Supplementary Material

Supplement to "Equilibrium computation in discrete network games"

(*Quantitative Economics*, Vol. 11, No. 4, November 2020, 1325–1347)

Michael P. Leung
Department of Economics, University of Southern California

This supplement is organized as follows. In Section SA.1, we state additional examples and remarks referred to in the main body of the paper. Next, Section SA.2 provides the proof of the main result, Theorem 1. Technical lemmas used to prove the result are stated in Section SA.3. Finally, Section SA.4 and Section SA.5 provide algorithms and complexity bounds for computing equilibria in undirected and directed network formation games, respectively.

SA.1. Additional examples and discussion

Example SA.1.1. This example illustrates the interpretation of Assumption 2 in the binary choice model. It is referenced in Example 4 of the main text. As in that example, consider the case of an exogenous network $A_{ij} \perp\!\!\!\perp \mathcal{R}_j^c$. Unlike that example, we instead consider the random connections model for $\boldsymbol{A}$ (Meester and Roy (1996)). Suppose $T_i = (\rho_i, \xi_i)$, where ρ_i has dimension d_ρ and ξ_i dimension d_ξ. Link formation is determined by

$$A_{ij} = g_n(T_i, T_j, \zeta_{ij}) \equiv \mathbf{1}\{V(r_n^{-1}\|\rho_i - \rho_j\|, \xi_i, \xi_j, \zeta_{ij}) > 0\},$$

where $V(\cdot)$ is a real-valued function, ρ_i is continuously distributed with density f, and r_n is a scaling constant defined below, analogous to ρ_n in Example 4. Leung and Moon (2019) proved a CLT for a generalization of this model allowing for strategic interactions. For simplicity, suppose $\zeta_{ij} \perp\!\!\!\perp T_i, T_j$. We assume $V(\cdot)$ is decreasing in its first component to capture homophily in ρ_i, which may represent, for example, geographic location. A special case is the *random geometric graph* where $g_n(\cdot)$ equals one if and only if $r_n^{-1}\|\rho_i - \rho_j\| < 1$, which states that agents only link with those in a fixed geographic neighborhood (Penrose (2003)).

Suppose $V(\cdot)$ is increasing in its second and third components. Let $\phi(\cdot \mid \rho)$ be the conditional density of ξ_i given $\rho_i = \rho$, and suppose there exists a density $\phi^*(\cdot)$ that stochastically dominates $\phi(\cdot \mid \rho)$ for all ρ. Then

$$\begin{aligned} &n\mathbf{P}(A_{ij} = 1 \mid T_i = (\rho, \xi)) \\ &\quad = n\int_{\mathbb{R}^{d_\rho}}\int_{\mathbb{R}^{d_\xi}} \mathbf{E}[g_n(T_i, T_j, \zeta_{ij}) \mid T_i = (\rho, \xi), \rho_j = \rho', \xi_j = \xi']\phi(\xi' \mid \rho')f(\rho')\,\mathrm{d}\xi'\,\mathrm{d}\rho' \end{aligned}$$

Michael P. Leung: leungm@usc.edu

Available at http://qeconomics.org. https://doi.org/10.3982/QE1386

$$= nr_n^{d_\rho} \int_{\mathbb{R}^{d_\rho}} \int_{\mathbb{R}^{d_\xi}} \mathbf{P}\big(V\big(\|\rho - \tilde{\rho}'\|, \xi, \xi', \zeta_{ij}\big) > 0\big) \\ \times \phi\big(\xi' \mid \rho + r_n(\rho' - \rho)\big) f\big(\rho + r_n(\rho' - \rho)\big)\, \mathrm{d}\xi'\, \mathrm{d}\tilde{\rho}',$$

where the second line uses the fact that $r_n^{-1}\|\rho - \rho'\| = \|\rho - (\rho + r_n^{-1}(\rho' - \rho))\|$ and the change of variables $\rho + r_n^{-1}(\rho' - \rho) \mapsto \tilde{\rho}'$.

Suppose $r_n = (\kappa/n)^{1/d_\rho}$, and let $\bar{f} = \sup_x f(x)$. Then the last line is bounded above by

$$\int_{\mathbb{R}^{d_\rho}} \int_{\mathbb{R}^{d_\xi}} \underbrace{\kappa \bar{f} \mathbf{P}\big(V\big(\|\tilde{\rho}'\|, \xi, \xi', \zeta_{ij}\big) > 0\big)}_{h((\rho,\xi),(\tilde{\rho}',\xi'))} \underbrace{\phi^*(\xi')\, \mathrm{d}\xi'\, \mathrm{d}\tilde{\rho}'}_{\mathrm{d}\mu((\tilde{\rho}',\xi'))},$$

where we replace $\|\rho - \tilde{\rho}'\|$ with $\|\tilde{\rho}'\|$, since the integral is over $\mathbb{R}^{d_\rho}$.

We have calculated that

$$\sup_n n\mathbf{P}(A_{ij} = 1 \mid T_i = t) \leq \int_{\mathbb{R}^{d_\rho}} \int_{\mathbb{R}^{d_\xi}} h\big(t, (\tilde{\rho}', \xi')\big) \phi^*(\xi')\, \mathrm{d}\xi'\, \mathrm{d}\tilde{\rho}'. \tag{SA.1.1}$$

Since types are independent, (2.10) holds. We therefore define the quantities in Assumption 2 as follows: $k = d_\rho$, $\varphi(t, t') = \mathbf{E}[\mathcal{R}_j^c] h(t, t')$, μ_k the Lebesgue measure on $\mathbb{R}^{d_\rho}$, and μ_{-k} the probability law induced by the density $\phi^*(\cdot)$. It follows that

$$\|\lambda\|_{\mathbf{m},k} = \mathbf{E}\big[\mathcal{R}_j^c\big] \\ \times \sup_{t_k \in \mathbb{R}^k} \left(\int_{\mathbb{R}^{d-k}} \left(\int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{d-k}} h\big((t_k, t_{-k}), (t'_k, t'_{-k})\big)^2 \phi^*(t'_{-k})\, \mathrm{d}t'_{-k} \right)^{1/2} \mathrm{d}t'_k \right)^2 \\ \times \phi^*(t_{-k})\, \mathrm{d}t_{-k} \right)^{1/2}.$$

Then by Jensen's inequality, $\|\lambda\|_{\mathbf{m},k} < 1$ implies

$$\mathbf{E}\big[\mathcal{R}_j^c\big] < \left(\sup_{t \in \mathbb{R}^d} \int_{\mathbb{R}^k} \int_{\mathbb{R}^{d-k}} h\big(t, (t'_k, t'_{-k})\big) \phi^*(t'_{-k})\, \mathrm{d}t'_{-k}\, \mathrm{d}t'_k \right)^{-1} \\ \leq \left(\sup_t \sup_n n\mathbf{P}(A_{ij} = 1 \mid T_i = t) \right)^{-1},$$

where the second inequality uses (SA.1.1). This implies that the strength of strategic interactions is bounded by the inverse of the limiting expected degree of any agent, which is analogous to (2.13).

Remark SA.1.1. We expand on the point made at the end of Example 4 regarding the behavioral implications of Assumption 2, (2.8), and (2.13). Consider a counterfactual policy intervention that increases the outcome of an agent i by one unit, and suppose her neighbors simultaneously myopically best respond once to the policy change. Then under (2.13), each of her neighbors' actions increase by β, so the total change in outcomes among i's neighbors is $\beta \sum_j A_{ij}$, which is less than one.

The row-normalized model under (2.8) has a similar implication. The total change in the outcome of i's neighbor j is now $\beta(\sum_k A_{jk})^{-1}$, so the total change in i's neighbors' outcomes is $\sum_j A_{ij}\beta(\sum_k A_{jk})^{-1}$. The average total change over all agents i counterfactually subjected to the policy (assuming for simplicity their network neighborhoods do not overlap) is

$$\frac{1}{n}\sum_i \sum_j A_{ij}\beta \frac{1}{\sum_k A_{jk}} = \beta \frac{1}{n}\sum_j \frac{\sum_i A_{ij}}{\sum_k A_{jk}} = \beta < 1.$$

Hence, the total change in neighbors' outcomes, on average, is less than one.

Finally, consider our model under Assumption 2. The total change in i's neighbors resulting from the policy intervention is at most $\sum_j A_{ij}\mathcal{R}_j^c$, since if $\mathcal{R}_j^c = 1$, then agent i's action is the same regardless of others' actions. Assumption 2 implies that this quantity is less than one in expectation.

Example SA.1.2. This example illustrates the interpretation of Assumption 2 in the multinomial choice setting. Consider Example 5, and suppose $K = 2$, $\beta_{\ell,k} = 0$ for all $\ell \neq k$, and $\beta_{\ell,\ell} \geq 0$ for all ℓ. That is, there are three elements in the choice set, and payoffs from choosing the ℓth action only depend on the fraction of friends choosing action ℓ and not any other action k. Then

$$\begin{aligned}\mathcal{R}_i^c = &\mathbf{1}\{-\beta_{2,2} \leq (X_{i2} - X_{i1})'\theta_2 + (\varepsilon_{i2} - \varepsilon_{i1}) \leq \beta_{1,1}\} \\ &\times \mathbf{1}\{-\beta_{1,1} \leq (X_{i1} - X_{i0})'\theta_2 + (\varepsilon_{i1} - \varepsilon_{i0}) \leq \beta_{0,0}\} \\ &\times \mathbf{1}\{-\beta_{2,2} \leq (X_{i2} - X_{i0})'\theta_2 + (\varepsilon_{i2} - \varepsilon_{i0}) \leq \beta_{0,0}\}.\end{aligned}$$

Suppose the network is realized according to the inhomogeneous random graph model of Example 4. Then following the calculations in that example, Assumption 2 is equivalent to (2.12). In the special case where the indicators in the previous equation are independent, $\mathbf{E}[\mathcal{R}_j^c] = \gamma_{21}\gamma_{10}\gamma_{20}$, where

$$\begin{aligned}\gamma_{\ell k} &= \mathbf{P}\big(-\beta_{\ell,\ell} \leq (X_{i\ell} - X_{ik})'\theta_2 + (\varepsilon_{i\ell} - \varepsilon_{ik}) \leq \beta_{k,k}\big) \\ &= \mathbf{P}\big((X_{i\ell} - X_{ik})'\theta_2 + \beta_{\ell,\ell} + (\varepsilon_{i\ell} - \varepsilon_{ik}) \geq 0\big) \\ &\quad - \mathbf{P}\big((X_{i\ell} - X_{ik})'\theta_2 - \beta_{k,k} + (\varepsilon_{i\ell} - \varepsilon_{ik}) \geq 0\big).\end{aligned}$$

This is the partial-equilibrium marginal effect of changing all of i's neighbors from action k to action ℓ on i's propensity to choose action ℓ. Hence, $\gamma_{\ell k}$ measures the strength of interactions and is equivalent to (2.9) in the case where $K = 1$ and payoffs are appropriately normalized.

SA.2. Proof of Theorem 1

Proof of Theorem 1. *Line 1.* Given $\boldsymbol{T}$, $\boldsymbol{A}$, computing $\boldsymbol{D}$ takes $O(n)$ evaluations of the payoff function, since we have to compute $\mathcal{R}_k^c$ for each agent k. Then, as discussed in Remark 1, computing the set of weakly connected components (strategic neighborhoods) takes $O(n+L)$ time, where L is the number of links in $\boldsymbol{D}$, assuming the graph is stored as an adjacency list. The expected number of links is $0.5\mathbf{E}[\sum_i \sum_j D_{ij}] \leq n^2\mathbf{E}[D_{ij}]$, which is $O(n)$ by Assumption 2. Hence, $L = O_p(n)$, so line 1 of the algorithm has complexity $O_p(n)$.[1]

Line 2. For each strategic neighborhood $\mathcal{S}(C)$, the algorithm verifies whether each element of $\mathcal{Y}(\mathcal{S}(C), \boldsymbol{T})$ is a Nash equilibrium. The size of this set is $2^{|C|}$. For each candidate action profile, we have to verify the equilibrium conditions by evaluating the payoff function for each agent in C, resulting in at most $|C|2^{|C|}$ evaluations for the whole set. Repeated for each neighborhood, the for-loop takes

$$\sum_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} |C|2^{|C|} \leq n \max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} 2^{|C|}$$

evaluations. By Lemma SA.2.1, which uses Assumptions 2 and 3, this quantity is $O_p(n^{1+q})$, where q is defined in the statement of this theorem.

Line 3. Under Assumption 1, we can apply Lemma SA.2.3, which yields

$$\left(\bigtimes_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} \mathcal{E}_{\mathrm{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})\big|_C \right) = \mathcal{E}_{\mathrm{NE}}(\boldsymbol{T}, \boldsymbol{A}). \tag{SA.2.1}$$

Therefore, the algorithm has the desired output. □

Lemma SA.2.1. *Under Assumptions* 2 *and* 3,

$$\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} 2^{|C|} = O_p\big(n^q\big),$$

where q is defined in Theorem 1.

Proof. Let $\beta = \|\lambda\|_{\mathbf{m},k}$. By Assumption 2, there exists $\varepsilon > 0$ such that $(1+\varepsilon)\beta < 1$. For any such ε and $m > 0$,

$$\begin{aligned}\mathbf{P}\Big(\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} 2^{|C|} > mn^q\Big) &\leq n\mathbf{P}\big(|C| > (q\log n + \log m)/\log 2\big)\\ &\leq cn^{1-q(\log 2)^{-1}\log((1+\varepsilon)\beta)^{-1}} m^{-(\log 2)^{-1}\log((1+\varepsilon)\beta)^{-1}}\end{aligned}$$

for some $c > 0$. The first line uses the union bound and the second Lemma SA.3.6 (which we can apply due to Assumptions 2 and 3). Since the exponent on m is negative in the last line, the last line is $o(1)$ as we take $m, n \to \infty$, if we pick $\varepsilon > 0$ such that the exponent on n is less than or equal to zero, or equivalently,

$$q(\log 2)^{-1} \log\big((1+\varepsilon)\beta\big)^{-1} \geq 1 \quad \Leftrightarrow \quad q \geq \frac{\log 2}{\log \dfrac{1}{(1+\varepsilon)\beta}}. \tag{SA.2.2}$$

[1] In the statement of the theorem, we measure work in terms of evaluations of the payoff function. A small complication to our argument is that depth-first search does not ever evaluate the payoff function. Instead, the unit of work is querying an adjacency list to reveal a neighbor of an agent in the graph. However, this will not have a higher order of complexity than evaluating a nontrivial payoff function, an assumption we implicitly maintain in this argument.

It thus suffices to establish that such an ε exists. Since $\varepsilon \in (0, \beta^{-1} - 1)$, the fraction on the right-hand side is a continuous function of ε with range $(\log 2/\log \beta^{-1}, \infty)$. Furthermore, $q > \log 2/\log \beta^{-1}$ by assumption, so by the intermediate value theorem, we can indeed choose ε such that (SA.2.2) holds. This prove the claim. □

LEMMA SA.2.2. *Let $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})$. Under Assumption 1,*

$$\mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)}) = \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})|_{\mathcal{S}(C)}.$$

PROOF. *Step 1.* We first prove that

$$\mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)}) \subseteq \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{Y} \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$ and $\boldsymbol{Y}' \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$. Construct an action profile $\boldsymbol{Y}^*$ by defining

$$Y_k^* = \begin{cases} Y_k & \text{if } k \in \mathcal{S}(C), \\ Y_k' & \text{otherwise.} \end{cases}$$

That is, we take $\boldsymbol{Y}'$ and replace the actions of all agents in $\mathcal{S}(C)$ with the actions dictated by $\boldsymbol{Y}$. It suffices to show $\boldsymbol{Y}^* \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$. For this purpose, fix two arbitrary agents $i \in \mathcal{S}(C)$ and $j \in \mathcal{N}_n \backslash \mathcal{S}(C)$.

We show that Y_i^* is a best response for i to $\boldsymbol{Y}^*$ in the sense that (2.1) holds. There are two cases to consider. First, suppose $i \in \mathcal{S}(C)\backslash C$. Then by definition of $\mathcal{S}(C)$, we have $\mathcal{R}_i^c = 0$, which means i's action is a best response regardless of the actions chosen by others. The second case to consider is $i \in C$. Then since $j \notin \mathcal{S}(C)$, i and j must not be connected in the network $\boldsymbol{D}$ because, by definition, C contains all agents k connected to i such that $\mathcal{R}_k^c = 1$, and $\mathcal{S}(C)$ adds to this set all agents k connected to i such that $\mathcal{R}_k^c = 0$. Since i and j are not connected, i's payoffs are not a function of j's action by Assumption 1. Since j is any arbitrary agent not in $\mathcal{S}(C)$, given that $\boldsymbol{Y} \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$, it follows that Y_i^* is a best response.

Next, we show that Y_j^* is a best response for j to $\boldsymbol{Y}^*$. Suppose, for i previously defined above, that $i \in \mathcal{S}(\mathcal{N}_n \backslash C)$. Then it must be the case that i's action is robust in the sense that $\mathcal{R}_i^c = 0$. Therefore, $Y_i' = Y_i$, so i's action under $\boldsymbol{Y}'$ is the same under $\boldsymbol{Y}^*$. On the other hand, consider $i \notin \mathcal{S}(\mathcal{N}_n \backslash C)$. Since $j \in \mathcal{N}_n \backslash C$ by assumption, we must have $A_{ij} = 0$, as argued in the previous paragraph. Then by Assumption 1, j's payoffs are not a function of i's action.

We have therefore established that (1) the only components of vectors $\boldsymbol{Y}'$ and $\boldsymbol{Y}^*$ that may differ are the actions of agents $i \notin \mathcal{S}(\mathcal{N}_n \backslash C)$, and (2) the payoffs of any $j \in \mathcal{N}_n \backslash C$ are not functions of the actions of such agents i. Then since $\boldsymbol{Y}' \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$, it follows that Y_j^* is a best response to $\boldsymbol{Y}^*$ for any $j \in \mathcal{N}_n \backslash C$, in particular for $j \in \mathcal{N}_n \backslash \mathcal{S}(C)$, which proves the desired claim.

Step 2. We prove that

$$\mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)}) \supseteq \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{Y} \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})|_{\mathcal{S}(C)}$. By definition, there exists $\boldsymbol{Y}' \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$ such that $\boldsymbol{Y} = \boldsymbol{Y}'_{\mathcal{S}(C)}$. Fix two arbitrary agents $i \in \mathcal{S}(C)$ and $j \in \mathcal{N}_n \backslash \mathcal{S}(C)$. There are two cases to consider. First, suppose $i \in \mathcal{S}(C) \backslash C$. Then by definition of $\mathcal{S}(C)$, we have $\mathcal{R}_i^c = 0$, which means i's action is optimal regardless of the actions chosen by others. The second case to consider is $i \in C$. Then since $j \notin \mathcal{S}(C)$, as argued in step 1, $A_{ij} = 0$, so i's payoffs are not a function of j's action. Since i, j are arbitrary, it follows that Y_i^* is optimal in the game where the set of agents is restricted to $\mathcal{S}(C)$. Hence, $\boldsymbol{Y} \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$, as desired. □

Lemma SA.2.3. *Under Assumption* 1, (SA.2.1) *holds.*

Proof. *Step 1.* Let $\boldsymbol{Y}^* \in \bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})} \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})|_C$. This set is well-defined in (2.7) because $\mathcal{C}(\boldsymbol{T}, \boldsymbol{A})$ partitions $\mathcal{N}_n$. Consider any $i \in \mathcal{N}_n$, and let C be the element of $\mathcal{C}(\boldsymbol{T}, \boldsymbol{A})$ containing i. We first prove that

$$Y^*_{\mathcal{S}(C)} \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)}). \tag{SA.2.3}$$

By construction, there exists $\boldsymbol{Y} \in \mathcal{E}_{NE}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$ such that the subvector of $Y^*_{\mathcal{S}(C)}$ on C equals Y_C. Furthermore, the subvector of $Y^*_{\mathcal{S}(C)}$ on $\mathcal{S}(C) \backslash C$ equals $Y_{\mathcal{S}(C) \backslash C}$ because agents in this set have robust actions by definition. Therefore, $Y^*_{\mathcal{S}(C)} = \boldsymbol{Y}$, which proves (SA.2.3).

This establishes that, in the game where the set of players is given by $\mathcal{S}(C)$, Y_i^* is a best response for i to $\boldsymbol{Y}^*_{\mathcal{S}(C)}$ in the sense of (2.1). In fact, Y_i^* is a best response to $\boldsymbol{Y}^*$ in the game with all n players. This is because i's payoffs are not a function of actions of agents $j \notin \mathcal{S}(C)$ by the second paragraph of the proof of Lemma SA.2.2. We have thus proved that $\boldsymbol{Y}^* \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$. Hence,

$$\bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})} \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})\big|_C \subseteq \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A}).$$

Step 2. We prove the $\supseteq$ direction. Let $\boldsymbol{Y}^* \in \mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$. For any $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})$, $Y^*_{\mathcal{S}(C)} \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$ by Lemma SA.2.2. Hence, $Y^*_C \in \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})|_C$, so

$$\boldsymbol{Y}^* \in \bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})} \mathcal{E}_{\text{NE}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})\big|_C,$$

as desired. □

SA.3. Tail bound for component sizes

This section proves an exponential tail bound on the size of any component of a certain random graph in the "subcritical" regime where an analog of Assumption 2 holds. The result is used to prove our main theorems on algorithmic complexity. The first lemma in Section SA.3.1 below generalizes Lemma 9 of Leung (2019) to models without homophily. Our proof also provides a complete argument that fills in some details skimmed over in the proof of the latter result.

We consider the following random graph model. Suppose each agent i is endowed with a type $\tau_i \in \mathbb{R}^d$, i.i.d. across agents with distribution Φ. Let $\boldsymbol{\tau} = (\tau_i)_{i=1}^n$ and $\boldsymbol{\zeta}$ be an $n \times$

n matrix with zeros on the diagonal, where the off-diagonal elements ζ_{ij} are distributed i.i.d. Let $\boldsymbol{\Gamma}$ be a directed network on $\mathcal{N}_n$ such that

$$\Gamma_{ij} = \gamma_n(\tau_i, \tau_j, \zeta_{ij})$$

for all $i, j \in \mathcal{N}_n$, where γ_n is a $\{0, 1\}$-valued function that may depend on the network size n. We apply the results of this section to networks $\boldsymbol{D}$ defined in Sections 2, SA.4, and SA.5, all of which have the same structure as $\boldsymbol{\Gamma}$.

Let $\mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$ be the set of strongly connected components of $\boldsymbol{\Gamma}$. We wish to obtain exponential tail bounds on the size of any $C \in \mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$. The technique is to stochastically bound component sizes by those of certain multitype branching processes and then show that the latter quantities have the required tail properties. Our results require the following assumptions.

Assumption SA.3.1. *Assumption* 2 *holds with* $\boldsymbol{\Gamma}$ *in place of* $\boldsymbol{D}$.

Assumption SA.3.2. *Assumption* 3 *holds with* $\boldsymbol{\Gamma}$ *in place of* $\boldsymbol{D}$.

SA.3.1 *Branching process*

Let μ and φ be defined as in Assumption SA.3.1. Denote by $\mathcal{P}_\varphi(t)$ a Poisson process on $\mathbb{R}^d$ with intensity $\varphi(t, t')\,\mathrm{d}\mu(t')$. Let $\mathfrak{X}_\varphi(t)$ be a multitype Galton–Walton branching process with type space $\mathbb{R}^d$, initialized at a particle of type t, where a particle of type t' is replaced in the next generation by a set of particles (the *offspring* of t') distributed according to $\mathcal{P}_\varphi(t')$. For a formal definition of this process see, for example, Mode (1971). In brief, this is a discrete-time process where the first generation consists of a single particle of type t. The particles in the second generation constitutes the offspring of t, characterized by types that are distributed according to $\mathcal{P}_\varphi(t)$. Let $t_1, \ldots, t_m$ be the types of the second generation. The third generation consists of the offspring of the second generation $\{\mathcal{P}_\varphi(t_i)\}_{i=1}^m$, which are i.i.d. processes. This process continues indefinitely.

This branching process is of interest because the expected number of offspring of a particle of type t is

$$\int_{\mathbb{R}^d} \varphi\big(t, t'\big)\,\mathrm{d}\mu\big(t'\big),$$

which is the upper bound on the conditional expected degree of an agent of type t in the graph $\boldsymbol{\Gamma}$ by Assumption SA.3.1.

Let $|\mathfrak{X}_\varphi(t)|$ be the total population of the branching process, that is, the total number of particles generated. We show that this stochastically dominates $|C_t|$ for any $C_t \in \mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$, for an appropriate choice of φ and t.

Lemma SA.3.1. *Let* $C_i \in \mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$ *be the component containing agent* $i \in \mathcal{N}_n$. *Under Assumptions* SA.3.1 *and* SA.3.2, *for any* $\varepsilon > 0$ *and* n *sufficiently large,* $|C_i|$ *is stochastically dominated by* $|\mathfrak{X}_{(1+\varepsilon)\varphi}(\tau_i)|$, *where* φ *is defined in Assumption* SA.3.1 *and* $(1+\varepsilon)\varphi$ *is the mapping* $(t, t') \mapsto (1+\varepsilon)\varphi(t, t')$.

Proof. *Step 1.* We explore C_i using a breadth-first search on $\boldsymbol{D}$ starting at i.[2] This is a discrete-time process, where at each time period $r = 0, 1, \dots$ we maintain the following three sets of agents: the set of removed agents $\mathcal{R}_r$, the set of active agents $\mathcal{A}_r$, and the set of unexplored agents $\mathcal{U}_r$. It will be convenient to represent $\mathcal{A}_r$ as a *queue* in the computer science sense.[3] The process evolves as follows:

- At time $r = 0$, initialize $\mathcal{A}_0 = \{i\}, \mathcal{U}_0 = \mathcal{N}_n \backslash \{i\}$, and $\mathcal{R}_0 = \emptyset$.
- At time $r = 1$, we activate the network neighbors of the only active agent i and then deactivate i, moving her to the removed set. That is, we update $\mathcal{A}_0$ to $\mathcal{A}_1$ by removing (dequeueing) i and then adding to the end of the queue (enqueueing) the set of elements $\{j \in \mathcal{N}_n : \Gamma_{ij} = 1\}$ in arbitrary order. Also we update $\mathcal{U}_1 = \mathcal{U}_0 \backslash \mathcal{A}_1$ and $\mathcal{R}_1 = \{i\}$.
- At time $r > 1$, we take the first agent in the queue $\mathcal{A}_{r-1}$, say j, activate all her unexplored neighbors, and deactivate and remove her. That is, we update $\mathcal{A}_{r-1}$ to $\mathcal{A}_r$ by dequeueing j and enqueueing $\{k \in \mathcal{N}_n : \Gamma_{jk} = 1, k \in \mathcal{U}_{r-1}\}$ in arbitrary order. Also, we update $\mathcal{U}_r = \mathcal{U}_{r-1} \backslash \mathcal{A}_r$ and $\mathcal{R}_r = \mathcal{R}_{r-1} \cup \{j\}$.

Note that at each time r, we explore $B_r(1) \equiv |\mathcal{A}_r| - |\mathcal{A}_{r-1}| + 1$ new agents, which are precisely those unexplored agents that are neighbors of the nominated agent j. Hence, the process explores C_i, and

$$1 + \sum_{r=1}^{\infty} B_r(1) = |C_i|. \tag{SA.3.1}$$

Step 2. We modify the breadth-first search to create a process whose size stochastically dominates it. This time we only need to maintain a queue of active agents $\mathcal{A}_r$ at each time period r. The process evolves as follows:

- At time $r = 0$, initialize $\mathcal{A}_0 = \{\tau_i\}$. Note that this process keeps track of agent types rather than labels, unlike the breadth-first search.
- At time $r > 0$, choose the first element in the queue $\mathcal{A}_{r-1}$, say τ^*. Generate $\{\tilde{\tau}_1\}_{i=1}^{n-1}$ i.i.d. with the same distribution as τ_i. Independently generate $\{\xi_i\}_{i=1}^{n-1}$ i.i.d., where ξ_j is Bernoulli with success probability $p_n(\tau^*, \tilde{\tau}_j)$, where $p_n(t, t') = \mathbf{P}(\Gamma_{ij} = 1 \mid \tau_i = t, \tau_j = t')$. Update $\mathcal{A}_{r-1}$ to $\mathcal{A}_r$ by dequeueing τ^* and enqueueing $\{\tilde{\tau}_j : \xi_j = 1, j \leq n\}$ in arbitrary order. Define $B_r(2) = \sum_{j \leq n-1} \xi_j$.

[2]The technique of exploring a component using a search process and approximating or bounding the latter by a branching process is a standard argument for obtaining stochastic bounds on component sizes of random graphs (e.g., Janson, Luczak, and Rucinski (2011, Chapter 3)). For some recent examples studying classes of graphs similar to ours, see Bollobás, Janson, and Riordan (2007) and Bollobás, Janson, and Riordan (2011). The specific line of argument we use is somewhat related to the coupling in Section 3 of the latter paper. However, to our knowledge, our result has not been previously established for the general class of graphs considered here.

[3]This is conceptualized as a horizontal collection of elements ordered from left to right with two associated operations. The `enqueue` operation adds an element to the right side of the queue. The `dequeue` operation removes an element from the left side of the queue.

By construction,

$$\sum_{r=1}^{s} B_r(2) \succeq \sum_{r=1}^{s} B_r(1) \quad \forall s > 1, \tag{SA.3.2}$$

where $\succeq$ denotes stochastic dominance. This is because (1) $\{\tilde{\tau}_j : \xi_j = 1, j \leq n-1\}$ has the same distribution as the set of types associated with neighbors of an agent with type τ^* in $\boldsymbol{\Gamma}$, and (2) the breadth-first search only restricts to "unexplored" agents when adding new agents to the active set at each time $r > 0$. In particular, note that $B_r(1)$ has the same distribution as $\sum_{j \leq n-|\mathcal{R}_{r-1}|} \xi_j$, conditional on the nominated node having type τ^*, which is clearly stochastically dominated by $\sum_{j \leq n-1} \xi_j$.

Step 3. We represent the process constructed in step 2 in a more convenient fashion in preparation for a stochastic dominance argument in step 4. Instead of defining $\mathcal{A}_r$ for $r > 0$ as in step 2, let us generate an independent binomial random variable N with $n-1$ trials and success probability

$$p^* = \sup_{t,t'} p_n(t, t').$$

Instead of $\xi_1, \xi_2, \ldots$, we generate $\tilde{\xi}_1, \tilde{\xi}_2, \ldots$ i.i.d. independently of N, where $\tilde{\xi}_j$ is Bernoulli with success probability $p_n(\tau^*, \tilde{\tau}_j)/p^*$. Finally, we construct $\mathcal{A}_r$ by removing τ^* from the queue $\mathcal{A}_{r-1}$ and adding $\{\tilde{\tau}_j : \tilde{\xi}_j = 1, j \leq N\}$ to the end of the queue in arbitrary order. Let $B_r(3) = \sum_{j \leq N} \tilde{\xi}_j$. Then by construction,

$$\sum_{r=1}^{s} B_r(3) \stackrel{d}{=} \sum_{r=1}^{s} B_r(2) \quad \forall s > 1. \tag{SA.3.3}$$

Step 4. We modify the process in step 3 to obtain one that stochastically dominates it in terms of size. Recall the definition of N and p^* from the previous step. Fix any $\varepsilon > 0$. Since $\sup_n p^* < 1$ by Assumption SA.3.2(a), N is a nondegenerate binomial random variable. It is well known that, for n sufficiently large, this is stochastically dominated by a Poisson random variable N' with intensity $(1+\varepsilon)(n-1)p^*$ (see, e.g., Bollobás, Janson, and Riordan (2007, proof of Theorem 12.5)). We may then couple N', N such that $N' \geq N$ a.s.

Consider a modification of the process in step 3 where we replace N with N', where N' is generated independently of all other quantities. Using the notation in the previous step, let $\eta(\tau^*)$ be the point process induced by the random set $\{\tilde{\tau}_j : \tilde{\xi}_j = 1, j \leq N'\}$ (see, e.g., Last and Penrose (2017, Definition 2.4)), where τ^* is the first element of the queue $\mathcal{A}_{r-1}$ (so the random variables in the set are generated at time r of the modified process). Let $B_r(4)$ be the number of elements in $\eta(\tau^*)$. Then

$$\sum_{r=1}^{s} B_r(4) \geq \sum_{r=1}^{s} B_r(3) \quad \text{a.s. } \forall s > 1 \tag{SA.3.4}$$

under the coupling.

Step 5. Let $\eta'(\tau^*)$ be the Poisson point process on $\mathbb{R}^d$ with intensity measure

$$(1+\varepsilon)(n-1)p_n(\tau^*, t')\,\mathrm{d}\Phi(t'), \tag{SA.3.5}$$

where Φ is the distribution of τ_i. By Lemma SA.3.2, $\eta(\tau^*)$ has the same distribution as $\eta'(\tau^*)$. By Assumption SA.3.1, the mean of the intensity measure (SA.3.5) is dominated by that of $(1+\varepsilon)\varphi(\tau^*, t')\,\mathrm{d}\mu(t')$. Thus, consider a modification of the process in step 4, where at each period $r > 0$, we replace $B_r(4)$ with $B_r(5) = |\mathcal{P}_{(1+\varepsilon)\varphi}(\tau^*)|$, where τ^* is the first element of the queue $\mathcal{A}_{r-1}$. Then

$$\sum_{r=1}^{s} B_r(5) \succeq \sum_{r=1}^{s} B_r(4) \quad \forall s > 1, \tag{SA.3.6}$$

Furthermore, since $\mathcal{P}_{(1+\varepsilon)\varphi}(\tau^*)$ is the same as the offspring distribution of the branching process $\mathfrak{X}_{(1+\varepsilon)\varphi}(\tau_i)$,

$$1 + \sum_{r=1}^{\infty} B_r(5) \overset{d}{=} \left|\mathfrak{X}_{(1+\varepsilon)\varphi}(\tau_i)\right|. \tag{SA.3.7}$$

This is quite evident from comparing the first few generations. For example, the second generation of the branching process has the same distribution as $B_1(5)$, as noted above. Conditional on the size of this generation being m, the third generation of the branching process has the same distribution as $B_2(5) + \cdots + B_{2+m-1}(5)$ by construction of the queue, and so on.

Therefore, combining (SA.3.1), (SA.3.2), (SA.3.3), (SA.3.4), (SA.3.6), and (SA.3.7), we have shown that

$$\left|\mathfrak{X}_{(1+\varepsilon)\varphi}(\tau_i)\right| \succeq |C_i|,$$

as desired. □

LEMMA SA.3.2. *Under the assumptions of Lemma SA.3.1, $\eta(\tau^*) \overset{d}{=} \eta'(\tau^*)$, where these quantities are defined in steps 4 and 5 of Lemma SA.3.1.*

PROOF. By Proposition 2.10(iii) of Last and Penrose (2017), it suffices to show equivalence of their respective Laplace functionals conditional on $\tau^* = t$. Let $u\colon \mathbb{R}^d \to [0,\infty]$ be a measurable function. Condition on $\tau^* = t$. The Laplace functional of $\eta(t)$ is

$$\begin{aligned}
&\mathbf{E}\exp\left\{-\int u(t')\,\mathrm{d}\eta(t')\right\} \\
&= \mathbf{E}\prod_{j=1}^{N'} \exp\{-\tilde{\xi}_j u(\tilde{\tau}_j)\} \\
&= \mathbf{E}\left(\int \mathbf{E}\big[\exp\{-\tilde{\xi}_j u(\tilde{\tau}_j)\} \,|\, \tau_j = t'\big]\,\mathrm{d}\Phi(t')\right)^{N'} \\
&= \mathbf{E}\left(\int \left[(e^{-u(t')} - 1)\frac{p_n(t,t')}{p^*} + 1\right]\mathrm{d}\Phi(t')\right)^{N'} \\
&= \exp\left\{\int (e^{-u(t')} - 1)(1+\varepsilon)(n-1)\,p_n(t,t')\,\mathrm{d}\Phi(t')\right\},
\end{aligned} \tag{SA.3.8}$$

where the last line uses the fact that

$$\mathbf{E}\big[w^{\mathrm{Poisson}(\lambda)}\big] = e^{\lambda(w-1)}$$

(e.g., Bollobás, Janson, and Riordan (2007, proof of Theorem 12.5)).

Let $\rho = \int p_n(t,t')\,\mathrm{d}\Phi(t')$. Let $\eta''(t)$ be the mixed binomial process with mixing distribution Poisson$((1+\varepsilon)(n-1)\rho)$ and sampling distribution equal to the probability measure $\mathcal{L}$ satisfying $\mathcal{L}(A) = \int_{t' \in A} \rho^{-1} p_n(t,t')\,\mathrm{d}\Phi(t')$ (Last and Penrose (2017, Definition 3.4)). By Proposition 3.5 of Last and Penrose (2017),

$$\eta''(t) \overset{d}{=} \eta'(t).$$

It remains to calculate the Laplace functional of $\eta''(t)$. For $N'' \sim$ Poisson$((1+\varepsilon)(n-1)\rho)$, this is given by

$$\begin{aligned}
&\mathbf{E}\exp\Big\{-\int u(t')\,\mathrm{d}\eta''(t')\Big\} \\
&= \mathbf{E}\Big(\int \exp\{-u(t')\}\rho^{-1} p_n(t,t')\,\mathrm{d}\Phi(t')\Big)^{N''} \\
&= \exp\Big\{\int e^{-u(t')}(1+\varepsilon)(n-1)p_n(t,t')\,\mathrm{d}\Phi(t') - (1+\varepsilon)(n-1)\rho\Big\},
\end{aligned}$$

which equals (SA.3.8). □

SA.3.2 *Exponential tail bound*

We next prove that the size of the branching process $\mathfrak{X}_{(1+\varepsilon)\varphi}(t)$ has exponential tails. For any constant α, let

$$J_\alpha(t) = \mathbf{E}\big[\alpha^{|\mathfrak{X}_{(1+\varepsilon)\varphi}(t)|}\big] \tag{SA.3.9}$$

and F be the functional satisfying

$$F(g)(t) = \int_{\mathbb{R}^d} g(t')\varphi(t,t')\,\mathrm{d}\mu(t')$$

for any bounded function $g\colon \mathbb{R}^d \to \mathbb{R}$. That is, F takes a function g as its argument, and $F(g)$ is a function with domain $\mathbb{R}^d$ and range $\mathbb{R}$. Let

$$\sigma_\varepsilon(t) = (1+\varepsilon)\int_{\mathbb{R}^k}\Big(\int_{\mathbb{R}^{d-k}} \varphi\big(t,(t'_k,t'_{-k})\big)^2\,\mathrm{d}\mu_{-k}(t'_{-k})\Big)^{1/2}\mathrm{d}\mu_k(t'_k).$$

Lemma SA.3.3. *Let $\beta = \|\lambda\|_{k,\mathbf{m}}$, the latter defined in Assumption SA.3.1. Under Assumptions SA.3.1 and SA.3.2, for any $\varepsilon \in (0, \beta^{-1}-1)$ and $\alpha \in (1, ((1+\varepsilon)\beta)^{-1})$,*

$$\sup_t J_\alpha(t) < \infty.$$

PROOF. We follow the proof of Theorem 2.5 in Turova (2012). We note for later that

$$\inf_t \sigma_0(t) > 0 \quad \text{and} \quad \sup_t \sigma_0(t) < \infty \tag{SA.3.10}$$

under Assumption SA.3.2.

We next construct a bounded function $h \geq 1$ that satisfies the conditions of Lemma SA.3.5. By Lemma SA.3.4, the existence of h implies the desired conclusion. For any constant c, define the function

$$\gamma_c = F\big(e^{c\sigma_\varepsilon} - 1\big).$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} \gamma_c(t) \leq & \underbrace{(1+\varepsilon)\int_{\mathbb{R}^k}\bigg(\int_{\mathbb{R}^{d-k}} \varphi\big(t,(t_k',t_{-k}')\big)^2 \,\mathrm{d}\mu_{-k}\big(t_{-k}'\big)\bigg)^{1/2} \mathrm{d}\mu_k\big(t_k'\big)}_{\sigma_\varepsilon(t)} \\ & \times \underbrace{\bigg(\int_{\mathbb{R}^{d-k}} \big(e^{c\sigma_\varepsilon((t_k',t_{-k}'))} - 1\big)^2 \,\mathrm{d}\mu_{-k}\big(t_{-k}'\big)\bigg)^{1/2}}_{K(c,t_k')}. \end{aligned} \tag{SA.3.11}$$

Note that, as a function of c, for any t, $K(c,t)$ is differentiable on $[0,a]$ for some $a > 0$ by dominated convergence and (SA.3.10). For c small, by the Cauchy–Schwarz inequality, there exists a universal constant $M > 0$ such that

$$K'\big(c,t_k'\big) \leq M\bigg(\int_{\mathbb{R}^{d-k}} \sigma_\varepsilon\big((t_k',t_{-k}')\big)^2 \exp\big\{2c\sigma_\varepsilon\big((t_k',t_{-k}')\big)\big\}\,\mathrm{d}\mu_{-k}\big(t_{-k}'\big)\bigg)^{1/2}.$$

For c small enough, this is finite by (SA.3.10). The previous equation yields

$$\lim_{c\downarrow 0} K'\big(c,t_k'\big) \leq \sup_{t_k'}\bigg(\int_{\mathbb{R}^{d-k}} \sigma_\varepsilon\big((t_k',t_{-k}')\big)^2 \,\mathrm{d}\mu_{-k}\big(t_{-k}'\big)\bigg)^{1/2} = (1+\varepsilon)\|\lambda\|_{k,\mathbf{m}},$$

Call the right-hand side β_ε. By Assumption SA.3.1, for ε sufficiently small, $\beta_\varepsilon < 1$. Then by the mean value theorem, for c sufficiently small and any t,

$$K(c,t) < \beta_\varepsilon c,$$

which, by (SA.3.11), implies

$$\gamma_c < \beta_\varepsilon c \sigma_\varepsilon. \tag{SA.3.12}$$

Fix $\alpha > 1$, and define

$$\tilde{\gamma}_c \equiv \alpha F\big(e^{\alpha c\sigma_\varepsilon} - 1\big) = \alpha\gamma_{\alpha c}.$$

By (SA.3.12), for c sufficiently small,

$$\tilde{\gamma}_c = \alpha\gamma_{\alpha c} \leq \alpha^2 c\beta_\varepsilon\sigma_\varepsilon. \tag{SA.3.13}$$

Now define the function

$$h = \alpha\big(e^{\alpha c \sigma_\varepsilon} - 1\big) + 1.$$

By (SA.3.10), $\sup_t h(t) < \infty$. To complete the proof, it suffices to show that

$$\alpha \exp\big\{F(h-1)\big\} \le h. \tag{SA.3.14}$$

Using (SA.3.13),

$$\alpha \exp\big\{F(h-1)\big\} = \alpha \exp\big\{\alpha F\big(e^{\alpha c \sigma_\varepsilon} - 1\big)\big\} = \alpha \exp\{\tilde{\gamma}_c\} \le \alpha \exp\big\{\alpha^2 c \beta_\varepsilon \sigma_\varepsilon\big\}. \tag{SA.3.15}$$

Suppose $\alpha \in (1, \delta/\beta_\varepsilon)$ for some $\delta \in (\beta_\varepsilon, 1)$. Then $\alpha \exp\{\alpha^2 c \beta_\varepsilon x\} \le \alpha \exp\{\alpha c \delta x\}$ for any $x \ge 0$. Under (SA.3.10), $\sigma_\varepsilon(t) > b$ for some positive b and all $t \in \mathbb{R}^d$. Therefore, there exists $\alpha \in (1, \delta/\beta_\varepsilon)$ such that

$$\alpha \exp\big\{\alpha^2 c \beta_\varepsilon \sigma_\varepsilon\big\} \le \alpha\left(\exp\{\alpha c \delta \sigma_\varepsilon\} - \frac{\alpha - 1}{\alpha}\right) = \alpha\big(\exp\{c\alpha\sigma_\varepsilon\} - 1\big) + 1 = h. \tag{SA.3.16}$$

Thus, (SA.3.15) and (SA.3.16) establish (SA.3.14). □

Lemma SA.3.4. *For any $\alpha \ge 1$, $J_\alpha(\cdot)$ is the minimal solution $f \ge 1$ to the functional fixed-point equation*

$$f = \alpha \exp\big\{F(f-1)\big\}. \tag{SA.3.17}$$

Proof. We first prove that $J_\alpha(\cdot)$ is a solution to (SA.3.17). This is a standard branching-process argument. We first construct a more convenient representation of $\mathfrak{X}_{(1+\varepsilon)\varphi}(t)$. Observe that we can represent the distribution of its second generation, namely $\mathcal{P}_{(1+\varepsilon)\varphi}(t)$, as follows. Partition $\mathbb{R}^d$ into cubes with side length one centered at integer-valued elements of $\mathbb{R}^d$. Label the elements of the partition arbitrarily $1, 2, \ldots$, and let Q_k be the cube associated with label k. For each partition k, let $\rho_k = \int_{t' \in Q_k} \varphi(t, t')\, \mathrm{d}\mu(t')$, and draw $N_k \sim \text{Poisson}((1+\varepsilon)\rho_k)$ independently across partitions. For each partition k, if $N_k > 0$, then conditional on N_k, draw types $\{\tilde{\tau}_i^k\}_{i=1}^{N_k}$ i.i.d. from the distribution μ_k satisfying $\mu_k(A) = \rho_k^{-1} \int_{t' \in A \cap Q_k} \varphi(t, t')\, \mathrm{d}\mu(t')$. If $N_k = 0$, then the set of types is $\emptyset$. Then the proper point process induced by the random set

$$\bigcup_{k=1}^{\infty} \{\tilde{\tau}_i^k\}_{i=1}^{N_k} \tag{SA.3.18}$$

(Last and Penrose (2017, Definition 2.4)) has the same distribution as $\mathcal{P}_{(1+\varepsilon)\varphi}(t)$ by the proof of Theorem 3.6 of Last and Penrose (2017). The set (SA.3.18) represents the types of particles associated with the second generation of the branching process.

Observe that the total population has the same distribution as the sum of the total populations of independent branching processes starting at initial particles of type t' for each t' in the second generation (SA.3.18). That is,

$$\big|\mathfrak{X}_{(1+\varepsilon)\varphi}(t)\big| \overset{d}{=} 1 + \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \big|\mathfrak{X}_{(1+\varepsilon)\varphi}\big(\tilde{\tau}_j^k\big)\big|,$$

where conditional on the realization of the second generation (SA.3.18), the associated branching processes $\mathfrak{X}_{(1+\varepsilon)\varphi}(\tilde{\tau}_j^k)$ are realized independently.

Using this new representation of the process, we have

$$\begin{aligned}
J_\alpha(t) &= \mathbf{E}\big[\alpha^{\sum_{k=1}^\infty \sum_{j=1}^{N_k} |\mathfrak{X}_{(1+\varepsilon)\varphi}(\tilde{\tau}_j^k)|+1}\big] \\
&= \mathbf{E}\left[\alpha \prod_{k=1}^\infty \big(\mathbf{E}\big[\alpha^{|\mathfrak{X}_{(1+\varepsilon)\varphi}(\tilde{\tau}_j^k)|+1} | N_k\big]\big)^{N_k}\right] \\
&= \mathbf{E}\left[\alpha \prod_{k=1}^\infty \left(\int_{\mathbb{R}^d} J_\alpha(t')\rho_k^{-1}\varphi(t,t')\,\mathrm{d}\mu_k(t')\right)^{N_k}\right] \\
&= \alpha \prod_{k=1}^\infty \exp\left\{(1+\varepsilon)\int_{\mathbb{R}^d} J_\alpha(t')\varphi(t,t')\,\mathrm{d}\mu_k(t') - (1+\varepsilon)\rho_k\right\} \\
&= \alpha \exp\left\{(1+\varepsilon)\int_{\mathbb{R}^d} \big(J_\alpha(t')-1\big)\varphi(t,t')\,\mathrm{d}\mu(t')\right\}.
\end{aligned}$$

The second and fourth lines use the monotone convergence theorem. The fourth line also uses the fact that $\mathbf{E}[w^{\text{Poisson}(\lambda)}] = e^{\lambda(w-1)}$ (e.g. Bollobás, Janson, and Riordan (2007, proof of Theorem 12.5)). The last line equals

$$\alpha \exp\big\{F\big(J_\alpha(t)-1\big)\big\}, \tag{SA.3.19}$$

as desired.

Next, we show that $J_\alpha(\cdot)$ is the minimal solution. Let Λ denote the functional $f \mapsto \alpha\exp\{F(f-1)\}$. Following the argument in the proof of Theorem 2.1 in Turova (2012),

$$J_\alpha = \lim_{k\to\infty} \Lambda^k(1) \equiv \lim_{k\to\infty} \underbrace{(\Lambda \circ \cdots \circ \Lambda)}_{k \text{ times}}(1).$$

Suppose, to obtain a contradiction, that there exists a solution f to (SA.3.17) such that $1 \le f < J_\alpha$ for some $\alpha \ge 1$. Since Λ is monotone,

$$\Lambda^k(1) \le \Lambda^k(f) \le f < J_\alpha = \lim_{k\to\infty} \Lambda^k(1).$$

Letting $k \to \infty$ on the left-hand side, we obtain a contradiction. Therefore, J_α is the minimal solution. □

LEMMA SA.3.5. *Let $\alpha \ge 1$. If there exists a bounded function $h\colon \mathbb{R}^d \to [1,\infty)$ such that*

$$h \ge \alpha \exp\big\{F(h-1)\big\},$$

then there exists a function $g\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $1 \le g(t) \le h(t)$ for all $t \in \mathbb{R}^d$ that solves (SA.3.17).

PROOF. We follow the proofs of Theorem 2.1 and Lemma 2.4 in Turova (2012). Let Λ denote the functional $f \mapsto \alpha\exp\{F(f-1)\}$. Since Λ is monotonic and $\Lambda h \le h$ by the assumption of this lemma, $h \geq \Lambda h \geq \Lambda^2 h \geq \cdots$. Since $h \geq 1$, $(\Lambda h)(t) = \alpha \exp\{(F(h-1))(t)\} \geq \alpha \geq 1$, which implies $\Lambda^k h \geq 1$. Hence the limit

$$h \geq g \equiv \lim_{k\to\infty} \Lambda^k h \geq 1,$$

exists. It remains to show that g is a solution to (SA.3.17). We have

$$\Lambda g = \alpha \exp\bigl\{F(g-1)\bigr\} = \alpha \exp\Bigl\{\lim_{k\to\infty} F\bigl(\Lambda^k h - 1\bigr)\Bigr\} = \lim_{k\to\infty} \Lambda\bigl(\Lambda^k h\bigr) = g,$$

where the second equality uses monotone convergence and the third uses the continuous mapping theorem. □

SA.3.3 *Main results*

The next lemmas provide exponential tail bounds on the size of and number of links in an arbitrary component in $\boldsymbol{\Gamma}$.

Lemma SA.3.6. *Let $\beta = \|\lambda\|_{\mathbf{m},k}$, the latter defined in Assumption SA.3.1. Let $C \in \mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$. Under Assumptions SA.3.1 and SA.3.2, there exists $c > 0$ such that for any $\varepsilon > 0$ and n sufficiently large,*

$$\mathbf{P}\bigl(|C| > R\bigr) \leq c\bigl((1+\varepsilon)\beta\bigr)^R.$$

Proof. Let C_i be the element of $\mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$ containing agent i. Choose $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$ and $(1+\varepsilon')\beta < 1$. Such an ε' exists by Assumption SA.3.1, which ensures $\beta < 1$. By Lemma SA.3.1, for n sufficiently large,

$$\mathbf{P}\bigl(|C_i| > R\bigr) \leq \mathbf{P}\bigl(\bigl|\mathfrak{X}_{(1+\varepsilon')\varphi}(\tau_i)\bigr| > R\bigr)$$

By Markov's inequality, for $\alpha = ((1+\varepsilon')\beta)^{-1}$,

$$\mathbf{P}\bigl(\bigl|\mathfrak{X}_{(1+\varepsilon)\varphi}(\tau_i)\bigr| > R\bigr) \leq \alpha^{-R} \sup_t J_\alpha(t),$$

where $J_\alpha(t)$ is defined in (SA.3.9). The right-hand side is bounded above by

$$c\bigl((1+\varepsilon')\beta\bigr)^R$$

for some positive constant c by Lemma SA.3.3, since $\alpha \geq 1$. This, in turn, is bounded above by $c((1+\varepsilon)\beta)^R$ by definition of ε'. □

Lemma SA.3.7. *Let $\beta = \|\lambda\|_{\mathbf{m},k}$, the latter defined in Assumption SA.3.1. Let $C \in \mathcal{C}^*(\boldsymbol{\tau}, \boldsymbol{\zeta})$. Under Assumptions SA.3.1 and SA.3.2, there exists $c > 0$ such that for any $\varepsilon > 0$ and n sufficiently large,*

$$\mathbf{P}\left(0.5 \sum_{i,j\in C} \Gamma_{ij} > R\right) \leq c\bigl((1+\varepsilon)\beta\bigr)^R.$$

Proof. Our strategy is to first construct a tree graph coupled to $\boldsymbol{\Gamma}$ that has at least the same number of links. Then we use the fact that the number of links in a tree is at most the number of nodes to reduce the problem to bounding the size of the tree graph. For this, we can use branching processes as in Lemma SA.3.6.

We use the breadth-first search in step 1 of the proof of Lemma SA.3.1 to construct a tree graph $\mathcal{T}$. Let i be an arbitrary node in C. As in that proof, at each time period $r = 0, 1, \ldots$ we maintain the set of removed agents $\mathcal{R}_r$, active agents $\mathcal{A}_r$, and unexplored agents $\mathcal{U}_r$, and we think of $\mathcal{A}_r$ is a queue. The process evolves as follows:

- At time $r = 0$, initialize $\mathcal{A}_0 = \{i\}, \mathcal{U}_0 = \mathcal{N}_n \backslash \{i\}$, and $\mathcal{R}_0 = \emptyset$. Let $\mathcal{T}$ be the network consisting of the singleton node i.

- At time $r = 1$, update $\mathcal{A}_0$ to $\mathcal{A}_1$ by dequeueing i and enqueueing $\{j \in \mathcal{N}_n : \Gamma_{ij} = 1\}$ in arbitrary order. Update $\mathcal{U}_1 = \mathcal{U}_0 \backslash \mathcal{A}_1$ and $\mathcal{R}_1 = \{i\}$. Add the enqueued set of nodes to $\mathcal{T}$, and link i to each node in this set.

- At time $r > 1$, take the first agent in the queue $\mathcal{A}_{r-1}$, say j. Update $\mathcal{A}_{r-1}$ to $\mathcal{A}_r$ by dequeueing j and enqueueing $\{k \in \mathcal{N}_n : \Gamma_{jk} = 1, k \in \mathcal{U}_{r-1}\}$ in arbitrary order. Update $\mathcal{U}_r = \mathcal{U}_{r-1} \backslash \mathcal{A}_r$ and $\mathcal{R}_r = \mathcal{R}_{r-1} \cup \{j\}$. Add the enqueued set of nodes to $\mathcal{T}$, and link i to each node in this set.

  Here is the key modification. Let $\mathcal{N}_j^* = \{k \in \mathcal{N}_n : \Gamma_{jk} = 1, k \notin \mathcal{U}_{r-1}\}$, the set of j's neighbors that were previously explored. In the original breadth-first search, these neighbors were ignored. Here, we instead use them to generate $|\mathcal{N}_j^*|$ new nodes in $\mathcal{T}$ by independently drawing their types from the conditional distribution of τ_1 given $\Gamma_{1j} = 1$ and τ_j. Add these new nodes to $\mathcal{T}$, and link j to each node in this set. Note that we do not add these artificially generated nodes to the unexplored set, so they are never revisited in the search, and hence will always remain leaf nodes in $\mathcal{T}$.

The purpose of the key modification is to ensure that each node in $\mathcal{T}$ that is not artificially generated (i.e., is also in C) has the same degree as in C, and furthermore, the distribution of j's neighbors' types is the same in C and $\mathcal{T}$ conditional on τ_j. Therefore,

$$0.5 \sum_{i,j} \Gamma_{ij} \leq 0.5 \sum_{i,j} \mathcal{T}_{ij} \leq |\mathcal{T}|, \tag{SA.3.20}$$

since the number of links in a tree equals the number of nodes minus one.

Following the proof of Lemma SA.3.1, $|\mathcal{T}|$ is stochastically dominated by $|\mathcal{X}_{(1+\varepsilon)\phi}(\tau_i)|$. As established in the proof of Lemma SA.3.6, there exists $c > 0$ such that for any $\varepsilon > 0$ and n sufficiently large,

$$\mathbf{P}\big(|\mathcal{T}| > R\big) \leq c\big((1+\varepsilon)\beta\big)^R. \tag{SA.3.21}$$

Then the result follows from (SA.3.20) and (SA.3.21). □

SA.4. Network formation games

This section applies the ideas in Section 2 to strategic network formation. We discuss an application of the algorithm to compute previously intractable bounds on the identified set using moment inequalities in Sheng (2016).

Let $\mathcal{N}_n = \{1, \ldots, n\}$ be a set of agents. Endow each agent $i \in \mathcal{N}_n$ with a type $T_i \in \mathbb{R}^{d_t}$, distributed i.i.d. across agents, and endow each agent tuple (i, j) with a random utility shock $\zeta_{ij} \in \mathbb{R}^{d_\zeta}$, i.i.d. across tuples. Note that $\zeta_{ij} \neq \zeta_{ji}$, and the two are independent. Let $\boldsymbol{T} = (T_i)_{i=1}^n$ be the type profile and $\boldsymbol{\zeta}$ the $n \times n$ matrix with ijth entry ζ_{ij} for $i \neq j$ and zeros on the diagonal.

This section considers undirected network formation, while Section SA.5 studies the directed case. Because we are modeling network formation, we need to specify agent preferences over networks. Let $U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ be the payoff i enjoys from $\boldsymbol{A}$. Let A_{-ij} be the adjacency matrix with ijth entry removed and (ℓ, A_{-ij}) the matrix $\boldsymbol{A}$ with the ijth entry replaced with the value ℓ. Define $V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = U_i((1, A_{-ij}), \boldsymbol{T}, \boldsymbol{\zeta}) - U_i((0, A_{-ij}), \boldsymbol{T}, \boldsymbol{\zeta})$, which is i's marginal utility from adding a link with agent j. It is often more convenient to specify the model in terms of marginal utilities, as we will do next (Boucher and Mourifié (2017), Graham (2016), Leung (2019), Menzel (2017), Sheng (2016)). We assume

$$V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = V_n\big(S_{ij}(\boldsymbol{A}, \boldsymbol{T}), T_i, T_j, \zeta_{ij}\big),$$

where $S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = S(A_{-ij}, T_i, T_j, T_{-ij})$ for some $\mathbb{R}^{d_s}$-valued function $S(\cdot)$, and T_{-ij} is the type profile $\boldsymbol{T}$ with entries T_i, T_j omitted. Hence, strategic interactions enter marginal utilities through the vector of statistics $S_{ij}(\boldsymbol{A}, \boldsymbol{T})$. Note that, as with $g_n(\cdot)$ in Section 2, $V_n(\cdot)$ may vary with the network size n, which will be important for network sparsity (see Remark 4).

Two standard solution concepts used in the literature are pairwise stability with transferable and nontransferable utility (Jackson (2010)). We next consider the former and defer the latter to Section SA.4.4. Define the joint surplus function

$$V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) \equiv V_n^*\big(S_{ij}^*(\boldsymbol{A}, \boldsymbol{T}), T_i, T_j, \zeta_{ij}, \zeta_{ji}\big) \equiv V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) + V_{ji}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}),$$

where $S_{ij}^*(\boldsymbol{A}, \boldsymbol{T}) = (S_{ij}(\boldsymbol{A}, \boldsymbol{T}), S_{ji}(\boldsymbol{A}, \boldsymbol{T}))$. A network $\boldsymbol{A}$ is pairwise stable if, for all $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$A_{ij} = \mathbf{1}\big\{V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) > 0\big\}. \tag{SA.4.1}$$

The idea is that, even if i's marginal utility from linking with j is negative, if j's marginal utility is positive and utility can be freely transferred between the agents, then j can compensate i for the loss of utility, so the link will form. Let $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ be the set of pairwise stable networks under transferable utility.

Example SA.4.1. Suppose $\boldsymbol{A}$ represents a friendship network, and let Z_i be a subvector of T_i representing race. Let

$$U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = (s_i + \gamma d_i)^\sigma + \sum_j A_{ij}\big(w_n(T_i, T_j)'\theta + \zeta_{ij}\big),$$

where $s_i = \sum_j A_{ij}\mathbf{1}\{Z_i = Z_j\}$ and $d_i = \sum_j A_{ij}\mathbf{1}\{Z_i \neq Z_j\}$ are respectively the number of same- and different-race friends. The first term captures returns to popularity. If $\sigma < 1$, then we have diminishing returns. If $\gamma < 1$, then individuals prefer friends of the same race. This term is motivated Currarini, Jackson, and Pin (2009). The second term captures costs/benefits of direct connections.

Marginal utilities are given by

$$V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = \big(s_{i,-j} + \gamma d_{i,-j} + \mathbf{1}\{Z_i = Z_j\} + \gamma \mathbf{1}\{Z_i \neq Z_j\}\big)^{\sigma} \\ - (s_{i,-j} + \gamma d_{i,-j})^{\sigma} + w_n(T_i, T_j)'\theta + \zeta_{ij},$$

where $s_{i,-j} = \sum_{k \neq j} A_{ik} \mathbf{1}\{Z_i = Z_k\}$ is the number of same-race friends with j excluded, and $d_{i,-j} = \sum_{k \neq j} A_{ik} \mathbf{1}\{Z_i \neq Z_k\}$. In this example, $S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = (s_{i,-j}, d_{i,-j})$.

Example SA.4.2. Dyadic regression models are commonly used in practice to study network formation. These specify a linear model of the joint surplus with no strategic interactions:

$$A_{ij} = \mathbf{1}\big\{w_n(T_i, T_j)'\beta + \nu_{ij} > 0\big\}, \tag{SA.4.2}$$

where $w_n(\cdot)$ can capture homophily in types, for example, $\mathbf{1}\{T_i = T_j\}$, and ν_{ij} corresponds to $\zeta_{ij} + \zeta_{ji}$. For example, Fafchamps and Gubert (2007) study the formation of risk-sharing networks between households in the rural Philippines and find evidence of geographic homophily.

Model (SA.4.1) may be viewed as a nonlinear generalization of dyadic regression that allows the latent index to be a function of network-dependent "regressors" $S^*_{ij}(\boldsymbol{A}, \boldsymbol{T})$. For example, consider the specification

$$V^*_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = w_n(T_i, T_j)'\theta_1 + \theta_2 \max_k A_{ik} A_{jk} + \nu_{ij}.$$

In this model, $S^*_{ij}(\boldsymbol{A}, \boldsymbol{T}) = \max_k A_{ik} A_{jk}$ captures a structural taste for transitivity (Graham (2016)). In Fafchamps and Gubert's setting, risk-sharing relationships between (i, k) and (j, k) may promote link formation between (i, j), since k ensures i against defaults by j and vice versa. This provides a strategic foundation for the well-known stylized fact that most real-world social networks exhibit unusual degrees of clustering (Jackson (2010)).

In the previous examples, $S_{ij}(\boldsymbol{A}, \boldsymbol{T})$ only depends on its arguments through the network neighbors of i and j. We next impose this restriction more generally. Recall that $\mathcal{N}(i)$ is the set of i's neighbors in $\boldsymbol{A}$, and let $\mathcal{N}(i, j) = (\mathcal{N}(i) \cup \mathcal{N}(j)) \backslash \{i, j\}$.

Assumption SA.4.1 (Local interactions). *There exists a function $\tilde{S}(\cdot)$ such that for all $n \in \mathbb{N}$ and $i, j \in \mathcal{N}_n$,*

$$S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = \tilde{S}(A_{\mathcal{N}(i,j)}, T_i, T_j, T_{\mathcal{N}(i,j)}).$$

This is analogous to Assumption 1 for graphical games. In Example SA.4.1, clearly $s_{i,-j}$ and $d_{i,-j}$ only depends on agents linked to i or j, and likewise with $\max_k A_{ik} A_{jk}$, $\sum_k A_{ik}$, $\sum_k A_{jk}$ in Example SA.4.2. Assumption SA.4.1 rules out models such as the connections model, where utility depends on agents further than two links away from the ego. However, most examples studied in the econometrics literature satisfy this assumption, as the main externalities of interest concern the impact of degree and the presence of common friends on link formation (Graham (2016), Leung (2019), Mele (2017), Menzel (2017), Ridder and Sheng (2017)).

Econometrician's information We assume the econometrician observes the type profile $\boldsymbol{T}$ and array of random utility shocks $\boldsymbol{\zeta}$, and given a known payoff function $U_i(\cdot)$ for each agent, her objective is to compute $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. Now, in practice, typically $T_i = (X_i, \varepsilon_i)$, where only ε_i is unobserved, and $U_i(\cdot)$ and the distributions of ε_i and ζ_{ij} are only known up to some finite-dimensional vector of parameters θ. However, for counterfactual exercises, a candidate value of θ is typically selected, which allows us to draw ε_i, ζ_{ij} to obtain the required inputs $\boldsymbol{T}, \boldsymbol{\zeta}$.

SA.4.1 *Strategic neighborhoods*

Similar to Algorithm 1, the main idea is to obtain $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ by computing the set of pairwise stable networks on each "strategic neighborhood." To define these neighborhoods in the network formation setting, we first need some notation. For any $G \subseteq \mathcal{N}_n$ and symmetric $n \times n$ matrix M, let $M_G = (M_{ij} : i, j \in G)$, the submatrix of M only containing rows and columns in G, which are ordered according to their original order in M. Let $\mathcal{A}(G)$ be the set of undirected networks on $G \subseteq \mathcal{N}_n$. Note that $\mathcal{E}_{\text{TU}}(T_G, \zeta_G)$ is the set of pairwise stable networks in the subgame where the set of players is G rather than $\mathcal{N}_n$.

Define the undirected network $\boldsymbol{D}$ on $\mathcal{N}_n$ such that for any $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$D_{ij} = \mathbf{1}\Big\{\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) \leq 0 \cap \sup_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big\}. \tag{SA.4.3}$$

To interpret this, note that if $\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0$ ($\sup_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) \leq 0$), then agents i and j are (not) linked in *any* pairwise stable network, since (not) forming a link is optimal regardless of the state of the ambient network. In either case, $D_{ij} = 0$, in which case we say that the A_{ij} is a *robust potential link*; otherwise, it is *nonrobust*.

Let $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta}) \subseteq \mathcal{N}_n$ be the set of components of $\boldsymbol{D}$. For any $G \subseteq \mathcal{N}_n$, define

$$\mathcal{S}(G) = G \cup \Big\{k \in \mathcal{N}_n : \max_{j \in G} \inf_s V_n^*(s, T_j, T_k, \zeta_{jk}, \zeta_{kj}) > 0\Big\}.$$

This adds to G all agents k connected to G through a robust link.

Definition SA.4.1. $\mathcal{S}(C)$ is a *strategic neighborhood* if $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$.

Example SA.4.3. In Figure SA.1, pairs of agents connected by thick lines have robust links, those connected by thin lines have nonrobust potential links, and unlinked pairs have robustly absent links. Then $\boldsymbol{D}$ has two components, which are the "islands" that result from deleting all thick lines: $\{1, 2, 3\}$ and $\{4, \dots, 8\}$. To obtain the strategic neighborhoods, we add to each component agents connected through thick lines, resulting in two such neighborhoods: $\{1, \dots, 4\}$ and $\{3, \dots, 8\}$.

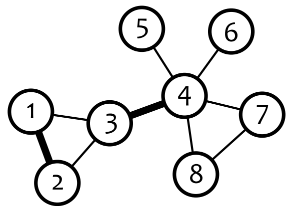

Figure SA.1. Thick lines denote robust links, thin lines denote nonrobust potential links, and unlinked pairs have robustly absent links.

Strategic neighborhoods have the following property, which is analogous to (2.4):

$$A_{\mathcal{S}(C)} \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \quad \forall \boldsymbol{A} \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta}). \tag{SA.4.4}$$

That is, if we take any pairwise stable network $\boldsymbol{A}$ and remove all other agents from the game except members of $\mathcal{S}(C)$, then the subnetwork of $\boldsymbol{A}$ on $\mathcal{S}(C)$ is still pairwise stable. A formal proof is given in Lemma SA.4.2. For intuition, consider Figure SA.1. Because the potential link between agents 7 and 8 is nonrobust, its pairwise stability depends on links formed by these agents, for example, A_{48}. The same story holds for this potential link, which depends on A_{34}. However, the latter is robust and therefore pairwise stable regardless of the state of, say, A_{13}. Hence, if the subnetwork on $\{3, \dots, 8\}$ is pairwise stable, it remains so after removing agents 1 and 2 from the game.

SA.4.2 *Algorithm*

Similar to Algorithm 1, we propose to compute $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ by exploiting (SA.4.4). We compute the sets of pairwise stable subnetworks on strategic neighborhoods and then appropriately combine these sets. Computation of $\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ for each $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ will be feasible using an exhaustive search over all possible subnetworks because $|\mathcal{S}(C)|$ grows logarithmically with n under our assumptions. To combine these sets to obtain $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$, we must account for the fact that strategic neighborhoods are not necessarily disjoint. For example, if $i, j \in \mathcal{S}(C) \cap \mathcal{S}(C')$ for two distinct components C, C', then it is unclear whether their equilibrium potential link ought to be dictated by profiles in $\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$ or $\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, A_{\mathcal{S}(C)})$. The main observation is that, since C, C' must be disjoint, it follows that the potential link between i and j is robust and, therefore, the same across all networks in these two sets.

To state the algorithm succinctly, some definitions are required. Recall the definition of $\pi(k; C)$ from (2.5). Let

$$\begin{aligned} &\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta}) \\ &\quad = \Big\{ A^* \in \mathcal{A}\big(\mathcal{S}(C)\big) : A^*_{\pi(i;C),\pi(j;C)} = 1 \text{ if } \inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0 \\ &\qquad \text{and } A^*_{\pi(i;C),\pi(j;C)} = 0 \text{ if } \sup_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) \le 0,\ \forall i, j \in \mathcal{S}(C), i \neq j \Big\}. \end{aligned} \tag{SA.4.5}$$

This is analogous to $\mathcal{Y}(\mathcal{S}(C), \boldsymbol{T})$ in Remark 1. It is the collection of networks on $\mathcal{A}(\mathcal{S}(C))$ such that, for each network in this set, (i, j)'s potential link is set to 1 (0) if $\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0$ ($\sup_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) \leq 0$). To compute the set of pairwise stable networks on $\mathcal{S}(C)$, it will be enough to search through this subset of networks rather than all possible networks.

For $G \subseteq H \subseteq \mathcal{N}_n$, let

$$\mathcal{E}_{\mathrm{TU}}(T_H, \zeta_H)|_G = \left\{\boldsymbol{A} \in \mathcal{A}(G) \colon \boldsymbol{A} = \boldsymbol{A}_G^* \text{ for some } \boldsymbol{A}^* \in \mathcal{E}_{TU}(T_H, \zeta_H)\right\},$$

which is the set of subnetworks on G obtained from a network in $\mathcal{E}_{\mathrm{TU}}(T_H, A_H)$. Define

$$\mathcal{B} = \left\{(i, j) \colon i, j \in \mathcal{N}_n, \{i, j\} \not\subseteq C \ \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})\right\}. \tag{SA.4.6}$$

This is the set of agent pairs that "bridge" two components. By definition, their potential links must be robust. Finally, let

$$\begin{aligned} &\bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\mathrm{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \\ &\quad = \Big\{\boldsymbol{A}^* \in \mathcal{A}(\mathcal{N}_n) \colon A_C^* \in \mathcal{E}_{\mathrm{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C \ \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta}), \\ &\qquad A_{ij}^* = \mathbf{1}\Big\{\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big\} \ \forall (i, j) \in \mathcal{B}\Big\}. \end{aligned} \tag{SA.4.7}$$

This is well-defined because any pair of agents must either be such that both are in different components (and, therefore, in $\mathcal{B}$) or the same component. With these definitions, we state our proposed procedure in Algorithm SA.1.

Remark SA.4.1 (Explanation of Algorithm SA.1). Line 1 of Algorithm SA.1 computes the connected components of $\boldsymbol{D}$. As discussed in Remark 1, this can be done in $O(n+L)$ time using depth-first search, where L is the number of links in $\boldsymbol{D}$. Line 2 runs an exhaustive search of $\mathcal{A}(\mathcal{S}(C))$ for each C. Line 3 shows how to assemble the equilibrium sets to obtain $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. For example, to obtain an arbitrary pairwise stable network, we take one pairwise stable subnetwork from $\mathcal{E}_{\mathrm{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C$ for each C and then connect them by linking between each pair of agents in different components that are robustly linked.

Remark SA.4.2 (Diagnostic for computational feasibility). In practice, the only computationally intensive step of the algorithm is exhaustive search over the links of C_1^*, the largest component of $\boldsymbol{D}$. A quick way to assess the feasibility of this step is to compute the number of links Δ in $D_{C_1^*}$. The runtime of exhaustive search is then 2^{Δ}, which is shown to be polynomial in n in the theorem below.

Remark SA.4.3 (Myopic best-response dynamics). When the objective is to obtain a single (arbitrary) equilibrium, say for the purposes of a simulation study, myopic best-response dynamics are commonly used. This is a discrete-time process initialized at an arbitrary starting network, where at each period, each pair of agents myopically reoptimizes their potential links according to (SA.4.1). Under strategic complements, this

Algorithm SA.1: Procedure for computing the set of pairwise stable networks.

Input: $\boldsymbol{T}, \boldsymbol{\zeta}, V_n^*(\cdot)$
Output: $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$

1 *Compute* $\boldsymbol{D}$ *and then* $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ *using depth-first search of* $\boldsymbol{D}$.
2 *Compute each* $\mathcal{E}_{TU}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ *using exhaustive search*:
for $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ **do**
  $\mathcal{E}_{\mathcal{S}(C)} \leftarrow \emptyset$
  for $\boldsymbol{A} \in \mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$ **do**
    if $A_{\pi(i;C),\pi(j;C)} = \mathbf{1}\{V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) > 0\}$ *for all* $i, j \in C$, $i \neq j$ **then**
      $\mathcal{E}_{\mathcal{S}(C)} \leftarrow \mathcal{E}_{\mathcal{S}(C)} \cup \{\boldsymbol{A}\}$
    end
  end
  $\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \leftarrow \mathcal{E}_{\mathcal{S}(C)}$
end
3 *Combine equilibrium sets*:
if $\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \neq \emptyset \; \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ **then**
  $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta}) \leftarrow \bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$
else $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta}) \leftarrow \emptyset$.

is guaranteed to converge to a pairwise stable network (Milgrom and Roberts (1990)). Otherwise, convergence holds if the starting network lies along an improving path to an equilibrium (Jackson and Watts (2002)). Our algorithm suggests a few practical improvements. First, the dynamics can be run separately, in parallel, on each strategic neighborhood. Second, it is faster to initialize the process at an element of $\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$, which fixes robust potential links at their optimal states in all equilibria. Then we can ignore pairs of agents in $\mathcal{B}$ in the dynamics, since their potential links are robust and already set to their optimal values.

THEOREM SA.4.1. *Suppose evaluating* D_{ij} *and* $V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ *have the same complexity for any* i, j. *Under Assumptions* SA.4.1 *and* SA.4.2, *the latter given in the next subsection, Algorithm* SA.1 *computes* $\mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ *in* $O_p(n^2 + n^{1+q})$ *evaluations of the joint surplus function for* $q > \log 2 / \log \|\lambda\|_{\mathbf{m},k}^{-1}$, *where* $\|\lambda\|_{\mathbf{m},k}$ *is defined in Assumption* SA.4.2.

PROOF. See Section SA.4.6. □

The complexity here differs slightly from that of Algorithm 1. This is because in the latter procedure, computing $\boldsymbol{D}$ only requires computing $\mathcal{R}_i^c$ for each i, whereas here, computing $\boldsymbol{D}$ requires calculating D_{ij} for each pair i, j, which therefore takes $O(n^2)$ steps. The assumption regarding the complexity of evaluating D_{ij} and $V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ plays the same role as the assumption discussed in Remark 5.

SA.4.3 *Assumptions*

We state conditions required for Theorem SA.4.1 analogous to Assumptions 2 and 3 in Section 2.3.

Assumption SA.4.2. *Assumptions* 2 *and* 3 *hold with* $\boldsymbol{D}$ *given by* (SA.4.3) *and* τ_i *replaced with* T_i *for all* i.

Leung (2019) Section 3.3 details how Assumption 2 in this setting is analogous to (2.8) and standard weak dependence conditions widely used in time series and spatial statistics. The remainder of this subsection discusses the economic interpretation of the condition.

Recall the definition of D_{ij} in (SA.4.3), and observe that

$$n\mathbf{E}[D_{ij}] = n\Big(\mathbf{P}\Big(\sup_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big) - \mathbf{P}\Big(\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big)\Big).$$

This measures the strength of strategic interactions because it corresponds to the partial-equilibrium change in (i,j)'s linking probability from changing their network dependent statistics $S_{ij}^*(\boldsymbol{A}, \boldsymbol{T})$ from their minimizing to their maximizing value. This is analogous to $\mathbf{E}[\mathcal{R}_j^c]$ in Section 2.3.[4] The next example shows that the analog of Assumption 2 in the network formation setting imposes a restriction on the strength of interactions.

Example SA.4.4. Consider the specification of $V_{ij}^*(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ in Example SA.4.2. Here,

$$D_{ij} = \mathbf{1}\big\{-\theta_2 < w_n(T_i, T_j)'\theta_1 + \nu_{ij} \le 0\big\}.$$

Suppose $T_i \in \mathbb{R}^2$ is continuously distributed, and $w_n(T_i, T_j)'\theta_1 = \theta_1 + \rho_n(T_i, T_j)$, where $\rho_n(T_i, T_j)$ equals $-\infty$ if $r_n^{-1}\|T_i - T_j\| \le 1$ and zero otherwise for $r_n = (\kappa/n)^2$. Here, we interpret T_i as geographic location, so the model imposes geographic homophily in the sense that agents only link with those for whom their scaled distance is at most 1 apart. Then

$$D_{ij} = \mathbf{1}\{-\theta_2 < \theta_1 + \nu_{ij} \le 0\}\mathbf{1}\big\{r_n^{-1}\|T_i - T_j\| \le 1\big\}. \tag{SA.4.8}$$

If $\nu_{ij} \perp\!\!\!\perp T_i, T_j$, then some algebra shows that

$$\|\lambda\|_{\mathbf{m},k} < 1 \quad \text{if and only if} \quad \gamma < \frac{1}{\kappa\pi}, \tag{SA.4.9}$$

where π is the universal constant and $\gamma = \mathbf{P}(-\theta_2 < \theta_1 + \nu_{ij} \le 0)$ (Leung (2019)). Observe that

$$\gamma = \mathbf{P}(\theta_1 + \theta_2 + \nu_{ij} > 0) - \mathbf{P}(\theta_1 + \nu_{ij} > 0),$$

which is the partial-equilibrium marginal effect of having a common friend on (i,j)'s linking probability, given that the scaled distance between the types of i and j does not

[4]The scaling by n ensures that $n\mathbf{E}[D_{ij}]$ has a nondegenerate limit, since sparsity implies $\mathbf{E}[D_{ij}] = O(n^{-1})$.

exceed 1. Furthermore, $\kappa\pi$ is the limiting expected number of agents within radius 1 of the ego, which is an upper bound on the asymptotic expected degree. Hence, γ measures the strength of strategic interactions, and (SA.4.9) is completely analogous to (2.13).[5]

REMARK SA.4.4. For the analog of Assumption 3 in the network formation setting, note that if $\varphi(t, t') = 0$ everywhere, then this corresponds to a model with no strategic interactions. This is because if $V_n^*(\cdot)$ does not vary in $S_{ij}^*(\boldsymbol{A}, \boldsymbol{T})$, then D_{ij} is necessarily zero. For models with strategic interactions, Assumption 3 imposes a mild nondegeneracy condition on the network $\boldsymbol{D}$; see the discussion following the statement of Assumption 3.

SA.4.4 *Nontransferable utility*

Consider the model in Section SA.4, and recall that $V_n(S_{ij}(\boldsymbol{A}, \boldsymbol{T}), T_i, T_j, \zeta_{ij})$ is agent i's marginal utility of adding a link with j. We next consider the model in which utility is nontransferable. We say $\boldsymbol{A}$ is pairwise stable if for all $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$A_{ij} = \mathbf{1}\big\{V_n(S_{ij}, T_i, T_j, \zeta_{ij}) > 0 \cap V_n(S_{ji}, T_j, T_i, \zeta_{ji}) > 0\big\} \tag{SA.4.10}$$

(Jackson (2010)). This states that a link forms between i and j if and only if both parties prefer it.

Let $\mathcal{E}_{\mathrm{NT}}(\boldsymbol{T}, \boldsymbol{\zeta})$ be the set of pairwise stable networks with nontransferable utility. We can employ Algorithm SA.1 to compute this set by appropriately redefining $\boldsymbol{D}$ and $\mathcal{S}(C)$. Toward this end, we first define the undirected network M on $\mathcal{N}_n$ such that for all $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$M_{ij} = \mathbf{1}\Big\{\sup_s V_n(s, T_i, T_j, \zeta_{ij}) > 0 \cap \sup_s V_n(s, T_j, T_i, \zeta_{ji}) > 0\Big\}. \tag{SA.4.11}$$

If $M_{ij} = 0$, then either i or j prefers not to form the link under any pairwise stable network. In this case, we say that A_{ij} is *robustly absent*. Let $\boldsymbol{D}$ be the undirected network on $\mathcal{N}_n$ such that for all $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$\begin{aligned} D_{ij} &= \max\{\tilde{D}_{ij}, \tilde{D}_{ji}\} M_{ij}, \quad \text{where} \\ \tilde{D}_{ij} &= \mathbf{1}\Big\{\inf_s V_n(s, T_i, T_j, \zeta_{ij}) \leq 0\Big\}. \end{aligned} \tag{SA.4.12}$$

To understand this definition, note that if $\tilde{D}_{ij} = 0$, then agent i prefers to form a link with j in any pairwise stable network. If in addition $\tilde{D}_{ji} = 0$, then we say that (i, j) form a *robust link*. If $\tilde{D}_{ij} M_{ij} = 1$, then whether i prefers to link with j may depend on links formed by other agents in the network. Hence if $D_{ij} = 1$, then we say that A_{ij} is a nonrobust potential link.

[5]Our only motivation for using this specification of $\rho_n(T_i, T_j)$ is that we can write D_{ij} in multiplicatively separable form (SA.4.8), which results in (SA.4.9). The latter allows us to draw a clear analogy with the more well-known condition (2.13). If we instead consider more empirically realistic specifications like $\rho_n(T_i, T_j) = r_n^{-1}\|T_i - T_j\|$, it is still possible to derive closed-form expressions for $\|\lambda\|_{\mathbf{m},k}$ in certain cases (see, e.g., Leung (2019, Appendix A)), but the expression does not decompose as nicely as (SA.4.9) and is therefore less immediately interpretable.

Let $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ be the set of components of $\boldsymbol{D}$, and

$$\mathcal{S}(G) = G \cup \Big\{k \in \mathcal{N}_n : \max_{j \in G}(1 - \tilde{D}_{jk})(1 - \tilde{D}_{kj}) = 1\Big\}.$$

This adds to G all agents k connected to G through a robust link. Then for $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$, $\mathcal{S}(C)$ is the strategic neighborhood associated with C. Next, define the set

$$\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta}) = \big\{\boldsymbol{A}^* \in \mathcal{A}\big(\mathcal{S}(C)\big) : \boldsymbol{A}^*_{\pi(i;C),\pi(j;C)} = 1 \text{ if } (1 - \tilde{D}_{ij})(1 - \tilde{D}_{ji}) = 1$$
$$\text{and } \boldsymbol{A}^*_{\pi(i;C),\pi(j;C)} = 0 \text{ if } M_{ij} = 0,\ \forall i, j \in \mathcal{S}(C), i \neq j\big\}.$$

This is the analog of (SA.4.5). We link (unlink) i and j in every network in this set if both agents prefer (not) to have the link regardless of the state of the ambient network. We use this notation in line 2 of the algorithm to fix the potential links of these agent pairs and only search over nonrobust potential links.

Using these new definitions, Algorithm SA.1 computes $\mathcal{E}_{\mathrm{NT}}(\boldsymbol{T}, \boldsymbol{\zeta})$ in the case of non-transferable utility. We omit the proof as it is essentially the same as the proof of Lemma SA.4.3, which considers the transferable-utility case. If Assumptions SA.4.1 and SA.4.2 hold for $\boldsymbol{D}$ given by (SA.4.12), then minor modifications of the proof of Theorem SA.4.1 establish that the algorithm outputs the equilibrium set in $O_p(n^2 + n^{1+q})$ evaluations of the marginal utility function. We conclude this subsection with an example illustrating the interpretation of Assumption 2 as a restriction on the strength of strategic interactions in the nontransferable utility setting.

Example SA.4.5. A bivariate logit analog of the dyadic regression model (SA.4.2) is

$$A_{ij} = \mathbf{1}\big\{w_n(T_i, T_j)'\beta + \zeta_{ij} > 0\big\}\mathbf{1}\big\{w_n(T_j, T_i)'\beta + \zeta_{ji} > 0\big\}.$$

Model (SA.4.10) is a nonlinear generalization of the bivariate logit that allows the latent indices to be functions of network-dependent "regressors" $S_{ij}(\boldsymbol{A}, \boldsymbol{T})$. For example, consider the specification of the marginal utility function

$$V_n\big(S_{ij}(\boldsymbol{A}, \boldsymbol{T}), T_i, T_j, \zeta_{ij}\big) = w_n(T_i, T_j)'\theta_1 + \theta_2 \max_k A_{ik}A_{jk} + \zeta_{ij}.$$

Here, $S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = \max_k A_{ik}A_{jk}$ captures a structural taste for transitivity, as discussed in Example SA.4.2.

As in Example SA.4.4, let us consider the geographic homophily model where $T_i \in \mathbb{R}^2$ is continuously distributed, and $w_n(T_i, T_j)'\theta_1 = \theta_1 + \rho_n(T_i, T_j)$, where $\rho_n(T_i, T_j)$ equals $-\infty$ if $r_n^{-1}\|T_i - T_j\| \leq 1$ and zero otherwise for $r_n = (\kappa/n)^2$. Then

$$D_{ij} = \mathbf{1}\big\{\{-\theta_2 < \theta_1 + \zeta_{ij} \leq 0 \cap \theta_1 + \theta_2 + \zeta_{ji} > 0\}$$
$$\cup \{-\theta_2 < \theta_1 + \zeta_{ji} \leq 0 \cap \theta_1 + \theta_2 + \zeta_{ij} > 0\}\big\}\mathbf{1}\big\{r_n^{-1}\|T_i - T_j\| \leq 1\big\}. \qquad \text{(SA.4.13)}$$

If $(\zeta_{ij}, \zeta_{ji}) \perp\!\!\!\perp (T_i, T_j)$, then

$$\|\lambda\|_{\mathbf{m},k} < 1 \quad \text{if and only if} \quad \gamma < \frac{1}{\kappa\pi},$$

where π is the universal constant and γ is the expectation of the first indicator on the right-hand side of (SA.4.13). To interpret this inequality, notice that $\tilde{\gamma} \equiv \mathbf{P}(-\theta_2 < \theta_1 + \zeta_{ij} \leq 0)$ is the partial-equilibrium marginal effect of having a common friend on i's propensity to link with j. Then if $\zeta_{ij} \perp\!\!\!\perp \zeta_{ji}$, we have $\gamma \geq \tilde{\gamma}^2$, so $\gamma < (\kappa\pi)^{-1}$ implies $\tilde{\gamma} < (\kappa\pi)^{-1/2}$, which is analogous to (2.13).

SA.4.5 *Application to Sheng (2016)*

Sheng (2016) studied estimation of network formation games under transferable and nontransferable utility. She assumes $T_i = (X_i, \varepsilon_i)$ with only X_i observed, and $U(\cdot)$ and the distribution of ζ_{ij} are known up to a finite-dimensional vector of parameters θ. Her paper provides tractable moment inequalities that conservatively bound the identified set of parameters. We discuss how our algorithm can be used to compute the sharp analogs of her bounds under additional assumptions.[6]

The moment inequalities are based on subnetwork moments $\mathbf{P}(A_G = a_G \mid X_G)$, where $G \subseteq \mathcal{N}_n$, $X_G = (X_i)_{i\in G}$, and a_G is a fixed network on G. For the case of transferable utility, Sheng showed that subnetwork moments are upper and lower bounded as follows:

$$\begin{aligned}&\mathbf{P}\big(a_G \in \mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G \cap \big|\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G\big| = 1 \mid X_G\big)\\ &\quad\leq \mathbf{P}(A_G = a_G \mid X_G) \leq \mathbf{P}\big(a_G \in \mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G \mid X_G\big). \end{aligned} \tag{SA.4.14}$$

These bounds are sharp under unrestricted equilibrium selection.[7] To compute the right-hand side, we have to simulate $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G$ for many draws of $\boldsymbol{\zeta}$. Checking whether a_G is an element of this set requires us to find an ambient network a_{-G} such that (a_G, a_{-G}) is pairwise stable. Computing the lower bound is even more difficult, since for each draw, we also have to verify that (a_G, a_{-G}) is the unique equilibrium. Thus, for n large, the bounds are generally intractable because of the need to compute $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G$. Sheng proposed new bounds that are computationally feasible but conservative relative to (SA.4.14).

We next show that if the model satisfies Assumption SA.4.2, then Algorithm SA.1 can be applied to feasibly simulate the sharp bounds (SA.4.14). First, we need some definitions. Fix $\boldsymbol{T}$, $\boldsymbol{\zeta}$, and the realization of $\boldsymbol{D}$ under these primitives. For any $i \in \mathcal{N}_n$, let C_i be i's component in $\boldsymbol{D}$. Define $G^* = \bigcup_{i\in G} \mathcal{S}(C_i)$, the union of the strategic neighborhoods of agents in G.

The key observation is that, by (SA.4.4), $a_G \in \mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G$, is equivalent to $a_G \in \mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}_{G^*}, \boldsymbol{\zeta}_{G^*})|_G$. The latter set is analogous to $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_G$, except G^* is the set of players, rather than $\mathcal{N}_n$. In other words, to simulate the upper bound, for each simulation draw, we only need to search through pairwise stable networks on the much smaller subset G^*, rather than $\mathcal{N}_n$ to find a network $a_{G^*\backslash G}$ on $G^*\backslash G$ such that $(a_G, a_{G^*\backslash G})$ is pairwise stable (and for the lower bound, uniquely so). This is feasible because under our assumptions, strategic neighborhood sizes have exponential tails, so G^* typically has manageable size.

[6] I thank a referee for suggesting this application.
[7] For the nontransferable case, replace $\mathcal{E}_{\mathrm{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ with $\mathcal{E}_{\mathrm{NT}}(\boldsymbol{T}, \boldsymbol{\zeta})$ in every expression in this subsection.

More precisely, we can simulate bounds in (SA.4.14) as follows. Given G, we simulate $\boldsymbol{\zeta}$ and compute G^* using line 1 of Algorithm SA.1. Then we use line 2 of the algorithm to compute $\mathcal{E}_{\text{TU}}(\boldsymbol{T}_{G^*}, \boldsymbol{\zeta}_{G^*})$, which is feasible under our assumptions by Theorem SA.4.1. For the upper bound, we keep simulation draws such that $a_G \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}_{G^*}, \boldsymbol{\zeta}_{G^*})|_G$. For the lower bound, we keep draws satisfying this and additionally satisfying $|\mathcal{E}_{\text{TU}}(\boldsymbol{T}_{G^*}, \boldsymbol{\zeta}_{G^*})|_G| = 1$. These sets of draws can be used to construct estimates for the bounds in (SA.4.14) that are consistent as the number of simulation draws diverge.

Regardless of whether Assumption SA.4.2 holds, computational feasibility relies on the size of the largest strategic neighborhood that comprises G^* being sufficiently small, since we are exhaustively searching all networks on $\mathcal{S}(C_i)$ for each $i \in G$ to find the equilibrium set. Our assumptions guarantee that the largest neighborhood is typically small, but in practice, one can diagnose feasibility by computing the size of this neighborhood first, which is a byproduct of line 1 of Algorithm SA.1; see Remark SA.4.2.

SA.4.6 *Proof of main result*

Proof of Theorem SA.4.1. *Line* 1. Given $\boldsymbol{T}, \boldsymbol{\zeta}$, computing $\boldsymbol{D}$ takes $O(n^2)$ evaluations of the joint surplus function, since we need to compute D_{ij} for each pair of agents. As discussed in Remark SA.4.1, computing $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ takes $O(n+L)$ time, where L is the number of links in $\boldsymbol{D}$. The expected number of links is $0.5\mathbf{E}[\sum_i \sum_j D_{ij}] \leq n^2 \mathbf{E}[D_{ij}]$, which is $O(n)$ by Assumption SA.4.2 (specifically, the analog of Assumption 2). Hence, $L = O_p(n)$, so line 1 of the algorithm requires $O_p(n^2)$ evaluations.

Line 2. For each strategic neighborhood $\mathcal{S}(C)$, the algorithm evaluates whether each network in $\mathcal{A}^*(\mathcal{S}(C), \boldsymbol{T}, \boldsymbol{\zeta})$ is pairwise stable. The size of this set is $2^{0.5\sum_{i,j\in C} D_{ij}}$. For each candidate network in this set, we need to verify the pairwise stability conditions by evaluating the joint surplus for each pair of agents (i, j) in C for which $D_{ij} = 1$. Hence, we require at most $(0.5\sum_{i,j\in C} D_{ij})2^{0.5\sum_{i,j\in C} D_{ij}}$ evaluations. These computations are repeated for every strategic neighborhood, resulting in a total of

$$\sum_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \left(0.5 \sum_{i,j\in C} D_{ij}\right) 2^{0.5\sum_{k,l\in C} D_{kl}} \leq 0.5 \sum_{i,j} D_{ij} \max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{k,l\in C} D_{kl}} \tag{SA.4.15}$$

evaluations. The inequality follows because $0.5\sum_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \sum_{i,j\in C} D_{ij}$ is the total number of links in $\boldsymbol{D}$. As shown above, $\sum_{i,j} D_{ij} = O_p(n)$. By Lemma SA.4.1, the max term is $O_p(n^q)$. Therefore, (SA.4.15) $= O_p(n^{1+q})$.

Line 3. Under Assumption SA.4.1, we can apply Lemma SA.4.3, which proves that the algorithm has the desired output. □

Lemma SA.4.1. *Under Assumption* SA.4.2,

$$\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{k,l\in C} D_{kl}} = O_p\big(n^q\big).$$

Proof. Let $\beta = \|\lambda\|_{\mathbf{m},k}$. By Assumption 2, there exists $\varepsilon > 0$ such that $(1+\varepsilon)\beta < 1$. Using Lemma SA.3.7, for any such ε and $m > 0$,

$$\mathbf{P}\Big(\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{A})} 2^{0.5\sum_{i,j\in C} D_{ij}} > mn^q\Big) \le cn^{1-q(\log 2)^{-1}\log((1+\varepsilon)\beta)^{-1}} m^{-(\log 2)^{-1}\log((1+\varepsilon)\beta)^{-1}}$$

for some $c > 0$, as in the proof of Lemma SA.2.1. As in that proof, this is $o(1)$ as $m, n \to \infty$ for q satisfying (SA.2.2). □

Lemma SA.4.2. *Consider the model of Section SA.4 under the solution concept of pairwise stability with transferable utility. Under Assumption SA.4.1, for any $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$,*

$$\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) = \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

Proof. We first prove that

$$\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \subseteq \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{A} \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ and $\boldsymbol{A}' \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. Construct a network $\boldsymbol{A}^*$ by defining

$$A^*_{k\ell} = \begin{cases} A_{k\ell} & \text{if } k, \ell \in \mathcal{S}(C), \\ A'_{k\ell} & \text{otherwise.} \end{cases}$$

That is, we take $\boldsymbol{A}'$ and replace the subnetwork on $\mathcal{S}(C)$ with $A_{\mathcal{S}(C)}$. It suffices to show $\boldsymbol{A}^* \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. For this purpose, fix arbitrary agents $i, j \in \mathcal{S}(C)$ and $k \in \mathcal{N}_n \backslash \mathcal{S}(C)$.

We show that A^*_{ij} is a "best response" for (i, j) to $\boldsymbol{A}^*$ in the sense that (SA.4.1) holds. There are two cases to consider. First suppose $\{i, j\}$ is not a subset of C. Then by definition of $\mathcal{S}(C)$, we have $\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0$, which means (i, j) form a link in any pairwise stable network (robust link). The second case to consider is $i, j \in C$. Then since $k \notin \mathcal{S}(C)$, we must have $\sup_s V_n^*(s, T_k, T_\ell, \zeta_{k\ell}, \zeta_{\ell k}) \le 0$ for any $\ell \in \{i, j\}$ by definition of $\mathcal{S}(C)$. That is, k is not linked with i or j in any pairwise stable network (robustly absent links). Then by Assumption SA.4.1, the joint surplus of (i, j) is not a function of k in the sense that removing k from the game does not affect the value of their joint surplus. Since k is any arbitrary agent not in $\mathcal{S}(C)$, given that $\boldsymbol{A} \in \mathcal{E}_{TU}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$, it follows that A^*_{ij} is a "best response" to $\boldsymbol{A}^*$, as desired.

Next, we show that, for any pair of agents $\{k, \ell\}$ that is not a subset of $\mathcal{S}(C)$, $A^*_{k\ell}$ is a "best response" to $\boldsymbol{A}^*$ in the sense that (SA.4.1) holds. Note that, necessarily, $\{k, \ell\} \subseteq \mathcal{N}_n \backslash C$. Suppose, for i, j previously defined above, that $i, j \in \mathcal{S}(\mathcal{N}_n \backslash C)$. Then it must be the case that (i, j) have a robust link. Therefore, $A'_{ij} = A_{ij}$, so (i, j)'s potential link is the same in $\boldsymbol{A}'$ and $\boldsymbol{A}^*$. On the other hand, suppose $\{i, j\}$ is not a subset of $\mathcal{S}(\mathcal{N}_n \backslash C)$. Since $k, \ell \subseteq \mathcal{N}_n \backslash C$, we must have $\sup_s V_n^*(s, T_p, T_q, \zeta_{pq}, \zeta_{qp}) \le 0$ for all $p \in \{i, j\}$ and $q \in \{k, \ell\}$, as argued in the previous paragraph. Then by Assumption 1, the joint surplus of (k, ℓ) is not a function of $\{i, j\}$ in the sense that removing $\{i, j\}$ from the game does not change the value of their joint surplus.

We have therefore established that (1) the only potential links of the networks $\boldsymbol{A}'$ and $\boldsymbol{A}^*$ that may differ are those of agents $i, j \not\subseteq \mathcal{S}(\mathcal{N}_n \backslash C)$, and (2) the joint surplus of any agent pair $\{k, \ell\} \subseteq \mathcal{N}_n \backslash C$ is not function of the links and types of such agents i, j under $\boldsymbol{A}^*$. Then since $\boldsymbol{A}' \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$, it follows that $A^*_{k\ell}$ is a "best response" to $\boldsymbol{A}^*$ for any $k, \ell \in \mathcal{N}_n \backslash C$ in the sense that (SA.4.1) holds, in particular for $k, \ell \nsubseteq \mathcal{S}(C)$, which proves the desired claim.

Step 2. We prove that

$$\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \supseteq \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{A} \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}$. By definition, there exists $\boldsymbol{A}' \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$ such that $\boldsymbol{A} = \boldsymbol{A}'_{\mathcal{S}(C)}$. Fix arbitrary agents $i, j \in \mathcal{S}(C)$ and $k \in \mathcal{N}_n \backslash \mathcal{S}(C)$. There are three cases to consider. First, suppose $\{i, j\} \nsubseteq C$. Then by definition of $\mathcal{S}(C)$, we have that $\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0$, so $A_{ij} = 1$ is pairwise stable regardless of how other links are formed. Second suppose $\{i, j\} \subseteq C$. Since $k \notin \mathcal{S}(C)$, then as argued in step 1, $A_{k\ell} = 0$ for any $\ell \in \{i, j\}$, so (i, j)'s joint surplus is not a function of j's links in $\boldsymbol{A}'$. Since i, j are arbitrary, it follows that, in the game where the set of agents is restricted to $\mathcal{S}(C)$, A_{ij} is a "best response" to $A_{\mathcal{S}(C)}$ in the sense that (SA.4.1) holds in this subgame. Hence, $\boldsymbol{A} \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$. □

Lemma SA.4.3. *Under Assumption* SA.4.1,

$$\bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C = \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta}).$$

Proof. *Step 1.* Let $\boldsymbol{A}^* \in \bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}$. This set is well-defined because for any pair of agents (i, j), either $i, j \in C$ for some $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$, or $(i, j) \in \mathcal{B}$. Consider any $i, j \in \mathcal{N}_n$. If $(i, j) \in \mathcal{B}$, then by definition of robustness, A^*_{ij} is a "best response" for (i, j) to $\boldsymbol{A}^*$ in the sense that (SA.4.1) holds.

On the other hand, suppose $i, j \in C$ for some $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$. We first prove that

$$A^*_{\mathcal{S}(C)} \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}). \tag{SA.4.16}$$

By construction, (1) there exists $\boldsymbol{A} \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ such that the subnetwork of $A^*_{\mathcal{S}(C)}$ on C equals A_C, and (2) $A_{k\ell} = A^*_{k\ell} = \mathbf{1}\{\inf_s V_n^*(s, T_k, T_\ell, \zeta_{k\ell}, \zeta_{\ell k}) > 0\}$ for any $k, \ell \in \mathcal{S}(C)$ such that $\{k, \ell\}$ is not a subset of C, since the potential link of any such pair is necessarily robust. Therefore, $A^*_{\mathcal{S}(C)} = \boldsymbol{A}$, which proves (SA.4.16).

This establishes that, in the game where the set of players is given by $\mathcal{S}(C)$, A^*_{ij} is a "best response" for (i, j) to $\boldsymbol{A}^*_{\mathcal{S}(C)}$ in the sense that (SA.4.1) holds. In fact, A^*_{ij} is a "best response" to $\boldsymbol{A}^*$ in the game with all n players. This is because, as argued in the second paragraph of the proof of Lemma SA.4.2, the joint surplus of (i, j) is not a function of the links of agents $k \notin \mathcal{S}(C)$ in the sense that removing such agents from the game does not affect the value of their joint surplus.

We have thus proved that $\boldsymbol{A}^* \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. Hence,

$$\bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C \subseteq \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta}).$$

Step 2. We prove the $\supseteq$ direction. Let $\boldsymbol{A}^* \in \mathcal{E}_{\text{TU}}(\boldsymbol{T}, \boldsymbol{\zeta})$. By definition of robustness, $A^*_{ij} = \mathbf{1}\{\inf_s V_n^*(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\}$ for any $i, j \in \mathcal{B}$. Let $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$. Then $A^*_{\mathcal{S}(C)} \in$

$\mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ by Lemma SA.2.2. Hence, $A^*_C \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C$, so

$$\boldsymbol{A}^* \in \Big\{\boldsymbol{A}^* \in \mathcal{A}(\mathcal{N}_n):\ A^*_C \in \mathcal{E}_{\text{TU}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C\ \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta}),$$
$$A^*_{ij} = \mathbf{1}\Big\{\inf_s V^*_n(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big\}\ \forall (i,j) \in \mathcal{B}\Big\},$$

as desired. □

SA.4.7 *Extension of Theorem 1*

We next generalize Theorem 1 to allow for strategic network formation. Suppose the network $\boldsymbol{A}$ either satisfies (SA.4.1) or (SA.4.10). Observe that $\boldsymbol{A}$ is a subnetwork of $\boldsymbol{M}$, whose ijth entry is defined by (SA.4.11) in the nontransferable case and by

$$M_{ij} = \mathbf{1}\Big\{\sup_s V^*_n(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big\}$$

in the transferable utility case. Let $\boldsymbol{D}^*$ be the directed network such that $D^*_{ij} = M_{ij}\mathcal{R}^c_j$ for all $i, j \in \mathcal{N}_n$ with $i \neq j$.

Theorem SA.4.2. *Suppose evaluating $\mathcal{R}^c_i$ and $U_i(S_i(\boldsymbol{Y}, \boldsymbol{T}, \boldsymbol{A}), T_i)$ have the same complexity for any i. Further suppose Assumptions 2 and 3 hold for $\boldsymbol{D}$ replaced with $\boldsymbol{D}^*$. Then Algorithm 1 computes $\mathcal{E}_{\text{NE}}(\boldsymbol{T}, \boldsymbol{A})$ in $O_p(n^{1+q})$ steps for q defined in Theorem 1.*

Proof. The proof is the same as that of Theorem 1, except we modify the proof for line 2 as follows. Let C_i be the strongly connected component of $\boldsymbol{D}$ containing agent i, that is, $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})$ and $i \in C$. Let C^*_i be the strongly connected component of $\boldsymbol{D}^*$ containing i. Then since $\boldsymbol{D}$ is a subnetwork of $\boldsymbol{D}^*$, $C_i \subseteq C^*_i$. Hence,

$$n \max_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{A})} 2^{|C|} \leq n \max_{i \in \mathcal{N}_n} 2^{|C^*_i|}.$$

Apply Lemma SA.2.1 to obtain an $O_p(n^{1+q})$ bound on the right-hand side. □

SA.5. Directed network formation

This section applies the ideas in Section 2 to the Bala and Goyal (2000) model of directed network formation. Define types T_i and random utility shocks ζ_{ij} as in Section SA.4. Let $U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ be the payoff i enjoys from the directed network $\boldsymbol{A}$ on $\mathcal{N}_n$. Agents each simultaneously choose the alters to which they would like to form directed links. That is, they unilaterally select the ith row of $\boldsymbol{A}$, denoted by A_i, keeping in mind that $A_{ii} = 0$, since we assume no self-links. Let A_{-i} be the adjacency matrix $\boldsymbol{A}$ with the ith row deleted. We say that $\boldsymbol{A}$ is *Nash stable* if

$$U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) > U_i\big((a_i, A_{-i}), \boldsymbol{T}, \boldsymbol{\zeta}\big) \quad \forall a_i \in \{0,1\}^{n-1}.\text{[8]} \qquad \text{(SA.5.1)}$$

[8]By (a_i, A_{-i}), we mean the following. Construct $\tilde{a}_i \in \{0,1\}^n$ from a_i by inserting a zero in the vector after the $(i-1)$th component. Then replace the ith row and column of $\boldsymbol{A}$ with $\tilde{a}_i$. The result is denoted (a_i, A_{-i}).

This differs from the pairwise stability concepts in two respects. First, there is no mutual consent. An agent i has sole discretion over the state of A_{ij} for any j, since links are directed. Second, an agent can deviate by simultaneously adding and/or removing any number of her directed links, whereas pairwise stability only allows an agent to unilaterally deviate by removing a single link. Let $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$ be the set of Nash stable networks.

A key difference between this solution concept and pairwise stability is that the action space here is $\{0, 1\}^{n-1}$. This makes computing $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$ more difficult because evaluating whether A_i is agent i's best response to A_{-i} potentially requires computing utilities for all other $2^{n-1} - 1$ alternatives. Equilibrium refinements in the undirected model, such as pairwise Nash stability, also share this problem, so the algorithm provided below will be useful for such extensions.

Define $V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = U_i((1, A_{-ij}), \boldsymbol{T}, \boldsymbol{\zeta}) - U_i((0, A_{-ij}), \boldsymbol{T}, \boldsymbol{\zeta})$, i's marginal utility from adding a link with agent j. As in Section SA.4, we assume

$$V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = V_n\big(S_{ij}(\boldsymbol{A}, \boldsymbol{T}), T_i, T_j, \zeta_{ij}\big),$$

where $S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = S(A_{-ij}, T_i, T_j, T_{-ij})$ for some $\mathbb{R}^{d_s}$-valued function $S(\cdot)$, and T_{-ij} is the type profile $\boldsymbol{T}$ with entries T_i, T_j omitted. Strategic interactions enter marginal utilities through $S_{ij}(\boldsymbol{T}, \boldsymbol{A})$.

Example SA.5.1. Consider the payoff function

$$U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = \sum_j A_{ij}\Bigg(w_n(T_i, T_j)'\theta_1 + \zeta_{ij} + \theta_2 A_{ji} + \theta_3 \sum_k A_{ik} A_{jk} + \theta_4 \sum_k A_{ki}\Bigg).$$

This is essentially a special case of the model studied in Ridder and Sheng (2017). The first two terms in the payoff function capture the direct benefits of the link A_{ij} for i. The remaining terms respectively capture reciprocity, a form of clustering (see Example SA.4.2), and popularity. Agent i's marginal utility of adding a link with j is

$$w_n(T_i, T_j)'\theta_1 + \zeta_{ij} + \theta_2 A_{ji} + \theta_3 \sum_k (A_{ik} A_{jk} + A_{ik} A_{jk}) + \theta_4 \sum_k A_{ki}.$$

Here, $S_{ij}(\boldsymbol{A}, \boldsymbol{T}) = (A_{ji}, \sum_k (A_{ik} A_{jk} + A_{ik} A_{jk}), \sum_k A_{ki})$.

The previous example satisfies a more general nonparametric restriction on strategic interactions that we state next.

Assumption SA.5.1 (Local interactions). *Let* $\mathcal{N}_{\text{in}}(i) = \{j \in \mathcal{N}_n \colon A_{ji} = 1\}$, $\mathcal{N}_{\text{out}}(i) = \{j \in \mathcal{N}_n \colon A_{ij} = 1\}$, *and*

$$\mathcal{N}^*(i) = \Bigg(\bigcup_{j \in \mathcal{N}_{\text{out}}(i)} \big(\mathcal{N}_{\text{in}}(j) \cup \mathcal{N}_{\text{out}}(j)\big)\Bigg) \cup \mathcal{N}^{\text{in}}(i).$$

For all $i \in \mathcal{N}_n$ *and* $n \in \mathbb{N}$, *there exists* $\tilde{U}_i(\cdot)$ *such that*

$$U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) = \tilde{U}_i(A_{\mathcal{N}^*(i)}, T_{\mathcal{N}^*(i)}, \zeta_{\mathcal{N}^*(i)}).$$

This says that i's payoffs only depend on the primitives through her in-neighbors ($\mathcal{N}_{\text{in}}(i)$) and in/out-neighbors of her out-neighbors ($\mathcal{N}_{\text{out}}(i)$). In Example SA.5.1, the statistic $\sum_j A_{ij}(w_n(T_i, T_j)'\theta_1 + \zeta_{ij} + \theta_2 A_{ji})$ is a function of i's out-neighbors, $\sum_j A_{ij} \sum_k A_{ki}$ i's in/out-neighbors, and $\sum_j \sum_k A_{ij} A_{ik} A_{jk}$ the in/out-neighbors of i's out-neighbors. Assumption SA.5.1 implies that the marginal utility function $V_n(\cdot)$ depends on the network only through the in/out-neighbors of i and j, which is analogous to Assumption SA.4.1.

SA.5.1 *Strategic neighborhoods*

Similar to the undirected case, the idea for computing $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$ is to compute Nash stable equilibria in smaller subgames involving only agents in a strategic neighborhood. We next define these neighborhoods for the directed case. Let $\boldsymbol{D}$ be the directed network on $\mathcal{N}_n$ such that for any $i, j \in \mathcal{N}_n$ with $i \neq j$,

$$D_{ij} = \mathbf{1}\Big\{\inf_s V_n(s, T_i, T_j, \zeta_{ij}) \leq 0 \cap \sup_s V_n(s, T_i, T_j, \zeta_{ij}) > 0\Big\}. \tag{SA.5.2}$$

This is similar to (SA.4.3) except defined using the marginal utility function rather than the joint surplus, since directed link formation does not require mutual consent. If $D_{ij} = 0$, then we say the potential link from i to j is *robust*. This is because if $\inf_s V_n(s, T_i, T_j, \zeta_{ij}) > 0$, then i prefers to link with j in any Nash stable network, what we call a *robust link*. If $\sup_s V_n(s, T_i, T_j, \zeta_{ij}) \leq 0$, i prefers not to link with j in any equilibrium, what we call a *robustly absent link*.

Let $\tilde{\boldsymbol{D}}$ be the undirected network on $\mathcal{N}_n$ obtained from $\boldsymbol{D}$ by ignoring the directionality of the links. That is, $\tilde{D}_{ij} = \max\{D_{ij}, D_{ji}\}$. Let $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ be the components of $\tilde{\boldsymbol{D}}$. For any $G \subseteq \mathcal{N}_n$, define

$$\mathcal{S}(G) = G \cup \Big\{k \in \mathcal{N}_n \colon \max_{j \in G} \max\Big\{\inf_s V_n(s, T_j, T_k, \zeta_{jk}), \inf_s V_n(s, T_k, T_j, \zeta_{kj})\Big\} > 0\Big\}.$$

This adds to G all agents k such that, for some $j \in G$, either k is robustly linked to j or the latter is robustly linked to k (or both). For any $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$, we call $\mathcal{S}(C)$ the *strategic neighborhood* associated with C.

As proven in Lemma SA.5.2, strategic neighborhoods have the following property, analogous to (SA.4.4):

$$A_{\mathcal{S}(C)} \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \quad \forall \boldsymbol{A} \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta}).$$

That is, if we take any Nash stable network $\boldsymbol{A}$ and remove all other agents from the game except members of $\mathcal{S}(C)$, then the subnetwork of $\boldsymbol{A}$ on $\mathcal{S}(C)$ is still Nash stable.

SA.5.2 *Algorithm*

As previously stated, an additional complication of Nash compared to pairwise stability is that to check the optimality of an agent's set of directed links in a given network, in principle we would have to compare her payoffs against those under all other $2^{n-1} - 1$

possible actions. The key observation is that we can substantially reduce the dimensionality of the space of alternative actions by fixing i's robust potential links and only considering through those that are nonrobust. Other than this complication, the basic idea of the algorithm is the same as the undirected case.

We first define the smaller action space it suffices to search over to verify Nash stability. Let $\mathcal{A}(G)$ be the set of directed networks on G, and define

$$\begin{aligned}\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta}) \\ &= \Big\{\boldsymbol{A}^* \in \mathcal{A}\big(\mathcal{S}(C)\big) \colon A^*_{\pi(i;C),\pi(j;C)} = 1 \text{ if } \inf_s V_n(s, T_i, T_j, \zeta_{ij}) > 0 \\ &\quad \text{and } A^*_{\pi(i;C),\pi(j;C)} = 0 \text{ if } \sup_s V_n(s, T_i, T_j, \zeta_{ij}) \le 0,\ \forall i, j \in \mathcal{S}(C), i \ne j\Big\}, \end{aligned} \tag{SA.5.3}$$

where $\pi(k; C)$ is defined in (2.5) (so $A^*_{\pi(i;C),\pi(j;C)}$ is simply the entry of $\boldsymbol{A}^*$ that corresponds to the potential link from i to j). This set is analogous to (SA.4.5). It is the set of directed networks on $\mathcal{S}(C)$, where the potential link from i to j is set to 1 in all networks if that link is robust and set to 0 if robustly absent. Let

$$\mathcal{A}^*_i(C, \boldsymbol{T}, \boldsymbol{\zeta}) = \big\{a \in \{0,1\}^{|\mathcal{S}(C)|} \colon a = A_i \text{ for some } \boldsymbol{A} \in \mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})\big\}.$$

This set collects the ith row of every network in $\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$, which corresponds to agent i's action. We can show that the number of elements in this set is $O_p(1)$ under a condition analogous to Assumption 2. Hence, searching through this set to verify Nash stability is feasible.

The remaining definitions are needed to show how to assemble Nash stable networks on strategic neighborhoods. They are entirely analogous to those in Section SA.4.2. For $H \subseteq \mathcal{N}_n$, note that $\mathcal{E}_{\text{NS}}(T_H, \zeta_H)$ is the set of directed Nash stable networks in the game consisting only of players in H. For $G \subseteq H \subseteq \mathcal{N}_n$, let

$$\mathcal{E}_{\text{NS}}(T_H, \zeta_H)|_G = \big\{\boldsymbol{A} \in \mathcal{A}(G) \colon \boldsymbol{A} = A^*_G \text{ for some } \boldsymbol{A}^* \in \mathcal{E}_{\text{NS}}(T_H, \zeta_H)\big\},$$

which is the set of subnetworks on G obtained from a network $\boldsymbol{A}^*$ on H. Define $\mathcal{B}$ as in (SA.4.6). By definition, for any pair of agents in this set, the potential links between them must be robust. Let

$$\begin{aligned}\bigtimes_{C \in \mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) = \Big\{&\boldsymbol{A}^* \in \mathcal{A}(\mathcal{N}_n) \colon A^*_C \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C\ \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta}), \\ &A^*_{ij} = \mathbf{1}\Big\{\inf_s V_n(s, T_i, T_j, \zeta_{ij}) > 0\Big\}\ \forall (i,j) \in \mathcal{B}\Big\},\end{aligned}$$

which is analogous to (SA.4.7). We state our proposed procedure in Algorithm SA.2.

Assumption SA.5.2. *Assumptions* 2 *and* 3 *hold with* $\boldsymbol{D}$ *given by* (SA.5.2) *and* τ_i *replaced with* T_i *for all* i.

Theorem SA.5.1. *Suppose evaluating* D_{ij} *and* $V_{ij}(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta})$ *have the same complexity for any* i, j. *Under Assumptions* SA.5.1 *and* SA.5.2, *Algorithm* SA.2 *computes* $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$ *in*

Algorithm SA.2: Procedure for computing the set of Nash stable networks.

Input: $\boldsymbol{T}, \boldsymbol{\zeta}, \{U_i(\cdot): i \in \mathcal{N}_n\}$

Output: $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$

1 *Compute* $\tilde{\boldsymbol{D}}$ *and then* $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ *using depth-first search of* $\tilde{\boldsymbol{D}}$.

2 *Compute each* $\mathcal{E}_{NS}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ *using exhaustive search*:

for $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ **do**
- $\mathcal{E}_{\mathcal{S}(C)} \leftarrow \emptyset$
- **for** $\boldsymbol{A} \in \mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$ **do**
  - **if** $U_i(\boldsymbol{A}, \boldsymbol{T}, \boldsymbol{\zeta}) > U_i((a_i, A_{-i}), \boldsymbol{T}, \boldsymbol{\zeta}) \; \forall a_i \in \mathcal{A}_i^*(C, \boldsymbol{T}, \boldsymbol{\zeta}), i \in C$ **then**
    - $\mathcal{E}_{\mathcal{S}(C)} \leftarrow \mathcal{E}_{\mathcal{S}(C)} \cup \{\boldsymbol{A}\}$
  - **end**
- **end**
- $\mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \leftarrow \mathcal{E}_{\mathcal{S}(C)}$

end

3 *Combine equilibrium sets*:

if $\mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \neq \emptyset \; \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ **then**
- $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta}) \leftarrow \bigtimes_{C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})} \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$

else $\mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta}) \leftarrow \emptyset$.

$O_p(\min\{n^{2+q}, n^2 + n^{1+3q}\})$ *evaluations of the payoff function for* $q > \log 2 / \log \|\lambda\|_{\mathbf{m},k}^{-1}$, *where* $\|\lambda\|_{\mathbf{m},k}$ *is defined in Assumption* SA.5.2.

Proof. See Section SA.5.3. □

Remark SA.5.1. Under sparsity (see Remark 4), for typical specifications, $S_{ij}(\boldsymbol{A}, \boldsymbol{T})$ will be asymptotically bounded. However, in many specifications, verifying Assumption 2 requires uniform boundedness, so that the supremum over $S_{ij}(\boldsymbol{A}, \boldsymbol{T})$ in the definition of $\boldsymbol{D}$ is finite with positive probability. Uniform boundedness is not satisfied in Example SA.5.1 due to the summations in the payoff function. This can be fixed by adding to the payoff function $C_i(A_i)$, a *capacity constraint* that equals $-\infty$ if i's degree exceeds some chosen value $\bar{C}$ and otherwise equals zero. Ridder and Sheng (2017) imposed uniform boundedness by scaling the sums in the payoff function in Example SA.5.1 by n^{-1}. Uniform boundedness is also maintained in Graham (2016), Leung (2019), and Menzel (2017).

SA.5.3 *Proof of main result*

Proof of Theorem SA.5.1. *Line* 1. Given $\boldsymbol{T}, \boldsymbol{\zeta}$, computing $\tilde{\boldsymbol{D}}$ takes $O(n^2)$ steps, since we need to compute D_{ij} for each pair of agents. As discussed in Remark SA.4.1, computing $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ takes $O(n + L)$ time, where L is the number of links in $\tilde{\boldsymbol{D}}$. The expected number of links is $0.5\mathbf{E}[\sum_i \sum_j \tilde{D}_{ij}] \leq n^2 \mathbf{E}[D_{ij} + D_{ji}]$, which is $O(n)$ by Assumption SA.5.2. Hence, $L = O_p(n^2)$, so line 1 of the algorithm requires $O_p(n^2)$ steps.

Line 2. For each component C, we have to iterate through every element of $\mathcal{A}^*(\mathcal{S}(C), \boldsymbol{T}, \boldsymbol{\zeta})$. The number of such elements is $2^{0.5\sum_{i,j\in C} D_{ij}}$. Then for each element of $\mathcal{A}^*(\mathcal{S}(C), \boldsymbol{T}, \boldsymbol{\zeta})$, the algorithm loops through all $|C|$ agents to verify Nash stability. Specifically, for each agent i, we need to evaluate $U_i((a_i, A_{-i}), \boldsymbol{T}, \boldsymbol{\zeta})$ for each $a_i \in \mathcal{A}_i^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$. Note that $|\mathcal{A}_i^*(C, \boldsymbol{T}, \boldsymbol{\zeta})| = 2^{\sum_{j\in C} D_{ij}}$, so for each element of $\mathcal{A}^*(C, \boldsymbol{T}, \boldsymbol{\zeta})$, the number of evaluations is order $\sum_{i\in C} 2^{\sum_{j\in C} D_{ij}}$. These computations are repeated for every component, resulting in a total of order

$$\sum_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{i,j\in C} D_{ij}} \sum_{i\in C} 2^{\sum_{j\in C} D_{ij}} \leq n\Big(\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{i,j\in C} D_{ij}}\Big)^3$$

evaluations. By Lemma SA.4.1, $\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{i,j\in C} D_{ij}} = O_p(n^q)$. Hence, the previous display is $O_p(n^{1+3q})$.

We can derive an alternative bound:

$$\sum_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{i,j\in C} D_{ij}} \sum_{i\in C} 2^{\sum_{j\in C} D_{ij}} \leq n\Big(\max_{C\in\mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} 2^{0.5\sum_{i,j\in C} D_{ij}}\Big) 2^{\max_{i\in\mathcal{N}_n}\sum_{j=1}^n D_{ij}}.$$

By Lemma SA.5.1 below, $2^{\max_{i\in\mathcal{N}_n}\sum_{j=1}^n D_{ij}} = O_p(n)$. Hence, the previous display is $O_p(n^{2+q})$.

Line 3. Under Assumption SA.5.1, we can apply Lemma SA.5.3, which proves that the algorithm has the desired output. □

LEMMA SA.5.1. *Under Assumption* SA.4.2,

$$2^{\max_{i\in\mathcal{N}_n}\sum_{j=1}^n D_{ij}} = O_p\big(n^{(1+c)\log 2}\big).$$

for any $c > 0$. *In particular, the left-hand side is* $O_p(n)$.

PROOF. Define $M_n = n^{(1+c)\log 2}$ for any $c > 0$. Note that for $c = 0.1$, $M_n < n$, which establishes the second claim.

By the union bound,

$$\mathbf{P}\big(2^{\max_{i\in\mathcal{N}_n}\sum_{j=1}^n D_{ij}} > M_n\big) \leq n\mathbf{P}\bigg(\sum_j D_{ij} > \frac{\log M_n}{\log 2}\bigg). \tag{SA.5.4}$$

Note that conditional on T_i, $\sum_j D_{ij}$ is a binomial random variable with mean $\mu_n(t) \equiv (n-1)\mathbf{E}[D_{ij} \mid T_i = t]$. A binomial concentration bound (Penrose (2003, Lemma 1.1)) yields

$$\begin{aligned}(\text{SA.5.4}) &\leq n\mathbf{E}\bigg[\exp\bigg\{-\mu_n(T_i)\bigg(1 + \frac{\log M_n}{\mu_n(T_i)\log 2}\bigg(\log\bigg(\frac{\log M_n}{\mu_n(T_i)\log 2} - 1\bigg)\bigg)\bigg)\bigg\}\bigg] \\ &= O\big(ne^{-\log M_n/\log 2}\mathbf{E}\big[e^{-\mu_n(T_i)}\big]\big). \end{aligned} \tag{SA.5.5}$$

By Assumption SA.4.2 (specifically, the analog of Assumption 2),

$$\mathbf{E}\big[e^{-\mu_n(T_i)}\big] \leq \exp\bigg\{\sup_t \int_{\mathbb{R}^d} \varphi\big(t, t'\big)\,\mathrm{d}\mu\big(t'\big)\bigg\},$$

which is finite by Assumption SA.4.2 (specifically, the analog of Assumption 3(b)). Then

$$(\text{SA.5.5}) = ne^{\log M_n^{-(\log 2)^{-1}}} = n^{-c}.$$

Therefore, (SA.5.4) $= o(1)$. □

LEMMA SA.5.2. *Under Assumption* SA.5.1, *for any* $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$,

$$\mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) = \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

PROOF. We first prove that

$$\mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \subseteq \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{A} \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ and $\boldsymbol{A}' \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$. Construct a network $\boldsymbol{A}^*$ by defining

$$A_{k\ell}^* = \begin{cases} A_{k\ell} & \text{if } k, \ell \in \mathcal{S}(C), \\ A'_{k\ell} & \text{otherwise.} \end{cases}$$

That is, we take $\boldsymbol{A}'$ and replace the subnetwork on $\mathcal{S}(C)$ with $A_{\mathcal{S}(C)}$. It suffices to show $\boldsymbol{A}^* \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$.

Consider an agent i, and let C_i be the element of $\mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$ containing i. We show that A_i^* is a "best response" for i to $\boldsymbol{A}^*$ in the sense that (SA.5.1) holds. For any agent $j \in \mathcal{N}_n \backslash \mathcal{S}(C_i)$, we have $\sup_s V_n(s, T_i, T_j, \zeta_{ij}) \leq 0$ and $\sup_s V_n(s, T_j, T_i, \zeta_{ji}) \leq 0$ by definition of strategic neighborhoods. Then by Assumption SA.5.1, agent i's payoff is not a function of any link in $\boldsymbol{A}^*$ involving j. Hence, to assess the optimality of A_i^*, we only need to consider the subvector restricted to $\mathcal{S}(C_i)$. But that subvector is a best response to the submatrix of $\boldsymbol{A}^*$ restricted to $\mathcal{S}(C_i)$ by definition of $\boldsymbol{A}$, so the claim follows.

Next, consider $j \in \mathcal{N}_n \backslash \mathcal{S}(C_i)$. We show that A_i^* is a "best response" for j to $\boldsymbol{A}^*$ in the sense that (SA.5.1) holds. There are three cases to consider. First, suppose $k \in \mathcal{N}_n \backslash \mathcal{S}(C_j)$. Then $\sup_s V_n(s, T_j, T_k, \zeta_{jk}) \leq 0$ and $\sup_s V_n(s, T_j, T_i, \zeta_{ji}) \leq 0$ by definition of strategic neighborhoods, so by Assumption SA.5.1, agent j's payoff is not a function of any link in $\boldsymbol{A}^*$ involving k. The second case is $k \in \mathcal{S}(C_j) \backslash \mathcal{S}(C_i)$. Then $A_{jk}^* = A'_{jk}$. The third case is $k \in \mathcal{S}(C_j) \cap \mathcal{S}(C_i)$. Of course, $k \notin C_i \cap C_j$, since these are disjoint sets by definition. So by construction of strategic neighborhoods, it must be that any link involving k and an agent $\ell \in \mathcal{S}(C_i) \cap \mathcal{S}(C_j)$ is robust, in which case $A_{k\ell} = A'_{k\ell} = A_{k\ell}^*$ and $A_{\ell k} = A'_{\ell k} = A_{\ell k}^*$ by definition of robustness.

We have therefore established that (1) the only potential links of the networks $\boldsymbol{A}'$ and $\boldsymbol{A}^*$ that may differ are those of agents $\{k, \ell\} \nsubseteq \mathcal{S}(\mathcal{N}_n \backslash C_i)$, and (2) the payoff of any agent $j \in \mathcal{N}_n \backslash C_i$ is not function of the links and types of such agents k, ℓ under $\boldsymbol{A}^*$. Then since $\boldsymbol{A}' \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$, it follows that A_i^* is a "best response" to $\boldsymbol{A}^*$.

Step 2. We prove that

$$\mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}) \supseteq \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}.$$

Let $\boldsymbol{A} \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})|_{\mathcal{S}(C)}$. By definition, there exists $\boldsymbol{A}' \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$ such that $\boldsymbol{A} = \boldsymbol{A}'_{\mathcal{S}(C)}$. Fix an arbitrary agent with associated component C_i and $j \in \mathcal{N}_n \backslash \mathcal{S}(C_i)$. Then

$\sup_s V_n(s, T_i, T_j, \zeta_{ij}) \leq 0$ and $\sup_s V_n(s, T_j, T_i, \zeta_{ji}) \leq 0$ by definition of strategic neighborhoods, so by Assumption SA.5.1, i's payoff is not a function of any link in $\boldsymbol{A}'$ involving j. Therefore, A'_i being a best response to $\boldsymbol{A}'$ (in the sense of (SA.5.1)) implies that the subvector of A'_i on $\mathcal{S}(C)$ is a best response to the subnetwork $\boldsymbol{A}'_{\mathcal{S}(C)}$ in the game only involving agents in $\mathcal{S}(C)$. But this subvector is exactly A_i, and the subnetwork is exactly $\boldsymbol{A}$, so the claim follows. □

Lemma SA.5.3. *Under Assumption* SA.4.1,

$$\bigtimes_{C \in \mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C = \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta}).$$

Proof. *Step 1.* Let $\boldsymbol{A}^* \in \bigtimes_{C \in \mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}$. This set is well-defined because for any pair of agents (i, j), either $i, j \in C$ for some $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$, or $(i, j) \in \mathcal{B}$. Consider any $i, j \in \mathcal{N}_n$. If $(i, j) \in \mathcal{B}$, then by definition of robustness, A^*_{ij} has the same value in any Nash stable network.

Suppose $i \in C$ for some $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$. We first prove that

$$A^*_{\mathcal{S}(C)} \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)}). \tag{SA.5.6}$$

By construction of $\boldsymbol{A}^*$, there exists $\boldsymbol{A} \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ such that $A^*_C = A_C$. Furthermore, for any $j, k \in \mathcal{S}(C)$ such that $\{j, k\} \nsubseteq C$, $A_{jk} = A^*_{jk}$ and $A_{kj} = A^*_{kj}$ are necessarily robust by definition of strategic neighborhoods. Therefore, $A^*_{\mathcal{S}(C)} = \boldsymbol{A}$, which proves (SA.5.6).

This establishes that, in the game where the set of players is given by $\mathcal{S}(C)$, the subvector of A^*_i on $\mathcal{S}(C)$ is a best response for i to $\boldsymbol{A}^*_{\mathcal{S}(C)}$ in the sense that (SA.5.1) holds. In fact, A^*_i is a best response to $\boldsymbol{A}^*$ in the game with all n players. This is because, as argued in the second paragraph of the proof of Lemma SA.5.2, i's payoff is not a function of the links in $\boldsymbol{A}^*$ involving agents $j \notin \mathcal{S}(C)$ by Assumption SA.5.1.

We have thus proved that $\boldsymbol{A}^* \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$. Hence,

$$\bigtimes_{C \in \mathcal{C}(\boldsymbol{T},\boldsymbol{\zeta})} \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C \subseteq \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta}).$$

Step 2. We prove the $\supseteq$ direction. Let $\boldsymbol{A}^* \in \mathcal{E}_{\text{NS}}(\boldsymbol{T}, \boldsymbol{\zeta})$. By definition of robustness, $A^*_{ij} = \mathbf{1}\{\inf_s V^*_n(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\}$ for any $i, j \in \mathcal{B}$. Let $C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta})$. Then $A^*_{\mathcal{S}(C)} \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})$ by Lemma SA.2.2. Hence, $A^*_C \in \mathcal{E}_{\text{NS}}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C$, so

$$\begin{aligned}\boldsymbol{A}^* \in \Big\{\boldsymbol{A}^* \in \mathcal{A}(\mathcal{N}_n)\colon\ & A^*_C \in \mathcal{E}_{NS}(T_{\mathcal{S}(C)}, \zeta_{\mathcal{S}(C)})|_C\ \forall C \in \mathcal{C}(\boldsymbol{T}, \boldsymbol{\zeta}),\\ & A^*_{ij} = \mathbf{1}\Big\{\inf_s V^*_n(s, T_i, T_j, \zeta_{ij}, \zeta_{ji}) > 0\Big\}\ \forall (i, j) \in \mathcal{B}\Big\},\end{aligned}$$

as desired. □

References

Bala, V. and S. Goyal (2000), "A noncooperative model of network formation." *Econometrica*, 68 (5), 1181–1229. [30]

Bollobás, B., S. Janson, and O. Riordan (2007), "The phase transition in inhomogeneous random graphs." *Random Structures and Algorithms*, 31 (1), 3–122. [8, 9, 11, 14]

Bollobás, B., S. Janson, and O. Riordan (2011), "Sparse random graphs with clustering." *Random Structures & Algorithms*, 38 (3), 269–323. [8]

Boucher, V. and I. Mourifié (2017), "My friend far far away: Asymptotic properties of pairwise stable networks." *Econometrics Journal*, 20 (3), 14–46. [17]

Currarini, S., M. Jackson, and P. Pin (2009), "An economic model of friendship: Homophily, minorities, and segregation." *Econometrica*, 77 (4), 1003–1045. [18]

Fafchamps, M. and F. Gubert (2007), "The formation of risk-sharing networks." *Journal of Economic Development*, 83, 326–350. [18]

Graham, B. (2016), "Homophily and transitivity in dynamic network formation." UC Berkeley working paper. [17, 18, 19, 34]

Jackson, M. (2010), *Social and Economic Networks*. Princeton University Press. [17, 18, 24]

Jackson, M. and A. Watts (2002), "The evolution of social and economic networks." *Journal of Economic Theory*, 106 (2), 265–295. http://linkinghub.elsevier.com/retrieve/pii/S0022053101929035. doi:10.1006/jeth.2001.2903. [22]

Janson, S., T. Luczak, and A. Rucinski (2011), *Random Graphs*, Vol. 45. John Wiley & Sons. [8]

Last, G. and M. Penrose (2017), *Lectures on the Poisson Process*, Vol. 7. Cambridge University Press. [9, 10, 11, 13]

Leung, M. (2019), "A weak law for moments of pairwise-stable networks." *Journal of Econometrics*, 210 (2), 310–326. [6, 17, 19, 23, 24, 34]

Leung, M. and R. Moon (2019), "Normal approximation in large network models." arXiv preprint arXiv:1904.11060v1. [1]

Meester, R. and R. Roy (1996), *Continuum Percolation*, Vol. 119. Cambridge University Press. [1]

Mele, A. (2017), "A structural model of dense network formation." *Econometrica*, 85 (3), 825–850. [19]

Menzel, K. (2017), "Strategic network formation with many agents." NYU working paper. [17, 19, 34]

Milgrom, P. and J. Roberts (1990), "Rationalizability, learning, and equilibrium in games with strategic complementarities." *Econometrica*, 1255–1277. [22]

Mode, C. (1971), *Multitype Branching Processes: Theory and Applications*, Vol. 34. American Elsevier Pub. Co. [7]

Penrose, M. (2003), *Random Geometric Graphs*. Oxford University Press. [1, 35]

Ridder, G. and S. Sheng (2017), "Estimation of large network formation games." UCLA working paper. [19, 31, 34]

Sheng, S. (2016), "A structural econometric analysis of network formation games." UCLA working paper. [16, 17, 26]

Turova, T. (2012), "Asymptotics for the size of the largest component scaled to "$\log n$" in inhomogeneous random graphs." *Arkiv för Matematik*, 51 (2), 371–403. [12, 14]

Co-editor Peter Arcidiacono handled this manuscript.

Manuscript received 10 July, 2019; final version accepted 22 June, 2020; available online 9 July, 2020.