\big) - W(w) \big|_+ \, \mathrm{d}Q(z')}_{\text{job offers by continuing firm}} \\ &+ \underbrace{\left(\chi \int x(z)\, \mathrm{d}\Gamma_0(z) + \delta\right)\big(U - W(w)\big)}_{\text{job destruction}}. \end{aligned} \tag{57}$$

The only difference with the baseline model is that $\mathrm{d}P(z)$ is replaced by $\mathrm{d}Q(z)$. Hence, the optimal behavior of workers remains as in the baseline model.

Firm Growth

The instantaneous change in employment at a firm of size N_t paying $w_t \geq \underline{w}$ with a vacancy rate λ_t is now given by

$$\mathrm{d}N_t = \widetilde{g}(w_t, \lambda_t) N_t \, \mathrm{d}t,$$

where the employment growth function can be expressed as

$$\widetilde{g}(w, \lambda) = \underbrace{\lambda u + \lambda(1-u)\widetilde{P}(w)}_{\equiv \widetilde{h}(w,\lambda)} - \underbrace{\big(\Lambda(1-u)\big(1 - \widetilde{Q}(w)\big) + \delta\big)}_{\equiv \widetilde{s}(w)}.$$

I define $\widetilde{h}(w, \lambda)$ and $\widetilde{s}(w)$ to be the hiring and separation rates, respectively. The hiring rate can be decomposed as the product of the vacancy rate and the vacancy yield (i.e., measure of hires per vacancy):

$$\widetilde{h}(w, \lambda) = \lambda \times \big(u + (1-u)\widetilde{P}(w)\big).$$

Unlike in the baseline model, firms now have two ways to increase their hiring: posting more vacancies or increasing their wage. The following lemma characterizes the employment growth function.

LEMMA 12: *The function $\widetilde{g}(w, \lambda)$ has the following properties*:

$$\begin{aligned}
\widetilde{g}_w(w, \lambda) &= \lambda(1-u)\widetilde{P}'(w) + \widetilde{\lambda}(w)(1-u)\widetilde{P}'(w), \\
\widetilde{g}_\lambda(w, \lambda) &= u + (1-u)\widetilde{P}(w), \\
\lim_{w\to\infty} \widetilde{g}(w, \lambda) &= \lambda - \delta.
\end{aligned}$$

Firm Problem

I now characterize the firm problem. In the baseline model, the Cobb–Douglas assumption ensures that the capital share is α for all firms. In the endogenous-λ extension, value-added per worker is $y = zk^\alpha - c(\lambda, z)$. To ensure that both models are completely comparable, I now assume that gross output per worker is exogenously given by $y(z) = zk(z)^\alpha$, where $k(z)$ is given by (15), and that payments to capital represent a fixed share α of value-added. The firm problem is thus given by

$$\begin{aligned}
rv(z) = \max_{w\geq b, \lambda\geq 0} \bigg\{&(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) \\
&+ \chi\bigg(\int v(x)\, \mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\}
\end{aligned} \tag{58}$$

for all firms who do not exit (i.e., $z \geq \underline{z}$).[23] The first-order condition for vacancy posting is

$$(1-\alpha)c_\lambda\big(\lambda(z), z\big) = \widetilde{g}_\lambda\big(w(z), \lambda(z)\big)v(z). \tag{62}$$

The first-order condition for wages is

$$1 = v(z)\widetilde{g}_w\big(w(z), \lambda(z)\big)$$

[23]The full firm problem can be expressed as a linear complementarity problem. Equations (10), (11), and (12) in the baseline model become:

$$rv(z) = \max_{w\geq b, \lambda\geq 0} \bigg\{(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) + \chi\bigg(\int v(x)\, \mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\}, \tag{59}$$

$$rv(z) \geq 0, \tag{60}$$

$$\begin{aligned}
0 = v(z)\bigg(&rv(z) - \max_{w\geq b, \lambda\geq 0} \bigg\{(1-\alpha)\big(y(z) - c(\lambda, z)\big) - w + v(z)\widetilde{g}(w, \lambda) \\
&+ \chi\bigg(\int v(x)\, \mathrm{d}\Gamma_0(x) - v(z)\bigg)\bigg\}\bigg).
\end{aligned} \tag{61}$$

$$= v(z)2\lambda(z)(1-u)\widetilde{P}'\big(w(z)\big),$$

where the second equality uses Lemma 12. Using the fact that $P'(z) = \widetilde{P}'(w(z))w'(z)$, we have that

$$w'(z) = v(z)2(1-u)\lambda(z)P'(z).$$

Using the boundary condition $w(\underline{z}) = \underline{w}$, we have that

$$w(z) = \underline{w} + 2(1-u)\int_{\underline{z}}^{z} v(x)\lambda(x)\,\mathrm{d}P(x). \tag{63}$$

Laws of Motion

The laws of motion (21) and (22) in the baseline model become:

$$\dot{u} = \underbrace{(\delta+\chi_x)(1-u)}_{\text{unemployment inflows}} - \underbrace{u\big(\mu_e + \Lambda(1-u)\big)}_{\text{unemployment outflows}}, \tag{64}$$

$$\dot{P}(z) = \underbrace{(1-u)\Lambda P(z)\big(Q(z)-1\big)}_{\text{job-to-job flows}} + \underbrace{\frac{u}{1-u}\mu_e\Gamma(z) + u\Lambda Q(z)}_{\text{employment inflows}}$$

$$- \underbrace{(\delta+\chi_x)P(z)}_{\substack{\text{employment}\\ \text{outflows}}} + \underbrace{\chi_s\big(\Gamma(z)-P(z)\big)}_{\text{productivity shocks}}. \tag{65}$$

Notice that the stationary employment-weighted productivity distribution $P(z)$ is no longer the solution to a quadratic equation, which means that it can no longer be solved analytically.

Functional Form

I assume the following function form for the recruitment cost function:

$$c(\lambda, z) = \frac{\underline{\lambda}^{-\theta}}{(1-\alpha)(1+\theta)}\lambda^{1+\theta}v(z).$$

Under this assumption, I obtain closed-form expression for some equilibrium objects.

LEMMA 13: *Denote the equilibrium vacancy yield (hires per vacancy) as $f(z) \equiv u + (1-u)P(z)$. The equilibrium objects $\lambda(z)$, Λ, $Q(z)$ are given by*

$$\lambda(z) = \underline{\lambda} f(z)^{\frac{1}{\theta}}, \tag{66}$$

$$\Lambda = \underline{\lambda}\frac{\theta}{1+\theta}\frac{1-u^{1+\frac{1}{\theta}}}{1-u}, \tag{67}$$

$$Q(z) = \frac{f(z)^{1+\frac{1}{\theta}} - u^{1+\frac{1}{\theta}}}{1-u^{1+\frac{1}{\theta}}}. \tag{68}$$

PROOF: The policy function $\lambda(z)$ is obtained directly from the first-order condition (62). To obtain an expression for Λ, $Q(z)$, I first define a function $\phi(z) = \int_0^z \frac{\lambda(x)}{\underline{\lambda}}\,\mathrm{d}P(x)$. We have

$$\begin{aligned}
\phi(z) &= \int_0^z f(z)^{\frac{1}{\theta}}\,\mathrm{d}P(z) \\
&= \frac{1}{1-u}\int_{f(0)}^{f(z)} x^{\frac{1}{\theta}}\,\mathrm{d}x \\
&= \frac{\theta}{1+\theta}\frac{f(z)^{1+\frac{1}{\theta}} - u^{1+\frac{1}{\theta}}}{1-u}.
\end{aligned}$$

Finally, evaluating the definitions (54) and (55) at $w = w(z)$, we have that $\Lambda = \lim_{z\to\infty}\underline{\lambda}\phi(z)$ and $Q(z) = \Lambda^{-1}\underline{\lambda}\phi(z)$. *Q.E.D.*

ASSUMPTION 1: *The rate condition in the baseline model* (*i.e.*, *Assumption* 2) *becomes*

$$\sup_z \lambda(z) < \chi + \delta \quad \Longleftrightarrow \quad \underline{\lambda} < \chi + \delta.$$

Defining the equilibrium hiring and separation rates $h(z) \equiv \widetilde{h}(\lambda(z), w(z))$, $s(z) = \widetilde{s}(w(z))$ as well as the vacancy-posting cost $c(z) \equiv c(\lambda(z), z)$, I obtain

$$h(z) = \underline{\lambda} f(z)^{1+\frac{1}{\theta}}, \tag{69}$$

$$s(z) = \Lambda(1-u)\big(1 - Q(z)\big) + \delta, \tag{70}$$

$$c(z) = \frac{1}{(1+\theta)(1-\alpha)} h(z) v(z). \tag{71}$$

Finally, firm-level labor shares are thus given by

$$\mathrm{LS}(z) = \frac{w(z)}{y(z) - \frac{1}{(1+\theta)(1-\alpha)} h(z) v(z)}.$$

B.4. *Partially-Segmented Labor Market Extension*

I now solve a stripped-down version of the baseline model with multiple industries. The model is static and only contains two workers and two firms in each industry. Instead of facing search frictions (which in the baseline model limit the firm-level growth rate of employment), firms face an exogenous capacity constraint (i.e., they can hire at most one worker).

The goal of the model is to show that the labor share in an industry is decreasing in its own level of productivity dispersion even when labor markets are not perfectly segmented along industry lines. I first solve the equilibrium under "perfectly-segmented labor market" and then solve the equilibrium under "partially-segmented labor markets."

Perfectly-Segmented Labor Market

There is a continuum of industries $j \in [0, 1]$. In each industry, there are two firms $i \in \{l, h\}$ that differ in productivity and two identical workers. Firm productivity is

$$z_{lj} = 1 - \sigma_j, \qquad z_{hj} = 1 + \sigma_j.$$

The average productivity is normalized to one and the parameter $0 < \sigma_j < 1$ represents productivity dispersion. The average level of productivity dispersion across industries is denoted $\overline{\sigma} \equiv \mathbb{E}\sigma_j$.

Firms compete à la Bertrand by posting wages subject to a capacity constraint (i.e., they can employ at most one worker). Workers observe the wages at both firms and choose to work at the firm with the highest wage. If both firms post the same wage, they each hire one worker. The problem of firm $i \in \{l, h\}$ is

$$\max_{w_i} z_i \min\{N_i, 1\} - w_i N_i,$$

$$\text{s.t.} \quad N_i = \begin{cases} 0 & \text{if } w_i < w_{-i}, \\ 1 & \text{if } w_i = w_{-i}, \\ 2 & \text{if } w_i > w_{-i}, \end{cases}$$

where the subscript $-i$ denotes the competitor of firm i. The outcome of Bertrand competition is well known: both firms offer a wage equal to the productivity of firm l. The following proposition expresses the equilibrium labor share as a function of productivity dispersion.

PROPOSITION 2—Perfectly-segmented labor markets: *The labor share in industry j is*

$$\text{LS}_j = 1 - \sigma_j.$$

PROOF: First, I compute the Nash equilibrium. The best response functions, which express the optimal wage of firm i as a function of the wage offered by firm $-i$ are

$$w_{lj}(w_{hj}) = \begin{cases} w_{hj} & \text{if } w_{hj} \leq z_{lj}, \\ 0 & \text{if } w_{hj} > z_{lj}, \end{cases} \qquad w_{hj}(w_{lj}) = \begin{cases} w_{lj} & \text{if } w_{lj} \leq z_{hj}, \\ 0 & \text{if } w_{lj} > z_{hj}. \end{cases}$$

Therefore, the only Nash equilibrium is $w_{lj} = w_{hj} = z_{lj}$ and the resulting labor allocation is $N_{lj} = N_{hj} = 1$. The resulting labor share in industry j is $\text{LS}_j = \frac{w_{lj}+w_{hj}}{z_{lj}+z_{hj}} = 1 - \sigma_j$. *Q.E.D.*

Importantly, the labor share is strictly decreasing in the level of productivity dispersion σ_j. Note that the emergence of profits in equilibrium is due to the combination of a capacity constraint and productivity dispersion. The capacity constraint has a similar effect to search frictions in the baseline model (i.e., high-productivity firms are not able to absorb all workers).

Partially-Segmented Labor Markets

I now consider the case where there is mobility of workers across industries (i.e., where firms sometimes hire workers located outside of their own industry).

Suppose that, with probability $1-\pi$, the high-productivity firm in industry j competes for workers with the low-productivity firm in another randomly-selected industry j'. Symmetrically, the high-productivity firm in industry j' competes with the low-productivity firm in industry j. With complementary probability π, firms in industry j compete with each other for workers in industry j, as in the perfectly-segmented case.

I interpret the parameter π as the amount of labor market segmentation. If $\pi = 1$, the labor market is perfectly segmented along industry lines: firms within an industry only compete with each other for workers. In contrast, if $\pi < 1$, the labor market is partially-segmented along industry lines: in some cases, firms compete for workers with firms in other industries.

Suppose that firms observe who they are competing with before posting wages. The following proposition expresses the expected equilibrium labor share as a function of own-industry productivity dispersion.

PROPOSITION 3—Partially-segmented labor markets: *The expected labor share in industry j conditional on own-industry productivity dispersion σ_j is*

$$\mathbb{E}(\mathrm{LS}_j \,|\, \sigma_j) = \pi(1-\sigma_j) + (1-\pi)\left(1 - \frac{1}{2}(\sigma_j - \overline{\sigma})\right).$$

PROOF: In industries j that have a segmented labor market, the labor share is exactly as in Proposition 2. In industries j in which the high-productivity firm competes for workers with the low-productivity firm in another industry j', the same logic applies (i.e., the low-productivity firm always posts a wage equal to its productivity and the high-productivity firm always posts a wage equal to the productivity of its competitor). In such industries, the labor share is $\frac{z_{lj}+z_{lj'}}{z_{lj}+z_{hj}} = 1 - \frac{1}{2}(\sigma_j + \sigma_{j'})$. The expected labor share in industry j conditional on own-industry productivity dispersion σ_j is therefore given by $\mathbb{E}(\mathrm{LS}_j \,|\, \sigma_j) = \pi(1-\sigma_j) + (1-\pi)(1 - \frac{1}{2}(\sigma_j - \overline{\sigma}))$. *Q.E.D.*

Two remarks are in order. First, industry labor shares are strictly decreasing in their level of productivity dispersion σ_j. Second, the relationship between labor share and own-industry productivity dispersion is weaker (in absolute value) when labor markets are not perfectly segmented along industry lines (i.e., $\pi < 1$).

APPENDIX C: SOLUTION ALGORITHM AND CALIBRATION

C.1. *Solution Algorithm*

I now present a brief but detailed description of the solution algorithm used to solve the baseline model.

Firm Problem

First, I combine the linear complementarity problem representation (LCP) of the firm problem (10), (11), (12) with the expressions for the wage schedule (16), capital per worker (15), and employment growth function $g(z)$ (18). I obtain the following equations:

$$rv(z) \geq (1-\alpha)y(z) - \underline{w} - \int_{\underline{z}}^{z} v(x)g'(x)\,dx + v(z)g(z) + \chi \int v(x)\Gamma_0(dx) - \chi v(z),$$

$$v(z) \geq 0,$$

$$v(z)\Bigg(rv(z)-\Bigg((1-\alpha)y(z)-\underline{w}-\int_{\underline{z}}^{z}v(x)g'(x)\,dx+v(z)g(z)$$
$$+\chi\int v(x)\Gamma_0(dx)-\chi v(z)\Bigg)\Bigg)=0.$$

To simplify notation, I define the the linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively, defined by the actions

$$\mathcal{A}f(z)=\int_{\underline{z}}^{z}f(x)g'(x)\,dx,\qquad \mathcal{B}f(z)=f(z)g(z),$$
$$\mathcal{C}f(z)=\int v(x)\Gamma_0(dx),\qquad \mathcal{I}f(z)=f(z).$$

The LCP can expressed in terms of the operators

$$rv(z)\geq(1-\alpha)y(z)-\underline{w}-\mathcal{A}v(z)+\mathcal{B}v(z)+\chi\mathcal{C}v(z)-\chi\mathcal{I}v(z),$$
$$v(z)\geq 0,$$
$$v(z)\big(rv(z)-\big((1-\alpha)y(z)-\underline{w}-\mathcal{A}v(z)+\mathcal{B}v(z)+\chi\mathcal{C}v(z)-\chi\mathcal{I}v(z)\big)\big)=0,$$

or more compactly as

$$v(z)\big(\mathcal{M}v(z)+q(z)\big)=0,\qquad \mathcal{M}v(z)+q(z)\geq 0,\quad v(z)\geq 0,$$

where $\mathcal{M}\equiv(r+\chi)\mathcal{I}+\mathcal{A}-\mathcal{B}-\chi\mathcal{C}$ and $q(z)\equiv\underline{w}-(1-\alpha)y(z)$. The threshold $\underline{z}$ does not appear anywhere but can recovered from the solution $v(z)$ as $\underline{z}\equiv\inf\{z:v(z)>0\}$. I now consider a discrete approximation of the system of equations over a grid $\{z_1,\dots,z_N\}$. Let $v\equiv(v(z_1),\dots,v(z_N))'$ and $y\equiv(y(z_1),\dots,y(z_N))'$. I approximate the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{I}$ by the finite difference method and obtain $N\times N$ matrices A, B, C, I, and M. The resulting system of equations is given by

$$v^T(Mv+q)=0,\qquad Mv+q\geq 0,\quad v\geq 0,$$

which is a plain-vanilla LCP that can be solved with standard routines.

Worker Problem

Define the excess value of working at a firm of productivity z as opposed to being unemployed by $V(z)\equiv W(z)-U$. Evaluating (3), (4), (5) and plugging in the optimal firm decisions (Proposition 2), unemployment rate (23), and employment-weighted distribution (24), I obtain

$$rV(z)=w(z)-b+(\chi-\mu)\int V(z')\,\mathrm{d}\Gamma_0(z')-\lambda(1-u)\int_0^z V(z')\,\mathrm{d}P(z')$$
$$-\lambda(1-u)\overline{P}(z)V(z)-(\delta+\chi)V(z),$$
$$V(z)\geq 0.$$

In operator form, the system is

$$\begin{aligned} rV(z) &= w(z) - b + (\chi - \mu)\mathcal{C}V(z) - \lambda(1-u)\mathcal{D}V(z) - \lambda(1-u)\mathcal{E}V(z) \\ &\quad - (\delta + \chi)\mathcal{I}V(z), \\ V(z) &\geq 0, \end{aligned}$$

where $\mathcal{C}, \mathcal{I}$ were defined earlier and

$$\mathcal{D}f(z) = \int_0^z f(z)\,\mathrm{d}P(z), \qquad \mathcal{E}f(z) = \big(1 - P(z)\big)f(z).$$

More compactly, we have

$$V(z)\big(\mathcal{M}V(z) + q(z)\big) = 0, \qquad \mathcal{M}V(z) + q(z) \geq 0, \quad V(z) \geq 0,$$

where

$$\mathcal{M} = (r + \chi + \delta)\mathcal{I} - (\chi - \mu)\mathcal{C} + \lambda(1-u)(\mathcal{D} + \mathcal{E}), \qquad q(z) = b - w(z)$$

I now consider a discrete approximation of the system of equations over a grid $\{z_1, \dots, z_N\}$. I approximate the operators $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{I}$ by the finite difference method and obtain $N \times N$ matrices C, D, E, I, and M. The resulting system of equations is given by

$$V^T(MV + q) = 0, \qquad MV + q \geq 0, \quad V \geq 0,$$

which again is a LCP that can be solved with standard routines. I recover the reservation wage as

$$\underline{w} = b + (\mu - \chi)CV.$$

Algorithm

The equilibrium allocation (unemployment rate, employment-weighted distribution) can be computed in closed form (3) but depends on the equilibrium thresholds $(\underline{z}, \underline{w})$. The algorithms consists of iterating over the worker and firm problem until convergence. The algorithm is initiated (step $b = 0$) with a guess $(\underline{z}^{(0)}, \underline{w}^{(0)})$. Then a typical step b consists of the following operations:

1. Using $\underline{z}^{(b-1)}$, $\underline{w}^{(b-1)}$, solve the firm problem. Collect the wage $\underline{w}^{(b)}$ and the threshold $\underline{z}^{(b)}$.
2. Using $\underline{z}^{(b)}$ and $\underline{w}^{(b)}$, solve the worker problem. Collect the reservation wage $\underline{w}^{(b)}$.
3. If $\max\{|\underline{z}^{(b)} - \underline{z}^{(b-1)}|, |\underline{w}^{(b)} - \underline{w}^{(b-1)}|\} < 10^{-6}$, stop. Otherwise, continue to step $b + 1$.

Solving the Firm Size Distribution

In Table V, I estimate a regression of firm labor share LS on size N in the data. To compute the model-implied regression coefficient, I need to solve for the joint-distribution of firm size N and productivity z. To do so, I use the Pareto extrapolation method developed in Gouin-Bonenfant and Toda (2022). Since the method applies to the discrete-time model, I discretize the model over short time intervals of one week.

C.2. *Solution Algorithm (Endogenous-λ Extension)*

I now present the solution algorithm used to solve the endogenous-λ model. I pay particular attention to where it differs from the baseline solution algorithm (see Appendix C.1).

Firm Problem

Define $c(z) \equiv c(\lambda(z), z)$. The firm problem can be solved exactly as in the baseline model, the only differences being the operator $\mathcal{A}$, which is now defined as

$$\mathcal{A}f(z) = \int_{\underline{z}}^{z} 2\lambda(x)(1-u)f(x)\,\mathrm{d}x,$$

and the function $q(z)$, which is now defined as

$$q(z) = \underline{w} + c(z) - (1-\alpha)y(z).$$

Worker Problem

As explained in Section B.3, the worker problem is nearly identical as in the baseline model, which can be discretized and solved numerically.

Unemployment and Employment-Weighted Productivity Distribution

I solve for the stationary employment-weighted productivity distribution $P(z)$ and unemployment rate u by iterating over their laws of motion (i.e., equations (64) and (65)) until convergence. Let $P_n^{(b)}$, $\lambda_n^{(b)}$ be the discrete approximation of $P(z_n)^{(b)}$, $\lambda(z_n)^{(b)}$ at iteration b, where z_n is a point on the productivity grid. Given $\lambda_n^{(b)}$, $u^{(b)}$, $P_n^{(b)}$, I use equations (54) and (55) combined with numerical integration to approximate $\{Q_1^{(b)}, Q_2^{(b)}, \dots\}$ and $\Lambda^{(b)}$.

The updating equations, where (b) denotes the iteration, are

$$\begin{aligned}
u^{(b+1)} &= u^{(b)} + \Delta\big[(\delta+\chi_x)\big(1-u^{(b)}\big) - u^{(b)}\big(\mu_e + \Lambda^{(b)}\big(1-u^{(b)}\big)\big)\big],\\
P_n^{(b+1)} &= P_n^{(b)} + \Delta\Bigg[\big(1-u^{(b+1)}\big)\Lambda^{(b)}P_n^{(b)}\big(Q_n^{(b)}-1\big) + \frac{u^{(b+1)}}{1-u^{(b+1)}}\mu_e\Gamma_n + u^{(b+1)}\Lambda^{(b)}Q_n^{(b)}\\
&\qquad - (\delta+\chi_x)P_n^{(b)} + \chi_s\big(\Gamma_n - P_n^{(b)}\big)\Bigg],\\
\lambda_n^{(b+1)} &= \frac{1}{c}\Big(\big(u^{(b+1)} + \big(1-u^{(b+1)}\big)P_n^{(b+1)}\big)\Big)^{\frac{1}{\vartheta}} v_n,
\end{aligned}$$

where I set the time step Δ to 1/12 (i.e., monthly). I iterate until $\max\{|u^{(b+1)} - u^{(b)}|, |P_n^{(b+1)} - P_n^{(b)}|\} < 10^{-6}$.

Algorithm

Similar to the algorithm for the baseline model, the algorithm consists of iterating over (i) the firm problem, (ii) the worker problem, and (iii) the unemployment and employment-weighted productivity distribution. The algorithm is initiated (step $b = 0$) with a guess $(\underline{z}^{(0)}, \underline{w}^{(0)})$ and $\lambda^{(0)} = (\lambda(z_1)^{(0)}, \dots, \lambda(z_N)^{(0)})'$.

TABLE C.I

CHOICE OF PRODUCTIVITY DISTRIBUTION.

Distribution	Dispersion	Skewness
Data	1.547	−0.046
Gamma	1.547	−0.027
Pareto	1.547	+0.465

Note: Dispersion is defined as the interdecile range of log labor productivity $\log P90 - \log P10$; skewness is defined as the Kelley skewness index, which is defined as $(\log P90 - 2\log P50 + \log P10)/(\log P90 - \log P10)$. The notation $P10$, $P50$, and $P90$ represent the 10th, 50th, and 90th percentiles of the distribution of log labor productivity (net of 3-digit NAICS industry and year fixed effects).

C.3. *Calibration*

Choice of Productivity Distribution

I now discuss why I use the Gamma distribution to model the distribution of labor productivity across firms, rather than a Pareto distribution considered in Proposition 6. In short, the reason is that the Pareto distribution implies too much skewness.

The first row of Table C.I contains the dispersion and skewness of log labor productivity in the data (as well as the precise definition of "dispersion" and "skewness"). The dispersion is 1.547 (which is the empirical target in the calibration exercise) and the skewness is small and slightly negative (−0.046). The second row contains the dispersion and skewness of log labor productivity in the baseline model calibrated using a Gamma distribution (see Section 3.1 for the calibration strategy). Notice the the (non-targeted) skewness coefficient is small and negative (−0.027), as in the data. The last row contains the dispersion and skewness in the baseline model calibrated with a Pareto distribution (see Section 5.4 for a discussion of the calibration strategy). The key observation is that the skewness is large and positive (+0.465). Hence, the Gamma distribution appears to be a more appropriate functional form for the empirical distribution of labor productivity.

Model-Implied Moments (for Calibration)

The unemployment rate is obtained directly from (23) while the the interdecile range of labor productivity is computed numerically. I now derive the formulas used to compute the five other moments in the model.

To compute the job creation and destruction rates by continuing firms analytically, it will be useful to obtain an implicit equation for the productivity threshold z_0 such that $g(z_0) = 0$. Straightforward algebra implies that

$$g(z_0) = 0 \quad \Longleftrightarrow \quad P(z_0) = \frac{\lambda(1-u) + \delta - \lambda u}{2\lambda(1-u)}.$$

The job creation rate by continuing firms is defined as $\int_{z_0}^{\infty} g(z)\,\mathrm{d}P(z)$, and can be expressed as

$$\int_{z_0}^{\infty} g(z)\,\mathrm{d}P(z) = \big(1 - P(z_0)\big)\big(\lambda u - \lambda(1-u) - \delta\big) + \lambda(1-u)\big(1 - P(z_0)^2\big).$$

Similarly, the job destruction rate by continuing firms is

$$-\int_{\underline{z}}^{z_0} g(z)\,\mathrm{d}P(z) = -P(z_0)\big(\lambda u - \lambda(1-u) - \delta\big) - \lambda(1-u)P(z_0)^2.$$

The measure of jobs created by entering firms over a period $[t, t+1)$ is $u\mu_e$ and the measure of jobs created by exiting firms is $(1-u)\chi_x$, where μ_e, χ_x are defined in (20). Therefore, the job creation (destruction) by entrants (exiters) are $\frac{u}{1-u}\mu_e$ and χ_x, respectively. As discussed in Section 3.1, the continuous-time growth rates g_c are transformed into discrete-time growth rates g_d using the formula $g_d = e^{g_c} - 1$.

Finally, to derive the autocorrelation of log labor productivity, I first approximate the law of motion for labor productivity y_t as

$$\log y_{t+1} \approx \begin{cases} \log y_t & \text{with prob. } e^{-\chi_s}, \\ x & \text{with prob. } 1 - e^{-\chi_s}, \end{cases}$$

where x is drawn from the stationary distribution of $\log y_t$ and χ_s is defined in (20). Denoting the stationary mean of $\log y_t$ as μ, notice that the autocorrelation of $\log y_t$ is

$$\begin{aligned} &\frac{\mathbb{E}(\log y_{t+1} - \mu)(\log y_t - \mu)}{\mathbb{E}(\log y_t - \mu)^2} \\ &\quad = \frac{e^{-\chi_s}\mathbb{E}(\log y_t - \mu)^2 + (1 - e^{-\chi_s})\mathbb{E}(\log y_t - \mu)(x - \mu)}{\mathbb{E}(\log y_t - \mu)^2} = e^{\chi_s}. \end{aligned}$$

The last step uses the fact that $\mathbb{E}(\log y_t - \mu)(x - \mu) = 0$ since x is independent of $\log y_t$.

I now derive expressions for the pass-through and separation elasticities (see Section 3.2 for definitions and context). To be consistent with the empirical evidence, the pass-through elasticity is computed using the OLS formula (i.e., $\text{cov}(\log w(z), \log y(z))/\text{var}(\log y(z))$). Similarly, the separation elasticity is given by $\text{cov}(\log S(z), \log w(z))/\text{var}(\log w(z))$, where $S(z) = 1 - e^{-\lambda(1-u)P(z)-\delta}$.[24] In both cases, z is drawn from the employment-weighted productivity distribution $P(z)$.

Jacobian Matrix

Let $\Lambda(\theta) = (\Lambda_1(\theta), \dots, \Lambda_m(\theta), \dots \Lambda_7(\theta))$ be the vector of model-implied moments, where $\theta = (\theta_1, \dots, \theta_p, \dots, \theta_6)$ is the vector of model parameters. To quantify the sensitivity of the model-implied moments to the choice of parameters, I compute the numerical derivative

$$\left.\frac{\partial \Lambda_m(\theta)}{\partial \theta_p}\right|_{\theta=\theta^*},$$

for all $(m, p) \in \{1, \dots, 7\} \times \{1, \dots, 6\}$, where θ^* is the vector of calibrated parameters. Table C.II reports the results.

C.4. *Alternative Calibrations*

Table C.III reports the model fit in the benchmark models and extensions.

[24]Recall that the flow of separations is given by exogenous lay-offs, which occur at Poisson rate δ, and endogenous separations, which occur at Poisson rate $\lambda(1-u)P(z)$

TABLE C.II

SENSITIVITY OF MODEL-IMPLIED MOMENTS TO PARAMETERS.

	Parameter					
Moment	λ	χ	μ	η	b	δ
Unemployment rate	0.117	−0.878	0.332	−0.001	0.496	1.810
Autocorrelation of log labor prod.	0.102	−1.170	0.192	−0.001	0.208	0.030
Interdecile range of log labor prod.	−1.445	5.953	−2.723	−0.204	−2.951	−0.421
Job creation (continuers)	0.289	−0.054	0.020	0.000	0.030	−0.403
Job creation (entrants)	0.026	−0.241	0.161	0.000	0.140	0.608
Job destruction (continuers)	0.184	0.174	−0.066	0.000	−0.098	0.187
Job destruction (exiters)	0.128	−0.458	0.242	−0.001	0.263	0.037

APPENDIX D: DATA

D.1. *Variables Construction*

I now describe the methodology used to construct the main firm-level variables (i.e., value-added, employment, average wage, and capital stock). I start with four variables constructed by Statistics Canada that are based on corporate tax return line items: gross profits, worker compensation, tangible capital assets, and intangible capital assets. The last two variables (tangible capital assets and intangible capital assets) represent book values of assets net of accumulated depreciation. First, I construct value added as

$$\text{value added} \equiv \text{gross profits} + \text{worker compensation}.$$

This approach is consistent with the income approach to measuring GDP which, in the corporate sector, sums the income which accrues to firm owners (gross operating income) and the income which accrues to workers (worker compensation). Labor share and labor productivity are defined as

$$\text{labor share} \equiv \text{worker compensation}/\text{value added},$$

$$\text{labor productivity} \equiv \text{value added}/\text{employment},$$

TABLE C.III

TARGETED MOMENTS.

		Model				
Moment	Data	Baseline	CM	BM	E-λ	Pareto
Unemployment rate	0.071	0.071	0.071	0.071	0.075	0.069
Autocorrelation of log labor prod.	0.810	0.806	0.806	–	0.803	0.806
Interdecile range of log labor prod.	1.547	1.547	1.547	1.547	1.544	1.547
Job creation (continuers)	0.061	0.055	0.055	–	0.050	0.055
Job creation (entrants)	0.019	0.022	0.022	–	0.024	0.022
Job destruction (continuers)	0.067	0.062	0.062	–	0.059	0.062
Job destruction (exiters)	0.016	0.016	0.016	–	0.015	0.015
EU transition rate	0.047	–	–	0.047	–	–

Note: "CM" refers to Coles–Mortensen; "BM" refers to Burdett–Mortensen; "Pareto" refers to the baseline model with a Pareto distribution; "E-λ" refers to the endogenous-λ extension of the baseline model.

TABLE D.I
SUMMARY STATISTICS (2000–2015).

Variable	Observations	Mean	Std. deviation
Employment	3,084,182	38.6	571.1
Value-added	3,084,182	2480.0	45,579.6
Capital stock	3,084,182	2733.4	90,548.0
Compensation per worker	3,084,182	37.8	27.6
Labor productivity	3,084,182	52.8	185.2

Note: All variables except employment are in thousands of 2002 Canadian dollars using the CPI deflator.

where employment is obtained by averaging the monthly number of employees throughout the year. The capital stock is the sum of tangible and intangible capital assets measured at book value net of accumulated depreciation.

Finally, I winsorize the labor share at the 0.1% level in the upper tail (i.e., labor share values above the 99.9 percentile are replaced by the value of the 99.9 percentile, which is approximately 20). I also remove from the main sample firm-year observations that either have negative value-added or missing values in employment, value-added, tangible capital assets, intangible capital assets or industry code. Table D.I contains summary statistics.

D.2. *Sample Validation*

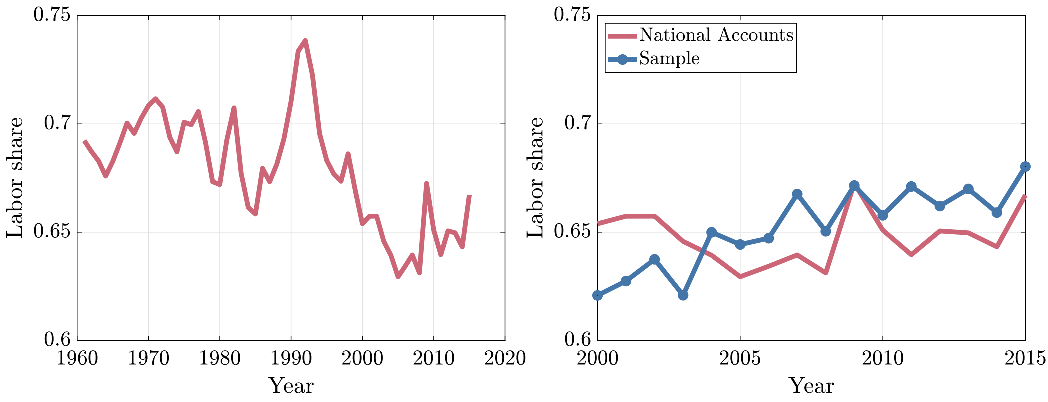
I now compare the aggregate labor share in the NALMF versus in the National Accounts. I use data from Statistics Canada (Table 380-0063) to compute the corporate sector labor share. As is standard, I assume that the components of income that are ambiguous (i.e., taxes net of subsidies and net mixed income) have a labor share equal to the aggregate labor share, which implies the following approximation:

$$\text{aggregate labor share} \approx \frac{\text{worker compensation}}{\text{worker compensation} + \text{gross operating surplus}}.$$

Figure D.1 plots the labor share in the National Accounts and in the main sample. A few remarks are in order. First, the Canadian labor share in the National Accounts has sustained a large decline over the course of the 1990s and early-2000s but has then somewhat recovered over the course of the 2005–2015 period. Second, the aggregate labor share in the main sample has a similar level and trend as the one in the National Account over the period where both data sets are available. Overall, the level and dynamics of the labor share in the NALMF is consistent with the aggregate data, except for the fact that it is too low in the first 4 years of the sample.

D.3. *Statistics by Labor Productivity Deciles*

Within each 2-digit NAICS industry-year bin, I sort firms by labor productivity (i.e., value-added per worker) and bin firms into deciles. Within each decile, I compute labor productivity, capital-output ratio, value-added share, and labor share. Finally, I average



(a) Aggregate labor share (b) Aggregate labor share

FIGURE D.1.—Labor share in the National Accounts and in the main sample.

over all industries within a year by applying value-added weights and compute a simple average of each variable over 2000–2015. Table D.II contains the resulting data. Note that labor productivity and capital-output ratio are expressed in relative term (i.e., as a ratio of the aggregate).

D.4. *Comparison Between Compustat and Census Data*

Table 3 in Barth, Bryson, Davis, and Freeman (2016) contains a measure of productivity dispersion in the U.S. The authors use data from the Census Bureau's Economic Census files and compute the variance of log revenue per worker in the cross-section of establishments. The data is based on quinquennial censuses of establishments, so they report values every 5 years. In Compustat, I compute the variance of log revenue per worker every 5 years from 1982 to 2007. I present the results in Table D.III. Overall, the levels are very comparable and both series show an increase in productivity dispersion despite differences in concept. The unit of observation in Compustat (as in the NALMF data) is a firm, while in Barth et al. (2016) it is an establishment. The main difference in trend appears to be from 2002 to 2007, where the increase in productivity dispersion is noticeably higher in Compustat.

TABLE D.II

MOMENTS BY LABOR PRODUCTIVITY DECILES.

	Labor productivity deciles									
Variable	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Labor productivity	0.164	0.354	0.434	0.518	0.596	0.692	0.783	0.973	1.278	2.889
Capital-output ratio	1.800	0.811	0.694	0.621	0.691	0.694	0.749	0.799	0.917	1.259
Value-added share	0.015	0.028	0.036	0.044	0.055	0.069	0.082	0.097	0.149	0.425
Labor share	1.578	1.126	0.954	0.911	0.858	0.802	0.761	0.742	0.667	0.439

TABLE D.III

PRODUCTIVITY DISPERSION IN COMPUSTAT VERSUS CENSUS DATA.

Years	1982	1987	1992	1997	2002	2007
Barth et al. (2016)	0.965	0.949	1.020	1.113	1.126	1.265
Compustat	0.950	1.024	1.048	1.185	1.162	1.474

D.5. *Industry-Level Data Set*

To construct the industry-level data set, I use the main sample (described in Appendix D.1) and sort firms by labor productivity within industry-year bins and assign to each firm a productivity quintile. An industry is defined according to 3-digit NAICS definitions. Within each industry-year bin, I compute the labor share and value-added shares. I restrict the sample to industry-year observations that have at least 100 firms in every year. The result is a balanced panel data set covering 69 industries over the 2000–2015 period. Table D.IV reports summary statistics.

APPENDIX E: QUANTIFYING THE THEORY: ROBUSTNESS CHECKS

E.1. *Assessing Cross-Sectional Moments*

Table E.I reproduces and extends the results presented in Table V. In specification (3), I instrument labor productivity with its 1-year lag. The coefficient on labor productivity goes up from −0.31 to −0.25. In specification (6), I regress labor share on labor productivity with the addition of the capital-output ratio as a control. The coefficient on labor productivity remains mostly unchanged and the coefficient on capital-output ratio is negative −0.025, as expected. Also, adding the capital-output output ratio increases the R^2 slightly, from 0.27 to 0.28. In specification (7), I use total worker compensation as an alternative measure of firm size (i.e., instead of employment). As with employment, the coefficient on size is close to zero and the R^2 is low.

The main takeaway is that differences in labor productivity across firms explain a large fraction of the variance in labor shares (the year and industry fixed effects explain less than 3% of the variance) and that, conditional on labor productivity, size, and capital-output ratio do not provide much additional predictive power. In the U.S. manufacturing sector, Kehrig and Vincent (2021) also find that differences in labor shares across firms are mostly explained by differences in labor productivity, rather than differences in capital intensity.

TABLE D.IV

SUMMARY STATISTICS (CROSS-INDUSTRY DATA SET).

Variable	Observations	Mean	Std. deviation
Labor share	1104	0.672	0.135
Productivity dispersion	1104	1.917	0.473

Note: Productivity dispersion is defined as the interdecile range of log labor productivity.

TABLE E.I

ASSESSING CROSS-SECTIONAL MOMENT (ADDITIONAL SPECIFICATIONS).

Labor share (log LS)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Value-added ($\log Y$)	−0.112						
	(0.001)						
Labor productivity ($\log y$)		−0.313	−0.248	−0.313		−0.320	
		(0.002)	(0.002)	(0.002)		(0.002)	
Employment ($\log N$)				0.009	−0.001		
				(0.001)	(0.001)		
Capital-output ($\log K/Y$)						−0.025	
						(0.000)	
Payroll ($\log wN$)							0.033
							(0.000)
Industry fixed effects	✓	✓	✓	✓	✓	✓	✓
Year fixed effects	✓	✓	✓	✓	✓	✓	✓
2SLS			✓				
Sample size	3,084,182	3,084,182	2,480,615	3,084,182	3,084,182	3,084,182	3,084,182
R^2	0.120	0.272	0.256	0.273	0.036	0.282	0.042

Note: Standard errors in parentheses are clustered at the firm level.

E.2. *Do High-Wage Firms Have a Lower Labor Share?*

In Section 3.2, I test (and find support for) a number of model predictions including (i) high-productivity firms have a lower labor share and (ii) high-productivity firms pay higher wages. Combining these two model predictions, one would also expect that that high-wage firms have a low labor share. However, when I estimate the cross-sectional relationship between labor share and wage in the data (where wage is defined as worker compensation per employee), I instead find that high-wage firms tend to have a slightly *higher* labor share.[25] In particular, the regression coefficient of log labor share on log wage (with year and industry fixed effects exactly as in Table V) is 0.11 in the data compared to −1.06 in the model (see the first row of Table E.II).

To understand why the model fails to match this particular moment, consider the following accounting decomposition. For any two random variables u and v, let $\beta_{u,v}$ denote the regression coefficient of the log of u on the log of v and let σ_u^2 denote the variance of the log of u.[26] Using the fact that the log of labor share is equal to the log of wage minus

TABLE E.II

WAGES AND LABOR SHARE IN THE CROSS-SECTION OF FIRMS.

Moment	Data	Model
$\beta_{\text{LS},w}$	0.11	−1.06
$\beta_{w,y}$	0.69	0.39
σ_y^2/σ_w^2	1.29	5.22

[25] Kehrig and Vincent (2021) also estimate a small (but negative) relationship between wage and labor share across establishments in the U.S. manufacturing sector (see their Figure VI).

[26] The regression coefficient is defined as $\beta_{u,v} \equiv \text{cov}(\log u, \log v)/\text{var}(\log v)$.

the log of labor productivity (i.e., $\log \mathrm{LS} = \log w - \log y$), we have that

$$\beta_{\mathrm{LS},w} = 1 - \beta_{w,y} \times \left(\sigma_y^2/\sigma_w^2\right).$$

Table E.II contains each component of the decomposition, both in the data (where all variables are net of year and industry fixed effects) and in the model. The regression coefficient of log wage on log productivity $\beta_{w,y}$ is 0.69 in the data, which is fairly close to the model's prediction of 0.39. In contrast, the ratio of variances σ_y^2/σ_w^2 is 1.29 in the data, while it is 5.22 in the model. Hence, the main reason why the model fails to match the empirical evidence on the relationship between productivity and wage $\beta_{\mathrm{LS},w}$ is because it underpredicts the variance of wages relative to the variance of labor productivity.

In the model, a single firm-level state variable (i.e., productivity z) determines the equilibrium wage that firms offer. In reality, there are other factors besides productivity that generate dispersion in average wages across firms, even within narrowly defined industry (i.e., the presence of a union, compensating wage differentials, etc.). To understand how such factors would bias the regression coefficient $\beta_{\mathrm{LS},w}$, consider the following stylized model extension. Suppose that wages are given by

$$w(z, \xi) = \xi w(z),$$

where $w(z)$ is the policy function in the baseline model and ξ is an exogenous wedge that is orthogonal to productivity z, has a cross-sectional mean of one, and has a log variance of σ_ξ^2. Under these assumptions, the regression coefficient of log labor share on log wages $\beta_{\mathrm{LS}(z,\xi),w(z,\xi)}$ is given by

$$\begin{aligned}\beta_{\mathrm{LS}(z,\xi),w(z,\xi)} &\equiv 1 - \beta_{w(z,\xi),y(z)} \times \left(\sigma_{y(z)}^2/\sigma_{w(z,\xi)}^2\right) \\ &= 1 - \beta_{w(z),y(z)} \times \left(\sigma_{y(z)}^2/\sigma_{w(z)}^2\right) \times \frac{\sigma_{w(z)}^2}{\sigma_{w(z)}^2 + \sigma_\xi^2}.\end{aligned}$$

A couple of remarks are in order. First, the presence of wedges ξ does not affect the regression coefficient of log wages on log labor productivity (i.e., $\beta_{w(z,\xi),y(z)} = \beta_{w(z),y(z)}$). This is a standard result: classical measurement errors in the independent variable does not bias the regression coefficient. Second, the ratio of variances has a multiplicative bias equal to $\sigma_w^2/(\sigma_w^2 + \sigma_\xi^2)$. For instance, as the the variance of the wedge σ_ξ^2 goes to infinity, the regression coefficient $\beta_{LS(z,\xi),w(z,\xi)}$ goes to one.

The key takeaway is that the presence of wedges ξ in the wage equation (i.e., unmodeled determinants of firm-level wages) biases the regression coefficient $\beta_{\mathrm{LS},w}$ (even its sign), but does not affect the estimated relationship between labor share and productivity $\beta_{\mathrm{LS},y}$. To match the relationship between labor share and wage in the data, one would need a richer model that generates more wage dispersion between firms.

E.3. *Moments by Labor Productivity Decile: Robustness Checks and Extensions*

I now describe various robustness checks and extensions of Figure 2 in Section 3.2, which plots the labor share by labor productivity decile.

Industry and Size Effects

In Figure E.1a, I group firms in the data into broad industry categories (i.e., goods producing, trade services, professional services, and other services). Similarly, Figure E.1b groups firms in the data into size groups (50–20, 20–100, 100–500, and 500+ employees). The main empirical findings hold within each of these groups: (i) there is a negative relationship between firm-level labor share and labor productivity and (ii) a large fraction of firms have a labor share above $1-\alpha$.

Measurement Error in Value-Added

To alleviate the concern that measurement error in value-added could generate a spurious negative relationship between labor share and labor productivity (see footnote 13), Figure E.1c reproduces Figure 2 by sorting firms according to their 1-year lagged labor productivity decile. The results are similar, the main difference being that the labor share of firms in the bottom decile of labor productivity is lower.

Benchmark Models and Endogenous-λ Extension

Figure E.1d plots the relationship between labor share and labor productivity in the calibrated benchmark models (i.e., Burdett–Mortensen and Coles–Mortensen). There are two takeaways. First, the Burdett–Mortensen and Coles–Mortensen models fail to match the high labor share (i.e., above $1-\alpha$) of low-productivity firms. Second, the labor share in the endogenous-λ model extension is very similar to the labor share in the baseline model. The main difference is that the labor share of high-productivity firms is higher (i.e., high-productivity firms exert less monopsony power).

Pareto Distribution

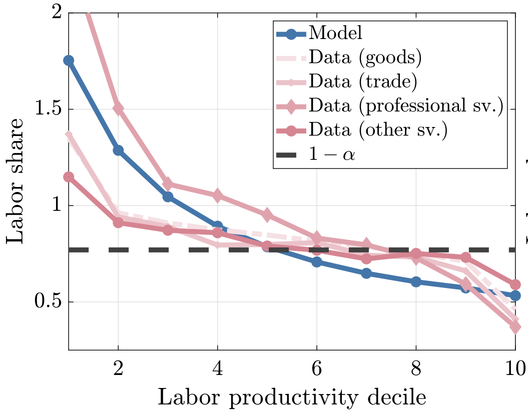
Figure E.1e plots the relationship between labor share and labor productivity in the baseline model calibrated with a Pareto distribution. The empirical fit is poor. In particular, the labor share in the bottom 9 deciles of labor productivity are much higher than in data. Moreover, the labor share in the top decile that is much too low.

Imputed Profit Share

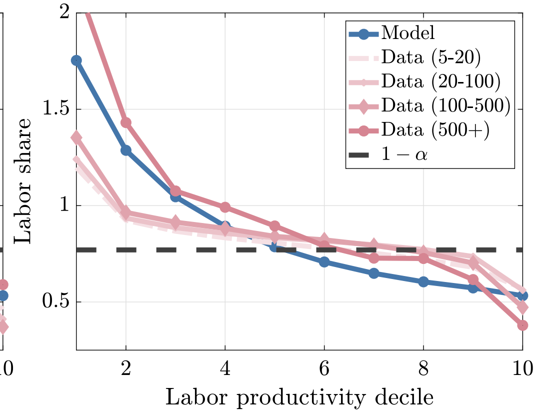
The empirical relationship between labor productivity and labor share could in principle be explained by the fact that firms differ in their capital share α, even within a narrowly-defined industry. Kehrig and Vincent (2021) find that, in the US manufacturing sector, differences in capital per worker explain a negligible fraction of the variation in labor shares across firms. I now conduct a similar exercise in my data. First, notice that if firms face the same user cost of capital R, the statistic that is informative about the relative capital share $R \times K/Y$ is the capital-output ratio K/Y. I impute the profit share Π/Y as

$$\underbrace{\Pi/Y}_{\text{profit share}} = \underbrace{1-\text{LS}}_{\text{nonlabor share}} - \underbrace{R \times K/Y}_{\text{capital share}}, \tag{72}$$

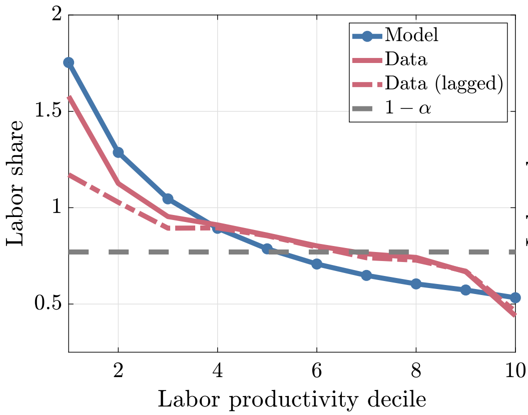
where both the nonlabor share $1-\text{LS}$ and the capital output ratio K/Y are directly observable. This approach has a long tradition in economics dating back to Jorgenson (1963) and has been recently used by Barkai (2020) using U.S. aggregate data. Figure E.1f provides a comparison of the (imputed) profit shares in the data and in the model. I find that the profit share increases with labor productivity. Both in the model and in the data, the bottom 5 deciles have a negative profit share while the top 5 have a positive one. Moreover, the qualitative results are not sensitive to the particular value of the user cost that I use. As a robustness check, I plot the profit shares in the data implied by a user cost of 0.11 and 0.31 instead of 0.21 as in the baseline model. The shape of the relationship between labor productivity and profit share remains mostly unchanged.



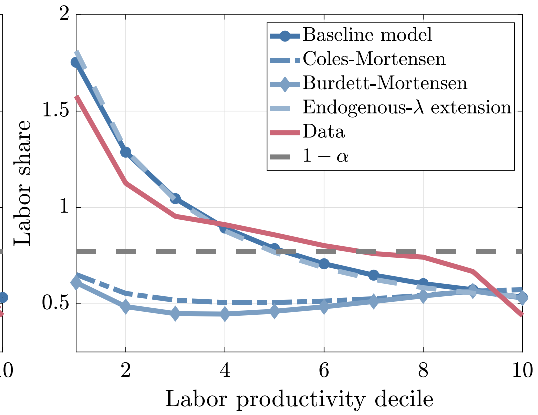
(a) Industry groups.



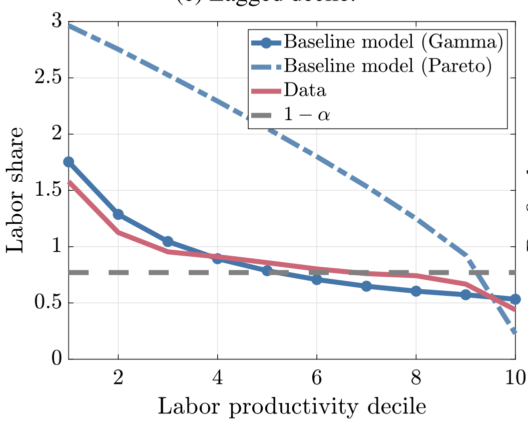
(b) Size groups.



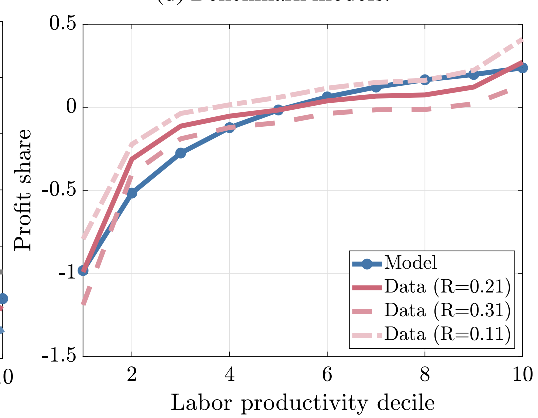
(c) Lagged decile.



(d) Benchmark models.



(e) Pareto distribution.



(f) Imputed profit share.

FIGURE E.1.—Moments by labor productivity decile: robustness checks and extensions.

TABLE E.III
CROSS-COUNTRY REGRESSIONS.

LS	(1)	(2)	(3)
σ	−0.077	−0.201	−0.119
	(0.026)	(0.035)	(0.085)
Year FE	✓	✓	
Industry FE	✓	✓	✓
Country FE		✓	
N	154	154	28
R^2	0.245	0.858	0.143

Note: Robust standard errors. "LS" denotes labor share; "σ" denotes productivity dispersion (i.e., the interdecile range of log labor productivity). For specifications (3), the windows are 2001–2006, 2006–2011, and 2010–2015. For specifications (4), the window is 2001–2011.

E.4. *Cross-Country Regressions*

In Table E.III, I estimate specifications (38), (39), and (40). In all specifications, I obtain negative coefficients ranging from −0.077 (specification 38) to −0.220 (specification 40 with 10-year differences). Despite the small sample size, I detect a clear negative relationship between labor share and productivity dispersion in all specifications.

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