Supplement to "On optimal inference in the linear IV model": Online Supplementary Material 1

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11. Outline

References to sections, theorems, and lemmas with section numbers less than 11 refer to sections and results in the main paper.

Section 12 of this Online Supplementary Material 1 (OSM1) provides expressions for the densities $f_Q(q; \beta_*, \beta_0, \lambda, \Omega)$, $f_{Q_1|Q_T}(q_1|q_T)$, and $f_Q(q; \rho_{uv}, \lambda_v)$, expressions for

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the POIS2 test statistic and critical value of AMS, and expressions for the one-to-one transformations between the reduced-form and structural variance matrices. Section 13 provides one-sided power bounds for invariant similar tests as $\beta_0 \to \pm\infty$, where β_0 denotes the null hypothesis value. Section 14 corrects (4.1) of AMS, which concerns the two-point weight function that defines AMS's two-sided AE power envelope.

Section 15 proves Lemma 6.1. Section 16 proves Theorem 5.1 and its Comment (v). Section 17 proves Theorem 6.2 and its Comment (iv), Corollary 6.3 and its Comment (ii), and Theorem 6.4. Section 18 proves Theorem 8.1. Section 19 proves Theorem 13.1 and Lemmas 14.1 and 14.2.

Section 20 computes the structural error variance matrices in scenarios 1 and 2 considered in (9) and (10) in Section 4.

Section 21 shows how the model is transformed to go from a testing problem of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$ for $\pi \in R^k$ and fixed Ω to a testing problem of $H_0 : \overline{\beta} = 0$ versus $H_1 : \overline{\beta} = \overline{\beta}_*$ for some $\overline{\pi} \in R^k$ and some fixed $\overline{\Omega}$ with diagonal elements equal to one. This links the model considered here to the model used in the Andrews, Moreira, and Stock (2006) (AMS) numerical work.

Section 22 shows how the model is transformed to go from a testing problem of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$ for $\pi \in R^k$ and fixed Ω to a testing problem of $H_0 : \overline{\beta} = \overline{\beta}_0$ versus $H_1 : \overline{\beta} = 0$ for some $\overline{\pi} \in R^k$ and some fixed $\overline{\Omega}$ with diagonal elements equal to one. These transformation results imply that there is no loss in generality in the numerical results of the paper to taking $\omega_1^2 = \omega_2^2 = 1$, $\beta_* = 0$, and $\rho_{uv} \in [0, 1]$ (rather than $\rho_{uv} \in [-1, 1]$).

Section 23 considers a variant of the CLR test, which we denote the CLR2$_n$ test, and computes probabilities that it has infinite length. It is not found to improve upon the CLR test.

Section 24 considers the linear IV model that allows for heteroskedasticity and autocorrelation in the errors, as in Moreira and Ridder (2017). It extends Theorem 5.1 to this model. Thus, it gives formulae for the probabilities that a CI has infinite right length, infinite left length, and infinite length in this model.

12. Definitions

12.1 *Densities of Q when $\beta = \beta_*$ and when $\beta_0 \to \pm\infty$*

In this subsection, we provide expressions for (i) the density $f_Q(q; \beta_*, \beta_0, \lambda, \Omega)$ of Q when the true value of β is β_*, and the null value β_0 is finite, (ii) the conditional density $f_{Q_1|Q_T}(q_1|q_T)$ of Q_1 given $Q_T = q_T$, and (iii) the limit of $f_Q(q; \beta_*, \beta_0, \lambda, \Omega)$ as $\beta_0 \to \pm\infty$.

Let

$$\xi_{\beta_*}(q) = \xi_{\beta_*}(q; \beta_0, \Omega) := c_{\beta_*}^2 q_S + 2c_{\beta_*} d_{\beta_*} q_{ST} + d_{\beta_*}^2 q_T, \tag{24}$$

where $c_{\beta_*} = c_{\beta_*}(\beta_0, \Omega)$ and $d_{\beta_*} = d_{\beta_*}(\beta_0, \Omega)$. As in Section 6, $f_Q(q; \beta_*, \beta_0, \lambda, \Omega)$ denotes the density of $Q := [S : T]'[S : T]$ when $[S : T]$ has the multivariate normal distribution in

(3) with $\beta = \beta_*$ and $\lambda = \mu_\pi' \mu_\pi$. This noncentral Wishart density is

$$f_Q(q; \beta_*, \beta_0, \lambda, \Omega) = K_1 \exp\bigl(-\lambda\bigl(c_{\beta_*}^2 + d_{\beta_*}^2\bigr)/2\bigr) \det(q)^{(k-3)/2} \exp\bigl(-(q_S + q_T)/2\bigr)$$
$$\times \bigl(\lambda \xi_{\beta_*}(q)\bigr)^{-(k-2)/4} I_{(k-2)/2}\bigl(\sqrt{\lambda \xi_{\beta_*}(q)}\bigr), \quad \text{where} \tag{25}$$
$$q = \begin{bmatrix} q_S & q_{ST} \\ q_{ST} & q_T \end{bmatrix}, \qquad q_1 = \begin{pmatrix} q_S \\ q_{ST} \end{pmatrix} \in R^+ \times R, \qquad q_T \in R^+,$$

$K_1^{-1} = 2^{(k+2)/2} pi^{1/2} \Gamma((k-1)/2)$, $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν, $pi = 3.1415\ldots$, and $\Gamma(\cdot)$ is the gamma function. This holds by Lemma 3(a) of AMS with $\beta = \beta_*$.

By Lemma 3(c) of AMS, the conditional density of Q_1 given $Q_T = q_T$ when $[S : T]$ is distributed as in (3) with $\beta = \beta_0$ is

$$f_{Q_1|Q_T}(q_1|q_T) := K_1 K_2^{-1} \exp(-q_S/2) \det(q)^{(k-3)/2} q_T^{-(k-2)/2}, \tag{26}$$

which does not depend on β_0, λ, or Ω.

By Lemma 6.1, the limit of $f_Q(q; \beta_*, \beta_0, \lambda, \Omega)$ as $\beta_0 \to \pm\infty$ is the density $f_Q(q; \rho_{uv}, \lambda_v)$. As in Section 6, $f_Q(q; \rho_{uv}, \lambda_v)$ denotes the density of $Q := [S : T]'[S : T]$ when $[S : T]$ has a multivariate normal distribution with means matrix in (18), all variances equal to one, and all covariances equal to zero. This is a noncentral Wishart density that has the following form:

$$f_Q(q; \rho_{uv}, \lambda_v) = K_1 \exp\bigl(-\lambda_v\bigl(1 + r_{uv}^2\bigr)/2\bigr) \det(q)^{(k-3)/2} \exp\bigl(-(q_S + q_T)/2\bigr)$$
$$\times \bigl(\lambda_v \xi(q; \rho_{uv})\bigr)^{-(k-2)/4} I_{(k-2)/2}\bigl(\sqrt{\lambda_v \xi(q; \rho_{uv})}\bigr), \quad \text{where} \tag{27}$$
$$\xi(q; \rho_{uv}) := q_S + 2 r_{uv} q_{ST} + r_{uv}^2 q_T.$$

This expression for the density holds by the proof of Lemma 3(a) of AMS with means matrix $\mu_\pi \cdot (1/\sigma_v, r_{uv}/\sigma_v)$ in place of the means matrix $\mu_\pi \cdot (c_\beta, d_\beta)$.

12.2 *POIS2 test*

Here, we define the POIS2$(q_1, q_T; \beta_0, \beta_*, \lambda)$ test statistic of AMS, which is analyzed in Section 6, and its conditional critical value $\kappa_{2,\beta_0}(q_T)$.

Given (β_*, λ), the parameters (β_{2*}, λ_2) are defined in (19), which is the same as (4.2) of AMS. By Corollary 1 of AMS, the optimal average-power test statistic against (β_*, λ) and (β_{2*}, λ_2) is

$$\text{POIS2}(Q; \beta_0, \beta_*, \lambda) := \frac{\psi(Q; \beta_0, \beta_*, \lambda) + \psi(Q; \beta_0, \beta_{2*}, \lambda_2)}{2\psi_2(Q_T; \beta_0, \beta_*, \lambda)}, \quad \text{where}$$
$$\psi(Q; \beta_0, \beta, \lambda) := \exp\bigl(-\lambda\bigl(c_\beta^2 + d_\beta^2\bigr)/2\bigr)\bigl(\lambda \xi_\beta(Q)\bigr)^{-(k-2)/4} I_{(k-2)/2}\bigl(\sqrt{\lambda \xi_\beta(Q)}\bigr), \tag{28}$$
$$\psi_2(Q_T; \beta_0, \beta, \lambda) := \exp\bigl(-\lambda d_\beta^2/2\bigr)\bigl(\lambda d_\beta^2 Q_T\bigr)^{-(k-2)/4} I_{(k-2)/2}\Bigl(\sqrt{\lambda d_\beta^2 Q_T}\Bigr),$$

Q and Q_T are defined in (4), $c_\beta = c_\beta(\beta, \Omega)$ and $d_\beta = d_\beta(\beta, \Omega)$ are defined in (3), $I_\nu(\cdot)$ is defined in (25), $\xi_\beta(Q)$ is defined in (24) with Q and β in place of q and β_*, and $\lambda := \mu_\pi' \mu_\pi$. Note that $\psi_2(Q_T; \beta_*, \lambda) = \psi_2(Q_T; \beta_{2*}, \lambda_2)$ by (19).

Let $\kappa_{2,\beta_0}(q_T)$ denote the conditional critical value of the POIS2$(Q; \beta_0, \beta_*, \lambda)$ test statistic. That is, $\kappa_{2,\beta_0}(q_T)$ is defined to satisfy

$$P_{Q_1|Q_T}\big(\text{POIS2}(Q; \beta_0, \beta_*, \lambda) > \kappa_{2,\beta_0}(q_T)|q_T\big) = \alpha \tag{29}$$

for all $q_T \geq 0$, where $P_{Q_1|Q_T}(\cdot|q_T)$ denotes probability under the density $f_{Q_1|Q_T}(\cdot|q_T)$ defined in (26). The critical value function $\kappa_{2,\beta_0}(\cdot)$ depends on $(\beta_0, \beta_*, \lambda, \Omega)$ and k (and (β_{2*}, λ_2) through (β_*, λ)).

12.3 *Structural and reduced-form variance matrices*

Let u_i, v_{1i}, and v_{2i} denote the ith elements of u, v_1, and v_2, respectively. We have

$$v_{1i} := u_i + v_{2i}\beta \quad \text{and} \quad \Omega = \begin{bmatrix} \omega_1^2 & \omega_{12} \\ \omega_{12} & \omega_2^2 \end{bmatrix}, \tag{30}$$

where β denotes the true value.

Given the true value β and some structural error variance matrix Σ, the corresponding reduced-form error variance matrix $\Omega(\beta, \Sigma)$ is

$$\begin{aligned} \Omega(\beta, \Sigma) &:= \text{Var}\left(\begin{pmatrix} v_{1i} \\ v_{2i} \end{pmatrix}\right) = \text{Var}\left(\begin{pmatrix} u_i + v_{2i}\beta \\ v_{2i} \end{pmatrix}\right) \\ &= \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \Sigma \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} \sigma_u^2 + 2\sigma_{uv}\beta + \sigma_v^2\beta^2 & \sigma_{uv} + \sigma_v^2\beta \\ \sigma_{uv} + \sigma_v^2\beta & \sigma_v^2 \end{bmatrix}, \quad \text{where} \\ \Sigma &= \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{bmatrix}. \end{aligned} \tag{31}$$

Given the true value β and the reduced-form error variance matrix Ω, the structural variance matrix $\Sigma(\beta, \Omega)$ is

$$\begin{aligned} \Sigma(\beta, \Omega) &:= \text{Var}\left(\begin{pmatrix} u_i \\ v_{2i} \end{pmatrix}\right) = \text{Var}\left(\begin{pmatrix} v_{1i} - v_{2i}\beta \\ v_{2i} \end{pmatrix}\right) \\ &= \begin{bmatrix} 1 & -\beta \\ 0 & 1 \end{bmatrix} \Omega \begin{bmatrix} 1 & 0 \\ -\beta & 1 \end{bmatrix} = \begin{bmatrix} \omega_1^2 - 2\omega_{12}\beta + \omega_2^2\beta^2 & \omega_{12} - \omega_2^2\beta \\ \omega_{12} - \omega_2^2\beta & \omega_2^2 \end{bmatrix}. \end{aligned} \tag{32}$$

Let $\sigma_u^2(\beta, \Omega)$, $\sigma_v^2(\beta, \Omega)$, and $\sigma_{uv}(\beta, \Omega)$ denote the $(1, 1)$, $(2, 2)$, and $(1, 2)$ elements of $\Sigma(\beta, \Omega)$. Let $\rho_{uv}(\beta, \Omega)$ denote the correlation implied by $\Sigma(\beta, \Omega)$.

In the asymptotics as $\beta_0 \to \pm\infty$, we fix β_* and Ω and consider the testing problem as $\beta_0 \to \pm\infty$. Rather than fixing Ω, one can equivalently fix the structural variance matrix when $\beta = \beta_*$, say at Σ_*. Given β_* and Σ_*, there is a unique reduced-form error variance matrix $\Omega = \Omega(\beta_*, \Sigma_*)$ defined using (31). Significant simplifications in certain formulae occur when they are expressed in terms of Σ_*, rather than Ω, for example, see Lemma 15.1(e) below.

For notational simplicity, we denote the $(1,1)$, $(2,2)$, and $(1,2)$ elements of Σ_* by σ_u^2, σ_v^2, and σ_{uv}, respectively, without any $*$ subscripts. As defined in (15), $\rho_{uv} := \sigma_{uv}/(\sigma_u \sigma_v)$. Thus, ρ_{uv} is the correlation between the structural and reduced-form errors u_i and v_{2i} when the true value of β is β_*. Note that ρ_{uv} does not change when (β_*, Σ_*) is fixed (or, equivalently, $(\beta_*, \Omega) = (\beta_*, \Omega(\beta_*, \Sigma_*))$ is fixed) and β_0 is changed. Also, note that $\sigma_v^2 = \omega_2^2$ because both denote the variance of v_{2i} under $\beta = \beta_*$ and $\beta = \beta_0$.

13. One-sided power bound as $\beta_0 \to \pm\infty$

In this section, we provide one-sided power bounds for invariant similar tests as $\beta_0 \to \pm\infty$ for fixed β_*. The approach is the same as in Andrews, Moreira, and Stock (2004) (AMS04) except that we consider $\beta_0 \to \pm\infty$; also see Mills, Moreira, and Vilela (2014).

13.1 *Point optimal invariant similar tests for fixed β_0 and β_**

First, we consider the point null and alternative hypotheses:

$$H_0 : \beta = \beta_0 \quad \text{and} \quad H_1 : \beta = \beta_*, \tag{33}$$

where $\pi \in R^k$ (or, equivalently, $\lambda \geq 0$) under H_0 and H_1.

Point optimal invariant similar (POIS) tests for any given null and alternative parameter values β_0 and β_*, respectively, and any given Ω are constructed in AMS04, Section 5. Surprisingly, the same test is found to be optimal for all values of π under H_1, that is, for all strengths of identification. The optimal test is constructed by determining the level α test that maximizes conditional power given $Q_T = q_T$ among tests that are invariant and have null rejection probability α conditional on $Q_T = q_T$, for each $q_T \in R$.

By AMS04 (Comment 2 to Corollary 2), the POIS test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$, for any $\pi \in R^k$ (or $\lambda \geq 0$) under H_1, rejects H_0 for large values of

$$\text{POIS}(Q; \beta_0, \beta_*) := Q_S + 2\frac{d_{\beta_*}(\beta_0, \Omega)}{c_{\beta_*}(\beta_0, \Omega)} Q_{ST}. \tag{34}$$

The critical value for the $\text{POIS}(Q; \beta_0, \beta_*)$ test is a conditional critical value given $Q_T = q_T$, which we denote by $\kappa_{\beta_0}(q_T)$. The critical value $\kappa_{\beta_0}(q_T)$ is defined to satisfy

$$P_{Q_1|Q_T}\big(\text{POIS}(Q; \beta_0, \beta_*) > \kappa_{\beta_0}(q_T) | q_T\big) = \alpha \tag{35}$$

for all $q_T \geq 0$, where $P_{Q_1|Q_T}(\cdot|q_T)$ denotes probability under the conditional density $f_{Q_1|Q_T}(q_1|q_T)$ defined in (26). Although the density $f_{Q_1|Q_T}(q_1|q_T)$ does not depend on β_0, $\kappa_{\beta_0}(q_T)$ depends on β_0, as well as (β_*, Ω, k), because $\text{POIS}(Q; \beta_0, \beta_*)$ does.

Note that, although the same $\text{POIS}(Q; \beta_0, \beta_*)$ test is best for all strengths of identification, that is, for all $\lambda = \mu_\pi' \mu_\pi > 0$, the power of this test depends on λ.

13.2 *One-sided power bound when* $\beta_0 \to \pm\infty$

Now we consider the best one-sided invariant similar test as $\beta_0 \to \pm\infty$ keeping (β_*, Ω) fixed. Lemma 15.1 below implies that

$$\lim_{\beta_0 \to \pm\infty} \frac{d_{\beta_*}(\beta_0, \Omega)}{c_{\beta_*}(\beta_0, \Omega)} = \left(\mp \frac{\rho_{uv}}{\sigma_v\left(1-\rho_{uv}^2\right)^{1/2}}\right)/(\mp 1/\sigma_v) = \frac{\rho_{uv}}{\left(1-\rho_{uv}^2\right)^{1/2}}, \tag{36}$$

where ρ_{uv}, defined in (15), is the correlation between the structural and reduced-form errors u_i and v_{2i} under β_*. Hence, the limit as $\beta_0 \to \pm\infty$ of the POIS$(Q; \beta_0, \beta_*)$ test statistic in (34) is

$$\mathrm{POIS}(Q; \infty, \rho_{uv}) := \lim_{\beta_0 \to \pm\infty}\left(Q_S + 2\frac{d_{\beta_*}(\beta_0, \Omega)}{c_{\beta_*}(\beta_0, \Omega)} Q_{ST}\right) = Q_S + 2\frac{\rho_{uv}}{\left(1-\rho_{uv}^2\right)^{1/2}} Q_{ST}. \tag{37}$$

Notice that (i) this limit is the same for $\beta_0 \to +\infty$ and $\beta_0 \to -\infty$, (ii) the POIS$(Q; \infty, \rho_{uv})$ statistic depends on $(\beta_*, \Omega) = (\beta_*, \Omega(\beta_*, \Sigma_*))$ only through $\rho_{uv} := \mathrm{Corr}(\Sigma_*)$, and (iii) when $\rho_{uv} = 0$, the POIS$(Q; \infty, \rho_{uv})$ statistic is the AR statistic (times k). Some intuition for result (iii) is that $EQ_{ST} = 0$ under the null and $\lim_{|\beta_0| \to \infty} EQ_{ST} = 0$ under any fixed alternative β_* when $\rho_{uv} = 0$ (see the discussion in Section 6.2). In consequence, Q_{ST} is not useful for distinguishing between H_0 and H_1 when $|\beta_0| \to \infty$ and $\rho_{uv} = 0$.

Let $\kappa_\infty(q_T)$ denote the conditional critical value of the POIS$(Q; \infty, \rho_{uv})$ test statistic. That is, $\kappa_\infty(q_T)$ is defined to satisfy

$$P_{Q_1|Q_T}\left(\mathrm{POIS}(Q; \infty, \rho_{uv}) > \kappa_\infty(q_T) | q_T\right) = \alpha \tag{38}$$

for all $q_T \geq 0$. The density $f_{Q_1|Q_T}(\cdot|q_T)$ of $P_{Q_1|Q_T}(\cdot|q_T)$ only depends on the number of IVs k; see (26). The critical value function $\kappa_\infty(\cdot)$ depends on ρ_{uv} and k.

Let $\phi_{\beta_0}(Q)$ denote a test of $H_0: \beta = \beta_0$ versus $H_1: \beta = \beta_*$ based on Q that rejects H_0 when $\phi_{\beta_0}(Q) = 1$. In most cases, a test depends on β_0 because the distribution of Q depends on β_0 (see (3) and (4)), and not because $\phi_{\beta_0}(\cdot)$ depends on β_0. For example, this is true of the AR, LM, and CLR tests in (6) and (7). However, we allow for dependence of $\phi_{\beta_0}(\cdot)$ on β_0 in the following result in order to cover all possible sequences of (nonrandomized) tests of $H_0: \beta = \beta_0$.

THEOREM 13.1. *Let* $\{\phi_{\beta_0}(Q) : \beta_0 \to \pm\infty\}$ *be any sequence of invariant similar level* α *tests of* $H_0: \beta = \beta_0$ *versus* $H_1: \beta = \beta_*$ *when* Q *has density* $f_Q(q; \beta, \beta_0, \lambda, \Omega)$ *for some* $\lambda \geq 0$ *and* Ω *is fixed and known. For fixed true* $(\beta_*, \lambda, \Omega)$, *the* POIS$(Q; \infty, \rho_{uv})$ *test satisfies*

$$\limsup_{\beta_0 \to \pm\infty} P_{\beta_*, \beta_0, \lambda, \Omega}\left(\phi_{\beta_0}(Q) = 1\right) \leq P_{\rho_{uv}, \lambda_v}\left(\mathrm{POIS}(Q; \infty, \rho_{uv}) > \kappa_\infty(Q_T)\right).$$

Comments. (i). Theorem 13.1 shows that the POIS$(Q; \infty, \rho_{uv})$ test provides an asymptotic power bound as $\beta_0 \to \pm\infty$ for any invariant similar test for any fixed $(\beta_*, \lambda, \Omega)$. This power bound is strictly less than one. The reason is that $\lim_{\beta_0 \to \pm\infty} |c_{\beta_*}(\beta_0, \Omega)| \nrightarrow \infty$. This is the same reason that the AR test does not have power that converges to one in this scenario; see Section 4. Hence, the bound in Theorem 13.1 is informative.

(ii). The power bound in Theorem 13.1 only depends on $(\beta_*, \lambda, \Omega)$ through ρ_{uv}, the magnitude of endogeneity under β_*, and λ_v, the concentration parameter.

(iii). As an alternative to the power bound given in Theorem 13.1, one might consider developing a formal limit of experiments result, for example, along the lines of van der Vaart (1998, Chapter 9). This approach does not appear to work for the sequence of experiments consisting of the two unconditional distributions of $[S : T]$ (or Q) for $\beta = \beta_0, \beta_*$ and indexed by β_0 as $\beta_0 \to \pm\infty$. The reason is that the likelihood ratio of these two distributions is asymptotically degenerate as $\beta_0 \to \pm\infty$ (either 0 or ∞ depending on which density is in the numerator) when the truth is taken to be $\beta = \beta_0$. This occurs because the length of the mean vector of T diverges to infinity as $\beta_0 \to \pm\infty$ (provided $\lambda = \mu_\pi' \mu_\pi > 0$) by (3) and Lemma 15.1(c) below. For the sequence of conditional distributions of Q given $Q_T = q_T$, it should be possible to obtain a formal limit of experiments result, but this would not very helpful because we are interested in the unconditional power of tests and a conditional limit of experiments result would not deliver this.

(iv). The proof of Theorem 13.1 is given in Section 19 below.

14. Equations (4.1) and (4.2) of AMS

This section corrects (4.1) of AMS, which concerns the two-point weight function that defines AMS's two-sided AE power envelope.

Equation (4.1) of AMS is:[10] given (β_*, λ), the second point (β_{2*}, λ_2) solves

$$\lambda_2^{1/2} c_{\beta_{2*}} = -\lambda^{1/2} c_{\beta_*} \quad (\neq 0) \quad \text{and} \quad \lambda_2^{1/2} d_{\beta_{2*}} = \lambda^{1/2} d_{\beta_*}. \tag{39}$$

AMS states that provided $\beta_* \neq \beta_{AR}$, the solutions to the two equations in (4.1) satisfy the two equations in (4.2) of AMS, which is the same as (19) and which we repeat here for convenience:[11]

$$\beta_{2*} = \beta_0 - \frac{d_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)} \quad \text{and} \quad \lambda_2 = \lambda \frac{\left(d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)\right)^2}{d_{\beta_0}^2}, \quad \text{where}$$
$$r_{\beta_0} := e_1' \Omega^{-1} a_0 \cdot \left(a_0' \Omega^{-1} a_0\right)^{-1/2} \quad \text{and} \quad e_1 := (1, 0)'. \tag{40}$$

Equation (4.2) is correct as stated, but (4.1) of AMS is not correct. More specifically, it is not complete. It should be: given (β_*, λ), the second point (β_{2*}, λ_2) solves either (39) or

$$\lambda_2^{1/2} c_{\beta_{2*}} = \lambda^{1/2} c_{\beta_*} \quad (\neq 0) \quad \text{and} \quad \lambda_2^{1/2} d_{\beta_{2*}} = -\lambda^{1/2} d_{\beta_*}. \tag{41}$$

For brevity, we write the "either or" conditions in (39) and (41) as

$$\lambda_2^{1/2} c_{\beta_{2*}} = \mp\lambda^{1/2} c_{\beta_*} \quad (\neq 0) \quad \text{and} \quad \lambda_2^{1/2} d_{\beta_{2*}} = \pm\lambda^{1/2} d_{\beta_*}. \tag{42}$$

[10] Note that (β_*, λ) and (β_{2*}, λ_2) in this paper correspond to (β^*, λ^*) and (β_2^*, λ_2^*) in AMS.

[11] The formulae in (19) and (40) only hold for $\beta_* \neq \beta_{AR}$, where $\beta_{AR} := (\omega_1^2 - \omega_{12}\beta_0)/(\omega_{12} - \omega_2^2\beta_0)$ provided $\omega_{12} - \omega_2^2\beta_0 \neq 0$ (which necessarily holds for $|\beta_0|$ sufficiently large because $\omega_2^2 > 0$).

The reason (4.1) of AMS needs to be augmented by (41) is that for some (β_*, λ), β_0, and Ω, (4.1) has no real solutions (β_{2*}, λ_2) and the expressions for (β_{2*}, λ_2) in (4.2) of AMS do not satisfy (4.1). Once (4.1) of AMS is augmented by (41), there exist real solutions (β_{2*}, λ_2) to the augmented conditions and they are given by the expressions in (4.2) of AMS, that is, by (40). This is established in the following lemma.

Lemma 14.1. *The conditions in* (42) *hold iff the conditions in* (4.2) *of AMS hold, that is, iff the conditions in* (40) *holds.*

With (4.1) of AMS replaced by (42), the results in Theorem 8(b) and (c) of AMS hold as stated. That is, the two-point weight function that satisfies (42) leads to a two-sided weighted average power (WAP) test that is asymptotically efficient under strong IVs. And, all other two-point weight functions lead to two-sided WAP tests that are not asymptotically efficient under strong IVs.

Lemma 14.2. *Under the assumptions of* Theorem 8 *of* AMS, *that is,* Assumptions SIV-LA *and* 1–4 *of* AMS, (a) *if* (β_{2*}, λ_2) *satisfies* (42), *then* $LR^*(\widehat{Q}_{1,n}, \widehat{Q}_{T,n}; \beta_*, \lambda) = e^{-\frac{1}{2}(\tau^*)^2} \times \cosh(\tau^* LM_n^{1/2}) + o_p(1)$, *where* $\tau^* = \lambda^{1/2} c_{\beta_*}$, *which is a strictly-increasing continuous function of* LM_n, *and* (b) *if* (β_{2*}, λ_2) *does not satisfy* (42), *then* $LR^*(\widehat{Q}_{1,n}, \widehat{Q}_{T,n}; \beta_*, \lambda) = \eta_2(Q_{ST,n}/Q_{T,n}^{1/2}) + o_p(1)$ *for a continuous function* $\eta_2(\cdot)$ *that is not even.*

Comments. (i). Lemma 14.2(a) is an extension of Theorem 8(b) of AMS; while Lemma 14.2(b) is a correction to Theorem 8(c) of AMS.

(ii). The proofs of Lemma 14.1 and 14.2 are given in Section 19 below.

Having augmented (4.1) by (41), the two-point weight function of AMS does not have the property that β_{2*} is necessarily on the opposite side of β_0 from β_*. However, it does have the properties that (i) for any (β_*, λ), (β_{2*}, λ_2) is the only point that yields a two-point WAP test that is asymptotic efficient in a two-sided sense under strong IVs, (ii) the marginal distributions of Q_S, Q_T, and Q_{ST} are the same under (β_*, λ) and (β_{2*}, λ_2), and (iii) the joint distribution of (Q_S, Q_{ST}, Q_T) under (β_*, λ) is the same as that of $(Q_S, -Q_{ST}, Q_T)$ under (β_{2*}, λ_2).

15. Proof of Lemma 6.1

The proof of Lemma 6.1 and other proofs below use the following lemma.

The distributions of $[S : T]$ under (β_0, Ω) and (β_*, Ω) depend on $c_\beta(\beta_0, \Omega)$ and $d_\beta(\beta_0, \Omega)$ for $\beta = \beta_0$ and β_*. The limits of these quantities as $\beta_0 \to \pm\infty$ are given in the following lemma.[12]

Lemma 15.1. *For fixed* β_* *and positive definite matrix* Ω, *we have*:

(a) $\lim_{\beta_0 \to \pm\infty} c_{\beta_0}(\beta_0, \Omega) = 0$.

(b) $\lim_{\beta_0 \to \pm\infty} c_{\beta_*}(\beta_0, \Omega) = \mp 1/\sigma_v$.

[12] Throughout, $\beta_0 \to \pm\infty$ means $\beta_0 \to \infty$ or $\beta_0 \to -\infty$.

(c) $\lim_{\beta_0 \to \pm\infty} d_{\beta_0}(\beta_0, \Omega) = \infty$.

(d) $d_{\beta_0}(\beta_0, \Omega)/|\beta_0| = \frac{\omega_2}{(\omega_1^2\omega_2^2 - \omega_{12}^2)^{1/2}} + o(1) = \frac{1}{\sigma_u(1-\rho_{uv}^2)^{1/2}} + o(1)$ *as* $|\beta_0| \to \infty$.

(e) $\lim_{\beta_0 \to \pm\infty} d_{\beta_*}(\beta_0, \Omega) = \pm \frac{\omega_2^2\beta_* - \omega_{12}}{\omega_2(\omega_1^2\omega_2^2 - \omega_{12}^2)^{1/2}} = \mp \frac{\rho_{uv}}{\sigma_v(1-\rho_{uv}^2)^{1/2}}$.

Comment. The limits in parts (d) and (e), expressed in terms of Σ_*, only depend on ρ_{uv}, σ_u, and σ_v and their functional forms are of a relatively simple multiplicative form. The latter provides additional simplifications of certain quantities that appear below.

Proof of Lemma 15.1. Part (a) holds because $c_{\beta_0}(\beta_0, \Omega) = 0$ for all β_0. Part (b) holds by the following calculations:

$$\begin{aligned}
\lim_{\beta_0 \to \pm\infty} c_{\beta_*}(\beta_0, \Omega) &= \lim_{\beta_0 \to \pm\infty} (\beta_* - \beta_0) \cdot \left(b_0'\Omega b_0\right)^{-1/2} \\
&= \lim_{\beta_0 \to \pm\infty} (\beta_* - \beta_0) \cdot \left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{-1/2} \\
&= \mp 1/\omega_2 \\
&= \mp 1/\sigma_v. \qquad (43)
\end{aligned}$$

Now, we establish part (e). Let $b_* := (1, -\beta_*)'$. We have

$$\begin{aligned}
\lim_{\beta_0 \to \pm\infty} d_{\beta_*}(\beta_0, \Omega) &= \lim_{\beta_0 \to \pm\infty} b_*'\Omega b_0 \cdot \left(b_0'\Omega b_0\right)^{-1/2} \det(\Omega)^{-1/2} \\
&= \lim_{\beta_0 \to \pm\infty} \frac{\omega_1^2 - \omega_{12}\beta_* - \omega_{12}\beta_0 + \omega_2^2\beta_*\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}} \\
&= \pm \frac{\omega_2^2\beta_* - \omega_{12}}{\omega_2\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}}. \qquad (44)
\end{aligned}$$

Next, we write the limit in (44) in terms of the elements of the structural error variance matrix Σ_*. The term in the square root in the denominator of (44) satisfies

$$\omega_1^2\omega_2^2 - \omega_{12}^2 = \left(\sigma_u^2 + 2\sigma_{uv}\beta_* + \sigma_v^2\beta_*^2\right)\sigma_v^2 - \left(\sigma_{uv} + \sigma_v^2\beta_*\right)^2 = \sigma_u^2\sigma_v^2 - \sigma_{uv}^2, \qquad (45)$$

where the first equality uses $\omega_2^2 = \sigma_v^2$ (since both denote the variance of v_{2i}), $\omega_1^2 = \sigma_u^2 + 2\sigma_{uv}\beta_* + \sigma_v^2\beta_*^2$, and $\omega_{12} = \sigma_{uv} + \sigma_v^2\beta_*$ (which both hold by (31) with $\beta = \beta_*$ and $\Sigma = \Sigma_*$), and the second equality holds by simple calculations. The limit in (44) in terms of the elements of Σ_* is

$$\pm \frac{\omega_2^2\beta_* - \omega_{12}}{\omega_2\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}} = \pm \frac{\sigma_v^2\beta_* - \left(\sigma_{uv} + \sigma_v^2\beta_*\right)}{\sigma_v\left(\sigma_u^2\sigma_v^2 - \sigma_{uv}^2\right)^{1/2}} = \mp \frac{\rho_{uv}}{\sigma_v\left(1 - \rho_{uv}^2\right)^{1/2}}, \qquad (46)$$

where the first equality uses (45), $\omega_2^2 = \sigma_v^2$, and $\omega_{12} = \sigma_{uv} + \sigma_v^2\beta_*$, and the second inequality holds by dividing the numerator and denominator by $\sigma_u\sigma_v$. This establishes part (e).

For part (c), we have

$$\begin{aligned}\lim_{\beta_0\to\pm\infty} d_{\beta_0}(\beta_0,\Omega) &= \lim_{\beta_0\to\pm\infty}\big(b_0'\Omega b_0\big)^{1/2}\det(\Omega)^{-1/2}\\ &= \lim_{\beta_0\to\pm\infty}\frac{\big(\omega_1^2-2\omega_{12}\beta_0+\omega_2^2\beta_0^2\big)^{1/2}}{\big(\omega_1^2\omega_2^2-\omega_{12}^2\big)^{1/2}}\\ &= \infty. \end{aligned} \tag{47}$$

Part (d) holds because, as $|\beta_0|\to\infty$, we have

$$\begin{aligned} d_{\beta_0}(\beta_0,\Omega)/|\beta_0| &= \frac{\big(\omega_1^2/\beta_0^2-2\omega_{12}/\beta_0+\omega_2^2\big)^{1/2}}{\big(\omega_1^2\omega_2^2-\omega_{12}^2\big)^{1/2}}\\ &= \frac{\omega_2}{\big(\omega_1^2\omega_2^2-\omega_{12}^2\big)^{1/2}}+o(1)\\ &= \frac{1}{\sigma_u\big(1-\rho_{uv}^2\big)^{1/2}}+o(1), \end{aligned} \tag{48}$$

where the last equality uses (45) and $\omega_2=\sigma_v$. □

Next, we prove Lemma 6.1, which states that for any fixed (β_*,λ,Ω), $\lim_{\beta_0\to\pm\infty} f_Q(q;\beta_*,\beta_0,\lambda,\Omega)=f_Q(q;\rho_{uv},\lambda_v)$.

PROOF OF LEMMA 6.1. By Lemma 15.1(b) and (e) and (17), we have $\lim_{\beta_0\to\pm\infty} c_{\beta_*}=\mp 1/\sigma_v$ and $\lim_{\beta_0\to\pm\infty} d_{\beta_*}=\mp r_{uv}/\sigma_v$. In consequence,

$$\begin{aligned} \lim_{\beta_0\to\pm\infty}\lambda\big(c_{\beta_*}^2+d_{\beta_*}^2\big) &= \lambda\big(1/\sigma_v^2\big)\big(1+r_{uv}^2\big)=\lambda_v\big(1+r_{uv}^2\big) \quad \text{and}\\ \lim_{\beta_0\to\pm\infty}\lambda\xi_{\beta_*}(q) &= \lim_{\beta_0\to\pm\infty}\lambda\big(c_{\beta_*}^2 q_S+2c_{\beta_*}d_{\beta_*}q_{ST}+d_{\beta_*}^2 q_T\big)\\ &= \lambda\big(1/\sigma_v^2\big)\big(q_S+2r_{uv}q_{ST}+r_{uv}^2 q_T\big)=\lambda_v\xi(q;\rho_{uv}), \end{aligned} \tag{49}$$

using the definitions of λ_v and $\xi(q;\rho_{uv})$ in (17) and (27), respectively, where the first equality in the third line uses $(\mp 1)(\mp r_{uv})=r_{uv}$. Combining this with (25) and (27) proves the result of the lemma. □

16. PROOF OF THEOREM 5.1

The proof of Theorem 5.1 uses the following lemma.[13] Let

$$\begin{aligned} S_{\pm\infty}(Y) &:= \big(Z'Z\big)^{-1/2}Z'Ye_2\cdot\frac{\mp 1}{\sigma_v},\\ T_{\pm\infty}(Y) &:= \big(Z'Z\big)^{-1/2}Z'Y\Omega^{-1}e_1\cdot\big(\pm\big(1-\rho_{uv}^2\big)^{1/2}\sigma_u\big), \quad \text{and} \end{aligned} \tag{50}$$

[13]The proof of Comment (v) to Theorem 5.1 is the same as that of Theorem 5.1(a) and (b) with $[S_{\beta_0}(Y),T_{\beta_0}(Y)]$ and $T_{\beta_0}(Y)$ in place of $Q_{\beta_0}(Y)$ and $Q_{T,\beta_0}(Y)$, respectively.

$$Q_{\pm\infty}(Y) := \begin{bmatrix} e_2' Y' P_Z Y e_2 \cdot \dfrac{1}{\sigma_v^2} & -e_2' Y' P_Z Y \Omega^{-1} e_1 \cdot \dfrac{\left(1-\rho_{uv}^2\right)^{1/2}\sigma_u}{\sigma_v} \\ -e_2' Y' P_Z Y \Omega^{-1} e_1 \cdot \dfrac{\left(1-\rho_{uv}^2\right)^{1/2}\sigma_u}{\sigma_v} & e_1' \Omega^{-1} Y' P_Z Y \Omega^{-1} e_1 \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \end{bmatrix},$$

where $\rho_{uv} := \mathrm{Corr}(u_i, v_{2i})$, $P_Z := Z(Z'Z)^{-1}Z'$, $e_1 := (1,0)'$, $e_2 := (0,1)'$, and $Q_{\pm\infty}(Y)$ is the same for $+\infty$ and $-\infty$. Let $Q_{T,\pm\infty}(Y)$ denote the $(2,2)$ element of $Q_{\pm\infty}(Y)$. As defined in (17), $r_{uv} = \rho_{uv}/(1-\rho_{uv}^2)^{1/2}$.

LEMMA 16.1. *For fixed β_* and positive definite matrix Ω, we have*

(a) $\lim_{\beta_0 \to \pm\infty} S_{\beta_0}(Y) = S_{\pm\infty}(Y)$,

(b) $S_{\pm\infty}(Y) \sim N(\mp \frac{1}{\sigma_v}\mu_\pi, I_k)$,

(c) $\lim_{\beta_0 \to \pm\infty} T_{\beta_0}(Y) = T_{\pm\infty}(Y) = (Z'Z)^{-1/2} Z' Y \Omega^{-1} e_1 \cdot (\pm(1-\rho_\Omega^2)^{1/2}\omega_1)$, *where* $\rho_\Omega := \mathrm{Corr}(v_{1i}, v_{2i})$,

(d) $T_{\pm\infty}(Y) \sim N(\mp \frac{r_{uv}}{\sigma_v}\mu_\pi, I_k)$,

(e) $S_{\pm\infty}(Y)$ *and* $T_{\pm\infty}(Y)$ *are independent,*

(f) $\lim_{\beta_0 \to \pm\infty} Q_{\beta_0}(Y) = Q_{\pm\infty}(Y)$, *and*

(g) $Q_{\pm\infty}(Y)$ *has a noncentral Wishart distribution with means matrix* $\mu_\pi(\frac{1}{\sigma_v}, \frac{r_{uv}}{\sigma_v}) \in R^{k\times 2}$, *identity variance matrix, and density given in* (27).

Comment. The convergence results in Lemma 16.1 hold for all realizations of Y.

PROOF OF THEOREM 5.1. First, we prove part (a). We have

$$\begin{aligned} &1\big(\mathrm{RLength}\big(\mathrm{CS}_\phi(Y)\big) = \infty\big) \\ &\quad = 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \mathrm{cv}\big(Q_{T,\beta_0}(Y)\big)\ \forall \beta_0 \ge K(Y) \text{ for some } K(Y) < \infty\big) \\ &\quad = \lim_{\beta_0 \to \infty} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \mathrm{cv}\big(Q_{T,\beta_0}(Y)\big)\big), \end{aligned} \tag{51}$$

where the second equality holds provided the limit as $\beta_0 \to \infty$ on the right-hand side (rhs) exists, the first equality holds by the definition of $\mathrm{CS}_\phi(Y)$ in (11)–(13) and the definition of $\mathrm{RLength}(\mathrm{CS}_\phi(Q)) = \infty$ in (14), and the second equality holds because its rhs equals one (when the rhs limit exists) iff $\mathcal{T}(Q_{\beta_0}(Y)) \le \mathrm{cv}(Q_{T,\beta_0}(Y))$ for $\forall \beta_0 \ge K(Y)$ for some $K(Y) < \infty$, which is the same as its left-hand side.

Now, we use the dominated convergence theorem (DCT) to show

$$\begin{aligned} &\lim_{\beta_0 \to \infty} E_{\beta_*,\pi,\Omega} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \mathrm{cv}\big(Q_{T,\beta_0}(Y)\big)\big) \\ &\quad = E_{\beta_*,\pi,\Omega} \lim_{\beta_0 \to \infty} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \mathrm{cv}\big(Q_{T,\beta_0}(Y)\big)\big). \end{aligned} \tag{52}$$

The DCT applies because (i) the indicator functions in (52) are dominated by the constant function equal to one, which is integrable, and (ii) $\lim_{\beta_0 \to \infty} 1(\mathcal{T}(Q_{\beta_0}(Y)) \le$

$\text{cv}(Q_{T,\beta_0}(Y))$ exists a.s.$[P_{\beta_*,\pi,\Omega}]$ and equals $1(\mathcal{T}(Q_{\pm\infty}(Y)) \le \text{cv}(Q_{T,\pm\infty}(Y))$ a.s.$[P_{\beta_*,\pi,\Omega}]$. The latter holds because the assumption that $\mathcal{T}(q)$ and $\text{cv}(q_T)$ are continuous at positive definite (pd) q and positive q_T, respectively, coupled with the result of Lemma 16.1(f) (that $Q_{\beta_0}(Y) \to Q_{\pm\infty}(Y)$ as $\beta_0 \to \infty$ for all sample realizations of Y, where $Q_{\pm\infty}(Y)$ is defined in (50)), imply that (a) $\lim_{\beta_0\to\infty} \mathcal{T}(Q_{\beta_0}(Y)) = \mathcal{T}(Q_{\pm\infty}(Y))$ for all realizations of Y for which $Q_{\pm\infty}(Y)$ is pd, (b) $\lim_{\beta_0\to\infty} \text{cv}(Q_{T,\beta_0}(Y)) = \text{cv}(Q_{T,\pm\infty}(Y))$ for all realizations of Y with $Q_{T,\pm\infty}(Y) > 0$, and hence (c) $\lim_{\beta_0\to\infty} 1(\mathcal{T}(Q_{\beta_0}(Y)) \le \text{cv}(Q_{T\beta_0}(Y)) = 1(\mathcal{T}(Q_{\pm\infty}(Y)) \le \text{cv}(Q_{T,\pm\infty}(Y))$ for all realizations of Y for which $\mathcal{T}(Q_{\pm\infty}(Y)) \ne \text{cv}(Q_{T,\pm\infty}(Y))$. We have $P_{\beta_*,\pi,\Omega}(\mathcal{T}(Q_{\pm\infty}(Y)) = \text{cv}(Q_{T,\pm\infty}(Y))) = 0$ by assumption, and $P_{\beta_*,\pi,\Omega}(Q_{\pm\infty}(Y)$ is pd & $Q_{T,\pm\infty}(Y) > 0) = 1$ (because $Q_{\pm\infty}(Y)$ has a noncentral Wishart distribution by Lemma 16.1(g)). Thus, condition (ii) above holds and the DCT applies.

Next, we have

$$\begin{aligned}
&1 - \lim_{\beta_0\to\infty} P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi(Q)=1\big)\\
&\quad= \lim_{\beta_0\to\infty} E_{\beta_*,\pi,\Omega} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \text{cv}\big(Q_{T,\beta_0}(Y)\big)\big)\\
&\quad= E_{\beta_*,\pi,\Omega} \lim_{\beta_0\to\infty} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \text{cv}\big(Q_{T,\beta_0}(Y)\big)\big)\\
&\quad= P_{\beta_*,\pi,\Omega}\big(\text{RLength}\big(\text{CS}_\phi(Y)\big) = \infty\big), \qquad (53)
\end{aligned}$$

where the first equality holds because the distribution of Q under $P_{\beta_*,\beta_0,\lambda,\Omega}(\cdot)$ equals the distribution of $Q_{\beta_0}(Y)$ under $P_{\beta_*,\pi,\Omega}(\cdot)$ and $\phi(Q) = 0$ iff $\mathcal{T}(Q_{\beta_0}) \le \text{cv}(Q_T)$ by (12), the second equality holds by (52), and the last equality holds by (51). Equation (53) establishes part (a).

The proof of part (b) is the same as that of part (a), but with LLength, $\forall\beta_0 \le -K(Y)$, and $\beta_0 \to -\infty$ in place of RLength, $\forall\beta_0 \ge K(Y)$, and $\beta_0 \to \infty$ respectively.

The proof of part (c) is as follows:

$$\begin{aligned}
&1\big(\text{RLength}\big(\text{CS}_\phi(Y)\big) = \infty \;\&\; \text{LLength}\big(\text{CS}_\phi(Y)\big) = \infty\big)\\
&\quad= 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \text{cv}\big(Q_{T\beta_0}(Y)\big)\; \forall\beta_0 \ge K(Y) \;\&\; \forall\beta_0 \le -K(Y) \text{ for some } K(Y) < \infty\big)\\
&\quad= \lim_{\beta_0\to\infty} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \text{cv}\big(Q_{T,\beta_0}(Y)\big) \;\&\; \mathcal{T}\big(Q_{-\beta_0}(Y)\big) \le \text{cv}\big(Q_{T,-\beta_0}(Y)\big)\big)\\
&\quad= 1(\mathcal{T}\big(Q_{\pm\infty}(Y)\big) \le \text{cv}\big(Q_{T,\pm\infty}(Y)\big)\\
&\quad= \lim_{\beta_0\to\infty} 1(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \text{cv}\big(Q_{T,\beta_0}(Y)\big), \qquad (54)
\end{aligned}$$

where the first two equalities hold for the same reasons as the equalities in (51), the third equality holds a.s.$[P_{\beta_*,\pi,\Omega}]$ by result (ii) that follows (52) and the same result with $-\beta_0$ in place of β_0 since $Q_{\pm\infty}(Y)$ is the same limit whether $\beta_0 \to \infty$ or $-\infty$, and the last equality holds by result (ii) that follows (52).

Now, we have

$$P_{\beta_*,\pi,\Omega}\big(\text{RLength}\big(\text{CS}_\phi(Y)\big) = \infty \;\&\; \text{LLength}\big(\text{CS}_\phi(Y)\big) = \infty\big)$$

$$= E_{\beta_*,\pi,\Omega} \lim_{\beta_0\to\infty} 1\big(\mathcal{T}\big(Q_{\beta_0}(Y)\big) \le \mathrm{cv}\big(Q_{T,\beta_0}(Y)\big)\big)$$

$$= 1 - \lim_{\beta_0\to\infty} P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi(Q)=1\big), \tag{55}$$

where the first equality holds by (54) and the second equality holds by the first three lines of (53). This establishes the equality in part (c) when $\beta_0 \to \infty$. The equality in part (c) when $\beta_0 \to -\infty$ holds because (54) and (55) hold with $\beta_0 \to \infty$ replaced by $\beta_0 \to -\infty$ since the indicator function on the rhs of the second equality in (54) depends on β_0 only through $|\beta_0|$. □

Proof of Lemma 16.1. Part (a) holds because

$$\lim_{\beta_0\to\pm\infty} S_{\beta_0}(Y) = \lim_{\beta_0\to\pm\infty} \big(Z'Z\big)^{-1/2} Z'Yb_0 \cdot \big(b_0'\Omega b_0\big)^{-1/2}$$

$$= \big(Z'Z\big)^{-1/2} Z'Y \lim_{\beta_0\to\pm\infty} \begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix} \big/ \big(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\big)^{1/2}$$

$$= \big(Z'Z\big)^{-1/2} Z'Ye_2(\mp 1/\sigma_v), \tag{56}$$

where $e_2 := (0,1)'$, the first equality holds by (3), the second equality holds because $b_0 := (1,-\beta_0)'$, and the third equality holds using $\omega_2 = \sigma_v$.

Next, we prove part (b). The statistic $S_{\pm\infty}(Y)$ has a multivariate normal distribution because it is a linear combination of multivariate normal random variables. The mean of $S_{\pm\infty}(Y)$ is

$$ES_{\pm\infty}(Y) = \big(Z'Z\big)^{-1/2} Z'Z[\pi\beta_* : \pi]e_2 \cdot \frac{\mp 1}{\sigma_v} = \big(Z'Z\big)^{1/2}\pi \cdot \frac{\mp 1}{\sigma_v} = \mu_\pi \cdot \frac{\mp 1}{\sigma_v}, \tag{57}$$

where the first equality holds using (2) with $a = (\beta_*, 1)'$ and (50). The variance matrix of $S_{\pm\infty}(Y)$ is

$$\mathrm{Var}\big(S_{\pm\infty}(Y)\big) = \mathrm{Var}\big(\big(Z'Z\big)^{-1/2} Z'Ye_2\big)/\sigma_v^2 = \mathrm{Var}\left(\sum_{i=1}^n \big(Z'Z\big)^{-1/2} Z_iY_i'e_2\right)\Big/\sigma_v^2$$

$$= \sum_{i=1}^n \mathrm{Var}\big(\big(Z'Z\big)^{-1/2} Z_iY_i'e_2\big)/\sigma_v^2 = \sum_{i=1}^n \big(Z'Z\big)^{-1/2} Z_iZ_i\big(Z'Z\big)^{-1/2} e_2'\Omega e_2/\sigma_v^2$$

$$= I_k, \tag{58}$$

where the third equality holds by independence across i and the last equality uses $\omega_2^2 = \sigma_v^2$. This establishes part (b).

To prove part (c), we have

$$\lim_{\beta_0\to\pm\infty} T_{\beta_0}(Y) = \lim_{\beta_0\to\pm\infty} \big(Z'Z\big)^{-1/2} Z'Y\Omega^{-1}a_0 \cdot \big(a_0'\Omega^{-1}a_0\big)^{-1/2}$$

$$= \big(Z'Z\big)^{-1/2} Z'Y\Omega^{-1} \lim_{\beta_0\to\pm\infty} \begin{pmatrix} \beta_0 \\ 1 \end{pmatrix} \big/ \big(\omega^{11}\beta_0^2 + 2\omega^{12}\beta_0 + \omega^{22}\big)^{1/2}$$

$$
\begin{aligned}
&= \left(Z'Z\right)^{-1/2} Z'Y\Omega^{-1}e_1 \cdot \left(\pm 1/\omega^{11}\right)^{1/2} \\
&= \left(Z'Z\right)^{-1/2} Z'Y\Omega^{-1}e_1 \cdot \left(\pm\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}/\omega_2\right), \qquad (59)
\end{aligned}
$$

where ω^{11}, ω^{12}, and ω^{22} denote the $(1,1)$, $(1,2)$, and $(2,2)$ elements of Ω^{-1}, respectively, $e_1 := (1,0)'$, the first equality holds by (3), the second equality holds because $a_0 := (\beta_0, 1)'$, and the fourth equality holds by the formula for ω^{11}. In addition, we have

$$
\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}/\omega_2 = \left(1-\rho_\Omega^2\right)^{1/2}\omega_1 = \left(1-\rho_{uv}^2\right)^{1/2}\sigma_u, \qquad (60)
$$

where the first equality uses $\rho_\Omega := \omega_{12}/(\omega_1\omega_2)$ and the second equality holds because $\omega_1^2\omega_2^2 - \omega_{12}^2 = \sigma_u^2\sigma_v^2 - \sigma_{uv}^2$ by (45) and $\omega_2 = \sigma_v$. Equations (59) and (60), combined with (50), establish part (c).

Now, we prove part (d). Like $S_{\pm\infty}(Y)$, $T_{\pm\infty}(Y)$ has a multivariate normal distribution. The mean of $T_{\pm\infty}(Y)$ is

$$
\begin{aligned}
ET_{\pm\infty}(Y) &= \left(Z'Z\right)^{-1/2} Z'Z[\pi\beta_* : \pi]\Omega^{-1}e_1 \cdot \left(\pm\left(1-\rho_{uv}^2\right)^{1/2}\sigma_u\right) \\
&= \left(Z'Z\right)^{1/2}\pi\left(\beta_*\omega^{11} + \omega^{12}\right) \cdot \left(\pm\left(1-\rho_{uv}^2\right)^{1/2}\sigma_u\right), \qquad (61)
\end{aligned}
$$

where the equality holds using (2) with $a = (\beta_*, 1)'$ and (50). In addition, we have

$$
\beta_*\omega^{11} + \omega^{12} = \frac{\beta_*\omega_2^2 - \omega_{12}}{\omega_1^2\omega_2^2 - \omega_{12}^2} = \frac{-\sigma_{uv}}{\sigma_u^2\sigma_v^2 - \sigma_{uv}^2} = \frac{-\rho_{uv}}{(1-\rho_{uv}^2)\sigma_u\sigma_v}, \qquad (62)
$$

where the second equality uses $\omega_1^2\omega_2^2 - \omega_{12}^2 = \sigma_u^2\sigma_v^2 - \sigma_{uv}^2$ by (45) and $\beta_*\omega_2^2 - \omega_{12} = -\sigma_{uv}$ by (32) with $\beta = \beta_*$. Combining (61) and (62) gives

$$
ET_{\pm\infty}(Y) = \mu_\pi \cdot \frac{\mp\rho_{uv}}{\sigma_v\left(1-\rho_{uv}^2\right)^{1/2}} = \mu_\pi \cdot \frac{\mp r_{uv}}{\sigma_v}. \qquad (63)
$$

The variance matrix of $T_{\pm\infty}(Y)$ is

$$
\begin{aligned}
\mathrm{Var}\left(T_{\pm\infty}(Y)\right) &= \mathrm{Var}\left(\left(Z'Z\right)^{-1/2} Z'Y\Omega^{-1}e_1\right) \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \\
&= \mathrm{Var}\left(\sum_{i=1}^n \left(Z'Z\right)^{-1/2} Z_iY_i'\Omega^{-1}e_1\right) \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \\
&= \sum_{i=1}^n \mathrm{Var}\left(\left(Z'Z\right)^{-1/2} Z_iY_i'\Omega^{-1}e_1\right) \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \\
&= \sum_{i=1}^n \left(Z'Z\right)^{-1/2} Z_iZ_i\left(Z'Z\right)^{-1/2} e_1'\Omega^{-1}e_1 \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \\
&= I_k \frac{\omega_2^2}{\omega_1^2\omega_2^2 - \omega_{12}^2} \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 \\
&= I_k \frac{\sigma_v^2}{\sigma_u^2\sigma_v^2 - \sigma_{uv}^2} \cdot \left(1-\rho_{uv}^2\right)\sigma_u^2 = I_k,
\end{aligned}
$$

where the first equality holds by (50), the third equality holds by independence across i, and the second last equality uses $\omega_1^2\omega_2^2 - \omega_{12}^2 = \sigma_u^2\sigma_v^2 - \sigma_{uv}^2$ by (45) and $\omega_2^2 = \sigma_v^2$.

Part (e) holds because

$$\begin{aligned}
&\operatorname{Cov}\big(S_{\pm\infty}(Y), T_{\pm\infty}(Y)\big) \\
&\quad = -\sum_{i=1}^{n} \operatorname{Cov}\big(\big(Z'Z\big)^{-1/2} Z_i Y_i' e_2, \big(Z'Z\big)^{-1/2} Z_i Y_i' \Omega^{-1} e_1\big) \cdot \big(1-\rho_{uv}^2\big)^{1/2} \sigma_u/\sigma_v \\
&\quad = \sum_{i=1}^{n} \big(Z'Z\big)^{-1/2} Z_i Z_i \big(Z'Z\big)^{-1/2} e_2' \Omega \Omega^{-1} e_1 \cdot \big(1-\rho_{uv}^2\big)^{1/2} \sigma_u/\sigma_v = 0^k. \qquad (64)
\end{aligned}$$

Part (f) follows from parts (a) and (c) of the lemma and (11).

Part (g) holds by the definition of the noncentral Wishart distribution and parts (b), (d), and (e) of the lemma. The density of $Q_{\pm\infty}(Y)$ equals the density in (27) because the noncentral Wishart density is invariant to a sign change in the means matrix. □

17. Proofs of Theorem 6.2, Corollary 6.3, and Theorem 6.4

The following lemma is used in the proof of Theorem 6.2. As above, let $P_{\beta_*,\beta_0,\lambda,\Omega}(\cdot)$ and $P_{\rho_{uv},\lambda_v}(\cdot)$ denote probabilities under the alternative hypothesis densities $f_Q(q;\beta_*,\beta_0,\lambda,\Omega)$ and $f_Q(q;\rho_{uv},\lambda_v)$, which are defined in Section 12.1. See (25) and (27) for explicit expressions for these noncentral Wishart densities.

Lemma 17.1. (a) $\lim_{\beta_0\to\pm\infty} P_{\beta_*,\beta_0,\lambda,\Omega}(\mathrm{POIS2}(Q;\beta_*,\beta_0,\lambda) > \kappa_{2,\beta_0}(Q_T)) = P_{\rho_{uv},\lambda_v}(\mathrm{POIS2}(Q;\infty,|\rho_{uv}|,\lambda_v) > \kappa_{2,\infty}(Q_T))$,

(b) $\lim_{\beta_0\to\pm\infty} P_{\beta_{2*},\beta_0,\lambda_2,\Omega}(\mathrm{POIS2}(Q;\beta_*,\beta_0,\lambda) > \kappa_{2,\beta_0}(Q_T)) = P_{-\rho_{uv},\lambda_v}(\mathrm{POIS2}(Q;\infty,|\rho_{uv}|,\lambda_v) > \kappa_{2,\infty}(Q_T))$,

(c) $P_{\rho_{uv},\lambda_v}(\mathrm{POIS2}(Q;\infty,|\rho_{uv}|,\lambda_v) > \kappa_{2,\infty}(Q_T)) = P_{-\rho_{uv},\lambda_v}(\mathrm{POIS2}(Q;\infty,|\rho_{uv}|,\lambda_v) > \kappa_{2,\infty}(Q_T))$,

(d) $\lim_{\beta_0\to\pm\infty} \beta_{2*} = -\beta_* + 2\frac{\omega_{12}}{\omega_2^2} = \beta_* + 2\frac{\sigma_u\rho_{uv}}{\sigma_v}$, *and*

(e) $\lim_{\beta_0\to\pm\infty} \lambda_2 = \lambda$.

The reason that Q has the density $f_Q(q;-\rho_{uv},\lambda_v)$ (defined in (27)) in the limit expression in Lemma 17.1(b) can be seen clearly from the following lemma.

Lemma 17.2. *For any fixed* (β_*,λ,Ω), $\lim_{\beta_0\to\pm\infty} f_Q(q;\beta_{2*},\beta_0,\lambda_2,\Omega) = f_Q(q;-\rho_{uv},\lambda_v)$ *for all* 2×2 *variance matrices* q, *where* β_{2*} *and* λ_2 *satisfy* (19) *and* ρ_{uv} *and* λ_v *are defined in* (15) *and* (17), *respectively.*

Proof of Lemma 17.2. Given (β^*,λ^*), suppose the second point (β_2^*,λ_2^*) solves (39). In this case, by Lemma 15.1(b) and (e), we have

$$\begin{aligned}
&\lim_{\beta_0\to\pm\infty} \lambda_2^{1/2} c_{\beta_{2*}}(\beta_0,\Omega) = \lim_{\beta_0\to\pm\infty} -\lambda^{1/2} c_{\beta_*}(\beta_0,\Omega) = \pm\lambda^{1/2}/\sigma_v = \pm\lambda_v^{1/2} \quad \text{and} \\
&\lim_{\beta_0\to\pm\infty} \lambda_2^{1/2} d_{\beta_{2*}}(\beta_0,\Omega) = \lim_{\beta_0\to\pm\infty} \lambda^{1/2} d_{\beta_*}(\beta_0,\Omega) = \mp\lambda^{1/2}\frac{\rho_{uv}}{\sigma_v\big(1-\rho_{uv}^2\big)^{1/2}} = \mp\lambda_v^{1/2} r_{uv}.
\end{aligned} \qquad (65)$$

Using (24), (27), and (65), we obtain

$$\begin{aligned}
&\lim_{\beta_0\to\pm\infty}\lambda_2\big(c^2_{\beta_{2*}}+d^2_{\beta_{2*}}\big)=\lambda_v\big(1+r^2_{uv}\big)\quad\text{and}\\
&\lim_{\beta_0\to\pm\infty}\lambda_2\xi_{\beta_{2*}}(q):=\lim_{\beta_0\to\pm\infty}\lambda_2\big(c^2_{\beta_{2*}}q_S+2c_{\beta_{2*}}d_{\beta_{2*}}q_{ST}+d^2_{\beta_{2*}}q_T\big)\\
&\qquad=\lambda_v\big(q_S-2r_{uv}q_{ST}+r^2_{uv}q_T\big)\\
&\qquad=:\lambda_v\xi(q;-\rho_{uv}).
\end{aligned}\tag{66}$$

On the other hand, given (β^*, λ^*), suppose the second point (β^*_2, λ^*_2) solves (41). In this case, the minus sign on the rhs side of the first equality on the first line of (65) disappears, the quantity on the rhs side of the last equality on the first line of (65) becomes $\mp\lambda_v^{1/2}$, a minus sign is added to the rhs side of the first equality on the second line of (65), and the quantity on the rhs side of the last equality on the second line of (65) becomes $\pm\lambda_v^{1/2}r_{uv}$. These changes leave $\lambda_2 c^2_{\beta_{2*}}$, $\lambda_2 d^2_{\beta_{2*}}$, and $\lambda_2 c_{\beta_{2*}}d_{\beta_{2*}}$ unchanged from the case where (β^*_2, λ^*_2) solves (39). Hence, (66) also holds when (β^*_2, λ^*_2) solves (41).

Combining (66) with (25) (with (β_{2*}, λ_2) in place of (β_*, λ)) and (27) proves the result of the lemma. □

Proof of Theorem 6.2. By Theorem 3 of AMS, for all $(\beta_*, \beta_0, \lambda, \Omega)$,

$$\begin{aligned}
&P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi_{\beta_0}(Q)=1\big)+P_{\beta_{2*},\beta_0,\lambda_2,\Omega}\big(\phi_{\beta_0}(Q)=1\big)\\
&\quad\le P_{\beta_*,\beta_0,\lambda,\Omega}\big(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(Q_T)\big)\\
&\qquad+P_{\beta_{2*},\beta_0,\lambda_2,\Omega}\big(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(Q_T)\big).
\end{aligned}\tag{67}$$

That is, the test on the rhs maximizes the two-point average power for testing $\beta=\beta_0$ against (β_*, λ) and (β_{2*}, λ_2) for fixed known Ω.

Equation (67) and Lemma 17.1(a)–(c) establish the result of Theorem 6.2 by taking the $\limsup_{\beta_0\to\pm\infty}$ of the left-hand side and the $\liminf_{\beta_0\to\pm\infty}$ of the rhs. □

The proof of Comment (iv) to Theorem 6.2 is the same as that of Theorem 6.2, but in place of (67) it uses the inequality in Theorem 1 of Chernozhukov, Hansen, and Jansson (2009), that is, $\int P_{\beta_*,\beta_0,\lambda,\mu_\pi/\|\mu_\pi\|,\Omega}(\phi_{\beta_0}(Q)=1)\,d\,\mathrm{Unif}(\mu_\pi/\|\mu_\pi\|)\le\int P_{\beta_*,\beta_0,\lambda,\mu_\pi/\|\mu_\pi\|,\Omega}(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(Q_T))\,d\,\mathrm{Unif}(\mu_\pi/\|\mu_\pi\|)$, plus the fact that the rhs expression equals $P_{\beta_*,\beta_0,\lambda,\Omega}(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(Q_T))$ because the distribution of Q only depends on μ_π through $\lambda=\mu'_\pi\mu_\pi$.

Proof of Lemma 17.1. To prove part (a), we write

$$\begin{aligned}
&P_{\beta_*,\beta_0,\lambda,\Omega}\big(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(Q_T)\big)\\
&\quad=\int\int 1\big(\mathrm{POIS2}(q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(q_T)\big)\phi_k(s-c_{\beta_*}\mu_\pi)\phi_k(t-d_{\beta_*}\mu_\pi)\,ds\,dt,\quad\text{and}\\
&P_{\rho_{uv},\lambda_v}\big(\mathrm{POIS2}\big(Q;\infty,|\rho_{uv}|,\lambda_v\big)>\kappa_{2,\infty}(Q_T)\big)
\end{aligned}\tag{68}$$

$$= \int \int 1\big(\mathrm{POIS2}\big(q; \infty, |\rho_{uv}|, \lambda_v\big) > \kappa_{2,\infty}(q_T)\big)\phi_k\big(s - (\mp 1/\sigma_v)\mu_\pi\big)$$
$$\times \phi_k\big(t - (\mp r_{uv}/\sigma_v)\mu_\pi\big)\, ds\, dt,$$

where $\phi_k(x)$ for $x \in R^k$ denotes the density of k i.i.d. standard normal random variables, $\lambda = \mu_\pi' \mu_\pi$, $s, t \in R^k$, $q = [s : t]'[s : t]$, $q_T = t't$, $c_{\beta_*} = c_{\beta_*}(\beta_0, \Omega)$, $d_{\beta_*} = d_{\beta_*}(\beta_0, \Omega)$, the $\mp$ signs in the last line are both $+$ or both $-$, and the integral in the last line is the same whether both $\mp$ signs are $+$ or $-$ (by a change of variables calculation).

We have

$$\lim_{\beta_0 \to \pm\infty} \phi_k\big(s - c_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)\phi_k\big(t - d_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)$$
$$= \phi_k\big(s - (\mp 1/\sigma_v)\mu_\pi\big)\phi_k\big(t - (\mp r_{uv}/\sigma_v)\mu_\pi\big) \tag{69}$$

for all $s, t \in R^k$, by Lemma 15.1(b) and (e) and the smoothness of the standard normal density function. By (20) and (28) and Lemma 15.1(b) and (e), we have

$$\lim_{\beta_0 \to \pm\infty} \mathrm{POIS2}(q; \beta_0, \beta_*, \lambda) = \mathrm{POIS2}\big(q; \infty, |\rho_{uv}|, \lambda_v\big) \tag{70}$$

for all for 2×2 variance matrices q, for given $(\beta_*, \lambda, \Omega)$. In addition, we show below that $\lim_{\beta_0 \to \pm\infty} \kappa_{2,\beta_0}(q_T) = \kappa_{2,\infty}(q_T)$ for all $q_T \geq 0$. Combining these results gives the following convergence result:

$$\lim_{\beta_0 \to \pm\infty} 1\big(\mathrm{POIS2}(q; \beta_0, \beta_*, \lambda) > \kappa_{2,\beta_0}(q_T)\big) \cdot \phi_k\big(s - c_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)\phi_k\big(t - d_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)$$
$$= 1\big(\mathrm{POIS2}\big(q; \infty, |\rho_{uv}|, \lambda_v\big) > \kappa_{2,\infty}(q_T)\big)$$
$$\cdot\, \phi_k\big(s - (\mp 1/\sigma_v)\mu_\pi\big)\phi_k\big(t - (\mp r_{uv}/\sigma_v)\mu_\pi\big) \tag{71}$$

for all $[s : t]$ for which $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) > \kappa_{2,\infty}(q_T)$ or $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) < \kappa_{2,\infty}(q_T)$, where $[s : t]$, q and (q_S, q_{ST}, q_T) are functionally related by $q = [s : t]'[s : t]$ and the definitions in (25).

Given Lebesgue measure on the set of points $(s', t')' \in R^{2k}$, the induced measure on $(q_S, q_{ST}, q_T) = (s's, s't, t't) \in R^3$ is absolutely continuous with respect to (wrt) Lebesgue measure on R^3 with positive density only for positive definite q. (This follows from change of variables calculations. These calculations are analogous to those used to show that if $[S : T]$ has the multivariate normal density $\phi_k(s - (\mp 1/\sigma_v)\mu_\pi)\phi_k(t - (\mp r_{uv}/\sigma_v)\mu_\pi)$, then Q has the density $f_Q(q; \rho_{uv}, \lambda_v)$, which, viewed as a function of (q_S, q_{ST}, q_T), is a density wrt Lebesgue measure on R^3 that is positive only for positive definite q.) The Lebesgue measure of the set of (q_S, q_{ST}, q_T) for which $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) = \kappa_{2,\infty}(q_T)$ is zero. (This holds because (i) the definition of $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v)$ in (20) implies that the Lebesgue measure of the set of (q_S, q_{ST}) for which $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) = \kappa_{2,\infty}(q_T)$ is zero for all $q_T \geq 0$ and (ii) the Lebesgue measure of the set of (q_S, q_{ST}, q_T) for which $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) = \kappa_{2,\infty}(q_T)$ is obtained by integrating the set in (i) over $q_T \in R$ subject to the constraint that q is positive definite.) In turn, this implies that the Lebesgue measure of the set of $(s', t')'$ for which $\mathrm{POIS2}(q; \infty, |\rho_{uv}|, \lambda_v) =$

$\kappa_{2,\infty}(q_T)$ is zero. Hence, (71) verifies the a.s. (wrt Lebesgue measure on R^{2k}) convergence condition required for the application of the DCT to obtain part (a) using (68).

Next, to verify the dominating function requirement of the DCT, we need to show that

$$\sup_{\beta_0 \in R} \left| \phi_k\big(s - c_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)\phi_k\big(t - d_{\beta_*}(\beta_0, \Omega)\mu_\pi\big)\right| \tag{72}$$

is integrable wrt Lebesgue measure on R^{2k} (since the indicator functions in (71) are bounded by one). For any $0 < c < \infty$ and $m \in R$, we have

$$\begin{aligned}
\int \sup_{|m| \le c} \exp\big(-(x-m)^2/2\big)\,dx &= 2\int_0^\infty \sup_{|m|\le c} \exp\big(-x^2/2 + mx - m^2/2\big)\,dx \\
&\le 2\int_0^\infty \exp\big(-x^2/2 + cx\big)\,dx \\
&= 2\int_0^\infty \exp\big(-(x-c)^2/2 + c^2/2\big)\,dx < \infty,
\end{aligned} \tag{73}$$

where the first equality holds by symmetry. This result yields the integrability of the dominating function in (72) because $\phi_k(\cdot)$ is a product of univariate standard normal densities and $\sup_{\beta_0 \in R} |c_{\beta_*}(\beta_0, \Omega)| < \infty$ and $\sup_{\beta_0\in R} |d_{\beta_*}(\beta_0, \Omega)| < \infty$ are finite by Lemma 15.1(b) and (e) and continuity of $c_{\beta_*}(\beta_0, \Omega)$ and $d_{\beta_*}(\beta_0, \Omega)$ in β_0.

Hence, the DCT applies and it yields part (a).

It remains to show $\lim_{\beta_0 \to \pm\infty} \kappa_{2,\beta_0}(q_T) = \kappa_{2,\infty}(q_T)$ for all $q_T \ge 0$. As noted above, $\lim_{\beta_0\to\pm\infty} \mathrm{POIS2}(q; \beta_0, \beta_*, \lambda) = \mathrm{POIS}(q; \infty, |\rho_{uv}|, \lambda_v)$ for all 2×2 variance matrices q. Hence, $1(\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda) \le x) \to 1(\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v) \le x)$ as $\beta_0 \to \pm\infty$ for all $x \in R$ for which $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v) \ne x$. We have $P_{Q_1|Q_T}(\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v) = x|q_T) = 0$ for all $q_T \ge 0$ by the absolute continuity of $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ (by the functional form of $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ and the absolute continuity of Q_1 under $P_{Q_1|Q_T}(\cdot|q_T)$, whose density is given in (26)). Thus, by the DCT, for all $x \in R$,

$$\begin{aligned}
&\lim_{\beta_0\to\pm\infty} P_{Q_1|Q_T}\big(\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda) \le x|q_T\big) \\
&\quad = P_{Q_1|Q_T}\big(\mathrm{POIS2}\big(Q; \infty, |\rho_{uv}|, \lambda_v\big) \le x|q_T\big) \quad \text{and} \\
&\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda) \to_d \mathrm{POIS2}\big(Q; \infty, |\rho_{uv}|.\lambda_v\big) \quad \text{as } \beta_0 \to \pm\infty \text{ under } P_{Q_1|Q_T}(\cdot|q_T).
\end{aligned} \tag{74}$$

The second line of (74), coupled with the fact that $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ has a strictly increasing distribution function at its $1-\alpha$ quantile under $P_{Q_1|Q_T}(\cdot|q_T)$ for all $q_T \ge 0$ (which is shown below), implies that the $1-\alpha$ quantile of $\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ (i.e., $\kappa_{2,\beta_0}(q_T)$) converges as $\beta_0 \to \pm\infty$ to the $1-\alpha$ quantile of $\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ (i.e., $\kappa_{2,\infty}(q_T)$). This can be proved by contradiction. First, suppose $\delta := \limsup_{j\to\infty} \kappa_{2,j}(q_T) - \kappa_{2,\infty}(q_T) > 0$ (where each $j \in R$ represents some value of β_0 here). Then there exists a subsequence $\{m_j : j \ge 1\}$ of $\{j : j \ge 1\}$

such that $\delta = \lim_{j\to\infty} \kappa_{2,m_j}(q_T) - \kappa_{2,\infty}(q_T)$. We have

$$\begin{aligned}\alpha &= \lim_{j\to\infty} P_{Q_1|Q_T}\big(\mathrm{POIS2}(Q; m_j, \beta_*, \lambda) > \kappa_{2,m_j}(q_T)|q_T\big)\\ &\leq \lim_{j\to\infty} P_{Q_1|Q_T}\big(\mathrm{POIS2}(Q; m_j, \beta_*, \lambda) > \kappa_{2,\infty}(q_T) + \delta/2|q_T\big)\\ &= P_{Q_1|Q_T}\big(\mathrm{POIS2}\big(Q; \infty, |\rho_{uv}|, \lambda_v\big) > \kappa_{2,\infty}(q_T) + \delta/2|q_T\big)\\ &< P_{Q_1|Q_T}\big(\mathrm{POIS2}\big(Q; \infty, |\rho_{uv}|, \lambda_v\big) > \kappa_{2,\infty}(q_T)|q_T\big)\\ &= \alpha, \end{aligned} \tag{75}$$

where the first equality holds by the definition of $\kappa_{2,\beta_0}(q_T)$, the first inequality holds by the expression above for δ, the second equality holds by the first line of (74) with $x = \kappa_{2,\infty}(q_T) + \delta/2$, the second inequality holds because $\delta > 0$ and the distribution function of $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ is strictly increasing at its $1-\alpha$ quantile $\kappa_{2,\infty}(q_T)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ for all $q_T \geq 0$, and the last equality holds by the definition of $\kappa_{2,\infty}(q_T)$. Equation (75) is a contradiction, so $\delta \leq 0$. An analogous argument shows that $\liminf_{\beta_0\to\infty} \kappa_{2,\beta_0}(q_T) - \kappa_{2,\infty}(q_T) < 0$ does not hold. Hence, $\lim_{\beta_0\to\infty} \kappa_{2,\beta_0}(q_T) = \kappa_{2,\infty}(q_T)$. An analogous argument shows that $\liminf_{\beta_0\to-\infty} \kappa_{2,\beta_0}(q_T) = \kappa_{2,\infty}(q_T)$.

It remains to show that the distribution function of $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ is strictly increasing at its $1-\alpha$ quantile $\kappa_{2,\infty}(q_T)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ for all $q_T \geq 0$. This holds because (i) $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ is a nonrandom strictly increasing function of $(\xi(Q; \rho_{uv}), \xi(Q; -\rho_{uv}))$ conditional on $T = t$ (specifically, $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v) = C_{q_T} \times \sum_{j=0}^{\infty}[(\lambda_v \xi(Q; \rho_{uv}))^j + (\lambda_v \xi(Q; -\rho_{uv}))^j]/(4^j j!\Gamma(\nu + j + 1))$, where C_{q_T} is a constant that may depend on q_T, $\nu := (k-2)/2$, and $\Gamma(\cdot)$ is the gamma function, by (20) and (4.8) of AMS, which provides an expression for the modified Bessel function of the first kind $I_\nu(x)$), (ii) $\xi(Q; \rho_{uv}) = (S + r_{uv}T)'(S + r_{uv}T)$ and $\xi(Q; -\rho_{uv}) = (S - r_{uv}T)'(S - r_{uv}T)$ have the same noncentral χ^2_k distribution conditional on $T = t$ (because $[S : T]$ has a multivariate normal distribution with means matrix given by (18) and identity variance matrix), (iii) $(\xi(Q; \rho_{uv}), \xi(Q; -\rho_{uv}))$ has a positive density on R^2_+ conditional on $T = t$ and also conditional on $Q_T = q_T$ (because the latter conditional density is the integral of the former conditional density over t such that $t't = q_T$), and hence, (iv) $\mathrm{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ has a positive density on R_+ conditional on q_T for all $q_T \geq 0$. This completes the proof of part (a).

The proof of part (b) is the same as that of part (a), but with (i) $-c_{\beta_*}$ and $\pm 1/\sigma_v$ in place of c_{β_*} and $\mp 1/\sigma_v$, respectively, in (68), (69), and (71), and (ii) π_2 in place of π, where

$$\begin{aligned}&\pi_2 := Me_{1,k}, \qquad e_{1,k} := (1,0,\ldots)' \in R^k, \qquad M := \frac{\lambda^{1/2} g(\beta_0, \beta_*, \Omega)}{\big(e'_{1,k} Z' Z e_{1,k}\big)^{1/2}},\\ &g(\beta_0, \beta_*, \Omega) := \frac{d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0}}, \quad \text{and} \quad \lambda_2 := \mu'_{\pi_2}\mu_{\pi_2}.\end{aligned} \tag{76}$$

As defined, λ_2 satisfies (19) because

$$\lambda_2 := \mu'_{\pi_2}\mu_{\pi_2} = \pi'_2 Z' Z \pi_2 = M^2 e'_{1,k} Z' Z e_{1,k} = \lambda g^2(\beta_0, \beta_*, \Omega). \tag{77}$$

In addition, $\lambda_2 \to \lambda$ as $\beta_0 \to \pm\infty$ by (81) below. With the above changes, the proof of part (a) establishes part (b).

Part (c) holds because the test statistic $\text{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$ and critical value $\kappa_{2,\infty}(Q_T)$ only depend on ρ_{uv} and q_{ST} through $|\rho_{uv}|$ and $|q_{ST}|$, respectively, and the density $f_Q(q; \rho_{uv}, \lambda_v)$ of Q only depends on the sign of ρ_{uv} through $r_{uv} q_{ST}$. In consequence, a change of variables from (q_S, q_{ST}, q_T) to $(q_S, -q_{ST}, q_T)$ establishes the result of part (c).

To prove part (d), we have

$$\begin{aligned} d_{\beta_0} &= \left(a_0' \Omega^{-1} a_0\right)^{1/2} = \frac{\omega_2^2 \beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2}{\omega_1^2 \omega_2^2 - \omega_{12}^2} \left(a_0' \Omega^{-1} a_0\right)^{-1/2} \quad \text{and} \\ r_{\beta_0} &= e_1' \Omega^{-1} a_0 \left(a_0' \Omega^{-1} a_0\right)^{-1/2} = \frac{\omega_2^2 \beta_0 - \omega_{12}}{\omega_1^2 \omega_2^2 - \omega_{12}^2} \left(a_0' \Omega^{-1} a_0\right)^{-1/2}, \end{aligned} \tag{78}$$

where the first equalities on lines one and two hold by (2.7) of AMS and (19), respectively. Next, we have

$$\begin{aligned} \beta_{2*} &= \beta_0 - \frac{d_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)} \\ &= \frac{d_{\beta_0}(2\beta_0 - \beta_*) + 2r_{\beta_0}(\beta_* - \beta_0)\beta_0}{d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)} \\ &= \frac{\left(\omega_2^2\beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2\right)(2\beta_0 - \beta_*) + 2\left(\omega_2^2\beta_0 - \omega_{12}\right)\left(\beta_*\beta_0 - \beta_0^2\right)}{\left(\omega_2^2\beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2\right) + 2\left(\omega_2^2\beta_0 - \omega_{12}\right)(\beta_* - \beta_0)} \\ &= \frac{\beta_0^2\left(-\omega_2^2\beta_* - 4\omega_{12} + 2\omega_2^2\beta_* + 2\omega_{12}\right) + O(\beta_0)}{\beta_0^2\left(\omega_2^2 - 2\omega_2^2\right) + O(\beta_0)} \\ &= \frac{\left(\omega_2^2\beta_* - 2\omega_{12}\right) + o(1)}{-\omega_2^2 + o(1)} \\ &= -\beta_* + \frac{2\omega_{12}}{\omega_2^2} + o(1), \end{aligned} \tag{79}$$

where the third equality uses (78) and the two terms involving β_0^3 in the numerator of the rhs of the third equality cancel. Next, we have

$$-\beta_* + \frac{2\omega_{12}}{\omega_2^2} = \frac{2\left(\omega_{12} - \omega_2^2\beta_*\right) + \omega_2^2\beta_*}{\omega_2^2} = \frac{2\sigma_{uv} + \sigma_v^2\beta_*}{\sigma_v^2} = \beta_* + 2\frac{\sigma_{uv}}{\sigma_v^2} = \beta_* + 2\frac{\sigma_u\rho_{uv}}{\sigma_v}, \tag{80}$$

where the second equality uses (32) with $\beta = \beta_*$ and $\omega_2^2 = \sigma_v^2$.

Next, we prove part (e). We have

$$\left(\frac{\lambda_2}{\lambda}\right)^{1/2} = \left|\frac{d_{\beta_0} + 2r_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0}}\right|$$

$$= \left| \frac{\omega_2^2\beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2 + 2(\omega_2^2\beta_0 - \omega_{12})(\beta_* - \beta_0)}{\omega_2^2\beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2} \right|$$
$$= \left| \frac{\beta_0^2(\omega_2^2 - 2\omega_2^2) + \beta_0(-2\omega_{12} + 2\omega_2^2\beta_* + 2\omega_{12}) + \omega_1^2 - 2\omega_{12}\beta_*}{\omega_2^2\beta_0^2 - 2\omega_{12}\beta_0 + \omega_1^2} \right|$$
$$= 1 + o(1), \tag{81}$$

where the first equality holds by (19) and the second equality uses (78). □

Proof of Corollary 6.3. We have

$$\begin{aligned}
&\big(P_{\beta_*,\lambda,\Omega}\big(\text{RLength}\big(\text{CS}_\phi(Y)\big)=\infty\big) + P_{\beta_{2*},\lambda_2,\Omega}\big(\text{RLength}\big(\text{CS}_\phi(Y)\big)=\infty\big)\big)/2 \\
&= 1 - \lim_{\beta_0\to\infty}\Big[P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi(Q)=1\big) + \lim_{\beta_0\to\infty} P_{\beta_{2*},\beta_0,\lambda_2,\Omega}\big(\phi(Q)=1\big)\Big]/2 \\
&\geq P_{\rho_{uv},\lambda_v}\big(\text{POIS2}\big(Q;\infty,|\rho_{uv}|,\lambda_v\big) > \kappa_{2,\infty}(Q_T)\big),
\end{aligned} \tag{82}$$

where the equality holds by Theorem 5.1(a) with (β_*, λ) and $(\beta_{2*}, \lambda_{2*})$, $P_{\beta_*,\lambda,\Omega}(\cdot)$ is equivalent to $P_{\beta_*,\pi,\Omega}(\cdot)$, which appears in Theorem 5.1(a) (because events determined by $\text{CS}_\phi(Y)$ only depend on π through λ, since $\text{CS}_\phi(Y)$ is based on rotation-invariant tests), and the inequality holds by Theorem 6.2(a). This establishes the first result of part (a).

The second result of part (a) holds by the same calculations as in (82), but with LLength and $\beta_0 \to -\infty$ in place of RLength and $\beta_0 \to \infty$, respectively, using Theorem 5.1(b) in place of Theorem 5.1(a).

Part (b) holds by combining Theorem 5.1(c) and Theorem 6.2 because, as noted in Comment (iii) to Theorem 6.2, the lim sup on the left-hand side in Theorem 6.2 is the average of two equal quantities. □

Next, we prove Comment (ii) to Corollary 6.3. The proof is the same as that of Corollary 6.3, but using

$$\begin{aligned}
&\int P_{\beta_*,\lambda,\mu_\pi/\|\mu_\pi\|,\Omega}\big(\text{RLength}\big(\text{CS}_\phi(Y)\big)=\infty\big)\, d\,\text{Unif}\big(\mu_\pi/\|\mu_\pi\|\big) \\
&= 1 - \lim_{\beta_0\to\infty} P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi(Q)=1\big)
\end{aligned} \tag{83}$$

and likewise with (β_{2*}, λ_2) in place of (β_*, λ) in place of the first equality in (82). The proof of (83) is the same as the proof of Theorem 5.1(a) but with $Q_{\beta_0}(Y)$ and $Q_{T,\beta_0}(Y)$ replaced by $[S_{\beta_0}(Y), T_{\beta_0}(Y)]$, and $T_{\beta_0}(Y)$, respectively, throughout the proof, with $E_{\beta_*,\pi,\Omega}(\cdot)$ replaced by $\int E_{\beta_*,\lambda,\mu_\pi/\|\mu_\pi\|,\Omega}(\cdot)\, d\,\text{Uniform}(\mu_\pi/\|\mu_\pi\|)$ in (52), and using Lemma 16.1(a) and (c) in place of Lemma 16.1(f) when verifying the limit property (ii) needed for the DCT following (52).

Proof of Theorem 6.4. The proof is quite similar to, but much simpler than, the proof of part (a) of Lemma 17.1 with $\text{POIS2}(q; \beta_0, \beta_*, \lambda) > \kappa_{2,\beta_0}(q_T)$ in (68) replaced by $q_S > \chi^2_{k,1-\alpha}/k$ for the AR test, $q^2_{ST}/q_T > \chi^2_{1,1-\alpha}$ for the LM test, and $q_S - q_T + ((q_S - q_T)^2 + 4q^2_{ST})^{1/2} > 2\kappa_{LR,\alpha}(q_T)$ for the CLR test. The proof is much simpler because for the latter three tests neither the test statistics nor the critical values depend on β_0. The parameter β_0, for which the limit as $\beta_0 \to \pm\infty$ is being considered, only enters through the multivariate normal densities in (68). The limits of these densities and an integrable dominating function for them have already been provided in the proof of Lemma 17.1(a). The indicator function that appears in (71) is bounded by one regardless of which test appears in the indicator function. In addition, $P_{\beta_*,\rho_{uv},\lambda_v}(AR = \chi^2_{k,1-\alpha}) = 0$ and $P_{\beta_*,\rho_{uv},\lambda_v}(LM = \chi^2_{1,1-\alpha}) = 0$ because the AR statistic has a noncentral χ^2_k distribution with noncentrality parameter λ_v under $P_{\beta_*,\rho_{uv},\lambda_v}$ (since $S \sim N(\mu_\pi/\sigma_v, I_k)$ by Lemma 6.1 and (18)) and the conditional distribution of the LM statistic given T under $P_{\beta_*,\rho_{uv},\lambda_v}$ is a noncentral χ^2 distribution.

Next, we show $P_{\beta_*,\rho_{uv},\lambda_v}(LR = \kappa_{LR,\alpha}(Q_T)) = 0$. Let $J = AR - LM$. Then $2LR = J + LM - Q_T + ((J + LM - Q_T)^2 + 4LM \cdot Q_T)^{1/2}$. We can write $Q = [S : T]'[S : T]$, where $[S : T]$ has a multivariate normal distribution with means matrix given by (18) and identity variance matrix. As shown below, conditional on $T = t$, LM and J have independent noncentral χ^2 distributions with 1 and $k - 1$ degrees of freedom, respectively. This implies that (i) the distribution of LR conditional on $T = t$ is absolutely continuous, (ii) $P_{\beta_*,\rho_{uv},\lambda_v}(LR = \kappa_{LR,\alpha}(Q_T)|T = t) = 0$ for all $t \in R^k$, and (iii) $P_{\beta_*,\rho_{uv},\lambda_v}(LR = \kappa_{LR,\alpha}(Q_T)) = 0$. It remains to show that conditional on $Q_T = q_T$, LM and J have independent noncentral χ^2 distributions. We can write $LM = S'P_TS$ and $J = S'(I_k - P_T)S$, where $P_T := T(T'T)^{-1}T'$ and S has a multivariate normal with identity variance matrix. This implies that P_TS and $(I_k - P_T)S$ are independent conditional on $T = t$ and LM and J have independent noncentral χ^2 distributions conditional on $T = t$ for all $t \in R^k$. This completes the proof. □

18. Proof of Theorem 8.1

The proof of Theorem 8.1(a) uses the following lemma.

Lemma 18.1. *Suppose* $b_{1x} = 1 + \delta_x/x$ *and* $b_{2x} = 1 - \delta_x/x$, *where* $\delta_x \to \delta_\infty \neq 0$ *as* $x \to \infty$, $K_{j1x} = (b_{jx}x)^\eta$ *for some* $\eta \in R$ *for* $j = 1, 2$, *and* $K_{j2x} \to K_\infty \in (0, \infty)$ *as* $x \to \infty$ *for* $j = 1, 2$. *Then* (a) *as* $x \to \infty$,

$$\log\left(K_{11x}K_{12x}e^{b_{1x}x} + K_{21x}K_{22x}e^{b_{2x}x}\right) - x - \eta\log x - \log K_\infty \\ \to \delta_\infty + \log\left(1 + e^{-2\delta_\infty}\right) \quad \textit{and}$$

(b) *the function* $s(y) := y + \log(1 + e^{-2y})$ *for* $y \in R$ *is infinitely differentiable, symmetric about zero, strictly increasing for* $y > 0$, *and hence, strictly increasing in* $|y|$ *for* $|y| > 0$.

Proof of Lemma 18.1. Part (a) holds by the following:

$$\log\left(K_{11x}K_{12x}e^{b_{1x}x} + K_{21x}K_{22x}e^{b_{2x}x}\right) - x - \eta\log x - \log K_\infty \\ = \log\left(K_{11x}K_{12x}e^{b_{1x}x}\left(1 + \frac{K_{21x}K_{22x}}{K_{11x}K_{12x}}e^{(b_{2x}-b_{1x})x}\right)\right) - x - \eta\log x - \log K_\infty$$

$$
\begin{aligned}
&= b_{1x}x + \log K_{11x} + \log(K_{12x}/K_\infty) + \log\left(1 + \frac{K_{21x}K_{22x}}{K_{11x}K_{12x}} e^{(b_{2x}-b_{1x})x}\right) - x - \eta \log x \\
&= \delta_x + \eta \log(b_{1x}) + \log(K_{12x}/K_\infty) + \log\left(1 + \frac{K_{21x}K_{22x}}{K_{11x}K_{12x}} e^{-2\delta_x}\right) \\
&\to \delta_\infty + \log\bigl(1 + e^{-2\delta_\infty}\bigr), \qquad (84)
\end{aligned}
$$

where the third equality uses $b_{1x}x - x = \delta_x$, $\log K_{11x} = \eta \log(b_{1x}x) = \eta \log(b_{1x}) + \eta \log(x)$, and $b_{2x} - b_{1x} = -2\delta_x/x$, and the convergence uses $\log(b_{1x}) = \log(1 + o(1)) \to 0$, $K_{12x}/K_\infty \to 1$, $K_{21x}/K_{11x} = (b_{2x}/b_{1x})^\eta = 1 + o(1)$, and $K_{22x}/K_{12x} \to 1$.

The function $s(y)$ is infinitely differentiable because $\log(x)$ and e^{-2y} are. The function $s(y)$ is symmetric about zero because

$$
\begin{aligned}
& y + \log\bigl(1 + e^{-2y}\bigr) = -y + \log\bigl(1 + e^{2y}\bigr) \\
\Leftrightarrow \quad & 2y = \log\bigl(1 + e^{2y}\bigr) - \log\bigl(1 + e^{-2y}\bigr) = \log\left(\frac{1 + e^{2y}}{1 + e^{-2y}}\right) = \log\bigl(e^{2y}\bigr) = 2y. \qquad (85)
\end{aligned}
$$

The function $s(y)$ is strictly increasing for $y > 0$ because

$$
\frac{d}{dy} s(y) = 1 - \frac{2e^{-2y}}{1 + e^{-2y}} = \frac{1 - e^{-2y}}{1 + e^{-2y}} = \frac{e^{2y} - 1}{e^{2y} + 1}, \qquad (86)
$$

which is positive for $y > 0$. We have $s(y) = s(|y|)$ because $s(y)$ is symmetric about zero, and $(d/d|y|)s(|y|) > 0$ for $|y| > 0$ by (86). Hence, $s(y)$ is strictly increasing in $|y|$ for $|y| > 0$. □

Proof of Theorem 8.1. Without loss in generality, we prove the results for the case where $\operatorname{sgn}(d_{\beta_*})$ is the same for all terms in the sequence as $\lambda d^2_{\beta_*} \to \infty$. Given (3), without loss of generality, we can suppose that

$$
S = c_{\beta_*}\mu_\pi + Z_S \quad \text{and} \quad T = d_{\beta_*}\mu_\pi + Z_T, \qquad (87)
$$

where Z_S and Z_T are independent $N(0^k, I_k)$ random vectors.

We prove part (c) first. The distribution of Q depends on μ_π only through λ. In consequence, without loss of generality, we can assume that $Y := \mu_\pi/\lambda^{1/2} \in R^k$ does not vary as $\lambda d^2_{\beta_*}$ and $\lambda^{1/2}c_{\beta_*}$ vary. The following establishes the a.s. convergence of the one-sided LM test statistic: as $\lambda d^2_{\beta_*} \to \infty$ and $\lambda^{1/2}c_{\beta_*} \to c_\infty$,

$$
\begin{aligned}
\frac{Q_{ST}}{Q_T^{1/2}} &= \frac{(c_{\beta_*}\mu_\pi + Z_S)'(d_{\beta_*}\mu_\pi + Z_T)}{\bigl((d_{\beta_*}\mu_\pi + Z_T)'(d_{\beta_*}\mu_\pi + Z_T)\bigr)^{1/2}} \\
&= \frac{(c_{\beta_*}\mu_\pi + Z_S)'(d_{\beta_*}\mu_\pi + Z_T)}{\bigl(d^2_{\beta_*}\lambda\bigr)^{1/2}\bigl(1 + o_{a.s.}(1)\bigr)} \\
&= \frac{\bigl(c_{\beta_*}\mu_\pi/\lambda^{1/2} + Z_S/\lambda^{1/2}\bigr)'(\operatorname{sgn}(d_{\beta_*})\mu_\pi + O_{a.s.}\bigl(1/|d_{\beta_*}|\bigr)}{\bigl(1 + o_{a.s.}(1)\bigr)}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\operatorname{sgn}(d_{\beta_*}) Y' Z_S + \operatorname{sgn}(d_{\beta_*}) \lambda^{1/2} c_{\beta_*} + O_{a.s.}\left(\frac{(\lambda c_{\beta_*}^2)^{1/2}}{(\lambda d_{\beta_*}^2)^{1/2}} \right) + O_{a.s.}\left(\frac{1}{(\lambda d_{\beta_*}^2)^{1/2}} \right) \right) \\
&\quad \times \left(1 + o_{a.s.}(1)\right) \\
&\to_{a.s.} \operatorname{sgn}(d_{\beta_*}) Y' Z_S + \operatorname{sgn}(d_{\beta_*}) c_\infty \\
&=: LM_{1\infty} \sim N\left(\operatorname{sgn}(d_{\beta_*}) c_\infty, 1\right),
\end{aligned} \tag{88}
$$

where the first equality holds by (4) and (87), the second equality holds using $d_{\beta_*}\mu_\pi + Z_T = (\lambda d_{\beta_*}^2)^{1/2}(d_{\beta_*}\mu_\pi/(\lambda d_{\beta_*}^2)^{1/2} + o_{a.s.}(1))$ since $\lambda d_{\beta_*}^2 \to \infty$, the convergence holds because $\lambda d_{\beta_*}^2 \to \infty$ and $\lambda^{1/2} c_{\beta_*} \to c_\infty$, and the limit random variable $LM_{1\infty}$ has a $N(\operatorname{sgn}(d_{\beta_*})c_\infty, 1)$ distribution because $\operatorname{sgn}(d_{\beta_*})Y'Z_S \sim N(0,1)$ (since $Z_S \sim N(0^k, I_k)$ and $\|Y\| = 1$).

The a.s. convergence in (88) implies convergence in distribution by the DCT applied to $1(Q_{ST}/Q_T^{1/2} \le y)$ for any fixed $y \in R$. In consequence, we have

$$
\begin{aligned}
P\left(LM > \chi^2_{1,1-\alpha}\right) &= P\left(\left(Q_{ST}/Q_T^{1/2}\right)^2 > \chi^2_{1,1-\alpha}\right) \\
\to \quad P\left(LM_{1\infty}^2 > \chi^2_{1,1-\alpha}\right) &= P\left(\chi^2_1(c_\infty^2) > \chi^2_{1,1-\alpha}\right)
\end{aligned} \tag{89}
$$

as $\lambda d_{\beta_*}^2 \to \infty$ and $\lambda^{1/2} c_{\beta_*} \to c_\infty$, which establishes part (c).

To prove Theorem 8.1(a), we apply Lemma 18.1 to a realization of the random vectors Z_S and Z_T with

$$
\begin{aligned}
x &:= \left(\lambda d_{\beta_*}^2 Q_T\right)^{1/2}, \\
b_{1x} x &:= \left(\lambda \xi_{\beta_*}(Q; \beta_0, \Omega)\right)^{1/2} := \lambda^{1/2}\left(c_{\beta_*}^2 Q_S + 2 c_{\beta_*} d_{\beta_*} Q_{ST} + d_{\beta_*}^2 Q_T\right)^{1/2}, \\
b_{2x} x &:= \lambda^{1/2}\left(c_{\beta_*}^2 Q_S - 2 c_{\beta_*} d_{\beta_*} Q_{ST} + d_{\beta_*}^2 Q_T\right)^{1/2}, \\
K_{11x} &:= (b_{1x} x)^{-(k-1)/2}, \\
K_{12x} &:= \frac{(b_{1x} x)^{1/2} I_{(k-2)/2}(b_{1x} x)}{e^{b_{1x} x}} \\
K_{21x} &:= (b_{2x} x)^{-(k-1)/2}, \quad \text{and} \\
K_{22x} &:= \frac{(b_{2x} x)^{1/2} I_{(k-2)/2}(b_{2x} x)}{e^{b_{2x} x}}.
\end{aligned} \tag{90}
$$

Thus, we take $\eta := -(k-1)/2$.

We have

$$
Q_T = (d_{\beta_*}\mu_\pi + Z_T)'(d_{\beta_*}\mu_\pi + Z_T) = \lambda d_{\beta_*}^2\left(1 + o_{a.s.}(1)\right). \tag{91}
$$

This implies that $x = (\lambda d_{\beta_*}^2)(1 + o_{a.s.}(1))$. Thus, $x \to \infty$ a.s. since $\lambda d_{\beta_*}^2 \to \infty$ by assumption.

The conditions $\lambda d_{\beta_*}^2 \to \infty$ and $\lambda^{1/2} c_{\beta_*} \to c_\infty \in R$ imply that $b_{1x} x \to \infty$ and $b_{2x} x \to \infty$ as $x \to \infty$. In consequence, by the properties of the modified Bessel function of the first kind, $I_{(k-2)/2}(x)$ for x large, for example, see Lebedev (1965, p. 136),

$$
\lim_{b_{1x} x \to \infty} K_{12x} = 1/(2\pi)^{1/2} \quad \text{and} \quad \lim_{b_{2x} x \to \infty} K_{22x} = 1/(2\pi)^{1/2}. \tag{92}
$$

Hence, the assumptions of Lemma 18.1 on K_{j2x} for $j=1,2$ hold with $K_\infty = 1/(2\pi)^{1/2}$.

Next, we have

$$\begin{aligned}
b_{1x} &= \big(\lambda c_{\beta_*}^2 Q_S + 2\lambda c_{\beta_*} d_{\beta_*} Q_{ST} + \lambda d_{\beta_*}^2 Q_T\big)^{1/2}/x \\
&= \left(1 + \frac{2\lambda c_{\beta_*} d_{\beta_*} Q_{ST}}{\big(\lambda d_{\beta_*}^2 Q_T\big)^{1/2} x} + \frac{\lambda c_{\beta_*}^2 Q_S}{x^2}\right)^{1/2} \\
&= \left(1 + \frac{2\lambda^{1/2} c_{\beta_*} \operatorname{sgn}(d_{\beta_*})}{x} \frac{Q_{ST}}{Q_T^{1/2}} + \frac{\lambda c_{\beta_*}^2 Q_S}{x^2}\right)^{1/2} \\
&= 1 + \big(1 + o_{a.s.}(1)\big)^{-1/2} \left(\frac{2\lambda^{1/2} c_{\beta_*} \operatorname{sgn}(d_{\beta_*})}{x} \frac{Q_{ST}}{Q_T^{1/2}} + \frac{\lambda c_{\beta_*}^2 Q_S}{x^2}\right),
\end{aligned} \tag{93}$$

where the fourth equality holds by the mean value theorem because $\lambda^{1/2} c_{\beta_*} = O(1)$, $x \to \infty$ a.s., and $Q_{ST}/Q_T^{1/2} = O(1)$ a.s. (by (88)) imply that the term in parentheses on the last line of (93) is $o_{a.s.}(1)$.

From (93), we have

$$\begin{aligned}
\delta_x &= \big(1 + o_{a.s.}(1)\big)^{-1/2} \left(2\lambda^{1/2} c_{\beta_*} \operatorname{sgn}(d_{\beta_*}) \frac{Q_{ST}}{Q_T^{1/2}} + \frac{\lambda c_{\beta_*}^2 Q_S}{x}\right) \\
&\to \quad 2c_\infty \operatorname{sgn}(d_{\beta_*}) LM_{1\infty} =: \delta_\infty \quad \text{a.s.}
\end{aligned} \tag{94}$$

using (88). This verifies the convergence condition of Lemma 18.1 on δ_x with $\delta_\infty \neq 0$ a.s. (by the absolute continuity of Z_S). Hence, Lemma 18.1 applies with $x, b_{1x}, \ldots$ as in (90).

Let ξ_{β_*} abbreviate $\xi_{\beta_*}(Q; \beta_0, \Omega) = c_{\beta_*}^2 Q_S + 2c_{\beta_*} d_{\beta_*} Q_{ST} + d_{\beta_*}^2 Q_T$. Let $\xi_{\beta_{2*}} = c_{\beta_*}^2 Q_S - 2c_{\beta_*} d_{\beta_*} Q_{ST} + d_{\beta_*}^2 Q_T$. So, $b_{1x} x = (\lambda \xi_{\beta_*})^{1/2}$ and $b_{2x} x = (\lambda \xi_{\beta_{2*}})^{1/2}$. Let

$$\begin{aligned}
\tau(\beta_*, \lambda, Q_T) &:= -\big(\lambda d_{\beta_*}^2 Q_T\big)^{1/2} + \frac{k-1}{2} \log\big(\big(\lambda d_{\beta_*}^2 Q_T\big)^{1/2}\big) - \log K_\infty \\
&= -x - \eta \log x - \log K_\infty,
\end{aligned} \tag{95}$$

where the equality holds using the definitions in (90) and $K_\infty = 1/(2\pi)^{1/2}$ by (92).

Given the definitions of $\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda)$ and $x, b_{1x}, \ldots$ in (28) and (90), respectively, Lemma 18.1(a) gives

$$\begin{aligned}
&\log\big(\mathrm{POIS2}(Q; \beta_0, \beta_*, \lambda)\big) + \log\big(2\psi_2(Q_T; \beta_0, \beta_*, \lambda)\big) + \tau(\beta_*, \lambda, Q_T) \\
&\quad = \log\big((\lambda \xi_{\beta_*})^{-(k-2)/4} I_{(k-2)/2}\big((\lambda \xi_{\beta_*})^{1/2}\big) + (\lambda \xi_{\beta_{2*}})^{-(k-2)/4} I_{(k-2)/2}\big((\lambda \xi_{\beta_{2*}})^{1/2}\big)\big) \\
&\qquad + \tau(\beta_*, \lambda, Q_T) \\
&\quad = \log\Bigg((\lambda \xi_{\beta_*})^{-(k-1)/4} \frac{(\lambda \xi_{\beta_*})^{1/4} I_{(k-2)/2}\big((\lambda \xi_{\beta_*})^{1/2}\big)}{e^{(\lambda \xi_{\beta_*})^{1/2}}} e^{(\lambda \xi_{\beta_*})^{1/2}} \\
&\qquad + (\lambda \xi_{\beta_{2*}})^{-(k-1)/4} \frac{(\lambda \xi_{\beta_{2*}})^{1/4} I_{(k-2)/2}\big((\lambda \xi_{\beta_{2*}})^{1/2}\big)}{e^{(\lambda \xi_{\beta_{2*}})^{1/2}}} e^{(\lambda \xi_{\beta_{2*}})^{1/2}}\Bigg) + \tau(\beta_*, \lambda, Q_T)
\end{aligned}$$

$$
\begin{aligned}
&= \log\bigl(K_{11x}K_{12x}e^{b_{1x}x} + K_{21x}K_{22x}e^{b_{2x}x}\bigr) - x - \eta \log x - \log K_\infty \\
&\to_{a.s.} \delta_\infty + \log\bigl(1 + e^{-2\delta_\infty}\bigr) \\
&= s(\delta_\infty) \\
&= s\bigl(2c_\infty|LM_{1\infty}|\bigr),
\end{aligned} \tag{96}
$$

where $\psi_2(Q_T;\beta_0,\beta_*,\lambda)$ is defined in (28), $LM^2_{1\infty} \sim \chi^2_1(c^2_\infty)$ is defined in (88), the first equality holds by the definition of $\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)$ in (28), the third equality uses the definitions in (90) and (95), the convergence holds by Lemma 18.1(a), the second last equality holds by the definition of $s(y)$ in Lemma 18.1(b), and the last equality holds because $\delta_\infty := 2c_\infty \operatorname{sgn}(d_{\beta_*})LM_{1\infty}$ (see (94)), and $s(y)$ is symmetric around zero by Lemma 18.1(b).

Applied to $1(\log(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)) + \log(2\psi_2(Q_T;\beta_0,\beta_*,\lambda)) + \tau(\beta_*,\lambda,Q_T) \le w)$ for any $w \in R$, equation (96) and the DCT give

$$
\begin{aligned}
&\log\bigl(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)\bigr) + \log\bigl(2\psi_2(Q_T;\beta_0,\beta_*,\lambda)\bigr) + \tau(\beta_*,\lambda,Q_T) \to_d s(\delta_\infty) \\
&\quad = s\bigl(2c_\infty|LM_{1\infty}|\bigr).
\end{aligned} \tag{97}
$$

Now we consider the behavior of the critical value function for the POIS2 test, $\kappa_{2,\beta_0}(q_T)$, where q_T denotes a realization of Q_T. We are interested in the power of the POIS2 test. So, we are interested in the behavior of $\kappa_{2,\beta_0}(q_T)$ for q_T sequences as $\lambda d^2_{\beta_*} \to \infty$ and $\lambda^{1/2}c_{\beta_*} \to c_\infty$ that are generated when the true parameters are (β_*,λ). This behavior is given in (91) to be $q_T = \lambda d^2_{\beta_*}(1+o(1))$ a.s. under (β_*,λ).

Up to this point in the proof, the parameters (β_*,λ) have played a duel role. First, they denote the parameter values against which the POIS2 test is designed to have optimal two-sided power and, hence, determine the form of the POIS2 test statistic. Second, they denote the true values of β and λ (because we are interested in the power of the POIS2 test when the (β_*,λ) values for which it is designed are the true values). Here, where we discuss the behavior of the critical value function $\kappa_{2,\beta_0}(\cdot)$, (β_*,λ) only play the former role. The true value of β is β_0 and the true value of λ we denote by λ_0. The function $\kappa_{2,\beta_0}(\cdot)$ depends on (β_*,λ) because the POIS2 test statistic does, but the null distribution that determines $\kappa_{2,\beta_0}(\cdot)$ does not depend on (β_*,λ). In spite of this, the values q_T which are of interest to us, do depend on (β_*,λ) as noted in the previous paragraph.

The function $\kappa_{2,\beta_0}(\cdot)$ is defined in (29). Its definition depends on the conditional null distribution of Q_1 given $Q_T = q_T$ whose density $f_{Q_1|Q_T}(\cdot|q_T)$ is given in (26). This density depends on k, but not on any other parameters, such as β_0, $\lambda_0 = \mu'_{\pi_0}\mu_{\pi_0}$, or Ω. In consequence, for the purposes of determining the properties of $\kappa_{2,\beta_0}(\cdot)$ we can suppose that $\beta_0 = 0$, $\mu_{\pi_0} = 1^k/\|1^k\|$, $\lambda_0 = 1$, and $\Omega = I_2$. In this case,

$$
S = Z_S \sim N\bigl(0^k, I_k\bigr), \qquad T = \mu_{\pi_0} + Z_T \sim N(\mu_{\pi_0}, I_k), \tag{98}
$$

and S and T are independent (using $d_{\beta_0}(\beta_0,\Omega) = b'_0\Omega b_0(b'_0\Omega b_0)^{-1/2}\det(\Omega)^{-1/2} = 1$ since $b_0 = (1,\beta_0)' = (1,0)'$).

We now show that $\kappa_{2,\beta_0}(q_T)$ satisfies

$$\begin{aligned}&\log\bigl(\kappa_{2,\beta_0}(q_T)\bigr)+\log\bigl(2\psi_2(q_T;\beta_0,\beta_*,\lambda)\bigr)+\tau(\beta_*,\lambda,q_T)\\&\quad\to s\bigl(2|c_\infty|\bigl(\chi^2_{1,1-\alpha}\bigr)^{1/2}\bigr)\quad\text{as } q_T\to\infty\end{aligned}\tag{99}$$

for any sequence of constants $q_T=\lambda d^2_{\beta_*}(1+o(1))$ as $\lambda d^2_{\beta_*}\to\infty$.

Suppose random variables $\{W_m: m\geq 1\}$ and W satisfy: (i) $W_m\to_d W$ as $m\to\infty$, (ii) W has a continuous and strictly increasing distribution function at its $1-\alpha$ quantile κ_∞, and (iii) $P(W_m>\kappa_m)=\alpha$ for all $m\geq 1$ for some constants $\{\kappa_m: m\geq 1\}$. Then $\kappa_m\to\kappa_\infty$. This holds because if $\limsup_{m\to\infty}\kappa_m>\kappa_\infty$, then there is a subsequence $\{v_m\}$ of $\{m\}$ such that $\lim_{m\to\infty}\kappa_{v_m}=\kappa_{\infty+}>\kappa_\infty$ and $\alpha=P(W_{v_m}>\kappa_{v_m})\to P(W>\kappa_{\infty+})<P(W>\kappa_\infty)=\alpha$, which is a contradiction, and likewise $\liminf_{m\to\infty}\kappa_m<\kappa_\infty$ leads to a contradiction.

We apply the result in the previous paragraph with (a) $\{W_m: m\geq 1\}$ given by $\log(\text{POIS2}(Q;\beta_0,\beta_*,\lambda))+\log(2\psi_2(q_T;\beta_0,\beta_*,\lambda))+\tau(\beta_*,\lambda,q_T)$ under the null hypothesis and conditional on $T=t$ with $t=1^k q_T^{1/2}/k^{1/2}$ for some sequence of constants $q_T=\lambda d^2_{\beta_*}(1+o(1))\to\infty$ as $\lambda d^2_{\beta_*}\to\infty$, (b) $W=s(2c_\infty|S'1^k/k^{1/2}|)$, where $S'1^k/k^{1/2}\sim N(0,1)$, (c) κ_m equal to $\log(\kappa_{2,\beta_0}(q_T))+\log(2\psi_2(q_T;\beta_0,\beta_*,\lambda))+\tau(\beta_*,\lambda,q_T)$, and (d) $\kappa_\infty=s(2|c_\infty|(\chi^2_{1,1-\alpha})^{1/2})$.

We need to show conditions (i)–(iii) above hold. Condition (ii) holds straightforwardly for W as in (b) given the normal distribution of S, the functional form of $s(y)$, and $c_\infty\neq 0$.

By definition of $\kappa_{2,\beta_0}(q_T)$, under the null hypothesis, $P_{Q_1|Q_T}(\text{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(q_T)|q_T)=\alpha$ for all $q_T\geq 0$; see (29). This implies that the invariant POIS2 test is similar. In turn, this implies that under the null hypothesis $P(\text{POIS2}(Q;\beta_0,\beta_*,\lambda)>\kappa_{2,\beta_0}(q_T)|T=t)=\alpha$ for all $t\in R^k$ because Theorem 1 of Moreira (2009) shows that any invariant similar test has null rejection probability α conditional on T. This verifies condition (iii) because the log function is monotone and the last two summands of W_m and κ_m defined in (a) and (c) above cancel.

Next, we show that condition (i) holds. Given (98) and $t=1^k q_T^{1/2}/k^{1/2}$, under the null and conditional on $T=t$, we have

$$\frac{Q_{ST}}{Q_T^{1/2}}=\frac{S't}{(t't)^{1/2}}=S'1^k/k^{1/2}\sim\chi^2_1,\tag{100}$$

which does not depend on $\lambda d^2_{\beta_*}$ or $\lambda^{1/2}c_{\beta_*}$. Hence, in place of the a.s. convergence result for $Q_{ST}/Q_T^{1/2}$ as $\lambda d^2_{\beta_*}\to\infty$ and $\lambda^{1/2}c_{\beta_*\to}c_\infty$ in (88), which applies under the alternative hypothesis with true parameters (β_*,λ), we have $Q_{ST}/Q_T^{1/2}=S'1^k/k^{1/2}$ under the null hypothesis for all $\lambda d^2_{\beta_*}$ and $\lambda^{1/2}c_{\beta_*}$. Using this in place of (88), the unconditional a.s. convergence result in (96), established in (90)–(96), goes through as a conditional on $T=t$ a.s. result without any further changes. In consequence, the convergence in distribution result in (97) also holds conditional on $T=t$ a.s., but with $s(2c_\infty|S'1^k/k^{1/2}|)$ in place of $s(2c_\infty|LM_{1\infty}|)$. This verifies condition (i).

Given that conditions (i)–(iii) hold, we obtain $\kappa_m\to\kappa_\infty$ as $\lambda d^2_{\beta_*}\to\infty$ for κ_m and κ_∞ defined in (c) and (d), respectively, above. This establishes (99).

Given (99), we have

$$
\begin{aligned}
&P_{\beta_*,\beta_0,\lambda,\Omega}\big(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda) > \kappa_{2,\beta_0}(Q_T)\big) \\
&\quad = P_{\beta_*,\beta_0,\lambda,\Omega}\big(\log\big(\mathrm{POIS2}(Q;\beta_0,\beta_*,\lambda)\big) + \log\big(2\psi_2(Q_T;\beta_0,\beta_*,\lambda)\big) + \tau(\beta_*,\lambda,Q_T) \\
&\quad > \log\big(\kappa_{2,\beta_0}(Q_T)\big) + \log\big(2\psi_2(Q_T;\beta_0,\beta_*,\lambda)\big) + \tau(\beta_*,\lambda,Q_T)\big) \\
&\quad \to_d P\big(s\big(2c_\infty|LM_{1\infty}|\big) > s\big(2c_\infty\big|\chi^2_{1,1-\alpha}\big|\big)\big) \\
&\quad = P\big(LM^2_{1\infty} > \chi^2_{1,1-\alpha}\big) \\
&\quad = P\big(\chi^2_1\big(c^2_\infty\big) > \chi^2_{1,1-\alpha}\big),
\end{aligned}
\tag{101}
$$

where the second last equality uses the fact that $s(y)$ is symmetric and strictly increasing for $y > 0$ by Lemma 18.1(b). Equation (101) establishes part (a) of the theorem.

Now we establish part (b) of the theorem. Let

$$
J := S' M_T S, \tag{102}
$$

where $M_T := I_k - P_T$ and $P_T := T(T'T)^{-1}T'$. It follows from (6) that

$$
LM = S' P_T S \quad \text{and} \quad Q_S = LM + J. \tag{103}
$$

By (91), $Q_T = \lambda d^2_{\beta_*}(1 + o_{a.s.}(1)) \to \infty$ a.s. as $\lambda d^2_{\beta_*} \to \infty$ when the true parameters are (β_*, λ). By (103) and some algebra, we have $(Q_S - Q_T)^2 + 4LM \cdot Q_T = (LM - J + Q_T)^2 + 4LM \cdot J$. This and the definition of LR in (6) give

$$
LR = \frac{1}{2}\Big(LM + J - Q_T + \sqrt{(LM - J + Q_T)^2 + 4LM \cdot J}\Big). \tag{104}
$$

Using a mean-value expansion of the square-root expression in (104) about $(LM - J + Q_T)^2$, we have

$$
\sqrt{(LM - J + Q_T)^2 + 4LM \cdot J} = LM - J + Q_T + (2\sqrt{\zeta})^{-1} 4LM \cdot J \tag{105}
$$

for an intermediate value ζ between $(LM - J + Q_T)^2$ and $(LM - J + Q_T)^2 + 4LM \cdot J$. It follows that

$$
LR = LM + o(1) \quad \text{a.s.} \tag{106}
$$

because $Q_T \to \infty$ a.s., $LM = O(1)$ a.s., and $J = O(1)$ a.s. as $\lambda d^2_{\beta_*} \to \infty$ and $\lambda^{1/2} c_{\beta_*} \to c_\infty \in R$, which imply that $(\sqrt{\zeta})^{-1} = o(1)$ a.s. These properties of LM and J hold because $LM = S'P_T S \le S'S$, $J = S'M_T S \le S'S$, and, using (87), we have $S'S = (c_{\beta_*}\mu_\pi + Z_S)'(c_{\beta_*}\mu_\pi + Z_S) = O(1)$ a.s. because $\|c_{\beta_*}\mu_\pi\|^2 = \lambda c^2_{\beta_*} = O(1)$ by assumption.

The critical value function for the CLR test, $\kappa_{LR,\alpha}(\cdot)$, depends only on k and α; see Lemma 3(c) and (3.5) in AMS. It is well known in the literature that $\kappa_{LR,\alpha}(\cdot)$ satisfies $\kappa_{LR,\alpha}(q_T) \to \chi^2_{1,1-\alpha}$ as $q_T \to \infty$, for example, see Moreira (2003, Proposition 1). Hence, we have

$$\begin{aligned} P_{\beta_*,\beta_0,\lambda,\Omega}\big(LR > \kappa_{LR,\alpha}(Q_T)\big) &= P_{\beta_*,\beta_0,\lambda,\Omega}\big(LM + o_{a.s.}(1) > \chi^2_{1,1-\alpha} + o_{a.s.}(1)\big) \\ &= P_{\beta_*,\beta_0,\lambda,\Omega}\big(LM + o_p(1) > \chi^2_{1,1-\alpha}\big) \\ &\to P\big(\chi^2_1(c^2_\infty) > \chi^2_{1,1-\alpha}\big) \end{aligned} \tag{107}$$

as $\lambda d^2_{\beta_*} \to \infty$ and $\lambda^{1/2} c_{\beta_*} \to c_\infty$, where the first equality holds by (106), $Q_T \to \infty$ a.s. by (91), and $\lim_{q_T\to\infty} \kappa_{LR,\alpha}(q_T) = \chi^2_{1,1-\alpha}$ and the convergence holds by part (c) of the theorem. This establishes part (b) of the theorem. □

Proof of Theorem 8.2. First, we establish part (a)(i) of the theorem. By (31) with $\beta = \beta_*$ and $\Sigma = \Sigma_*$, we have

$$\Omega(\beta_*, \Sigma_*) = \begin{bmatrix} \omega_1^2 & \omega_{12} \\ \omega_{12} & \omega_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_u^2 + 2\sigma_{uv}\beta_* + \sigma_v^2\beta_*^2 & \sigma_{uv} + \sigma_v^2\beta_* \\ \sigma_{uv} + \sigma_v^2\beta_* & \sigma_v^2 \end{bmatrix}. \tag{108}$$

Using this, we obtain, as $\rho_{uv} \to \pm 1$,

$$\begin{aligned} c_{\beta_*} &= c_{\beta_*}\big(\beta_0, \Omega(\beta_*, \Sigma_*)\big) \\ &= (\beta_* - \beta_0)\big(\omega_1^2 - 2\beta_0\omega_{12} + \omega_2^2\beta_0^2\big)^{-1/2} \\ &= (\beta_* - \beta_0)\big(\sigma_u^2 + 2\sigma_{uv}\beta_* + \sigma_v^2\beta_*^2 - 2\beta_0\big(\sigma_{uv} + \sigma_v^2\beta_*\big) + \sigma_v^2\beta_0^2\big)^{-1/2} \\ &= (\beta_* - \beta_0)\big(\sigma_u^2 + 2(\beta_* - \beta_0)\sigma_u\sigma_v\rho_{uv} + (\beta_* - \beta_0)^2\sigma_v^2\big)^{-1/2} \\ &\to (\beta_* - \beta_0)\big(\sigma_u^2 \pm 2(\beta_* - \beta_0)\sigma_u\sigma_v + (\beta_* - \beta_0)^2\sigma_v^2\big)^{-1/2} \\ &= (\beta_* - \beta_0)/\big|\sigma_u \pm (\beta_* - \beta_0)\sigma_v\big|, \end{aligned} \tag{109}$$

where the second equality uses (3), the convergence only holds if $\sigma_u \pm (\beta_* - \beta_0)\sigma_v \neq 0$, and the fourth equality uses $\sigma_{uv} = \sigma_u\sigma_v\rho_{uv}$. This proves part (a)(i).

To prove part (a)(ii), we have

$$\begin{aligned} d_{\beta_*} = d_{\beta_*}\big(\beta_0, \Omega(\beta_*, \Sigma_*)\big) &= b_*'\Omega b_0\big(b_0'\Omega b_0\big)^{-1/2}\det(\Omega)^{-1/2} \\ &= \big(\omega_1^2 - \omega_{12}(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_*\big)\cdot\big(\omega_1^2 - 2\beta_0\omega_{12} + \omega_2^2\beta_0^2\big)^{-1/2} \\ &\quad \cdot\big(\omega_1^2\omega_2^2 - \omega_{12}^2\big)^{-1/2}, \end{aligned} \tag{110}$$

where the second equality holds by (3). The second multiplicand on the rhs of (110) converges to $|\sigma_u \pm (\beta_* - \beta_0)\sigma_v|^{-1}$ provided $\sigma_u \pm (\beta_* - \beta_0)\sigma_v \neq 0$ by the calculations in (109).

The first multiplicand on the rhs of (110) satisfies, as $\rho_{uv} \to \pm 1$,

$$\begin{aligned} &\omega_1^2 - \omega_{12}(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_* \\ &\quad = \sigma_u^2 + 2\sigma_{uv}\beta_* + \sigma_v^2\beta_*^2 - \big(\sigma_{uv} + \sigma_v^2\beta_*\big)(\beta_0 + \beta_*) + \sigma_v^2\beta_0\beta_* \end{aligned}$$

$$
\begin{aligned}
&= \sigma_u^2 + \sigma_u\sigma_v\rho_{uv}(\beta_* - \beta_0) \\
&\to \sigma_u\big(\sigma_u \pm \sigma_v(\beta_* - \beta_0)\big), \qquad (111)
\end{aligned}
$$

where the first equality uses (108) and the second equality holds by simple algebra and $\sigma_{uv} = \sigma_u\sigma_v\rho_{uv}$.

The reciprocal of the square of the third multiplicand on the rhs of (110) satisfies, as $\rho_{uv} \to \pm 1$,

$$
\begin{aligned}
\omega_1^2\omega_2^2 - \omega_{12}^2 &= \big(\sigma_u^2 + 2\sigma_u\sigma_v\rho_{uv}\beta_* + \sigma_v^2\beta_*^2\big)\sigma_v^2 - \big(\sigma_u\sigma_v\rho_{uv} + \sigma_v^2\beta_*\big)^2 \\
&\to \big(\sigma_u^2 \pm 2\sigma_u\sigma_v\beta_* + \sigma_v^2\beta_*^2\big)\sigma_v^2 - \big(\pm\sigma_u\sigma_v + \sigma_v^2\beta_*\big)^2 \\
&= (\sigma_u \pm \sigma_v\beta_*)^2\sigma_v^2 - (\pm\sigma_u + \sigma_v\beta_*)^2\sigma_v^2 \\
&= 0, \qquad (112)
\end{aligned}
$$

where the first equality holds by (108) and $\sigma_{uv} = \sigma_u\sigma_v\rho_{uv}$.

Combining (110)–(112) and $\lambda > 0$ proves part (a)(ii).

Next, we establish part (b) of the theorem. Using the definition of $c_\beta(\beta_0, \Omega)$ in (3), we have

$$
\begin{aligned}
\lim_{\rho_\Omega \to \pm 1} c_{\beta_*}(\beta_0, \Omega) &= \lim_{\rho_\Omega \to \pm 1} (\beta_* - \beta_0)\big(b_0'\Omega b_0\big)^{-1/2} \\
&= \lim_{\rho_\Omega \to \pm 1} (\beta_* - \beta_0)\big(\omega_1^2 - 2\beta_0\omega_1\omega_2\rho_\Omega + \omega_2^2\beta_0^2\big)^{-1/2} \\
&= (\beta_* - \beta_0)/|\omega_1 \mp \omega_2\beta_0|, \qquad (113)
\end{aligned}
$$

where the third equality holds provided $\omega_1 \mp \omega_2\beta_0 \neq 0$. This establishes part (b)(i) of the theorem.

Using the definition of $d_\beta(\beta_0, \Omega)$ in (3) and $b_* := (1, \beta_*)'$, we have

$$
\begin{aligned}
\lim_{\rho_\Omega \to \pm 1} d_{\beta_*}(\beta_0, \Omega) &= \lim_{\rho_\Omega \to \pm 1} b_*'\Omega b_0\big(b_0'\Omega b_0\big)^{-1/2}\det(\Omega)^{-1/2} \\
&= \lim_{\rho_\Omega \to \pm 1} \big(\omega_1^2 - \omega_1\omega_2\rho_\Omega(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_*\big) \\
&\quad \cdot \big(\omega_1^2 - 2\beta_0\omega_1\omega_2\rho_\Omega + \omega_2^2\beta_0^2\big)^{-1/2} \cdot \big(\omega_1^2\omega_2^2 - \omega_1^2\omega_2^2\rho_\Omega^2\big)^{-1/2} \\
&= (\omega_1 \mp \omega_2\beta_0)(\omega_1 \mp \omega_2\beta_*) \cdot \frac{1}{|\omega_1 \mp \omega_2\beta_0|} \cdot \frac{1}{\omega_1\omega_2} \cdot \lim_{\rho_\Omega \to \pm 1} \frac{1}{\big(1 - \rho_\Omega^2\big)^{1/2}} \\
&= \operatorname{sgn}\big((\omega_1 \mp \omega_2\beta_0)(\omega_1 \mp \omega_2\beta_*)\big) \cdot \infty, \qquad (114)
\end{aligned}
$$

where the third and fourth equalities hold provided $\omega_1 \mp \omega_2\beta_0 \neq 0$ and $\omega_1 \mp \omega_2\beta_* \neq 0$. This and $\lambda > 0$ establish part (b)(ii) of the theorem.

Part (c)(i) is proved as follows:

$$c_{\beta_*} = \frac{\beta_* - \beta_0}{\left(\sigma_u^2 + 2(\beta_* - \beta_0)\sigma_u\sigma_v\rho_{uv} + (\beta_* - \beta_0)^2\sigma_v^2\right)^{1/2}}$$
$$\to \mp\frac{1}{\sigma_v} \quad \text{as } (\rho_{uv}, \beta_0) \to (1, \pm\infty), \tag{115}$$

where the first equality holds by (109) and the convergence holds by considering only the dominant β_0 terms. The same result holds as $(\rho_{uv}, \beta_0) \to (1, \pm\infty)$ because ρ_{uv} enters the middle expression in (115) only through a term that does not affect the limit.

Part (c)(ii) is proved using the expression for d_{β_*} in (110). By (112), the third multiplicand in (110), which does not depend on β_0, diverges to infinity when $\rho_{uv} \to 1$ or -1. The product of the first two multiplicands on the rhs of (110) equals

$$\frac{\omega_1^2 - \omega_{12}(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_*}{\left(\omega_1^2 - 2\beta_0\omega_{12} + \omega_2^2\beta_0^2\right)^{1/2}} = \frac{\sigma_u^2 + \sigma_u\sigma_v\rho_{uv}(\beta_* - \beta_0)}{\left(\sigma_u^2 + 2(\beta_* - \beta_0)\sigma_u\sigma_v\rho_{uv} + (\beta_* - \beta_0)^2\sigma_v^2\right)^{1/2}}$$
$$\to \mp\frac{\sigma_u\sigma_v}{\sigma_v} = \mp\sigma_u \quad \text{as } (\rho_{uv}, \beta_0) \to (1, \pm\infty), \tag{116}$$

where the equality uses the calculations in the first three lines of (109) and (111) and the convergence holds by considering only the dominant β_0 terms. When $(\rho_{uv}, \beta_0) \to (-1, \pm\infty)$, the limit in (116) is $\pm\sigma_u$ because ρ_{uv} enters multiplicatively in the dominant β_0 term in the numerator. In both cases, the product of the first two multiplicands on the rhs of (110) converges to a nonzero constant and the third multiplicand diverges to infinity. Hence, d_{β_*} diverges to $+\infty$ or $-\infty$ and $\lambda d_{\beta_*}^2 \to \infty$ since $\lambda > 0$, which completes the proof.

Part (d)(i) holds because

$$c_{\beta_*} = \frac{\beta_* - \beta_0}{\left(\omega_1^2 - 2\beta_0\omega_1\omega_2\rho_\Omega + \omega_2^2\beta_0^2\right)^{1/2}} \to \mp\frac{1}{\omega_2} \quad \text{as } (\rho_\Omega, \beta_0) \to (1, \pm\infty), \tag{117}$$

where the equality uses (113). The same convergence holds as $(\rho_\Omega, \beta_0) \to (1, \pm\infty)$ because ρ_{uv} enters the middle expression in (117) only through a term that does not affect the limit.

Part (d)(ii) is proved using the expression for d_{β_*} in (114):

$$d_{\beta_*} = \frac{\left(\omega_1^2 - \omega_1\omega_2\rho_\Omega(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_*\right)}{\left(\omega_1^2 - 2\beta_0\omega_1\omega_2\rho_\Omega + \omega_2^2\beta_0^2\right)^{1/2}} \cdot \left(\omega_1^2\omega_2^2 - \omega_1^2\omega_2^2\rho_\Omega^2\right)^{-1/2},$$
$$\frac{\left(\omega_1^2 - \omega_1\omega_2\rho_\Omega(\beta_0 + \beta_*) + \omega_2^2\beta_0\beta_*\right)}{\left(\omega_1^2 - 2\beta_0\omega_1\omega_2\rho_\Omega + \omega_2^2\beta_0^2\right)^{1/2}} \to \frac{\pm\left(\omega_2^2\beta_* - \omega_1\omega_2\right)}{\omega_2} = \mp(\omega_1 - \omega_2\beta_*), \quad \text{and} \tag{118}$$
$$\left(\omega_1^2\omega_2^2 - \omega_1^2\omega_2^2\rho_\Omega^2\right)^{-1/2} \to \infty \quad \text{as } (\rho_\Omega, \beta_0) \to (1, \pm\infty).$$

Hence, $\lambda d_{\beta_*}^2 \to \infty$ as $(\rho_\Omega, \beta_0) \to (1, \pm\infty)$ provided $\omega_1 - \omega_2\beta_* \neq 0$. When $(\rho_\Omega, \beta_0) \to (-1, \pm\infty)$, the limit in the second line of (118) is $\pm(\omega_2^2\beta_* + \omega_1\omega_2)/\omega_2 = \pm(\omega_1 + \omega_2\beta_*)$, and hence, $\lambda d_{\beta_*}^2 \to \infty$ provided $\omega_1 + \omega_2\beta_* \neq 0$, which completes the proof. □

19. Proofs of Theorem 13.1 and Lemmas 14.1 and 14.2

Proof of Theorem 13.1. By Corollary 2 and Comment 2 to Corollary 2 of Andrews, Moreira, and Stock (2004), for all $(\beta_*, \beta_0, \lambda, \Omega)$,

$$P_{\beta_*,\beta_0,\lambda,\Omega}\big(\phi_{\beta_0}(Q)=1\big) \leq P_{\beta_*,\beta_0,\lambda,\Omega}\big(\text{POIS}(Q; \beta_0, \beta_*) > \kappa_{\beta_0}(Q_T)\big). \tag{119}$$

That is, the test on the rhs is the (one-sided) POIS test for testing $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$ for fixed known Ω and any $\lambda \geq 0$ under H_1.

We use the DCT to show

$$\begin{aligned} &\lim_{\beta_0\to\pm\infty} P_{\beta_*,\beta_0,\lambda,\Omega}\big(\text{POIS}(Q; \beta_0, \beta_*) > \kappa_{\beta_0}(Q_T)\big) \\ &\quad = P_{\rho_{uv},\lambda_v}\big(\text{POIS}(Q; \infty, \rho_{uv}) > \kappa_\infty(Q_T)\big). \end{aligned} \tag{120}$$

Equations (119) and (120) imply that the result of Theorem 13.1 holds.

By (34), (37), and Lemma 15.1(b) and (e),

$$\lim_{\beta_0\to\pm\infty} \text{POIS}(q; \beta_0, \beta_*) = \text{POIS}(q; \infty, \rho_{uv}) \tag{121}$$

for all 2×2 variance matrices q, for given (β_*, π, Ω).

The proof of (120) is the same as the proof of Lemma 17.1(a), but with $\text{POIS}(Q; \beta_0, \beta_*)$, $\kappa_{\beta_0}(Q_T)$, $\text{POIS}(Q; \infty, \rho_{uv})$, and $\kappa_\infty(Q_T)$ in place of $\text{POIS2}(Q; \beta_0, \beta_*, \lambda)$, $\kappa_{2,\beta_0}(Q_T)$, $\text{POIS2}(Q; \infty, |\rho_{uv}|, \lambda_v)$, and $\kappa_{2,\infty}(Q_T)$, respectively, using (121) in place of (70), and using the results (established below) that (i) the Lebesgue measure of the set of (q_S, q_{ST}, q_T) for which $\text{POIS}(q; \infty, \rho_{uv}) = \kappa_\infty(q_T)$ is zero, (ii) $P_{Q_1|Q_T}(\text{POIS}(Q; \infty, \rho_{uv}) = x|q_T) = 0$ for all $q_T \geq 0$, and (iii) the distribution function of $\text{POIS}(Q; \infty, \rho_{uv})$ is strictly increasing at its $1-\alpha$ quantile $\kappa_\infty(q_T)$ under $P_{Q_1|Q_T}(\cdot|q_T)$ for all $q_T \geq 0$.

Condition (i) holds because (a) $\text{POIS}(q; \infty, \rho_{uv}) = q_S + 2r_{uv}q_{ST}$ (see (37)) implies that the Lebesgue measure of the set of (q_S, q_{ST}) for which $q_S + 2r_{uv}q_{ST} = \kappa_\infty(q_T)$ is zero for all q_T and (b) the Lebesgue measure of the set of (q_S, q_{ST}, q_T) for which $q_S + 2r_{uv}q_{ST} = \kappa_\infty(q_T)$ is obtained by integrating the set in (a) over $q_T \in R$ subject to the constraint that q is positive definite.

Condition (ii) holds by the absolute continuity of $\text{POIS}(Q; \infty, \rho_{uv})$ under $P_{Q_1|Q_T}(\cdot|q_T)$ (by the functional form of $\text{POIS}(Q; \infty, \rho_{uv})$ and the absolute continuity of Q_1 under $P_{Q_1|Q_T}(\cdot|q_T)$, whose density is given in (26)).

Condition (iii) holds because we can write $\text{POIS}(Q; \infty, \rho_{uv}) = S'S + 2r_{uv}S'T = (S + r_{uv}T)'(S + r_{uv}T) - r_{uv}^2T'T$, where $[S : T]$ has a multivariate normal distribution with means matrix given by (18) and identity variance matrix, and hence, $\text{POIS}(Q; \infty, \rho_{uv})$ has a shifted noncentral χ^2 distribution conditional on $T = t$. In consequence, it has a positive density on $(r_{uv}^2t't, \infty) = (r_{uv}^2q_T, \infty)$ conditional on $T = t$ and also conditional on $Q_T = q_T$ (because the latter conditional density is the integral of the former conditional density over t such that $t't = q_T$). This completes the proof. □

Proof of Lemma 14.1. First, we show that (42) implies the equation for λ_2 in (40). By the expression $d_\beta = a'\Omega^{-1}a_0(a_0'\Omega^{-1}a_0)^{-1/2}$ given in (2.7) in AMS, where $a := (\beta, 1)'$ and

$a_0 := (\beta_0, 1)'$, for any $\beta \in R$,

$$\begin{aligned} d_\beta - d_{\beta_0} &= (a - a_0)'\Omega^{-1} a_0 \big(a_0' \Omega^{-1} a_0\big)^{-1/2} \\ &= (\beta - \beta_0) e_1' \Omega^{-1} a_0 \big(a_0' \Omega^{-1} a_0\big)^{-1/2} := (\beta - \beta_0) r_{\beta_0}, \end{aligned} \tag{122}$$

where $e_1 := (1, 0)'$ and the last equality holds by the definition of r_{β_0}.

Substituting (122) into the second equation in (42) gives

$$\begin{aligned} &\lambda_2^{1/2} d_{\beta_{2*}} = \pm \lambda^{1/2} d_{\beta_*} \\ \text{iff} \quad &\lambda_2^{1/2} \big(d_{\beta_0} + r_{\beta_0}(\beta_{2*} - \beta_0)\big) = \pm \lambda^{1/2} \big(d_{\beta_0} + r_{\beta_0}(\beta_* - \beta_0)\big) \\ \text{iff} \quad &\lambda_2^{1/2} d_{\beta_0} = \pm \lambda^{1/2} \big(d_{\beta_0} + r_{\beta_0}(\beta_* - \beta_0)\big) - r_{\beta_0} \lambda_2^{1/2} (\beta_{2*} - \beta_0). \end{aligned} \tag{123}$$

Given the definition of c_β in (3), the first equation in (42) can be written as

$$\lambda_2^{1/2}(\beta_{2*} - \beta_0) = \mp \lambda^{1/2} (\beta_* - \beta_0). \tag{124}$$

Substituting this into (123) yields

$$\begin{aligned} &\lambda_2^{1/2} d_{\beta_{2*}} = \pm \lambda^{1/2} d_{\beta_*} \\ \text{iff} \quad &\lambda_2^{1/2} d_{\beta_0} = \pm \lambda^{1/2} \big(d_{\beta_0} + 2 r_{\beta_0}(\beta_* - \beta_0)\big) \\ \text{iff} \quad &\lambda_2^{1/2} = \pm \lambda^{1/2} \frac{d_{\beta_0} + 2 r_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0}}. \end{aligned} \tag{125}$$

The square of the equation in the last line in (125) is the equation for λ_2 in (40).

Next, we show that (42) implies the equation for β_{2*} in (40). Using (124), the first equation in (42) can be written as

$$\beta_{2*} = \beta_0 \mp \frac{\lambda^{1/2}}{\lambda_2^{1/2}} (\beta_* - \beta_0). \tag{126}$$

This combined with the equation for $\lambda^{1/2}/\lambda_2^{1/2}$ obtained from the last line of (125) gives

$$\beta_{2*} = \beta_0 - \frac{d_{\beta_0}}{d_{\beta_0} + 2 r_{\beta_0}(\beta_* - \beta_0)} (\beta_* - \beta_0), \tag{127}$$

where a minus sign appears because the $\mp$ sign in (126) gets multiplied by the $\pm$ sign in the last line of (125), which yields a minus sign in both cases. Equation (127) is the same as the first condition in (40). This completes the proof that (42) implies (40).

Now, we prove the converse. We suppose (40) holds. Taking the square root of the second equation in (40) gives

$$\lambda_2^{1/2} = \pm \lambda^{1/2} \frac{d_{\beta_0} + 2 r_{\beta_0}(\beta_* - \beta_0)}{d_{\beta_0}}, \tag{128}$$

where the $\pm$ sign means that this equation holds either with $+$ or with $-$. Substituting this into the first equation in (40) gives (126), which is the same as (124), and (124) is the first equation in (42).

The second equation in (42) is given by (123). Given that the first equation in (42) holds, the second equation in (42) is given in (125). The last line of (125) holds by (128). This completes the proof that (40) implies (42). □

Proof of Lemma 14.2. The proof of part (a) of the lemma is essentially the same as that of Theorem 8(b) in AMS. The only change is to note that when (β_{2*}, λ_2) satisfies (41), we have $\tau^* = \tau_2^*$, $\delta^* = -\delta_2^*$, and $\delta_{\max} = |\delta^*| = |\delta_2^*|$ (using the notation in AMS). Because $\delta_{\max} = |\delta^*| = |\delta_2^*|$, we obtain $\sqrt{\delta^2} - \sqrt{\delta_{\max}^2} = 0$ and the remainder of the proof of Theorem 8(b) goes through as is.

The proof of part (b) of the lemma is quite similar to the proof of Theorem 8(c) of AMS. The latter proof first considers the case where "(β_{2*}, λ_2) does not satisfy the second condition of (39)." This needs to be changed to "(β_{2*}, λ_2) does not satisfy the second condition of (39) or (41)." With this change, the rest of that part of the proof of Theorem 8(c) goes through unchanged.

The remaining cases (where both (39) and (41) fail) to consider are (i) when the second condition in (39) holds and the first condition in (39) fails and (ii) when the second condition in (41) holds and the first condition in (41) fails. These are mutually exclusive scenarios because the second conditions in (39) and (41) are incompatible. The proof of Theorem 8(c) of AMS considers case (i) and proves the result of Theorem 8(c) for that case. The proof of Theorem 8(c) for case (ii) is quite similar to that for case (i) using (A.21) in AMS because $\delta^* = -\delta_2^*$, $\delta_{\max} = |\delta^*| = |\delta_2^*| > 0$, and $\tau^* \neq \tau_2^*$ imply that $\operatorname{sgn}(\delta^*) = -\operatorname{sgn}(\delta_2^*)$ and $\tau^* \operatorname{sgn}(\delta^*) \neq -\tau_2^* \operatorname{sgn}(\delta_2^*)$. This last inequality shows that the expression in (A.21) in AMS is a continuous function of $Q_{ST} Q_T^{-1/2}$ that is not even. (Note that (A.21) in AMS has a typo: the quantity $\tau_2^* \operatorname{sgn}(\delta^*)$ in its second summand should be $\tau_2^* \operatorname{sgn}(\delta_2^*)$.) □

20. Structural error variance matrices under distant alternatives and distant null hypotheses

Here, we compute the structural error variance matrices in scenarios 1 and 2 considered in (9) and (10) in Section 4. By design, the reduced-form variance matrix Ω is the same for β_0 and $\beta_{*,}$ and hence, does not vary between these two scenarios.

In scenario 1 in (9), the structural error variance matrix under H_0 is $\Sigma(\beta_0, \Omega)$, defined in (32). Under $H_1 : \beta = \beta_*$, as $|\beta_*| \to \infty$, we have

$$\begin{aligned} \lim_{\beta_* \to \pm\infty} \rho_{uv}(\beta_*, \Omega) &= \lim_{\beta_* \to \pm\infty} \frac{\omega_{12} - \omega_2^2 \beta_*}{\left(\omega_1^2 - 2\omega_{12}\beta_* + \omega_2^2 \beta_*^2\right)^{1/2} \omega_2} = \mp 1 \quad \text{and} \\ \lim_{|\beta_*| \to \infty} \sigma_u^2(\beta_*, \Omega)/\sigma_v^2(\beta_*, \Omega) &= \frac{\omega_1^2 - 2\omega_{12}\beta_* + \omega_2^2 \beta_*^2}{\omega_2^2} = \infty, \end{aligned} \tag{129}$$

where $\rho_{uv}(\beta_*, \Omega)$, $\sigma_u^2(\beta_*, \Omega)$, and $\sigma_v^2(\beta_*, \Omega)$ are defined just below (32). Equation (129) shows that, for standard power envelope calculations, when the alternative hypothesis value β_* is large in absolute value the structural variance matrix under H_1 exhibits correlation close to one in absolute value and a large ratio of structural to reduced-form variances.

In scenario 2 in (10), the structural error variance error matrix under H_* is $\Sigma(\beta_*, \Omega)$. Under $H_0 : \beta = \beta_0$, by exactly the same argument as in (129) with β_0 in place of β_*, we obtain

$$\lim_{\beta_0 \to \pm\infty} \rho_{uv}(\beta_0, \Omega) = \mp 1 \quad \text{and} \quad \lim_{|\beta_0| \to \infty} \sigma_u^2(\beta_0, \Omega)/\sigma_v^2(\beta_0, \Omega) = \infty. \tag{130}$$

So, in scenario 2, when the null hypothesis value β_0 is large in absolute value the structural variance matrix under H_0 exhibits correlation close to one in absolute value and a large ratio of structural to reduced-form variances.

From a testing perspective, it is natural and time honored to fix the null hypothesis value β_0 and consider power as the alternative hypothesis value β_* varies. On the other hand, a confidence set is the set of null hypothesis values β_0 for which one does not reject $H_0 : \beta = \beta_0$. Hence, for a given true value β_*, the false coverage probabilities of the confidence set equal one minus its power as one varies $H_0 : \beta = \beta_0$. Thus, from the confidence set perspective, it is natural to fix β_* and consider power as β_0 varies.

21. Transformation of the β_0 versus β_* testing problem to a 0 versus $\overline{\beta}_*$ testing problem

In this section, we transform the general testing problem of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$ for $\pi \in R^k$ and fixed Ω to a testing problem of $H_0 : \overline{\beta} = 0$ versus $H_1 : \overline{\beta} = \overline{\beta}_*$ for some $\overline{\pi} \in R^k$ and some fixed $\overline{\Omega}$ whose diagonal elements equal one. This is done using the transformations given footnotes 7 and 8 of AMS, which argue that there is no loss in generality in the AMS numerical results to take $\omega_1^2 = \omega_2^2 = 1$ and $\beta_0 = 0$. These results help link the numerical work done in this paper with that done in AMS.

Starting with the model in (1), we transform the model based on (y_1, y_2) with parameters (β, π) and fixed reduced-form variance matrix Ω to a model based on $(\widetilde{y}_1, y_2)$ with parameters $(\widetilde{\beta}, \pi)$ and fixed reduced-form variance matrix $\widetilde{\Omega}$, where

$$\begin{aligned}
\widetilde{y}_1 &:= y_1 - y_2\beta_0, \\
\widetilde{\beta} &:= \beta - \beta_0, \quad \text{and} \\
\widetilde{\Omega} &:= \operatorname{Var}\left(\begin{pmatrix} \widetilde{y}_1 \\ y_2 \end{pmatrix}\right) = \operatorname{Var}\left(\begin{bmatrix} 1 & -\beta_0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \\
&= \begin{bmatrix} \omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2 & \omega_{12} - \omega_2^2\beta_0 \\ \omega_{12} - \omega_2^2\beta_0 & \omega_2^2 \end{bmatrix}.
\end{aligned} \tag{131}$$

The transformed testing problem is $H_0 : \widetilde{\beta} = 0$ versus $H_1 : \widetilde{\beta} = \widetilde{\beta}_*$, where $\widetilde{\beta}_* = \beta_* - \beta_0$, with parameter π and reduced-form variance matrix $\widetilde{\Omega}$.

The matrix $\widetilde{\Omega}$ does not have diagonal elements equal to one, so we transform the model based on $(\widetilde{y}_1, y_2)$ with parameters $(\widetilde{\beta}, \pi)$ and fixed reduced-form variance matrix $\widetilde{\Omega}$ to a model based on $(\overline{y}_1, \overline{y}_2)$ with parameters $(\overline{\beta}, \overline{\pi})$ and fixed reduced-form variance matrix $\overline{\Omega}$, where[14]

$$\begin{aligned}
\overline{y}_1 &:= \frac{\widetilde{y}_1}{\widetilde{\omega}_1} = \frac{y_1 - y_2\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}} \\
\overline{y}_2 &:= \frac{1}{\widetilde{\omega}_2} y_2 = \frac{1}{\omega_2} y_2, \\
\overline{\beta} &:= \frac{\widetilde{\omega}_2}{\widetilde{\omega}_1}\widetilde{\beta} = \frac{\omega_2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}}(\beta - \beta_0), \quad \text{and} \\
\overline{\pi} &:= \frac{1}{\widetilde{\omega}_2}\pi = \frac{1}{\omega_2}\pi.
\end{aligned} \tag{132}$$

In addition, we have

$$\begin{aligned}
\overline{\Omega} &:= \operatorname{Var}\begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} = \operatorname{Var}\left(\begin{bmatrix} 1/\widetilde{\omega}_1 & 0 \\ 0 & 1/\widetilde{\omega}_2 \end{bmatrix}\begin{pmatrix} \widetilde{y}_1 \\ y_2 \end{pmatrix}\right) \\
&= \begin{bmatrix} 1/\widetilde{\omega}_1 & 0 \\ 0 & 1/\widetilde{\omega}_2 \end{bmatrix}\widetilde{\Omega}\begin{bmatrix} 1/\widetilde{\omega}_1 & 0 \\ 0 & 1/\widetilde{\omega}_2 \end{bmatrix} \\
&= \begin{bmatrix} 1/\widetilde{\omega}_1 & 0 \\ 0 & 1/\omega_2 \end{bmatrix}\begin{bmatrix} \omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2 & \omega_{12} - \omega_2^2\beta_0 \\ \omega_{12} - \omega_2^2\beta_0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} 1/\widetilde{\omega}_1 & 0 \\ 0 & 1/\omega_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \dfrac{\omega_{12} - \omega_2^2\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\omega_2} \\ \dfrac{\omega_{12} - \omega_2^2\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\omega_2} & 1 \end{bmatrix}.
\end{aligned} \tag{133}$$

The transformed testing problem is $H_0 : \overline{\beta} = 0$ versus $H_1 : \overline{\beta} = \overline{\beta}_*$, where

$$\overline{\beta}_* = \frac{\omega_2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}}(\beta_* - \beta_0), \tag{134}$$

with parameter $\overline{\pi}$ and reduced-form variance matrix $\overline{\Omega}$.

Now, we consider the limit as $\beta_0 \to \pm\infty$ of the original model and see what it yields in terms of the transformed model. We have

$$\lim_{\beta_0 \to \pm\infty} \overline{\beta}_* = \mp 1 \quad \text{and} \lim_{\beta_0 \to \pm\infty} \overline{\Omega} = \begin{bmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{bmatrix}. \tag{135}$$

[14] The formula $\overline{\beta} := (\widetilde{\omega}_2/\widetilde{\omega}_1)\widetilde{\beta}$ in (132) comes from $\overline{y}_1 := \widetilde{y}_1/\widetilde{\omega}_1 = (y_2\widetilde{\beta} + u)/\widetilde{\omega}_1 = y_2\widetilde{\beta}/\widetilde{\omega}_1 + u/\widetilde{\omega}_1 = (y_2/\widetilde{\omega}_2)\widetilde{\beta}(\widetilde{\omega}_2/\widetilde{\omega}_1) + u/\widetilde{\omega}_1 = \overline{y}_2\overline{\beta} + \overline{u}$, where the last equality holds when $\overline{\beta} := (\widetilde{\omega}_2/\widetilde{\omega}_1)\widetilde{\beta}$ and $\overline{u} := u/\widetilde{\omega}_1$.

So, the asymptotic testing problem as $\beta_0 \to \pm\infty$ in terms of a model with a null hypothesis $\overline{\beta}$ value of 0 and a reduced-form variance matrix $\overline{\Omega}$ with ones on the diagonal is a test of $H_0 : \overline{\beta} = 0$ versus $H_1 : \overline{\beta} = \mp 1$.

We get the same expression for the limits as $\beta_0 \to \pm\infty$ of $c_{\beta_*}(\beta_0, \Omega)$ and $d_{\beta_*}(\beta_0, \Omega)$ written in terms of the transformed parameters $(\overline{\beta}_0, \overline{\beta}_*, \overline{\pi}, \overline{\Omega})$ as in Lemma 15.1 except they are multiplied by σ_v. This occurs because $\mu_{\overline{\pi}} = \mu_\pi/\sigma_v$. In consequence, the limits as $\beta_0 \to \pm\infty$ of $c_{\beta_*}(\beta_0, \Omega)\mu_\pi$ and $d_{\beta_*}(\beta_0, \Omega)\mu_\pi$ written in terms of the transformed parameters $(\overline{\beta}_0, \overline{\beta}_*, \overline{\pi}, \overline{\Omega})$ are the same as their limits without any transformation.

Lemma 21.1. *Let $\overline{\beta}_* = \overline{\beta}_*(\beta_0)$ and $\overline{\Omega} = \overline{\Omega}(\beta_0)$ be defined in (134) and (133), respectively. Let $\overline{\beta}_0(\beta_0) = 0$.*

(a) $\lim_{\beta_0 \to \pm\infty} c_{\overline{\beta}_*(\beta_0)}(\overline{\beta}_0(\beta_0), \overline{\Omega}(\beta_0)) = \mp 1$.

(b) $\lim_{\beta_0 \to \pm\infty} d_{\overline{\beta}_*(\beta_0)}(\overline{\beta}_0(\beta_0), \overline{\Omega}(\beta_0)) = \mp \frac{\rho_{uv}}{(1-\rho_{uv}^2)^{1/2}}$.

Comment. (i). By Lemmas 15.1 and 21.1, the distributions of all of the tests considered in this paper are the same in the model in Section 2 when β_* and Ω are fixed and the null hypothesis value β_0 satisfies $\beta_0 \to \pm\infty$, and in the transformed model of this section when the null hypothesis $\overline{\beta}_0$ is fixed at 0 and the alternative hypothesis value $\overline{\beta}_* = \overline{\beta}_*(\beta_0)$ and the reduced-form variance $\overline{\Omega} = \overline{\Omega}(\beta_0)$ converge as in (135) as $\beta_0 \to \pm\infty$. (This uses the fact that $\sigma_v = 1$ in Lemma 21.1.)

(ii). AMS footnote 5 notes that there is a special parameter value $\beta = \beta_{AR}$ at which the one-sided point optimal invariant similar test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_{AR}$ is the (two-sided) AR test. In footnote 5, β_{AR} is defined to be $\beta_{AR} = \frac{\omega_1^2 - \omega_{12}\beta_0}{\omega_{12} - \omega_2^2\beta_0}$. If we compute β_{AR} for the transformed model $(\overline{y}_1, \overline{y}_2)$ with parameters $(\overline{\beta}, \overline{\pi}, \overline{\Omega})$, where $\overline{\beta}_0 = 0$, we obtain

$$\overline{\beta}_{AR} = \frac{\overline{\omega}_1^2 - \overline{\omega}_{12}\overline{\beta}_0}{\overline{\omega}_{12} - \overline{\omega}_2^2\overline{\beta}_0} = \frac{1}{\overline{\omega}_{12}} = \mp 1, \tag{136}$$

which is the same as the limit of $\overline{\beta}_* = \overline{\beta}_*(\beta_0)$ as $\beta_0 \to \pm\infty$ in (132).

Proof of Lemma 21.1. First, we prove part (a). We have

$$\begin{aligned}
c_{\overline{\beta}_*}(\overline{\beta}_0, \overline{\Omega}) &= (\overline{\beta}_* - \overline{\beta}_0)\big(\overline{b}_0'\overline{\Omega}\overline{b}_0\big)^{-1/2} \\
&= \frac{\omega_2}{\big(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\big)^{1/2}}(\beta_* - \beta_0)\big(1 - 2\overline{\omega}_{12}\overline{\beta}_0 + \overline{\beta}_0^2\big)^{-1/2} \\
&= \frac{\omega_2(\beta_* - \beta_0)}{\big(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\big)^{1/2}} \\
&\to \mp 1 \quad \text{as } \beta_0 \to \pm\infty,
\end{aligned} \tag{137}$$

where the second equality uses (134) and the third equality uses $\overline{\beta}_0 = 0$. Hence, $c_{\overline{\beta}_*}(\overline{\beta}_0, \overline{\Omega})\mu_{\overline{\pi}} \to \mp(1/\sigma_v)\mu_\pi$ as $\beta_0 \to \pm\infty$ using the expression for $\overline{\pi}$ in (132) and $\omega_2 = \sigma_v$.

Next, we prove part (b). Let $\overline{b}_* = (1, -\overline{\beta}_*)'$ and $\overline{b}_0 = (1, -\overline{\beta}_0)'$. We have

$$\begin{aligned} \det(\overline{\Omega}) &= 1 - \overline{\omega}_{12}^2, \\ \overline{\omega}_{12} &= \frac{\omega_{12} - \omega_2^2\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\omega_2}, \quad \text{and} \\ \overline{b}_*'\overline{\Omega b}_0\left(\overline{b}_0'\overline{\Omega b}_0\right)^{-1/2} &= \frac{1 - \overline{\omega}_{12}\overline{\beta}_0 - \overline{\omega}_{12}\overline{\beta}_* + \overline{\beta}_0\overline{\beta}_*}{\left(1 - 2\overline{\omega}_{12}\overline{\beta}_0 + \overline{\beta}_0^2\right)^{1/2}} = 1 - \overline{\omega}_{12}\overline{\beta}_*, \end{aligned} \tag{138}$$

where the second equality on the third line uses $\overline{\beta}_0 = 0$. Next, we have

$$\begin{aligned} 1 - \overline{\omega}_{12}\overline{\beta}_* &= 1 - \frac{\omega_{12} - \omega_2^2\beta_0}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\omega_2}\frac{\omega_2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}}(\beta_* - \beta_0) \\ &= 1 - \frac{\left(\omega_{12} - \omega_2^2\beta_0\right)(\beta_* - \beta_0)}{\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2} \\ &= \frac{\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2 - \omega_{12}\beta_* + \omega_{12}\beta_0 + \omega_2^2\beta_0\beta_* - \omega_2^2\beta_0^2}{\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2} \\ &= \frac{\omega_1^2 - \omega_{12}\beta_0 - \omega_{12}\beta_* + \omega_2^2\beta_0\beta_*}{\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2}, \end{aligned} \tag{139}$$

where the first equality uses (133) and (134).

In addition, we have

$$\begin{aligned} 1 - \overline{\omega}_{12}^2 &= 1 - \frac{\left(\omega_{12} - \omega_2^2\beta_0\right)^2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)\omega_2^2} \\ &= \frac{\omega_1^2\omega_2^2 - 2\omega_{12}\omega_2^2\beta_0 + \omega_2^4\beta_0^2 - \omega_{12}^2 + 2\omega_{12}\omega_2^2\beta_0 - \omega^4\beta_0^2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)\omega_2^2} \\ &= \frac{\omega_1^2\omega_2^2 - \omega_{12}^2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)\omega_2^2}, \end{aligned} \tag{140}$$

where the first equality uses (138).

Using (138)–(140), we have

$$\begin{aligned} d_{\overline{\beta}_*}(\overline{\beta}_0, \overline{\Omega}) &= \overline{b}_*'\overline{\Omega b}_0\left(\overline{b}_0'\overline{\Omega b}_0\right)^{-1/2}\det(\overline{\Omega})^{-1/2} \\ &= \frac{\omega_1^2 - \omega_{12}\beta_0 - \omega_{12}\beta_* + \omega_2^2\beta_0\beta_*}{\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2}\left(\frac{\omega_1^2\omega_2^2 - \omega_{12}^2}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)\omega_2^2}\right)^{-1/2} \\ &= \frac{\left(\omega_1^2 - \omega_{12}\beta_0 - \omega_{12}\beta_* + \omega_2^2\beta_0\beta_*\right)}{\left(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2\right)^{1/2}\left(\omega_1^2\omega_2^2 - \omega_{12}^2\right)^{1/2}}\omega_2. \end{aligned} \tag{141}$$

The rhs of (141) is the same as the expression on the second line of (44) multiplied by $\omega_2 = \sigma_v$. In consequence, the calculations in (44)–(46) give the result of part (a) of Lemma 21.1. □

22. Transformation of the β_0 versus β_* testing problem to a $\overline{\beta}_0$ versus 0 testing problem

In this section, we transform the general testing problem of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$ for $\pi \in R^k$ and fixed reduced-form variance matrix Ω to a testing problem of $H_0 : \overline{\beta} = \overline{\beta}_0$ versus $H_1 : \overline{\beta} = 0$ for some $\overline{\pi} \in R^k$ and some fixed $\overline{\Omega}$ with diagonal elements equal to one. These transformation results imply that there is no loss in generality in the numerical results of the paper to taking $\omega_1^2 = \omega_2^2 = 1$ and $\beta_* = 0$. We also show that there is no loss in generality in the numerical results of the paper to taking $\rho_{uv} \in [0, 1]$, rather than $\rho_{uv} \in [-1, 1]$, where ρ_{uv} is the structural variance matrix correlation defined in (15).

We consider the same transformations as in Section 21, but with β_* in place of β_0 in (131)–(133) and with the roles of β_* and β_0 reversed in (134) and (135). The transformed testing problem given the transformations in (131) (with β_* in place of β_0) is H_0: $\widetilde{\beta} = \widetilde{\beta}_0$ versus $H_1 : \widetilde{\beta} = 0$, where $\widetilde{\beta}_0 = \beta_0 - \beta_*$, with parameter π and reduced-form variance matrix $\widetilde{\Omega}$. The transformed testing problem given the transformations in (131)–(133) (with β_* in place of β_0) is H_0: $\overline{\beta} = \overline{\beta}_0$ versus $H_1 : \overline{\beta} = 0$, where $\widetilde{\beta}_0 = \beta_0 - \beta_*$, with parameters $\overline{\beta}$, $\overline{\pi}$, and $\overline{\Omega}$ defined in (132) and (133) (with the roles of β_* and β_0 reversed).

For example, a scenario in which a typical test has high power in the original scenario of testing $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$, such as $\beta_0 = 0$ and $|\beta_*|$ large, gets transformed into the testing problem of H_0: $\overline{\beta} = \overline{\beta}_0$ versus $H_1 : \overline{\beta} = 0$ with correlation $\overline{\omega}_{12}$ (the $(1, 2)$ element of $\overline{\Omega}$) close to ± 1, because by (135) (with the roles of β_* and β_0 reversed) we have

$$\lim_{\beta_* \to \pm\infty} \overline{\Omega} = \begin{bmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{bmatrix}. \tag{142}$$

In this case, we also have $\lim_{\beta_* \to \pm\infty} \overline{\beta}_0 = \mp 1$ by (135). Also, note that the reduced-form and structural variances matrices are equal when the alternative hypothesis holds in the testing problem H_0: $\overline{\beta} = \overline{\beta}_0$ versus $H_1 : \overline{\beta} = 0$, so the result in (142) also applies to the structural variance matrix $\Sigma(\overline{\beta}, \overline{\Omega})$ when $\overline{\beta} = 0$ whose correlation we denote by $\overline{\rho}_{uv}$, that is, $\lim_{\beta_* \to \pm\infty} \overline{\rho}_{uv} = \mp 1$. Here, the parameter $\overline{\rho}_{uv}$ is the parameter ρ_{uv} that appears in the tables in the paper. These results are useful in showing how the numerical results of the paper apply to general hypotheses of the form $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_*$.

Next, we show that there is no loss in generality in the numerical results of the paper to taking $\rho_{uv} \in [0, 1]$. We consider the hypotheses $H_0 : \beta = \beta_0$ versus $H_1 : \beta = 0$, as in the numerical results in the paper. When the true β equals 0 and Ω has ones on its diagonal, the reduced-form and structural variance matrices are equal; see (32). Hence, the correlation ω_{12} given by Ω equals the structural variance correlation ρ_{uv} in power calculations in the paper, and it suffices to show that there is no loss in generality in the numerical results of the paper to taking $\omega_{12} \in [0, 1]$.

By (3), the distributions of S and T only depend on $c_\beta(\beta_0, \Omega)$, $d_\beta(\beta_0, \Omega)$, and $\mu_\pi := (Z'Z)^{1/2}\pi$. The vector μ_π does not depend on β, β_0, or Ω. First, note that ω_{12} enters

$c_\beta(\beta_0, \Omega) := (\beta - \beta_0)(b_0'\Omega b_0)^{-1/2} = (\beta - \beta_0)(\omega_1^2 - 2\omega_{12}\beta_0 + \omega_2^2\beta_0^2)^{-1/2}$ only through $\omega_{12}\beta_0$. In consequence, the distribution of S is the same under (β_0, ω_{12}) as under $(-\beta_0, -\omega_{12})$. Second, by (2.8) of AMS, $d_\beta(\beta_0, \Omega)$ can be written as $b'\Omega b_0(b_0'\Omega b_0)^{-1/2} \times \det(\Omega)^{-1/2}$, where $b := (1, -\beta)'$. The distribution of T when $\beta = 0$ depends on $d_0(\beta_0, \Omega) = (1 - \omega_{12}\beta_0)(b_0'\Omega b_0)^{-1/2}\det(\Omega)^{-1/2}$. The first two multiplicands depend on ω_{12} only through $\omega_{12}\beta_0$ and the third multiplicand only depends on ω_{12} through ω_{12}^2 (because $\det(\Omega) = 1 - \omega_{12}^2$). In addition, S and T are independent. Hence, the distribution of $[S : T]$ for given (β_0, ω_{12}) when $\beta = 0$ equals its distribution under $(-\beta_0, -\omega_{12})$ when $\beta = 0$. Thus, the power of a test of $H_0 : \beta = \beta_0$ versus $H_1 : \beta = 0$ when $\omega_{12} < 0$ equals its power for testing $H_0 : \beta = -\beta_0$ versus $H_1 : \beta = 0$ for $-\omega_{12} > 0$.

23. Unknown variance CLR test

In this section, we consider a different form of the CLR test to see whether it has smaller probabilities of infinite length than the CLR test defined in (6) and (7).[15] By Moreira (2003, pp. 1036, 1045), the likelihood ratio statistic under the assumption that the reduced-form variance matrix is unknown is

$$\begin{aligned} LR_U &:= \frac{n}{2}\ln\left(1 + \frac{b_0'YP_ZYb_0}{(n-k)b_0'\widehat{\Omega}b_0}\right) - \frac{n}{2}\ln\left(1 + \frac{\lambda_{\min}(\widehat{\Omega}^{-1/2}YP_ZY\widehat{\Omega}^{-1/2})}{n-k}\right), \quad \text{where} \\ \widehat{\Omega} &:= YM_ZY/(n-k). \end{aligned} \tag{143}$$

(Note that Moreira (2003) denotes the statistic LR_U by LR and the statistic LR in (6) above by LR_0.)

The probabilities that the CLR test has infinite length (given in Table 1 in Section 7) are computed under the assumption that Ω is known. If we made comparisons of these results to analogous results for the conditional test that employs the statistic LR_U (combined with the same conditional critical value as in (7)), the comparisons would be misleading because LR_U does not make use of the known value of Ω. To obtain a fair comparison, we alter the LR_U statistic by replacing $\widehat{\Omega}$ by Ω. The resulting statistic is

$$\begin{aligned} LR2_n &:= \frac{n}{2}\ln\left(1 + \frac{b_0'YP_ZYb_0}{(n-k)b_0'\Omega b_0}\right) - \frac{n}{2}\ln\left(1 + \frac{\lambda_{\min}(\Omega^{-1/2}YP_ZY\Omega^{-1/2})}{n-k}\right) \\ &= \frac{n}{2}\ln\left(1 + \frac{Q_S}{(n-k)}\right) - \frac{n}{2}\ln\left(1 + \frac{Q_S - LR}{n-k}\right), \end{aligned} \tag{144}$$

where the second equality holds by the definition of Q_S in (3) and (4) and the expression $LR_0 = \overline{S}'\overline{S} - \overline{\lambda}_{\min}$ on p. 1033 of Moreira (2003), which in the notation of this paper is $LR = Q_S - \overline{\lambda}_{\min}$ for $\overline{\lambda}_{\min} := \lambda_{\min}(\Omega^{-1/2}YP_ZY\Omega^{-1/2})$ by p. 1045 of Moreira (2003).

The conditional critical value for this statistic is the same as that in (7). We call the resulting test the CLR2_n test. Somewhat confusingly, or perhaps paradoxically, the form of the $LR2_n$ statistic is determined by assuming Ω is unknown, which yields a test that depends on an estimator $\widehat{\Omega}$ of Ω, which we then replace by Ω, which yields a test for

[15] We thank Marcelo Moreira for suggesting that we consider the CLR2_n tests considered in this section.

the case where Ω is known. Note that the $LR2_n$ statistic depends on n, whereas the LR statistic in (6) does not.

Table SM-VI in the Online Supplementary Material 2 reports differences in the probabilities that the CLR2_n and CLR CIs have infinite length for the same k, λ, and ρ_{uv} values as in Table I, for three values of n: $n = 100, 500$, and 1000. Note that the data generating process depends only on k, λ, and ρ_{uv}, and not on n. The quantity n only enters through the form of the $LR2_n$ statistic.

The results in Table SM-VI show that the CLR2_n and CLR CIs perform very similarly. This is especially true for $n = 500$ and 1000 in which cases all differences are less than 0.005. For $n = 100$, the differences exceed 0.005 in some scenarios where ρ_{uv} is small (0, 0.3, and 0.5) and k is large ($k \geq 10$ for $\rho_{uv} = 0.0, 0.3$ and $k \geq 20$ for $\rho_{uv} = 0.5$). The largest difference is 0.0235 and is achieved when $n = 100$, $\rho_{uv} = 0$, $k = 40$, and $\lambda = 20$.

Based on these results, we do not find that the CLR2_n test improves on the CLR test in terms of its probabilities of having infinite length. The differences between the CLR2_n and CLR tests are quite small, especially for $n = 500$ and 1000.

24. Heteroskedastic and autocorrelated model

Theorem 5.1 gives formulae for the probabilities that certain CIs have infinite right length, infinite left length, and infinite length in the homoskedastic Gaussian linear IV model. In this section, we extend these results to the Gaussian linear IV model that allows for heteroskedasticity and autocorrelation (HC) in the errors. We use the specification and notation in Moreira and Ridder (2017). The reduced-form model is $Y = Z\pi a' + V$, as in (2), but without the assumption that the rows of V are i.i.d. with distribution Ω. Rather, we assume that

$$\operatorname{vec}(\widetilde{V}) := \operatorname{vec}\big((Z'Z)^{-1/2} Z'V\big) \sim N(0, \Sigma), \tag{145}$$

where $\widetilde{V} \in R^{k\times 2}$ and Σ is a positive definite $2k \times 2k$ matrix. The matrix Σ can be consistently estimated. In consequence, we focus on the case where Σ is known. Let $P_1 := Z(Z'Z)^{-1/2} \in R^{n\times k}$ and let $P_2 \in R^{n\times(n-k)}$ be such that $P := [P_1 : P_2]$ is orthogonal. A one-to-one transformation of Y is $(P_1'Y, P_2'Y)$. The matrix $P_2'Y$ is ancillary and the variance of V is only restricted by $\operatorname{Var}(\operatorname{vec}(P_1'V)) = \Sigma$. In consequence, we only consider tests that are a function of $P_1'Y$. We have

$$R := P_1'Y = \mu_\pi a' + \widetilde{V}, \quad \text{where } \mu_\pi := (Z'Z)^{1/2}\pi \text{ and } a := (\beta, 1)'. \tag{146}$$

For a given null hypothesis value β_0, a one-to-one transformation of R is $(S_{\beta_0}(R), T_{\beta_0}(R))$, where

$$\begin{aligned} S_{\beta_0}(R) &:= \big[(b_0' \otimes I_k)\Sigma(b_0 \otimes I_k)\big]^{-1/2}(b_0' \otimes I_k)\operatorname{vec}(R), \\ T_{\beta_0}(R) &:= \big[(a_0' \otimes I_k)\Sigma^{-1}(a_0 \otimes I_k)\big]^{-1/2}(a_0' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R), \end{aligned} \tag{147}$$

$a_0 := (\beta_0, 1)'$, and $b_0 := (1, -\beta_0)'$. The statistics $S_{\beta_0}(R)$ and $T_{\beta_0}(R)$ are independent. Their distributions are

$$\begin{aligned} S_{\beta_0}(R) &\sim N\big((\beta-\beta_0)C_{\beta_0}\mu_\pi, I_k\big) \quad \text{and} \\ T_{\beta_0}(R) &\sim N(D_\beta \mu_\pi, I_k), \quad \text{where} \\ C_{\beta_0} &:= \big[\big(b_0' \otimes I_k\big)\Sigma(b_0 \otimes I_k)\big]^{-1/2} \quad \text{and} \\ D_\beta &:= \big[\big(a_0' \otimes I_k\big)\Sigma^{-1}(a_0 \otimes I_k)\big]^{-1/2}\big(a_0' \otimes I_k\big)\Sigma^{-1}(a \otimes I_k). \end{aligned} \tag{148}$$

As shown in the following lemma, the limits of $S_{\beta_0}(R)$ and $T_{\beta_0}(R)$ as $\beta_0 \to \pm\infty$ are

$$\begin{aligned} S_{\pm\infty}(R) &:= \mp\Sigma_{22}^{-1/2}R_2 \quad \text{and} \\ T_{\pm\infty}(R) &:= \pm\big(\Sigma^{11}\big)^{-1/2}\big(e_1' \otimes I_k\big)\Sigma^{-1}\operatorname{vec}(R), \end{aligned} \tag{149}$$

where R_2 denotes the second column of R, Σ_{22} denotes the lower right $k \times k$ block of Σ, Σ^{11} denotes the upper left $k \times k$ block of Σ^{-1}, and $e_1 := (1, 0)'$.

LEMMA 24.1. *For fixed true value $\beta = \beta_*$ and positive definite matrix Σ, we have*

(a) $\lim_{\beta_0 \to \pm\infty} S_{\beta_0}(R) = S_{\pm\infty}(R)$,

(b) $S_{\pm\infty}(R) \sim N(\mp\Sigma_{22}^{-1/2}\mu_\pi, I_k)$,

(c) $\lim_{\beta_0 \to \pm\infty} T_{\beta_0}(R) = T_{\pm\infty}(R)$,

(d) $T_{\pm\infty}(R) \sim N(\pm(\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(\mu_\pi a_*'), I_k)$, *where $a_* := (\beta_*, 1)'$, and*

(e) $S_{\pm\infty}(R)$ *and* $T_{\pm\infty}(R)$ *are independent.*

Comments. (i). The convergence results in Lemma 24.1 hold for all realizations of R.

(ii). In the homoskedastic case, where $\Sigma = \Omega \otimes I_k$, we have $S_{\pm\infty}(R) = S_{\pm\infty}(Y)$ and $T_{\pm\infty}(R) = T_{\pm\infty}(Y)$, where $S_{\pm\infty}(Y)$ and $T_{\pm\infty}(Y)$ are defined in (50) for the homoskedastic model.

These results hold by the following calculations. In the homoskedastic case, $\Sigma_{22} = \omega_2^2 I_k = \sigma_v^2 I_k$, where ω_2^2 denotes the $(2,2)$ element of Ω and $\sigma_v^2 := \operatorname{Var}(v_{2i})$. This yields $S_{\pm\infty}(R) = \mp(1/\sigma_v)R_2 = \mp(1/\sigma_v)(Z'Z)^{-1/2}Z'Ye_2 := S_{\pm\infty}(Y)$. In the homoskedastic case, $\Sigma^{11} = \omega^{11}I_k$, where ω^{11} denotes the $(1,1)$ element of Ω^{-1}, $\Sigma^{-1} = \Omega^{-1} \otimes I_k$, and $(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R) = (e_1'\Omega^{-1} \otimes I_k)\operatorname{vec}(R) = R\Omega^{-1}e_1$, where the last equality uses the formula $\operatorname{vec}(ABC) = (C' \otimes A)\operatorname{vec}(B)$. We have $\omega^{11} = \omega_2^2/(\omega_1^2\omega_2^2 - \omega_{12}^2)$ by the formula for the inverse of a 2×2 matrix, $\omega_1^2\omega_2^2 - \omega_{12}^2 = \sigma_u^2\sigma_v^2 - \sigma_{uv}^2 = \sigma_u^2\sigma_v^2(1-\rho_{uv}^2)$, where the first equality holds by (45), and $(\omega^{11})^{-1/2} = \sigma_u\sigma_v(1-\rho_{uv}^2)^{1/2}/\omega_2 = \sigma_u(1-\rho_{uv}^2)^{1/2}$, where the last equality uses $\sigma_v = \omega_2$. Putting these results together gives $T_{\pm\infty}(R) := \pm(\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R) = \pm\sigma_u(1-\rho_{uv}^2)^{1/2}R\Omega^{-1}e_1 = (Z'Z)^{-1/2}Z'Y\Omega^{-1}e_1 \cdot (\pm(1-\rho_{uv}^2)^{1/2}\sigma_u) := T_{\pm\infty}(R)$.

Let $P_{\beta_*,\pi,\Sigma}(\cdot)$ denote the probability distribution of R when β_*, π, Σ are the true values.

The HC model analogue of Theorem 5.1 is the following.

THEOREM 24.2. *Suppose* $\mathrm{CS}_\phi(R)$ *is a* CS *based on level* α *tests* $\phi(S_{\beta_0}(R), T_{\beta_0}(R))$ *whose test statistic and critical value functions,* $\mathcal{T}(s,t)$ *and* $\mathrm{cv}(t)$*, respectively, are continuous at all* $k \times 2$ *matrices* $[s:t]$ *and* k *vectors* t*,* $P_{\beta_*,\pi,\Sigma}(\mathcal{T}(S_c(R), T_c(R)) = \mathrm{cv}(T_c(R))) = 0$ *for* $c = +\infty$ *in parts* (a) *and* (c) *below and* $c = -\infty$ *in part* (b) *below. Then, for all* (β_*, π, Σ) *with* Σ *positive definite,*

(a) $P_{\beta_*,\pi,\Sigma}(\mathrm{RLength}(\mathrm{CS}_\phi(R)) = \infty) = 1 - \lim_{\beta_0 \to \infty} P_{\beta_*,\pi,\Sigma}(\phi(S_{\beta_0}(R), T_{\beta_0}(R)) = 1)$,

(b) $P_{\beta_*,\pi,\Sigma}(\mathrm{LLength}(\mathrm{CS}_\phi(R)) = \infty) = 1 - \lim_{\beta_0 \to -\infty} P_{\beta_*,\pi,\Sigma}(\phi(S_{\beta_0}(R), T_{\beta_0}(R)) = 1)$, *and*

(c) *if* $\mathcal{T}(S_c(R), T_c(R)) \leq \mathrm{cv}(T_c(R))$ *for* $c = +\infty$ *iff the same inequality holds for* $c = -\infty$ *a.s., then* $P_{\beta_*,\pi,\Sigma}(\mathrm{Length}(\mathrm{CS}_\phi(R)) = \infty) = 1 - \lim_{\beta_0 \to \pm\infty} P_{\beta_*,\pi,\Sigma}(\phi(S_{\beta_0}(R), T_{\beta_0}(R)) = 1)$.

PROOF OF THEOREM 24.2. The proof is essentially the same as that for Theorem 5.1 with (i) $(S_{\beta_0}(R), T_{\beta_0}(R))$ and $T_{\beta_0}(R)$ in place of $Q_{\beta_0}(Y)$ and $Q_{T,\beta_0}(Y)$, respectively, using (ii) Lemma 24.1 in place of Lemma 16.1, and using (iii) the assumption of the Theorem that "$\mathcal{T}(s,t)$ and $\mathrm{cv}(t)$ are continuous at all $k \times 2$ matrices $[s:t]$ and k vectors t," in place of the assumption of Theorem 5.1 that "$\mathcal{T}(q)$ and $\mathrm{cv}(q_T)$ are continuous at all positive definite 2×2 matrices q and positive constants q_T." (In the argument following (52) in the proof of Theorem 5.1, the latter assumption is combined with the result of Lemma 16.1(g), which implies that $Q_\infty(Y)$ is pd a.s. and $Q_{T,\infty}(Y) > 0$ a.s. In contrast, in the proof of the present theorem, this part of the argument is not needed because there is no restriction to positive definite matrices q and positive constants q_T.) In the proof of part (c), the second last equality in (54) in the proof of Theorem 5.1 holds (with the changes listed in (i)–(iii) above) because the assumption imposed in part (c) of the present theorem is the same as condition (iii) stated immediately above (54). □

PROOF OF LEMMA 24.1. We prove part (a) first. Dividing the components of $S_{\beta_0}(R)$ in (147) by $|\beta_0|$, we obtain

$$S_{\beta_0}(R) = \left[\left((b_0/|\beta_0|)' \otimes I_k\right)\Sigma\left((b_0/|\beta_0|) \otimes I_k\right)\right]^{-1/2}\left((b_0/|\beta_0|)' \otimes I_k\right)\mathrm{vec}(R). \tag{150}$$

We have

$$\begin{aligned} &\lim_{\beta_0 \pm\infty}\left((b_0/|\beta_0|)' \otimes I_k\right)\mathrm{vec}(R) = \left((0, \mp 1) \otimes I_k\right)\mathrm{vec}(R) = \mp R_2 \quad \text{and} \\ &\lim_{\beta_0 \pm\infty}\left((b_0/|\beta_0|)' \otimes I_k\right)\Sigma\left((b_0/|\beta_0|) \otimes I_k\right) = \left((0, \mp 1) \otimes I_k\right)\Sigma\left((0, \mp 1) \otimes I_k\right) = \Sigma_{22}, \end{aligned} \tag{151}$$

using $b_0 := (1, -\beta_0)'$, where R_2 denotes the second column of R. Combining (150) and (151) and using the positive definiteness of Σ_{22} gives $\lim_{\beta_0 \pm\infty} S_{\beta_0}(R) = \mp\Sigma_{22}^{-1/2}R_2 := S_{\pm\infty}(R)$, which proves part (a).

Part (b) holds by the definition of $S_{\pm\infty}(R)$ in (149) because $R_2 \sim N(\mu_\pi, \Sigma_{22})$ by (145) and (146).

To prove part (c), we divide the components of $T_{\beta_0}(R)$ in (147) by $|\beta_0|$ to obtain

$$T_{\beta_0}(R) = \left[\left((a_0/|\beta_0|)' \otimes I_k\right)\Sigma^{-1}\left((a_0/|\beta_0|) \otimes I_k\right)\right]^{-1/2}\left((a_0/|\beta_0|)' \otimes I_k\right)\Sigma^{-1}\mathrm{vec}(R), \tag{152}$$

where $a_0 = (\beta_0, 1)'$. We have

$$\begin{aligned} &\lim_{\beta_0 \pm\infty} \big((a_0/|\beta_0|)' \otimes I_k\big)\Sigma^{-1}\operatorname{vec}(R) = \pm\big((1,0)\otimes I_k\big)\Sigma^{-1}\operatorname{vec}(R) \quad \text{and} \\ &\lim_{\beta_0 \pm\infty} \big((a_0/|\beta_0|)' \otimes I_k\big)\Sigma^{-1}\big((a_0/|\beta_0|)\otimes I_k\big) = \big((\pm 1,0)\otimes I_k\big)\Sigma^{-1}\big((\pm 1,0)\otimes I_k\big) = \Sigma^{11}, \end{aligned} \tag{153}$$

where Σ^{11} denotes the upper left $k \times k$ block of Σ^{-1}. Combining (152) and (153) and using the positive definiteness of Σ^{-1} gives $\lim_{\beta_0 \pm\infty} T_{\beta_0}(R) = \pm(\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R) := T_{\pm\infty}(R)$, which establishes part (c) of the lemma.

Part (d) holds by the definition of $T_{\pm\infty}(R)$ in (149) because $R = \mu_\pi a_*' + \widetilde{V}$ when $\beta = \beta_*$ by (146), $\operatorname{vec}(\widetilde{V}) \sim N(0, \Sigma)$ by (145), and

$$\begin{aligned} \operatorname{Var}\big(T_{\pm\infty}(R)\big) &= \operatorname{Var}\big((\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R)\big) \\ &= (\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\Sigma\Sigma^{-1}(e_1 \otimes I_k)(\Sigma^{11})^{-1/2} \\ &= I_k. \end{aligned} \tag{154}$$

Part (e) holds because $S_{\pm\infty}(R)$ and $T_{\pm\infty}(R)$ are jointly normal with covariance

$$\begin{aligned} \operatorname{Cov}\big(S_{\pm\infty}(R), T_{\pm\infty}(R)\big) &= \operatorname{Cov}\big(\mp \Sigma_{22}^{-1/2}(e_2' \otimes I_k)\operatorname{vec}(R), \pm(\Sigma^{11})^{-1/2}(e_1' \otimes I_k)\Sigma^{-1}\operatorname{vec}(R)\big) \\ &= -\Sigma_{22}^{-1/2}(e_2' \otimes I_k)\operatorname{Var}\big(\operatorname{vec}(R)\big)\Sigma^{-1}(e_1 \otimes I_k)(\Sigma^{11})^{-1/2} \\ &= I_k. \end{aligned} \tag{155}$$

This implies that $S_{\pm\infty}(R)$ and $T_{\pm\infty}(R)$ are independent, which proves part (e). □

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