SUPPLEMENT TO ‘PERSUASION MEETS DELEGATION’

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## APPENDIX A: COMPACTIFICATION IN DETERMINISTIC DELEGATION

Analogously to Section 5.2, this section shows that, under Inada-type assumptions, standard deterministic delegation and deterministic delegation with outside option can be represented as our deterministic delegation problem.

In standard delegation, the set of decisions is $Y_0 = (\underline{y}_0, \overline{y}_0)$. In delegation with outside option, the set of decisions is $Y_0 = [\underline{y}, \overline{y}_0)$, and the agent can always choose the outside option $\underline{y}$.

Say that a delegation set $B \subset Y_0$ is *undominated by* $V_0 : X \to \mathbb{R}$ if

$$V(y_B^*(x), x) \geq V_0(x), \quad \text{for some } x \in X.$$

Propositions $1'$ and $2'$ are the deterministic counterparts of Propositions 1 and 2.

PROPOSITION $1'$: *Suppose that* $Y_0 = (\underline{y}_0, \overline{y}_0) \subseteq \mathbb{R}$, *and*

$$\max_{x\in X} U(y,x) \to -\infty \quad \textit{and} \quad \max_{x\in X} V(y,x) \to -\infty \quad \textit{as } y \to \underline{y}_0 \textit{ and } y \to \overline{y}_0. \tag{56}$$

*For each continuous* $V_0$, *there exist* $\underline{y}, \overline{y} \in Y_0$ *(with* $u(\underline{y}, \underline{x}) < 0 < u(\overline{y}, \overline{x})$*) such that the following holds. For each delegation set* $B$ *that is undominated by* $V_0$, *there exists another delegation set* $\tilde{B} \subset Y = [\underline{y}, \overline{y}]$ *with* $\{\underline{y}, \overline{y}\} \subset \tilde{B}$ *such that* $y_{\tilde{B}}^*(x) = y_B^*(x)$ *for almost all* $x \in X$.

PROOF: Define

$$Z = \left\{ y \in Y_0 : \max_{x\in X} V(y,x) \geq \min_{x\in X} V_0(x) \right\}.$$

If $Z$ is empty, then every $B \subset Y_0$ is dominated by $V_0$, and the proposition holds trivially. Assume henceforth that $Z$ is nonempty. Let $\underline{z} = \inf Z$ and $\overline{z} = \sup Z$. By (56), compactness of $X$, and continuity of $V$ and $V_0$, we have $\underline{y}_0 < \underline{z} \leq \overline{z} < \overline{y}_0$.

Next, let $\varepsilon > 0$ and define

$$\tilde{Y} = \left\{ y \in Y_0 : \max_{x\in X} U(y,x) \geq \min_{x\in X,\, z\in[\underline{z},\overline{z}]} U(z,x) - \varepsilon \right\}.$$

As $[\underline{z}, \overline{z}] \subset \tilde{Y}$, $\tilde{Y}$ is nonempty. Let $\underline{y} = \inf \tilde{Y}$ and $\overline{y} = \sup \tilde{Y}$. By (56), compactness of $X$ and $[\underline{z}, \overline{z}]$, and continuity of $U$, we have $\underline{z} > \underline{y} > \underline{y}_0$ and $\overline{z} < \overline{y} < \overline{y}_0$. Let $Y = [\underline{y}, \overline{y}]$. Thus, each type of the agent strictly prefers every decision in $[\underline{z}, \overline{z}]$ to every decision outside of $Y$. For each $B \subset Y_0$ undominated by $V_0$, let $\tilde{B} = (B \cap Y) \cup \{\underline{y}, \overline{y}\}$. Then, by (50) and strict aggregate downcrossing of $u$ in $x$, we have $y_{\tilde{B}}^*(x) = y_B^*(x)$ for almost all $x \in X$. *Q.E.D.*

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PROPOSITION 2′: *Suppose that* $Y_0 = [\underline{y}, \overline{y}_0) \subset \mathbb{R}$ *and*

$$\max_{x \in X} U(y, x) \to -\infty \quad \textit{as } y \to \overline{y}_0. \tag{57}$$

*Then there exists* $\overline{y} \in Y_0$ *(with* $u(\overline{y}, \overline{x}) < 0$*) such that the following holds. For each delegation set* $B$ *with* $\underline{y} \in B$*, there exists another delegation set* $\tilde{B} \subset Y = [\underline{y}, \overline{y}]$ *with* $\{\underline{y}, \overline{y}\} \subset \tilde{B}$ *such that* $y^*_{\tilde{B}}(x) = y^*_B(x)$ *for almost all* $x \in X$.

PROOF: Let $\varepsilon > 0$ and define

$$\overline{y} = \inf \left\{ y \in Y_0 : \max_{x \in X} U(y, x) \geq \min_{x \in X} U(\underline{y}, x) - \varepsilon \right\}.$$

By (57), compactness of $X$ and continuity of $U$, we have $\underline{y} < \overline{y} < \overline{y}_0$. Let $Y = [\underline{y}, \overline{y}]$. Thus, each type of the agent strictly prefers decision $\underline{y}$ to every decision outside of $Y$. For each $B \subset Y_0$ with $\underline{y} \in B$, let $\tilde{B} = (B \cap Y) \cup \{\overline{y}\}$. Then, by (50) and strict aggregate downcrossing of $u$ in $x$, we have $y^*_{\tilde{B}}(x) = y^*_B(x)$ for almost all $x \in X$. *Q.E.D.*

## APPENDIX B: PROOFS

### B.1. *Proof of Lemma 4*

Consider a sequence of intervals $(\underline{x}_n, \overline{x}_n) \subset Y$ with $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$Q_n(x_{n+1}|x_n) = \begin{cases} \delta_{x_n}, & \text{if } x_n \notin (\underline{x}_n, \overline{x}_n), \\ \frac{\overline{x}_n - x_n}{\overline{x}_n - \underline{x}_n} \delta_{\underline{x}_n} + \frac{x_n - \underline{x}_n}{\overline{x}_n - \underline{x}_n} \delta_{\overline{x}_n}, & \text{if } x_n \in (\underline{x}_n, \overline{x}_n), \end{cases}$$

where $\delta_x$, with $x \in Y$, denotes the degenerate distribution at $x$. Let $H_1 = H$, and for each $n \in \mathbb{N}$, let $H_{n+1}(x_{n+1}) = \int Q_n(x_{n+1}|x_n) H_n(\mathrm{d}x_n)$. This construction gives a finitely supported conditional distribution $P_n(x_{n+1}|x)$ such that $\int P_n(x_{n+1}|x) H(\mathrm{d}x) = H_{n+1}(x_{n+1})$ for all $x_{n+1} \in Y$, and $\int x_{n+1} P_n(\mathrm{d}x_{n+1}|x) = x$ for all $x \in Y$. In the proof of their Theorem 4.1, Müller and Rüschendorf (2001) show that a sequence of intervals $(\underline{x}_n, \overline{x}_n)$ can be chosen in such a way that $P_n(\cdot|x)$ converges weakly to $P(\cdot|x)$ such that $\int P(y|x) H(\mathrm{d}x) = F(y)$ for all $y \in Y$, and $\int y P(\mathrm{d}y|x) = x$ for all $x \in Y$.

Since the likelihood ratio is closed with respect to weak convergence (e.g., Müller and Stoyan, 2002, Theorem 1.4.9), it remains to show that $P_n(x_{n+1}|x)$ increases in $x$ with respect to the likelihood ratio order. Since $P_n(x_{n+1}|x)$ has a finite support, by induction, it suffices to show that, for all intervals $(\underline{z}, \overline{z}) \subset Y$, all finite sets $Z \subset Y$, and all discrete probability densities $h_1(\cdot)$ and $h_2(\cdot)$ supported on $Z$ that are ordered with respect to the likelihood ratio order,

$$h_1(y_1) h_2(y_2) \geq h_2(y_1) h_1(y_2), \quad \text{for all } y_2 > y_1, \tag{58}$$

we have that discrete probability densities $\tilde{h}_l(\cdot)$, with $l = 1, 2$, supported on $\tilde{Z} = Z \cup \{\underline{z}, \overline{z}\} \setminus (\underline{z}, \overline{z})$ and defined by

$$\tilde{h}_l(y) = \begin{cases} \sum_{\tilde{z} \in Z \cap [\underline{z}, \overline{z}]} h_l(\tilde{z}) \frac{\overline{z} - \tilde{z}}{\overline{z} - \underline{z}}, & \text{if } y = \underline{z}, \\ h_l(y), & \text{if } y \notin [\underline{z}, \overline{z}], \\ \sum_{\tilde{z} \in Z \cap [\underline{z}, \overline{z}]} h_l(\tilde{z}) \frac{\tilde{z} - \underline{z}}{\overline{z} - \underline{z}}, & \text{if } y = \overline{z}, \end{cases} \tag{59}$$

are also ordered with respect to the likelihood ratio order,

$$\tilde{h}_1(y_1)\tilde{h}_2(y_2) \geq \tilde{h}_2(y_1)\tilde{h}_1(y_2), \quad \text{for all } y_2 > y_1,$$

This follows from direct calculations for all possible cases with $y_2 > y_1$. The only non-trivial case is where $y_1 = \underline{z}$ and $y_2 = \overline{z}$. In this case, we have

$$\begin{aligned}\tilde{h}_1(\underline{z})\tilde{h}_2(\overline{z}) - \tilde{h}_2(\underline{z})\tilde{h}_1(\overline{z}) &= \left(\sum_{\tilde{z}\in Z\cap[\underline{z},\overline{z}]} h_1(\tilde{z})\frac{\overline{z}-\tilde{z}}{\overline{z}-\underline{z}}\right)\left(\sum_{\tilde{z}\in Z\cap[\underline{z},\overline{z}]} h_2(\tilde{z})\frac{\tilde{z}-\underline{z}}{\overline{z}-\underline{z}}\right)\\ &\quad - \left(\sum_{\tilde{z}\in Z\cap[\underline{z},\overline{z}]} h_2(\tilde{z})\frac{\overline{z}-\tilde{z}}{\overline{z}-\underline{z}}\right)\left(\sum_{\tilde{z}\in Z\cap[\underline{z},\overline{z}]} h_1(\tilde{z})\frac{\tilde{z}-\underline{z}}{\overline{z}-\underline{z}}\right)\\ &= \sum_{y_1,y_2\in Z\cap[\underline{z},\overline{z}]:\ y_1<y_2} (h_1(y_1)h_2(y_2) - h_2(y_1)h_1(y_2))\frac{y_2-y_1}{\overline{z}-\underline{z}} \geq 0,\end{aligned}$$

where the first equality is by (59), the second equality is by rearrangement, and the inequality is by (58). *Q.E.D.*

### B.2. *Proofs of Claims 1–4*

PROOF OF CLAIM 1: It suffices to show that $H_{\hat{\pi}}(x) = H(x)$ and $U_{\hat{\pi}}(x) = \int_x^{\overline{y}}(1-H(\tilde{x}))\mathrm{d}\tilde{x}$ for all $x \in Y$. Indeed, for all $x \in Y$, we have

$$H_{\hat{\pi}}(x) = \int_Y (1-\hat{\pi}(y,x))F(\mathrm{d}y) = \int_Y P(x|y)F(\mathrm{d}y) = H(x),$$

where the first equality is by definition, the second equality is by (28), and the third equality is by (25) and Bayes' rule. Moreover, for all $x \in Y$, we have

$$\begin{aligned}U_{\hat{\pi}}(x) &= \int_Y (y-x)\hat{\pi}(y,x)F(\mathrm{d}y) = \int_Y (y-x)(1-P(x|y))F(\mathrm{d}y)\\ &= \int_Y (y-x)\left(\int_{(x,\overline{y}]} P(\mathrm{d}\tilde{x}|y)\right)F(\mathrm{d}y) = \int_{(x,\overline{y}]}\int_Y (y-x)P(\mathrm{d}y|\tilde{x})H(\mathrm{d}\tilde{x})\\ &= \int_{(x,\overline{y}]} (\tilde{x}-x)H(\mathrm{d}\tilde{x}) = \int_{(x,\overline{y}]} (1-H(\tilde{x}))\mathrm{d}\tilde{x},\end{aligned}$$

where the first equality is by the definition, the second equality is by (28), the third equality is by the Leibniz rule, the fourth equality is by Bayes' rule, the fifth equality is by (26), and the sixth equality is by integration by parts. *Q.E.D.*

PROOF OF CLAIM 2: Let $\pi_I$ satisfy (IC$_I$). Since $\pi_I$ is right-continuous in $t$, it follows that $\tilde{\pi}$ given by $\tilde{\pi}(y,x) = \pi_I(F(y),G(x))$ for all $(y,x) \in Y \times Y$ satisfies (IC) for all $x, \hat{x} \in [\underline{x}, \overline{x})$. By the standard argument, (IC) with $(x,\hat{x}) \in \{(x_1,x_2),(x_2,x_1)\}$ and $(\hat{a}_0,\hat{a}_1) = (0,1)$ yields

$$-(1-H_{\tilde{\pi}}(x_1))(x_2-x_1) \leq U_{\tilde{\pi}}(x_2) - U_{\tilde{\pi}}(x_1) \leq -(1-H_{\tilde{\pi}}(x_2))(x_2-x_1),$$
$$\text{for all } \underline{x} \leq x_1 < x_2 < \overline{x}.$$

Hence, $H_{\tilde{\pi}}$ is increasing on $[\underline{x}, \overline{x})$, and, by the envelope theorem,

$$U_{\tilde{\pi}}(x_2) - U_{\tilde{\pi}}(x_1) = -\int_{x_1}^{x_2} (1 - H_{\tilde{\pi}}(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } \underline{x} \le x_1 < x_2 < \overline{x}. \tag{60}$$

Since $H_{\tilde{\pi}}$ and $U_{\tilde{\pi}}$ are monotone on $[\underline{x}, \overline{x})$, the left limits $H_{\tilde{\pi}}(\overline{x}_-) = \lim_{x \uparrow \overline{x}} H_{\tilde{\pi}}(x)$ and $U_{\tilde{\pi}}(\overline{x}_-) = \lim_{x \uparrow \overline{x}} U_{\tilde{\pi}}(x)$ exist. Moreover, there exists a left-continuous function $\phi : Y \to [0,1]$ such that $H_{\tilde{\pi}}(\overline{x}_-) = \int_Y (1 - \phi(y))F(\mathrm{d}y)$ and $U_{\tilde{\pi}}(\overline{x}_-) = \int_Y (y - \overline{x})\phi(y)F(\mathrm{d}y)$.

Let

$$\underline{x}^* = \min\left\{x \in [\underline{y}, \underline{x}] : U_{\tilde{\pi}}(\underline{x}) + (1 - H_{\tilde{\pi}}(\underline{x}))(\underline{x} - x) \ge \int_Y (y - x)F(\mathrm{d}y)\right\},$$
$$\overline{x}^* = \max\left\{x \in [\overline{x}, \overline{y}] : U_{\tilde{\pi}}(\overline{x}_-) + (1 - H_{\tilde{\pi}}(\overline{x}_-))(\overline{x} - x) \ge 0\right\},$$

which are well-defined because $U_{\tilde{\pi}}(\underline{x}) \ge \int_Y (y - x)F(\mathrm{d}y)$ by (IC) with $(\hat{a}_0, \hat{a}_1) = (1,1)$ and $U_{\tilde{\pi}}(\overline{x}_-) \ge 0$ by (IC) with $(\hat{a}_0, \hat{a}_1) = (0,0)$. Consider $\pi$ given for each $(y,x) \in Y \times Y$ by

$$\pi(y,x) = \begin{cases} 1, & \text{if } x \in [\underline{y}, \underline{x}^*) \\ \tilde{\pi}(y, \underline{x}), & \text{if } x \in [\underline{x}^*, \underline{x}), \\ \tilde{\pi}(y, x), & \text{if } x \in [\underline{x}, \overline{x}), \\ \phi(y), & \text{if } x \in [\overline{x}, \overline{x}^*), \\ 0, & \text{if } x \in [\overline{x}^*, \overline{y}], \end{cases} \tag{61}$$

Since $\pi(y,x) = \tilde{\pi}(y,x)$ for all $(y,x) \in Y \times [\underline{x}, \overline{x})$, we have $H_\pi(x) = H_{\tilde{\pi}}(x)$ and $U_\pi(x) = U_{\tilde{\pi}}(x)$ for all $x \in [\underline{x}, \overline{x})$. By (61) and the monotonicity of $H_{\tilde{\pi}}$ on $[\underline{x}, \overline{x})$, $H_\pi$ is increasing on $Y$.

Next, we have:

$$U_\pi(x) = 0 = \int_x^{\overline{y}} (1 - H_\pi(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } x \in [\overline{x}^*, \overline{y}], \tag{62}$$

where the equalities are by the definition of $\pi$, $H_\pi$, and $U_\pi$;

$$U_\pi(x) = (\overline{x}^* - x)(1 - H_\pi(\overline{x})) = \int_x^{\overline{y}} (1 - H_\pi(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } x \in [\overline{x}, \overline{x}^*), \tag{63}$$

where the first equality is by the definition of $\pi$, $H_\pi$, $U_\pi$, and $\overline{x}^*$, and the second is by (62);

$$U_\pi(x) = U_\pi(\overline{x}) + \int_x^{\overline{x}} (1 - H_{\tilde{\pi}}(\tilde{x}))\mathrm{d}\tilde{x} = \int_x^{\overline{y}} (1 - H_{\tilde{\pi}}(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } x \in [\underline{x}, \overline{x}), \tag{64}$$

where the first equality is by (60) and the definition of $\pi(y, \overline{x})$, and the second is by (63);

$$U_\pi(x) = U_\pi(\underline{x}) + (\underline{x} - x)(1 - H(\overline{x})) = \int_x^{\overline{y}} (1 - H_{\tilde{\pi}}(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } x \in [\underline{x}^*, \underline{x}), \tag{65}$$

where the first equality is by the definition of $\pi$, $H_\pi$, and $U_\pi$, and the second is by (64);

$$U_\pi(x) = U_\pi(\underline{x}^*) + (\underline{x}^* - x)(1 - H(\underline{y})) = \int_x^{\overline{y}} (1 - H_{\tilde{\pi}}(\tilde{x}))\mathrm{d}\tilde{x}, \quad \text{for all } x \in [\underline{y}, \underline{x}^*), \tag{66}$$

where the first equality is by the definition of $\pi$, $H_\pi$, $U_\pi$, and $\underline{x}^*$, and the second is by (65);

$$U_\pi(\underline{y}) = \int_Y (y - \underline{y}) F(\mathrm{d}y) = \int_Y (1 - F(y))\mathrm{d}y, \tag{67}$$

where the first equality is by the definition of $\pi$ and $U_\pi$, and the second is by integration by parts. Since $\pi$ satisfies (24), it satisfies (IC) by Lemma 3. *Q.E.D.*

PROOF OF CLAIM 3: Let $j = 0, 1$ and let $\pi \in \Pi_j$. The principal's interim utility is

$$\begin{aligned} V_\pi(x) &= \alpha \int_x^{\overline{y}} (1 - H_\pi(\tilde{x}))\mathrm{d}\tilde{x} + (\alpha x - d(x))(1 - H_\pi(x)) \\ &= -\alpha x \int_{(x,\overline{y}]} H_\pi(\mathrm{d}\tilde{x}) + \alpha \int_{(x,\overline{y}]} \tilde{x} H_\pi(\mathrm{d}\tilde{x}) + (\alpha x - d(x)) \int_{(x,\overline{y}]} H_\pi(\mathrm{d}\tilde{x}) \\ &= \int_{(x,\overline{y}]} (\alpha\tilde{x} - d(x)) H_\pi(\mathrm{d}\tilde{x}), \quad \text{for all } x \in X, \end{aligned} \tag{68}$$

where the first equality is by (22), Lemmas 1 and 3, and Claim 2, the second equality is by integration by parts, and the last equality is by rearrangement. So, the principal's expected utility is

$$\begin{aligned} W(\pi) &= \int_X \left( \int_{(x,\overline{y}]} (\alpha\tilde{x} - d(x)) H_\pi(\mathrm{d}\tilde{x}) \right) g(x)\mathrm{d}x \\ &= \int_X \left( \int_{\underline{x}}^{\tilde{x}} (\alpha\tilde{x} - d(x)) g(x)\mathrm{d}x \right) H_\pi(\mathrm{d}\tilde{x}) + \int_{(\overline{x},\overline{y}]} \left( \int_X (\alpha\tilde{x} - d(x)) g(x)\mathrm{d}x \right) H_\pi(\mathrm{d}\tilde{x}) \\ &= \int_Y \nu(\tilde{x}) H_\pi(\mathrm{d}\tilde{x}), \end{aligned}$$

where the first equality is by (68) and the definition of $W(\pi)$, the second equality is by Fubini's theorem, and the third equality is by the definition of $\nu$. *Q.E.D.*

PROOF OF CLAIM 4: Let $j = 0, 1$ and let $\pi \in \Pi_j$.

*If.* Suppose that $H_\pi$ maximizes $\int_Y \nu(x) H(\mathrm{d}x)$ over distributions $H$ that satisfy (MPS). We show that the principal's expected utility is higher under $\pi$ than under any $\tilde{\pi} \in \Pi_j$. As $\tilde{\pi} \in \Pi_j$, it satisfies ($IC_0$)–($IC_2$). Observe that $H_\pi$ and $H_{\tilde{\pi}}$ satisfy (MPS) by Lemmas 1 and 3, Claim 2, and condition (23). The claim follows from

$$W(\tilde{\pi}) = \int_Y \nu(x) H_{\tilde{\pi}}(\mathrm{d}x) \leq \int_Y \nu(x) H_\pi(\mathrm{d}x) = W(\pi),$$

where the equalities hold by Claim 3, and the inequality holds by the optimality of $H_\pi$. Hence $\pi$ is optimal on $\Pi_j$.

*Only if.* Suppose that $\pi$ is optimal on $\Pi_j$. Consider any distribution $H$ on $Y$ that satisfies (MPS). By Lemma 1 and Claim 1, there exists a (monotone) delegation mechanism $\hat{\pi}$ on $Y \times Y$ with $H_{\hat{\pi}} = H$ that satisfies ($IC_0$)–($IC_2$) for all $x, \hat{x} \in Y$. We have

$$\int_Y \nu(x) H_\pi(\mathrm{d}x) = W(\pi) \geq W(\hat{\pi}) = \int_Y \nu(x) H_{\hat{\pi}}(\mathrm{d}x) = \int_Y \nu(x) H(\mathrm{d}x),$$

where the first and second equalities are by Claim 3, the inequality is by the optimality of $\pi$, and the third equality is by $H_{\hat{\pi}} = H$. Thus, $H_\pi$ maximizes $\int_Y \nu(x) H(\mathrm{d}x)$ over distributions $H$ that satisfy (MPS). *Q.E.D.*

### B.3. *Proofs of Propositions 1 and 2*

To simplify notation, we prove Propositions 1 and 2 using the original decision variable $s$ with $S = \mathbb{R}$ in standard delegation and $S = [\underline{s}, \infty)$ in delegation with outside option. When we refer to constraints ($\text{IC}_0$)–($\text{IC}_2$) and conditions (33)–(34), it is understood that they are expressed in variable $s$ rather than $y$.

We first introduce some notations and prove a lemma. Let $k : X \to \mathbb{R}$ and $\ell : X \to \mathbb{R}$ be continuous functions. Define

$$Z = \left\{ s \in S : \max_{x \in X} \big( k(x)s - \ell(x) - C(s) \big) \geq 0 \right\}, \qquad \underline{z} = \inf Z, \quad \overline{z} = \sup Z. \tag{69}$$

Note that $\underline{z}$ and $\overline{z}$ can be infinite if $Z$ is unbounded or empty. Also, define

$$\Lambda = \left\{ \lambda \in \Delta(S) : \max_{x \in X} \mathbb{E}_\lambda \big[ k(x)s - \ell(x) - C(s) \big] \geq 0 \right\}. \tag{70}$$

CLAIM 6: *If $\underline{z}$ and $\overline{z}$ are finite, and*

$$\max_{x \in X} \big( k(x)\underline{z} - \ell(x) - C(\underline{z}) \big) = 0 \quad \textit{and} \quad \max_{x \in X} \big( k(x)\overline{z} - \ell(x) - C(\overline{z}) \big) = 0, \tag{71}$$

*then, for each $\lambda \in \Lambda$, there exists $\hat{\lambda} \in \Lambda$ such that $\operatorname{supp}(\hat{\lambda}) \subset [\underline{z}, \overline{z}]$, $\mathbb{E}_{\hat{\lambda}}[s] = \mathbb{E}_\lambda[s]$, and $\mathbb{E}_{\hat{\lambda}}[C(s)] = \mathbb{E}_\lambda[C(s)]$.*

PROOF: Let $L(s) = \max_{x \in X} \big( k(x)s - \ell(x) \big)$. For all $s \in S$ and all $\lambda \in \Delta(S)$, we have

$$\begin{aligned} &\max_{x \in X} \big( k(x)s - \ell(x) - C(s) \big) = L(s) - C(s), \\ &\max_{x \in X} \mathbb{E}_\lambda \big[ k(x)s - \ell(x) - C(s) \big] = L(\mathbb{E}_\lambda[s]) - \mathbb{E}_\lambda[C(s)]. \end{aligned} \tag{72}$$

Fix $\lambda \in \Lambda$. We have

$$0 \leq L(\mathbb{E}_\lambda[s])] - \mathbb{E}_\lambda[C(s)] \leq L(\mathbb{E}_\lambda[s])] - C(\mathbb{E}_\lambda[s]),$$

where the first inequality is by (72) and $\lambda \in \Lambda$, and the second inequality is by the convexity of $C$. Then, by (72) and the definition of $\underline{z}$ and $\overline{z}$, $\mathbb{E}_\lambda[s] \in [\underline{z}, \overline{z}]$. Moreover, if $\mathbb{E}_\lambda[s] = \underline{z}$ or $\mathbb{E}_\lambda[s] = \overline{z}$, then $\lambda = \delta_z$, so $\hat{\lambda} = \lambda$ is as required. Assume henceforth that $\underline{z} < \mathbb{E}_\lambda[s] < \overline{z}$. Let

$$\theta = \frac{\mathbb{E}_\lambda[s] - \underline{z}}{\overline{z} - \underline{z}} \quad \text{and} \quad \tau = \frac{(1-\theta)C(\underline{z}) + \theta C(\overline{z}) - \mathbb{E}_\lambda[C(s)]}{(1-\theta)C(\underline{z}) + \theta C(\overline{z}) - C(\mathbb{E}_\lambda[s])}.$$

We claim that $\tau \in [0,1]$. Indeed, the numerator is smaller than the denominator because $C$ is convex. Moreover, the denominator is strictly positive because $\underline{z} < \mathbb{E}_\lambda[s] < \overline{z}$ and $C$ is strictly convex. Finally, the numerator is positive, because

$$\begin{aligned}(1-\theta)C(\underline{z})+\theta C(\overline{z})-\mathbb{E}_{\lambda}[C(s)] &= (1-\theta)L(\underline{z})+\theta L(\overline{z})-\mathbb{E}_{\lambda}[C(s)] \\ &\geq (1-\theta)L(\underline{z})+\theta L(\overline{z})-L(\mathbb{E}_{\lambda}[s])\geq 0,\end{aligned}$$

where the equality is by (71), the first inequality is by the definition of $\Lambda$, and the second inequality is by the convexity of $L(s)$ and the definition of $\theta$, which implies that $(1-\theta)\underline{z}+\theta\overline{z}=\mathbb{E}_{\lambda}[s]$.

Let $\hat{\lambda}\in\Lambda$ be given by

$$\hat{\lambda}=\tau\delta_{\mathbb{E}_{\lambda}[s]}+(1-\tau)(1-\theta)\delta_{\underline{z}}+(1-\tau)\theta\delta_{\overline{z}}.$$

By construction, $\operatorname{supp}(\hat{\lambda})\subset[\underline{z},\overline{z}]$. Moreover, by the definition of $\theta$ and $\tau$, we have $\mathbb{E}_{\hat{\lambda}}[s]=\mathbb{E}_{\lambda}[s]$ and $\mathbb{E}_{\hat{\lambda}}[C(s)]=\mathbb{E}_{\lambda}[C(s)]$. *Q.E.D.*

PROOF OF PROPOSITION 1: Let $k(x)=d(x)/\alpha$ and $\ell(x)=V_0(x)/\alpha$. By (29), we have

$$V(s,x)-V_0(x)=\alpha\big(k(x)s-\ell(x)-C(s)\big).$$

Observe that $Z\subset\mathbb{R}$ and $\Lambda\subset\Delta(\mathbb{R})$, given by (69) and (70), are the sets of deterministic and stochastic decisions that are undominated by $V_0$.

We now show that $\underline{z}$ and $\overline{z}$, given by (69), are finite. If $Z$ is empty, that is, $V(s,x)-V_0(x)<0$ for all $s\in\mathbb{R}$ and all $x\in X$, then there is no mechanism that is undominated by $V_0$, and the proposition holds trivially. Assume henceforth that $Z$ is nonempty, and thus $\underline{z}\leq\overline{z}$. By (33) and continuity of $d$ and $V_0$, we have $\underline{z}>-\infty$ and $\overline{z}<\infty$.

Next, define

$$\underline{s}^*=\inf\left\{s\in\mathbb{R}:\max_{x\in X,\,z\in[\underline{z},\overline{z}]}(U(s,x)-U(z,x))\geq 0\right\},$$

$$\overline{s}^*=\sup\left\{s\in\mathbb{R}:\max_{x\in X,\,z\in[\underline{z},\overline{z}]}(U(s,x)-U(z,x))\geq 0\right\}.$$

By (33) and compactness of $[\underline{z},\overline{z}]$, we have $\underline{z}\geq\underline{s}^*>-\infty$ and $\overline{z}\leq\overline{s}^*<\infty$. Let $S^*=[\underline{s}^*,\overline{s}^*]$. By the definition of $S^*$, each type of the agent strictly prefers every decision in $[\underline{z},\overline{z}]$ — and, thus, every lottery over $[\underline{z},\overline{z}]$ — to every decision $s\notin S^*$.

Let $\pi$ be a mechanism that satisfies ($IC_0$) and is undominated by $V_0$. Because $\underline{z}$ and $\overline{z}$ are finite and $S=\mathbb{R}$, condition (71) of Claim 6 is satisfied. Thus, by Claim 6, there exists a mechanism $\tilde{\pi}$ such that $\operatorname{supp}(\tilde{\pi})\subset[\underline{z},\overline{z}]\subset S^*$, and

$$\mathbb{E}_{\tilde{\pi}(\cdot|x)}[s]=\mathbb{E}_{\pi(\cdot|x)}[s]\quad\text{and}\quad\mathbb{E}_{\tilde{\pi}(\cdot|x)}[C(s)]=\mathbb{E}_{\pi(\cdot|x)}[C(s)],\qquad\text{for all } x\in X.$$

By (29), equalities (31) hold for all $x\in X$. Finally, let $\varepsilon_1,\varepsilon_2>0$ be such that $c(\underline{s}^*-\varepsilon_1)<\underline{x}$ and $c(\overline{s}^*+\varepsilon_2)>\overline{x}$, and let $Y=[\underline{y},\overline{y}]=[c(\underline{s}^*-\varepsilon_1),c(\overline{s}^*+\varepsilon_2)]$. By the definition of $\overline{s}^*$ and $\underline{s}^*$ and (33), $\tilde{\pi}$ satisfies ($IC_1$) and ($IC_2$) with strict inequalities. *Q.E.D.*

PROOF OF PROPOSITION 2: Let $k(x)=x$ and $\ell(x)=x\underline{s}-C(\underline{s})$. Thus, by (29), we have

$$U(s,x)-U(\underline{s},x)=k(x)s-\ell(x)-C(s). \tag{73}$$

Observe that $Z \subset [\underline{s}, \infty]$ and $\Lambda \subset \Delta([\underline{s}, \infty))$, given by (69) and (70), are the sets of deterministic and stochastic decisions that are preferred to the outside option $\underline{s}$ by at least one type of the agent.

Let $\pi$ be a mechanism that satisfies ($IC_0$)–($IC_1$). Observe that $\underline{z} = \underline{s}$, and, by (34), we have $\overline{z} < \infty$. Then, by $S = [\underline{s}, \infty)$, (69), and (73), the condition (71) of Claim 6 is satisfied. Thus, by Claim 6, there exists a mechanism $\tilde{\pi}$ such that $\operatorname{supp}(\tilde{\pi}) \subset [\underline{z}, \overline{z}]$, and

$$\mathbb{E}_{\tilde{\pi}(\cdot|x)}[s] = \mathbb{E}_{\pi(\cdot|x)}[s] \quad \text{and} \quad \mathbb{E}_{\tilde{\pi}(\cdot|x)}[C(s)] = \mathbb{E}_{\pi_D(\cdot|x)}[C(s)], \qquad \text{for all } x \in X.$$

By (29), equalities (31) holds for all $x \in X$. Finally, let $\varepsilon > 0$ be such that $c(\overline{z} + \varepsilon) > \overline{x}$, and let $Y = [\underline{y}, \overline{y}] = [c(\underline{z}), c(\overline{z} + \varepsilon)]$. By the definition of $\overline{z}$ and (34), $\tilde{\pi}$ satisfies ($IC_2$) with strict inequality. *Q.E.D.*

## B.4. *Proof of Theorem 3*

Let $j = 0, 1$ and let $\pi \in \Pi_j$. We first prove three simple claims.

CLAIM 7: *Let $\nu' \in \boldsymbol{\nu}'$. If $p$ satisfies (40), then it is continuous and convex.*

PROOF: Let $p$ be given by (40). Then $|\nu'(x)| \le L$ for all $x \in Y$ with $L \in \mathbb{R}$ given by

$$L = \sup_{x \in Y} |(\alpha x - d(x))g(x) + \alpha G(x)|,$$

where $L \in \mathbb{R}$, because $d$ is continuous, $g$ is càdlàg, and $Y$ is compact. Hence, by (40), for each $y \in Y$, there exists a converging sequence $x_n$ in $\operatorname{supp}(H_\pi)$, with converging $\nu'(x_n)$, such that

$$p(z) \ge \lim_{n \to \infty} (\nu(x_n) + \nu'(x_n)(z - x_n)), \quad \text{for all } z \in Y, \text{ with equality at } z = y. \tag{74}$$

Then, $p$ is continuous, because, by (74), for all $z \in Y$,

$$\begin{aligned} p(y) - p(z) &\le \lim_{n \to \infty} (\nu(x_n) + \nu'(x_n)(y - x_n) - \nu(x_n) - \nu'(x_n)(z - x_n)) \\ &= \lim_{n \to \infty} \nu'(x_n)(y - z) \le L|y - z|. \end{aligned}$$

Also, $p$ is convex, since, by (74), for all $z, z' \in Y$ and all $\rho \in [0, 1]$ with $\rho z + (1 - \rho) z' = y$,

$$p(y) - \rho p(z) - (1 - \rho) p(z') \le \lim_{n \to \infty} \nu'(x_n)(y - \rho z - (1 - \rho) z') = 0. \qquad \textit{Q.E.D.}$$

CLAIM 8: *If $p$ is convex and satisfies (41) and (42), then, for all distributions $H$ on $Y$ that satisfy (MPS), we have*

$$\int_Y \nu(x) H(\mathrm{d}x) \le \int_Y p(x) H(\mathrm{d}x) \le \int_Y p(y) F(\mathrm{d}y) = \int_Y \nu(x) H_\pi(\mathrm{d}x). \tag{75}$$

PROOF: Let $p$ be convex and satisfy (41) and (42), and let $H$ be a distribution that satisfies (MPS). The first inequality holds by (41), the second inequality holds because $p$ is convex and $H$ satisfies (MPS), and the equality holds by (42). *Q.E.D.*

CLAIM 9: *If $p$ is continuous and convex, and satisfies (41) and (42), then there exists a selection $\nu'$ from $\boldsymbol{\nu}'$ such that $p_\pi$ given by (40) satisfies $p(y) \ge p_\pi(y)$ for all $y \in Y$.*

PROOF: Let $p$ be continuous and convex, and satisfy (41) and (42). Observe that for $H = H_\pi$, all inequalities in (75) hold with equality. Hence, by the continuity of $\nu$ and $p$, we have

$$p(x) = \nu(x), \quad \text{for all } x \in \text{supp}(H_\pi). \tag{76}$$

Fix $x \in \text{supp}(H_\pi)$ such that $x < \overline{y}$. For all $y \in (x, \overline{y}]$ and all $\varepsilon \in (0,1]$, we have

$$\frac{p(y) - p(x)}{y - x} \geq \frac{p(x + \varepsilon(y - x)) - p(x)}{\varepsilon(y - x)} \geq \frac{\nu(x + \varepsilon(y - x)) - \nu(x)}{\varepsilon(y - x)},$$

where the first inequality is by the convexity of $p$ and the second inequality is by (41) and (76). Taking the limit $\varepsilon \downarrow 0$ implies that

$$\frac{p(y) - p(x)}{y - x} \geq \nu'(x_+), \quad \text{for all } y \in (x, \overline{y}].$$

Since $p$ is convex, taking the limit $y \downarrow x$ implies that $p'(x_+)$ is well defined and satisfies $p'(x_+) \geq \nu'(x_+)$. By a symmetric argument, for all $x \in \text{supp}(H_\pi)$ such that $x > \underline{y}$, we have that $p'(x_-)$ is well defined and satisfies $p'(x_-) \leq \nu'(x_-)$.

If $x = \underline{y} \in \text{supp}(H_\pi)$, then

$$p(y) \geq p(\underline{y}) + p'(\underline{y}_+)(y - \underline{y}) \geq \nu(\underline{y}) + \nu'(\underline{y})(y - \underline{y}), \quad \text{for all } y \in Y,$$

where the first inequality is by the convexity of $p$ and the second inequality is by (41) and $p'(\underline{y}_+) \geq \nu'(\underline{y}_+) = \nu'(\underline{y})$. Similarly, if $x = \overline{y} \in \text{supp}(H_\pi)$, then

$$p(y) \geq p(\overline{y}) + p'(\overline{y}_-)(y - \overline{y}) \geq \nu(\overline{y}) + \nu'(\overline{y})(y - \overline{y}), \quad \text{for all } y \in Y.$$

Finally, consider $x \in \text{supp}(H_\pi)$ such that $x \in (\underline{y}, \overline{y})$. By the convexity of $p$, we have $p'(x_-) \leq p'(x_+)$. In both cases $\nu'(x_-) < \nu'(x_+)$ and $\nu'(x_-) \geq \nu'(x_+)$, the inequalities $p'(x_-) \leq \nu'(x_-)$, $p'(x_+) \geq \nu'(x_+)$, and $p'(x_-) \leq p'(x_+)$ imply that there exists $\nu'(x) \in \boldsymbol{\nu}'(x)$ such that $p'(x_-) \leq \nu'(x) \leq p'(x_+)$. Then, by the convexity of $p$ and (41), we have

$$p(y) \geq p(x) - p'(x_-)(x - y) \geq \nu(x) - \nu'(x)(x - y), \quad \text{for all } y < x,$$
$$p(y) \geq p(x) + p'(x_+)(y - x) \geq \nu(x) + \nu'(x)(y - x), \quad \text{for all } y > x.$$

In sum, there exists a selection $\nu'$ from $\boldsymbol{\nu}'$ such that

$$p(y) \geq \nu(x) + \nu'(x)(y - x), \quad \text{for all } x \in \text{supp}(H_\pi) \text{ and all } y \in Y.$$

Thus, $p_\pi$ given by (40) satisfies $p \geq p_\pi$. *Q.E.D.*

We now prove Theorem 3.

*If.* Suppose that there exists a selection $\nu'$ from $\boldsymbol{\nu}'$ such that $p$ given by (40) satisfies (41) and (42). Then $\pi$ is optimal by Claims 4, 7, and 8.

*Only if.* Suppose that $\pi$ is optimal. By Claim 4, $H_\pi$ maximizes $\int_Y \nu(x) H(\mathrm{d}x)$ over distributions $H$ that satisfy (MPS). Thus, since $\nu$ given by (36) is Lipschitz continuous, Theorem 2 in Dworczak and Martini (2019) implies that there exists a continuous and convex function $p$ on $Y$ that satisfies (41) and (42). Next, by Claim 9, there exists a selection $\nu'$ from $\boldsymbol{\nu}'$ such that $p_\pi$ given by (40) satisfies $p(y) \geq p_\pi(y)$ for all $y \in Y$. Then,

$$\int_Y p(y) F(\mathrm{d}y) = \int_Y \nu(x) H_\pi(\mathrm{d}x) \leq \int_Y p_\pi(x) H_\pi(\mathrm{d}x) \leq \int_Y p_\pi(y) F(\mathrm{d}y) \leq \int_Y p(y) F(\mathrm{d}y),$$

where the equality holds by (42), the first inequality holds because $p_\pi(x) \geq \nu(x)$ for all $x \in \operatorname{supp}(H_\pi)$ by (40), the second inequality holds because $p_\pi$ is convex by Claim 7 and $H_\pi$ satisfies (MPS), and the last inequality holds by $p \geq p_\pi$. So all inequalities hold with equality. Thus, $p = p_\pi$, by the continuity of $p$ and $p_\pi$. *Q.E.D.*

### B.5. *Proof of Remark 2*

Let $j = 0, 1$ and let $\pi \in \Pi_j$ be monotone. The marginal distributions of $J_\pi$ are $F$ and $H_\pi$, because

$$
\begin{aligned}
J_\pi(y, \overline{y}) &= \int_{\underline{y}}^{y} (1 - \pi(\tilde{y}|\overline{y})) F(\mathrm{d}\tilde{y}) = \int_{\underline{y}}^{y} F(\mathrm{d}\tilde{y}) = F(y), \quad \text{for all } y \in Y, \\
J_\pi(\overline{y}, x) &= \int_{\underline{y}}^{\overline{y}} (1 - \pi(y|x)) F(\mathrm{d}y) = H_\pi(x), \quad \text{for all } x \in Y,
\end{aligned}
\tag{77}
$$

where the first equalities in both lines hold by (43), the second equality in the first line holds by ($\mathrm{IC}_1$), and the second equality in the second line holds by (38).

Consider $\pi_P$ given by $\pi_P(x|y) = \pi(y|x)$ for all $y \in Y$ and all $x \in Y$. By assumption, $\pi(y|x)$ is a monotone delegation mechanism, so it is increasing and left-continuous in $y$ and decreasing and right-continuous in $x$, and satisfies $\pi(\tilde{y}|\overline{y}) = 0$ by ($\mathrm{IC}_1$). Thus, $\pi_P$ is a monotone persuasion mechanism. By the definition of persuasion mechanisms and (43),

$$
J_\pi(y, x) = \int_{\underline{y}}^{y} (1 - \pi_P(x|\tilde{y})) F(\mathrm{d}\tilde{y}) = \mathbb{P}(\text{state} < y, \text{decision} \leq x), \quad \text{for all } (y, x) \in Y \times Y.
$$

By Lemmas 1 and 2 and Claim 2, for all $\hat{x} \in Y$ and $H_\pi$-almost all $x \in Y$, we have

$$
\int_Y \left( \int_{\underline{y}}^{x} (y - \tilde{x}) \mathrm{d}\tilde{x} \right) \pi_P(\mathrm{d}y|x) \geq \int_Y \left( \int_{\underline{y}}^{\hat{x}} (y - \tilde{x}) \mathrm{d}\tilde{x} \right) \pi_P(\mathrm{d}y|x). \tag{78}
$$

Condition (78) implies the first-order condition

$$
\int_Y (y - x) \pi_P(\mathrm{d}y|x) = 0, \quad \text{for } H_\pi\text{-almost all } x \in Y.
$$

Then, for all functions $\phi : Y \to \mathbb{R}$, we have

$$
\int_{Y \times Y} \phi(x)(y - x) J_\pi(\mathrm{d}y, \mathrm{d}x) = \int_X \phi(x) \int_Y (y - x) \left(-\pi_P(\mathrm{d}y|x)\right) H_\pi(\mathrm{d}x) = 0. \tag{79}
$$

Fix $p \in P_\pi$. Then

$$
\begin{aligned}
\int_Y \nu(x) H_\pi(\mathrm{d}x) &= \int_{Y \times Y} \nu(x) J_\pi(\mathrm{d}y, \mathrm{d}x) = \int_{Y \times Y} (\nu(x) + \nu'(x)(y - x)) J_\pi(\mathrm{d}y, \mathrm{d}x) \\
&\leq \int_{Y \times Y} p(y) J_\pi(\mathrm{d}y, \mathrm{d}x) = \int_Y p(y) F(\mathrm{d}y),
\end{aligned}
\tag{80}
$$

where the first and last equalities are by (77), the second equality is by (79), and the inequality is by (40). Thus, (42) holds if and only if the inequality holds with equality, which is equivalent to (45).

Let $j=0,1$, let $\pi \in \Pi_j$ be deterministic, and let $B \subset Y$ be a corresponding compact delegation set. Then an extension of $\pi$ from $Y \times X$ to $Y \times Y$ that satisfies ($IC_0$)–($IC_2$) for all $x, \hat{x} \in Y$ is given by

$$\pi(y|x) = \mathbf{1}\{x_B^*(y) > x\}, \quad \text{for all } (y,x) \in Y \times Y. \tag{81}$$

Then, by (38), (43) and (81),

$$H_\pi(x) = \int_Y \mathbf{1}\{x_B^*(y) \le x\} F(\mathrm{d}y) \quad \text{and} \quad J_\pi(y,x) = \int_{\underline{y}}^{y} \mathbf{1}\{x_B^*(\tilde{y}) \le x\} F(\mathrm{d}\tilde{y}),$$
$$\text{for all } y \in Y \text{ and all } x \in Y.$$

Since $f(y) > 0$ for all $y \in Y$ and $H_\pi$ is a distribution of $x_B^*(y)$ where $y$ has distribution $F$, we have

$$x_B^*(y) \in \operatorname{supp}(H_\pi), \quad \text{for all } y \in Y. \tag{82}$$

We thus obtain

$$\begin{aligned}\int_Y \nu(x) H_\pi(\mathrm{d}x) &= \int_{Y\times Y} \nu(x) J_\pi(\mathrm{d}y, \mathrm{d}x) = \int_{Y \times Y} (\nu(x) + \nu'(x)(y-x)) J_\pi(\mathrm{d}y, \mathrm{d}x) \\ &= \int_Y (\nu(x_B^*(y)) + \nu'(x_B^*(y))(y - x_B^*(y))) F(\mathrm{d}y) \le \int_Y p(y) F(\mathrm{d}y),\end{aligned} \tag{83}$$

where the first equality is by (77), the second equality is by (79), the third equality is because $J_\pi$ is a joint distribution of $(y, x_B^*(y))$ where $y$ has distribution $F$, and the inequality is by (40) and (82). Thus, (42) holds if and only if inequality (83) holds with equality, which is equivalent to (46) by (40) and the continuity of $p$ on $Y$. *Q.E.D.*

### B.6. *Proof of Corollary 1*

We first prove Corollary 1 for delegation with outside option, and then explain how the proof changes in standard delegation.

*Only if.* Suppose that delegation set $\{\underline{y}\} \cup [y^*, \overline{y}_0)$ is optimal. As follows from Proposition 2, there exists $Y = [\underline{y}, \overline{y}] \subset [\underline{y}, \overline{y}_0)$, such that the agent's best response for all $x \in X$ is the same under delegation sets $\{\underline{y}\} \cup [y^*, \overline{y}_0)$ and $B = \{\underline{y}\} \cup [y^*, \overline{y}]$. By (44), we have

$$x_B^*(y) = \begin{cases} y, & \text{if } y \ge y^*, \\ z^*, & \text{if } y < y^*, \end{cases} \quad \text{where} \quad z^* = \frac{1}{F(y^*) - F(\underline{y})} \int_{\underline{y}}^{y^*} F(\mathrm{d}y).$$

Hence, by (46), we have

$$p(y) = \begin{cases} \nu(y), & \text{for all } y \in (y^*, \overline{y}], \\ \nu(z^*) + \nu'(z^*)(y - z^*), & \text{for all } y \in [\underline{y}, y^*]. \end{cases} \tag{84}$$

We thus obtain (a) by (40) and Claim 7, and (b) by (41).

*If.* Suppose that conditions (a) and (b) hold. Then, $p$ given by (84) satisfies (40), (41), and (46), so delegation set $\{\underline{y}\} \cup [y^*, \overline{y}_0)$ is optimal.

In standard delegation, for the *only if* part, we instead apply Proposition 1 with $V_0$ given by (32) and observe that $z^* \in (\underline{y}, \underline{x})$, because the agent with any type $x \in X$ strictly prefers $y^*$ to $\underline{y}$. The rest of the proof is the same. *Q.E.D.*

### B.7. *Proof of Corollary 2*

Let $B \in \mathcal{B}$ be a monotone partition in persuasion. Since $u_P$ is strictly aggregate downcrossing in $x$, $x^*_B(y)$ given by (49) is uniquely defined for all $y \in Y$. Redefine $x^*_B(\underline{y}) = \underline{x}$, which is w.l.o.g. because state $y = \underline{y}$ occurs with zero probability and is always revealed, as $\underline{z}_B(\underline{y}) = \overline{z}_B(\underline{y}) = \underline{y}$ by definition. The corresponding persuasion mechanism is given by

$$\pi_P(x|y) = \mathbf{1}\{x^*_B(y) > x\}, \quad \text{for all } y \in Y \text{ and all } x \in X.$$

Observe that (i) $\pi_P$ is left-continuous in $y$, because partition intervals $(\underline{z}_B(y), \overline{z}_B(y)]$ are closed on the right; (ii) $\pi_P$ is right-continuous in $x$ because $\pi_P$ is defined using a strict inequality; (iii) $\pi_P$ satisfies the normalization $\pi_P(x|\underline{y}) = 0$ for all $x \in X$ because $x^*_B(\underline{y}) = \underline{x}$; (iv) $\pi_P$ is monotone, because $u_P$ is upcrossing in $y$ and thus $x^*_B(y)$ is increasing in $y$; (v) $\pi_P$ is incentive-compatible, because the agent's decision is optimal for each partition element, except possibly for $\{\underline{y}\}$. In sum, mechanism $\pi_P$ is monotone, deterministic, and incentive-compatible.

Consider deterministic mechanisms $\pi_D$ and $\pi_I$ given by $\pi_D(y|x) = \pi_I(y,x) = \pi_P(x|y)$ for all $y \in Y$ and all $x \in X$. Note that

$$\pi_I(y,x) = \pi_D(y|x) = \mathbf{1}\{y^*(x) < y\}, \quad \text{for all } y \in Y \text{ and all } x \in X,$$

$$\text{where} \quad y^*(x) = \begin{cases} \inf\{y \in Y : x^*_B(y) > x\}, & \text{if } x < x^*_B(\overline{y}), \\ \overline{y}, & \text{if } x \geq x^*_B(\overline{y}). \end{cases}$$

By Theorem 1, $\pi_D$ and $\pi_I$ are incentive-compatible and satisfy $W_P(\pi_P) = W_D(\pi_D) = W_I(\pi_I)$. Since $\pi_D$ is incentive-compatible, we have, for almost all $x \in X$,

$$U_D(y^*(x), x) \geq \max\{\sup_{\hat{x} \in X} U_D(y^*(\hat{x}), x), U_D(\underline{y}, x), U_D(\overline{y}, x)\} = \max_{y \in B} U_D(y, x),$$

and thus $y^*(x)$ satisfies (50). Next, by strict aggregate downcrossing of $u_D$ in $x$, $y^*_B$ that satisfies (50) is uniquely defined for almost all $x$, so $y^* = y^*_B$ almost everywhere. Similarly, since $\pi_I$ is incentive-compatible, we have, for almost all $x \in X$,

$$\int_{y^*(x)}^{\overline{y}} u_I(y,x) f(y) \mathrm{d}y \geq \max_{(a_0, a_1, b) \in \{0,1\}^2 \times B} a_0 \int_{\underline{y}}^{b} u_I(y,x) f(y) \mathrm{d}y + a_1 \int_{b}^{\overline{y}} u_I(y,x) f(y) \mathrm{d}y,$$

and thus $a^*(y,x) = \mathbf{1}\{y > y^*(x)\}$ satisfies (51). Next, by strict aggregate downcrossing of $u_I$ in $x$, for $a^*_B$ that satisfies (51), we have $a^* = a^*_B$ almost everywhere. *Q.E.D.*

### B.8. *Proof of Proposition 3 and Remark 3*

By Corollary 2, it suffices to show that if (52) holds, then full disclosure maximizes the principal's expected utility in the equivalent persuasion problem where the utilities are given by (47). Under full disclosure, each state $y \in Y$ is revealed, so the principal chooses decision $x^*(y)$. As decisions $x \notin [x^*(\underline{y}), x^*(\overline{y})]$ can never be chosen, w.l.o.g., assume that $X = [\underline{x}, \overline{x}] = [x^*(\underline{y}), x^*(\overline{y})]$. For each $x \in X$, let $y^*(x) \in Y$ be such that $u(y^*(x), x) = 0$.

CLAIM 10: *Define a function* $q : X \to \mathbb{R}$ *by*

$$q(x) = \begin{cases} 0, & \text{if } x = \underline{x} \text{ and } u(\underline{y}, \underline{x}) < 0 \text{ or } x = \overline{x} \text{ and } u(\overline{y}, \overline{x}) > 0, \\ -\frac{v(y^*(x), x) g(x)}{u_x(y^*(x), x)}, & \text{otherwise}. \end{cases} \tag{85}$$

*Condition* (52) *holds iff*

$$E(y,x) = V_P(y,x^*(y)) - V_P(y,x) - q(x)u(y,x) \geq 0, \text{ for all } y \in Y \text{ and all } x \in X. \quad (86)$$

PROOF: Suppose that (86) holds. By rearrangement, for all $y_1, y_2 \in Y$ and all $x \in X$ such that $u(y_1,x) < 0 < u(y_2,x)$, we have

$$\frac{V_P(y_2,x^*(y_2)) - V_P(y_2,x)}{u(y_2,x)} \geq q(x) \geq \frac{V_P(y_1,x) - V_P(y_1,x^*(y_1))}{-u(y_1,x)}, \quad (87)$$

yielding (52) by (47). Conversely, suppose that (52) holds. There are four cases to consider depending on whether $u(\underline{y},\underline{x})$ and $u(\overline{y},\overline{x})$ are equal to 0. By symmetry, it suffices to consider the case $u(\underline{y},\underline{x}) < 0 = u(\overline{y},\overline{x})$. Fix any $x \in [\underline{x},\overline{x})$. By (47) and (52), we have

$$\inf_{y_2 > y^*(x)} \frac{V_P(y_2,x^*(y_2)) - V_P(y_2,x)}{u(y_2,x)} \geq \sup_{y_1 < y^*(x)} \frac{V_P(y_1,x) - V_P(y_1,x^*(y_1))}{-u(y_1,x)}. \quad (88)$$

Hence there exists $q(x) \in \mathbb{R}$ bounded above by the left-hand side of (88) and below by the right-hand side of (88), so (87) holds for all $y_1, y_2 \in Y$ such that $u(y_1,x) < 0 < u(y_2,x)$. If $x = \underline{x}$, the right-hand side of (87) is 0, so (87) holds with $q(\underline{x}) = 0$. If $x \in (\underline{x},\overline{x})$, L'Hôpital's rule for $y_2 \downarrow y^*(x)$ and $y_1 \uparrow y^*(x)$ implies that $q(x) = -v(y^*(x),x)g(x)/u_x(y^*(x),x)$. Rearranging (87) yields $E(y,x) \geq 0$ for all $y \neq y^*(x)$, and thus for all $y \in Y$ by continuity in $y$. Moreover, by continuity in $x$, we have $E(y,\overline{x}) \geq 0$ for all $y \in Y$. *Q.E.D.*

For each incentive-compatible persuasion mechanism $\pi_P$, define

$$J_{\pi_P}(y,x) = \mathbb{P}(\text{state} < y, \text{decision} \leq x) = \int_{\underline{y}}^{y} (1 - \pi_P(x|\tilde{y}))F(\mathrm{d}\tilde{y}), \quad \text{for all } (y,x) \in Y \times X.$$

If (52) holds, then so does (86), by Claim 10. Then, the principal gets a higher expected utility under full disclosure than under $\pi_P$, because

$$\begin{aligned}\int V_P(y,x)(-\pi_P(\mathrm{d}x|y)F(\mathrm{d}y) &= \int V_P(y,x)J_{\pi_P}(\mathrm{d}y,\mathrm{d}x)\\ &= \int (V_P(y,x) + q(x)u(y,x))J_{\pi_P}(\mathrm{d}y,\mathrm{d}x)\\ &\leq \int V_P(y,x^*(y))J_{\pi_P}(\mathrm{d}y,\mathrm{d}x) = \int V_P(y,x^*(y))F(\mathrm{d}y),\end{aligned}$$

where the first and last inequalities are by the definition of $J_{\pi_P}$, the second equality is by incentive compatibility of $\pi_P$ and the definition of $q(\underline{x})$ and $q(\overline{x})$ in (85), and the inequality is by (86).

We now prove Remark 3. Suppose that $u(y,x) = y - x$ and (53) holds. There are four cases to consider depending on whether $\underline{y}$ and $\overline{y}$ are equal to $\underline{x}$ and $\overline{x}$. By symmetry, it suffices to consider the case $\underline{y} < \underline{x}$ and $\overline{y} = \overline{x}$. By (85), $q(x) = v(x,x)g(x)$ for $x \in (\underline{x},\overline{x}]$ and $q(\underline{x}) = 0$. Denote $\kappa = \min_{y,x \in Y \times X} v_y(y,x)$. Note that $E(x,x) = 0$. By Claim 10, (52) holds if $E(y,x) \geq$

0 for all $(y,x)$. If $x=\underline{x}$, then $E(y,\underline{x})=0$ for $y\leq\underline{x}$ and $E(y,\underline{x})\geq 0$ for $y>\underline{x}$, because

$$E_y(y,\underline{x})=v(y,y)g(y)+\int_{\underline{x}}^{y} v_y(y,\tilde{x})g(\tilde{x})\mathrm{d}\tilde{x}\geq v(y,y)g(y)+\kappa G(y)-\kappa G(\underline{x})$$
$$\geq v(y,y)g(y)+\kappa G(y)-\kappa G(\underline{x})-v(\underline{x},\underline{x})g(\underline{x})\geq 0,$$

where the first inequality is by the definition of $\kappa$, and the second and third inequalities are by (53). If $x\in(\underline{x},\overline{x}]$ and $y\in(\underline{x},\overline{x}]$, then $E(y,x)\geq 0$ because, for $y\geq(\leq)x$, we have

$$E_y(y,x)=v(y,y)g(y)+\int_{x}^{y} v_y(y,\tilde{x})g(\tilde{x})\mathrm{d}\tilde{x}-v(x,x)g(x)$$
$$\geq(\leq)v(y,y)g(y)+\kappa G(y)-\kappa G(x)-v(x,x)g(x)\geq(\leq)0,$$

where the first inequality is by the definition of $\kappa$, and the second inequality is by (53). Finally, if $x\in(\underline{x},\overline{x}]$ and $y\in[\underline{y},\underline{x}]$, then $E(y,x)\geq 0$ because

$$E_y(y,x)=-\int_{\underline{x}}^{x} v_y(y,\tilde{x})g(\tilde{x})\mathrm{d}\tilde{x}-v(x,x)g(x)\leq\kappa G(\underline{x})-\kappa G(x)-v(x,x)g(x)$$
$$\leq v(\underline{x},\underline{x})g(\underline{x})+\kappa G(\underline{x})-\kappa G(x)-v(x,x)g(x)\leq 0,$$

where the first inequality is by the definition of $\kappa$, and the second and third inequalities are by (53). *Q.E.D.*

### B.9. *Proof of Proposition 4*

By Corollary 2, it suffices to characterize optimal monotone partitions in the equivalent persuasion problem where

$$U_P(s,t)=c(s)t-\frac{t^2}{2}\quad\text{and}\quad V_P(s,t)=e(c(s))t-\frac{\beta t^2}{2}. \tag{89}$$

A monotone partition $B\in\mathcal{B}$ is represented by a countable set of pooling intervals $(\underline{b}_i,\overline{b}_i]$. The remaining states $\tilde{B}=S\setminus\left(\bigcup_i(\underline{b}_i,\overline{b}_i]\right)$ are revealed.

Let $m_i=\int_{\underline{b}_i}^{\overline{b}_i} c(s)\mathrm{d}s/(\overline{b}_i-\underline{b}_i)$. Note that the left derivative of $\eta_B$ is

$$\eta_B'(s)=\begin{cases} e(c(s))-\frac{\beta c(s)}{2}, & \text{if } s\in\tilde{B},\\ \frac{1}{\overline{b}_i-\underline{b}_i}\int_{\underline{b}_i}^{\overline{b}_i}\left(e(c(\tilde{s}))-\frac{\beta c(\tilde{s})}{2}\right)\mathrm{d}\tilde{s}, & \text{if } s\in(\underline{b}_i,\overline{b}_i].\end{cases} \tag{90}$$

We have

$$
\begin{aligned}
W_P(B) &= \int_{\tilde{B}} V_P(s,s)\mathrm{d}s + \sum_i \int_{\underline{b}_i}^{\bar{b}_i} V_P(s,m_i)\,\mathrm{d}s \\
&= \int_{\tilde{B}} \left( e(c(s))c(s) - \frac{\beta(c(s))^2}{2} \right) \mathrm{d}s + \sum_i \int_{\underline{b}_i}^{\bar{b}_i} \left( e(c(s))m_i - \frac{\beta m_i^2}{2} \right) \mathrm{d}s \\
&= \int_{\tilde{B}} c(s)\eta_B(\mathrm{d}s) + \sum_i \int_{\underline{b}_i}^{\bar{b}_i} c(s)\eta_B(\mathrm{d}s) = \int_S c(s)\eta_B(\mathrm{d}s) \\
&= \eta_B(1) - \int_S \eta_B(s)c(\mathrm{d}s) \leq \eta(1) - \int_S \operatorname{conv}\eta(s)c(\mathrm{d}s),
\end{aligned} \tag{91}
$$

where the first equality is by the definition of $W_P$, the second equality is by (89), the third equality is by (90) and $\int_{\underline{b}_i}^{\bar{b}_i} m_i \mathrm{d}s = \int_{\underline{b}_i}^{\bar{b}_i} c(s)\mathrm{d}s$, the fourth equality is by $\tilde{B} \cup \left(\bigcup_i (\underline{b}_i, \bar{b}_i]\right) = S$, the fifth equality is by integration by parts and normalizations $c(0) = 0$ and $c(1) = 1$, and the inequality is by the definitions of $\eta_B$ and $\operatorname{conv}\eta$, and by $\eta_B(1) = \eta(1)$. Finally, let $B^*$ be such that $\operatorname{conv}\eta = \eta_{B^*}$. That is, $\operatorname{conv}\eta$ is linear on $(\underline{b}_i^*, \bar{b}_i^*]$ for each $i$ and $\operatorname{conv}\eta = \eta$ on $\tilde{B}^*$. By (91) and the strict monotonicity of $c$, we have $W_P(B) \leq W_P(B^*)$ for $B \in \mathcal{B}$, with equality if and only if $\eta_B = \operatorname{conv}\eta$. *Q.E.D.*

### B.10. *Proofs of Claim 5 and Corollary 3*

PROOF OF CLAIM 5: Suppose that $U(1,x) \geq U(0,x)$, that is, $1 - x \leq p(1)$. Then

$$
\frac{\partial V(s,x)}{\partial s} = \beta p(s) - (1-\beta)p'(s)s - \beta(1-x) \geq \beta p(s) - (1-\beta)p'(s)s - \beta p(1) \geq 0,
$$

where the first inequality is by $1 - x \leq p(1)$, and the second inequality is by $p(s) \geq p(1)$, $p'(s) \leq 0$, and $\beta \in [0,1]$. *Q.E.D.*

PROOF OF COROLLARY 3: *Part (1).* We have

$$
\begin{aligned}
\nu'(y) &= \left(\beta - \frac{k}{k+1}\right) G(y) - \frac{k}{k+1} g(y)y, \\
\nu''(y) &= g(y)y \left( \left(\beta - \frac{2k}{k+1}\right) \frac{1}{y} - \left(\frac{k}{k+1}\right) \frac{g'(y)}{g(y)} \right).
\end{aligned}
$$

Given (1), the expression in the parentheses is increasing. Corollary 1 implies that there exists $s^* \in [0,1]$ such that delegation set $\{0\} \cup [s^*, 1]$ is optimal.

*Part (2).* We have

$$
\begin{aligned}
\eta'(s) &= e(c(s)) - \frac{\beta}{2}c(s) = \frac{\beta}{2}(1 - p(s)) + \left(1 - \frac{\beta}{2}\right) p'(s)s, \\
\eta''(s) &= (1-\beta)p'(s) + \left(1 - \frac{\beta}{2}\right) p''(s)s = p'(s) \left(1 - \beta + \left(1 - \frac{\beta}{2}\right) \frac{p''(s)s}{p'(s)}\right).
\end{aligned}
$$

Given (2), the expression in the parentheses is decreasing, and $p'(s) < 0$. Proposition 4 implies that there exists $s^* \in [0,1]$ such that delegation set $\{0\} \cup [s^*, 1]$ is optimal. *Q.E.D.*

## APPENDIX C: COUNTEREXAMPLES

### C.1. *Failure of Equivalence Without Single-Crossing Utilities*

First, we show that Lemma 1 does not hold if $u_D(s,t)$ is not upcrossing in $s$. Consider a delegation problem with $(u_D, v_D)$ given by

$$u_D(s,t) = \begin{cases} 1, & (s,t) \in [0, \frac{1}{3}] \times [\frac{1}{2}, 1], \\ -1, & (s,t) \in (\frac{1}{3}, 1] \times [\frac{1}{2}, 1], \\ 0, & \text{otherwise}, \end{cases} \quad v_D(s,t) = \begin{cases} 3, & (s,t) \in (\frac{1}{3}, 1] \times [0, \frac{1}{2}), \\ -1, & \text{otherwise}. \end{cases}$$

Note that $u_D(s,t)$ is not upcrossing in $s$. Let

$$\pi_D(s,t) = \begin{cases} 1, & (s,t) \in (\frac{1}{3}, 1] \times [0, \frac{1}{2}), \\ 0, & \text{otherwise}. \end{cases}$$

Mechanism $\pi_D$ satisfies ($\text{IC}_D$). However, in the discriminatory disclosure problem with $(u_I, v_I) = (u_D, v_D)$, mechanism $\pi_I = \pi_D$ violates ($\text{IC}_I$). Indeed, the agent with a type $t \in [1/2, 1]$ strictly prefers to misreport his type and choose the action opposite to the recommendation. Moreover, the principal's expected utility of $1$ attained by incentive-compatible delegation mechanism $\pi_D$ is not attained by any incentive-compatible disclosure mechanism.

Second, we show that Lemma 2 does not hold if $u_I(s,t)$ is not aggregate downcrossing in $t$. Consider a discriminatory disclosure problem with $(u_I, v_I)$ given by

$$u_I(s,t) = \begin{cases} -1, & (s,t) \in [0, \frac{1}{3}] \times [\frac{1}{2}, 1], \\ 1, & (s,t) \in (\frac{1}{3}, 1] \times [\frac{1}{2}, 1], \\ 0, & \text{otherwise}, \end{cases} \quad v_I(s,t) = \begin{cases} 3, & (s,t) \in [0, \frac{1}{3}] \times [0, \frac{1}{2}), \\ -1, & \text{otherwise}. \end{cases}$$

Note that $u_I(s,t)$ is not aggregate downcrossing in $t$ (although it satisfies a weaker notion of aggregate downcrossing defined by Karlin and Rubin, 1956). Let

$$\pi_I(s,t) = \begin{cases} 1, & (s,t) \in (0, \frac{1}{3}] \times [0, \frac{1}{2}), \\ \frac{1}{2}, & (s,t) \in (\frac{1}{3}, 1] \times [0, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Mechanism $\pi_I$ satisfies ($\text{IC}_I$). However, in the persuasion problem with $(u_P, v_P) = (u_I, v_I)$, mechanism $\pi_P = \pi_I$ violates ($\text{IC}_P$). Indeed, when $s \in (1/3, 1]$, the agent is recommended decision $t = 0$ with probability $1/2$. But, conditional on this recommendation, the agent infers that $s \in (1/3, 1]$, in which case he strictly prefers $t = 1$. Moreover, the principal's expected utility of $1/6$ attained by incentive-compatible disclosure mechanism $\pi_I$ is not attained by any incentive-compatible persuasion mechanism.

### C.2. *Suboptimality of Monotone Mechanisms*

First, we show that the principal's expected utility can be strictly higher under non-monotone disclosure and persuasion mechanisms than under any delegation mechanism. Consider a discriminatory disclosure problem with $(u_I, v_I)$ given by

$$u_I(s,t) = 0, \;\; (s,t) \in [0,1] \times [0,1], \;\; \text{and} \;\; v_I(s,t) = \begin{cases} 1, & (s,t) \in [0, \frac{1}{2}] \times [0, 1], \\ -1, & (s,t) \in (\frac{1}{2}, 1] \times [0, 1]. \end{cases}$$

The principal's maximum utility of $1/2$ is attained by the first-best disclosure mechanism

$$\pi_I(s,t)=\begin{cases}1, & (s,t)\in(0,\frac{1}{2}]\times[0,1),\\ 0, & \text{otherwise.}\end{cases}$$

In the persuasion problem with $(u_P,v_P)=(u_I,v_I)$, mechanism $\pi_P=\pi_I$ also maximizes the principal's expected utility. Now consider the delegation problem with $(u_D,v_D)=(u_I,v_I)$. Note that $\pi_D(s|t)=\pi_I(s,t)$ is not a well-defined delegation mechanism, because $\pi_I$ is not increasing in $s$. Moreover, since $V_D(s,t)=\max\{-s,s-1\}\leq 0$ for all $(s,t)$, the principal's expected utility in delegation is at most 0.

Second, we show that the principal's expected utility can be strictly higher under non-monotone disclosure and delegation mechanisms than under any persuasion mechanism. Consider a discriminatory disclosure problem with $(u_I,v_I)$ given by

$$u_I(s,t)=0,\ \ (s,t)\in[0,1]\times[0,1],\ \ \text{and}\ \ v_I(s,t)=\begin{cases}-1, & (s,t)\in[0,1]\times[0,\frac{1}{2}),\\ 1, & (s,t)\in[0,1]\times[\frac{1}{2},1].\end{cases}$$

The principal's maximum utility of $1/2$ is attained by the first-best disclosure mechanism

$$\pi_I(s,t)=\begin{cases}1, & (s,t)\in(0,1]\times[\frac{1}{2},1),\\ 0, & \text{otherwise.}\end{cases}$$

In the delegation problem with $(u_D,v_D)=(u_I,v_I)$, mechanism $\pi_D=\pi_I$ also maximizes the principal's expected utility. Now consider the persuasion problem with $(u_P,v_P)=(u_I,v_I)$. Note that $\pi_P(t|s)=\pi_I(s,t)$ is not a well-defined persuasion mechanism, because $\pi_I$ is not decreasing in $t$. Moreover, since $V_P(s,t)=\max\{-t,t-1\}\leq 0$ for all $(s,t)$, the principal's expected utility in persuasion is at most 0.

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