Supplement to "A generalized approach to indeterminacy in linear rational expectations models"

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Appendix

Appendix A presents the proof of both Theorem 1 and Theorem 2. In Appendix B, we show that the estimation of the model in Galí (2021) delivers the same posterior distributions of the model parameters regardless of which forecast errors we include in our representation. We also report the Raftery–Lewis diagnostics for each parameter chain in Galí (2021). Appendix C reports the Raftery–Lewis diagnostics for each parameter chain in Lubik and Schorfheide (2004) using the hybrid ("Mixture") and random walk ("Random walk") algorithms. Finally, Appendix D provides detailed suggestions on the practical implementation of our method.

Appendix A: Proof of Theorem 1 and Theorem 2

A.1 *Equivalence under determinacy*

This section considers the case in which the original LRE is determinate, and shows the equivalence of the solution obtained using the proposed augmented representation with the one from the standard solution method described in Sims (2001).

A.1.1 *Canonical solution* Consider the LRE model in (5) and reported in the following equation:

$$\underset{k\times k}{\Gamma_0}\ \underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\ \underset{k\times 1}{X_{t-1}} + \underset{k\times l}{\Psi}\ \underset{l\times 1}{\varepsilon_t} + \underset{k\times p}{\Pi}\ \underset{p\times 1}{\eta_t}. \tag{S1}$$

The method described in Sims (2001) delivers a solution, if it exists, by following four steps. First, Sims (2001) showed how to write the model in the form

$$SZ'X_t = TZ'X_{t-1} + Q\Psi\varepsilon_t + Q\Pi\eta_t, \tag{S2}$$

where $\Gamma_0 = Q'SZ'$ and $\Gamma_1 = Q'TZ'$ result from the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$, and the $k \times k$ matrices Q and Z are orthonormal, upper triangular and possibly complex. Also, the diagonal elements of S and T contain the generalized eigenvalues of $\{\Gamma_0, \Gamma_1\}$.

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Second, given that the QZ decomposition is not unique, Sims' algorithm chooses a decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in an increasing order, that is,

$$|t_{jj}|/|s_{jj}| \geq |t_{ii}|/|s_{ii}| \quad \text{for } j > i.$$

The algorithm then partitions the matrices S, T, Q, and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad Z' = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where the first block corresponds to the system of equations for which $|t_{jj}|/|s_{jj}| \leq 1$ and the second block groups the equations which are characterized by explosive roots, $|t_{jj}|/|s_{jj}| > 1$.

The third step imposes conditions on the second, explosive block to guarantee the existence of at least one bounded solution. Defining the transformed variables

$$\xi_t \equiv Z' X_t = \begin{bmatrix} \underset{(k-n)\times 1}{\xi_{1,t}} \\ \underset{n\times 1}{\xi_{2,t}} \end{bmatrix},$$

where n is the number of explosive roots, and the transformed parameters

$$\widetilde{\Psi} \equiv Q'\Psi, \quad \text{and} \quad \widetilde{\Pi} \equiv Q'\Pi,$$

the second block can be written as

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1}(\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t).$$

As this system of equations contains the explosive roots of the original system, then a bounded solution, if it exists, will set

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{S3}$$

$$\underset{n\times \ell\ell\times 1}{\widetilde{\Psi}_2} \; \underset{}{\varepsilon_t} + \underset{n\times pp\times 1}{\widetilde{\Pi}_2} \; \underset{}{\eta_t} = 0, \tag{S4}$$

where n also denotes the number of equations in (S4). A necessary condition for the existence of a solution requires that the number of unstable roots (n) equals the number of expectational variables (p). In this section, we are considering the solution under determinacy, and this guarantees that there are no degrees of indeterminacy $m^*(\theta) = 0$. The sufficient condition then requires that the columns of the matrix $\widetilde{\Pi}_2$ are linearly independent so that there is at least one bounded solution. In that case, the matrix $\widetilde{\Pi}_2$ is a square, nonsingular matrix and equation (S4) imposes linear restrictions on the forecast errors, η_t, as a function of the fundamental shocks, ε_t,

$$\eta_t = -\widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \varepsilon_t. \tag{S5}$$

The fourth and last step finds the solution for the endogenous variables, X_t, by combining the restrictions in (S3) and (S5) with the system of stable equations in the first block,

$$\begin{aligned}\xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t)\\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\big(\widetilde{\Psi}_1 - \widetilde{\Pi}_1\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\big)\varepsilon_t.\end{aligned} \tag{S6}$$

Using the algorithm by Sims (2001), we can describe the solution under determinacy of the LRE model in (S1) with equations (S3), (S5), and (S6).

A.1.2 *Augmented representation* We now consider the methodology proposed in this paper, and we augment the LRE model in (S1) with the following system of m equations:

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \quad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix},$$

where Φ is a $m \times m$ diagonal matrix. As the original model in (S1) is determinate, then we assume that all the diagonal elements α_i belong to the interval $[1, 2]$. Therefore, we are appending a system of stable equations, and we show that the solution for the endogenous variables, X_t, is equivalent to the one found in Section A.1.1. Defining the augmented vector of endogenous variables, $\hat{X}_t \equiv (X_t, \omega_t)'$ and the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the representation that we propose takes the form

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{S7}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix}\Gamma_0 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix}\Gamma_1 & \mathbf{0}\\ \mathbf{0} & \Phi\end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix}\Psi & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix}\Pi_n & \Pi_f\\ 0 & -\mathbf{I}\end{bmatrix},$$

and without loss of generality the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p-m)$ and $k \times m$.

We can find a solution to the augmented representation in (S7) by using Sims' algorithm. Similarly to the previous section, we follow the four steps which describe the algorithm. First, the solution algorithm performs the QZ decomposition of the matrices $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and the augmented representation takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{S8}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$, and

$$\hat{S} = \begin{bmatrix}S_{11} & \mathbf{0} & S_{12}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & S_{22}\end{bmatrix}, \qquad \hat{T} = \begin{bmatrix}T_{11} & \mathbf{0} & T_{12}\\ \mathbf{0} & \Phi & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & T_{22}\end{bmatrix}, \qquad \hat{Z}' = \begin{bmatrix}Z_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ Z_2 & \mathbf{0}\end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix}Q_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ Q_2 & \mathbf{0}\end{bmatrix}.$$

Importantly, note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z, Q\}$ found in the previous section using the canonical solution method.

Second, the algorithm chooses a QZ decomposition which groups the equations in a stable and an explosive block. Because this section assumes that the original model is determinate and that the diagonal elements of the matrix Φ are within the unit circle, the explosive block corresponds to the third system of equations in (S8), which is characterized by explosive roots. Recalling the definition of the matrices $\hat{\Psi}$ and $\hat{\Pi}$, the system of equations in the third block is

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1} (\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t). \tag{S9}$$

The third step imposes conditions to guarantee the existence of a bounded solution. However, the explosive block in (S9) is identical to the system of equations found in the previous section. Therefore, the algorithm imposes the same restrictions to guarantee the existence of a bounded solution, that is,

$$\xi_{2,0} = 0 \tag{S10}$$

and as found earlier

$$\eta_t = -\widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \varepsilon_t. \tag{S11}$$

Finally, the last step combines these restrictions with the system of equations in the stable block which corresponds to the first and second systems of equations in (S8),

$$\xi_{1,t} = S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \big(\widetilde{\Psi}_1 - \widetilde{\Pi}_1 \widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2\big) \varepsilon_t, \tag{S12}$$

$$\omega_t = \Phi \omega_{t-1} + \nu_t - \eta_{f,t}. \tag{S13}$$

Recalling that $\xi_t \equiv Z' X_t$, the solution in (S10)$\sim$(S13) obtained for the augmented representation of the LRE model delivers the same solution for the endogenous variables of interest, X_t, found in the previous section and defined in equations (S3), (S5), and (S6).

A.2 *Equivalence under indeterminacy*

This section shows the equivalence of the solutions obtained for a LRE model under indeterminacy using the proposed augmented representation and the methodology of Lubik and Schorfheide (2003, 2004).

A.2.1 *Lubik and Schorfheide (2003)* As in Section A.1, we consider the LRE model in (S1) and reported below as (S14)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{S14}$$

In this section, we assume that the model is indeterminate, and we present the method used by Lubik and Schorfheide (2003). The authors implement the first two

steps of the algorithm by Sims (2001) and described in Section A.1.1.[1] They proceed by first applying the QZ decomposition to the LRE model in (S14) and then ordering the resulting system of equations in a stable and an explosive block as defined in equation (S2). However, their approach differs in the third step when the algorithm imposes restrictions to guarantee the existence of a bounded solution. In particular, the restrictions in (S3) and (S4) reported below as (S15) and (S16) require that

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{S15}$$

$$\underset{n\times \ell\ell\times 1}{\widetilde{\Psi}_2}\ \varepsilon_t + \underset{n\times p}{\widetilde{\Pi}_2}\ \underset{p\times 1}{\eta_t} = 0. \tag{S16}$$

Nevertheless, it is clear that the system of equation in (S16) is indeterminate as the number of forecast errors exceeds the number of explosive roots ($p > n$). Equivalently, there are less equations (n) than the number of variables to solve for (p). To characterize the full set of solutions to equation (S16), Lubik and Schorfheide (2003) decompose the matrix $\tilde{\Pi}_2$ using the following singular value decomposition:

$$\underset{n\times p}{\widetilde{\Pi}_2} \equiv \underset{n\times n}{U}\begin{bmatrix}\underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}}\end{bmatrix}\underset{p\times p}{V'},$$

where m represents the degrees of indeterminacy. Given the partition $\underset{p\times p}{V} \equiv \begin{bmatrix}\underset{p\times n}{V_1} & \underset{p\times m}{V_2}\end{bmatrix}$, equation (S16) can be written as

$$\underset{n\times n}{D_{11}^{-1}}\ \underset{n\times n}{U'}\ \underset{n\times \ell\ell\times 1}{\widetilde{\Psi}_2}\ \varepsilon_t + \underset{n\times p}{V_1'}\ \underset{p\times 1}{\eta_t} = 0. \tag{S17}$$

Given that the system is indeterminate, Lubik and Schorfheide (2003) append additional m equations,

$$\underset{m\times \ell_\ell\times 1}{\widetilde{M}}\ \varepsilon_t + \underset{m\times m}{M_\zeta}\ \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2'}\ \underset{p\times 1}{\eta_t}\,. \tag{S18}$$

The $m\times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero, covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamental shocks, ε_t, that is,

$$E[\zeta_t] = 0, \qquad E\left[\zeta_t \varepsilon_t'\right] = 0, \qquad E\left[\zeta_t \zeta_t'\right] = \Omega_{\zeta\zeta}.$$

The matrix $\widetilde{M}$ captures the correlation of the forecast errors, η_t, with fundamentals, ε_t, and Lubik and Schorfheide (2003) choose the normalization $M_\zeta = I_m$. The reason to append the system of equations in (S18) to the equations in (S17) is to exploit the properties of the orthonormal matrix V. Premultiplying the system by the matrix V and recalling that $V * V' = I$, the expectational errors can be written as a function of the fundamental shocks, ε_t, and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \Big(-\underset{p\times n}{V_1}\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U_1'}\underset{n\times n\ n\times \ell}{\widetilde{\Psi}_2} + \underset{p\times m}{V_2}\underset{m\times \ell}{\widetilde{M}}\Big)\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}\,.$$

[1] It is relevant to mention that in this section the matrices obtained from the QZ decomposition and the ordering of the equations into a stable and an explosive block differ from those in Section A.1 both in terms of their dimensionality and the elements constituting them. However, we opted to use the same notation for simplicity.

More compactly,

$$\underset{p\times 1}{\eta_t} = (\underset{p\times n n\times \ell}{V_1\, N} + \underset{p\times m m\times \ell}{V_2\, \widetilde{M}})\,\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m m\times 1}{V_2\, \zeta_t}\,, \tag{S19}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}}\,\underset{n\times n n\times \ell}{U_1'\,\widetilde{\Psi}_2}$$

is a function of the parameters of the model. Given the restriction in (S15) and (S19), the fourth step of the algorithm combines these equations with the system of stable equations in the first block as in Section A.1.1,

$$\begin{aligned}\xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t)\\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1 + \widetilde{\Pi}_1V_1N + \widetilde{\Pi}_1V_2\widetilde{M})\varepsilon_t + S_{11}^{-1}(\widetilde{\Pi}_1V_2)\zeta_t.\end{aligned} \tag{S20}$$

Using the method in Lubik and Schorfheide (2003), we can describe the solution for the original LRE model under indeterminacy with equations (S15), (S19), and (S20).

A.2.2 *Augmented representation* We now consider the augmented representation as in (S7) and reported below as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{S21}$$

where $\hat{X}_t \equiv (X_t, \omega_t)'$, $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$ and

$$\hat{\Gamma}_0 \equiv \begin{bmatrix}\Gamma_0 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix}\Gamma_1 & \mathbf{0}\\ \mathbf{0} & \Phi\end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix}\Psi & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix}\Pi_n & \Pi_f\\ 0 & -\mathbf{I}\end{bmatrix}.$$

where the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality.

The novelty of our approach is that, given our representation, we can easily obtain the solution by using Sims' algorithm even when the original LRE is assumed to be indeterminate. It is enough to assume that the auxiliary processes ω_t are characterized by explosive roots, or equivalently that the diagonal elements of the matrix Φ are outside the unit circle. This approach guarantees that the Blanchard–Kahn condition for the augmented representation is satisfied and, given the analytic form that we propose for the auxiliary processes, we show that the solution for the endogenous variables of interest, X_t, is equivalent to the method of Lubik and Schorfheide (2003).

To show this result, we simply apply the four steps of the algorithm described in Sims (2001) to the proposed augmented representation. First, the QZ decomposition of (S21) takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{S22}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition[2] of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\begin{aligned} \hat{S} &= \begin{bmatrix} S_{11} & S_{12} & \mathbf{0} \\ \mathbf{0} & S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{0} \\ \mathbf{0} & T_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \end{bmatrix}, \\ \hat{Z}' &= \begin{bmatrix} Z_1 & \mathbf{0} \\ Z_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \end{aligned} \tag{S23}$$

Note that in the expression above the auxiliary matrix Φ is in the lower (explosive) block because of our simplifying assumption that $m^*(\theta) = m$. When $m^*(\theta) < m$, part of the matrix Φ would belong in the stable block. As mentioned above, we made this simplifying assumption without loss of generality and only to simplify the exposition.

Second, the QZ decomposition chosen by the algorithm groups the explosive dynamics of the model in the second and third system of equations in (S22), which are reported below as (S24)

$$\begin{bmatrix} S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi_2 \\ \omega_t \end{bmatrix} = \begin{bmatrix} T_{22} & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix} \begin{bmatrix} \xi_{2,t-1} \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} (\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t). \tag{S24}$$

In the third step, the following restrictions are imposed:

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{S25}$$

$$\underset{m\times 1}{\omega_0} = 0, \tag{S26}$$

$$\begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} (\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t) = 0. \tag{S27}$$

Recalling the definition of $\hat{\Psi}$ and $\hat{\Pi}$ in (S21), then equation (S27) can be written as

$$\underbrace{\begin{bmatrix} \widetilde{\Psi}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{p\times(\ell+m)} \underset{(\ell+m)\times 1}{\hat{\varepsilon}_t} + \underbrace{\begin{bmatrix} \widetilde{\Pi}_{n,2} & \widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{p\times p} \underset{p\times 1}{\eta_t} = 0, \tag{S28}$$

where $\widetilde{\Psi} \equiv Q'\Psi$ and $\widetilde{\Pi} \equiv Q'\Pi$. Equation (S28) shows transparently how the explosive auxiliary process that we append in our augmented representation helps to solve the original LRE model under indeterminacy. The system of equations in (S28) is determinate, as the number of equations defined by the explosive roots of the system equals the number of expectational errors of the model. Thus, the necessary condition for the existence of a bounded solution for the augmented representation is satisfied. Assuming that the columns of the matrix associated with the vector of nonfundamental shocks,

[2]Note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}^T, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z^T, Q\}$ found from the QZ decomposition using the methodology of Lubik and Schorfheide (2003).

η_t, are linearly independent, we can impose linear restrictions on the forecast errors as a function of the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$,

$$\eta_t = -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 & \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \hat{\varepsilon}_t.$$

More compactly,

$$\eta_t = C_1\varepsilon_t + C_2\nu_t, \tag{S29}$$

where $C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix}$ and $C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}$ are a function of the structural parameters of the model.

The last step of Sims' algorithm combines the restrictions in (S25), (S26), and (S29) with the stationary block derived from the QZ decomposition in (S22),

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1 + \widetilde{\Pi}_1C_1)\varepsilon_t + S_{11}^{-1}(\widetilde{\Pi}_1C_2)\nu_t. \end{aligned} \tag{S30}$$

A.3 *Mapping of normalization in Lubik and Schorfheide (2004) to Bianchi–Nicolò*

We prove the equivalence between the parametrization of the Lubik–Schorfheide indeterminate equilibrium $\theta^{\text{LS}} \in \Theta^{\text{LS}}$ and the Bianchi–Nicolò equilibrium parametrized by $\theta^{\text{BN}} \in \Theta^{\text{BN}}$. In particular, we show that there is a unique mapping between the linear restrictions imposed in each of the two methodologies on the forecast errors to guarantee the existence of at least a bounded solution. As shown in Section A.2.1, the method by Lubik and Schorfheide (2003) imposes the following restrictions on the nonfundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks introduced in their specification, ζ_t,

$$\underset{p\times 1}{\eta_t} = (\underset{p\times n}{V_1}\underset{n\times \ell}{N} + \underset{p\times m}{V_2}\underset{m\times \ell}{\widetilde{M}})\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}. \tag{S31}$$

Using the methodology proposed in this paper, Section A.2.2 shows that the restrictions on the nonfundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks, ν_t, are

$$\underset{p\times 1}{\eta_t} = \underset{p\times \ell}{C_1}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2}\underset{m\times 1}{\nu_t}, \tag{S32}$$

where

$$C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}.$$

Post-multiplying equation (S31) and (S32) by ε_t' and taking expectations on both sides,

$$\underset{p\times l}{\Omega_{\eta\varepsilon}} = \underset{p\times n}{V_1}\underset{n\times \ell}{N}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{V_2}\underset{m\times \ell}{\widetilde{M}}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}},$$

$$\underset{p\times l}{\Omega_{\eta\varepsilon}} = \underset{p\times \ell}{C_1}\,\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{C_2}\,\underset{m\times l}{\Omega_{\nu\varepsilon}}.$$

Premultiplying by V_2' and equating the equations,

$$\underset{m\times \ell}{\widetilde{M}}\,\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} = \big(\underset{m\times p}{V_2'}\,\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2'}\,\underset{p\times n}{V_1}\,\underset{n\times \ell}{N}\big)\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{m\times p}{V_2'}\,\underset{p\times m}{C_2}\,\underset{m\times l}{\Omega_{\nu\varepsilon}}. \tag{S33}$$

Using the properties of the vec operator, the following result holds:

$$\begin{aligned}\underset{(m\times\ell)\times 1}{\mathrm{vec}(\widetilde{M})} = \underset{(m\times\ell)\times(m\times\ell)}{(\Omega_{\varepsilon\varepsilon}\otimes I_m)^{-1}}\Big[\underset{(m\times\ell)\times\ell^2}{\big[I_l\otimes\big(V_2'C_1 - V_2'V_1N\big)\big]}\underset{\ell^2\times 1}{\mathrm{vec}(\Omega_{\varepsilon\varepsilon})} \\ + \underset{(m\times\ell)\times(m\times\ell)}{\big(I_l\otimes V_2'C_2\big)}\underset{(m\times\ell)\times 1}{\mathrm{vec}(\Omega_{\nu\varepsilon})}\Big].\end{aligned} \tag{S34}$$

Equation (S34) is the first relevant equation to show the mapping between the representation in Lubik and Schorfheide (2003) and our representation. For a given variance-covariance matrix of the exogenous shocks, $\Omega_{\varepsilon\varepsilon}$, that is common between the two representations, equation (S34) tells us that the covariance structure, $\Omega_{\nu\varepsilon}$, of the sunspot shock in our representation with the exogenous shocks has a unique mapping to the matrix, $\widetilde{M}$, in Lubik and Schorfheide (2003). Clearly, equation (S33) can also be used to derive the mapping from their representation to our method.

We now show how to derive the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Considering again equation (S31) and (S32), we post-multiply by ζ_t' and take expectations on both sides,

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\zeta}},\\ \underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\zeta}}.\end{aligned}$$

Premultiplying both equations by V_2' and equating them,

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{\Omega_{\zeta\nu}}\big(\underset{m\times m}{V_2'C_2}\big)'. \tag{S35}$$

Finally, to obtain an expression for $\Omega_{\zeta\nu}$, we post-multiply equation (S31) and (S32) by ν_t' and taking expectations

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\nu}} &= \big(\underset{p\times n}{V_1}\,\underset{n\times \ell}{N} + \underset{p\times m}{V_2}\,\underset{m\times \ell}{\widetilde{M}}\big)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\nu}},\\ \underset{p\times m}{\Omega_{\eta\nu}} &= \underset{p\times \ell}{C_1}\,\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\nu}}.\end{aligned}$$

Premultiplying both equations by V_2' and solving for $\Omega_{\zeta\nu}$,

$$\underset{m\times m}{\Omega_{\zeta\nu}} = \big(\underset{m\times p}{V_2'}\,\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2'}\,\underset{p\times n}{V_1}\,\underset{n\times \ell}{N} - \underset{m\times \ell'}{\widetilde{M}}\big)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \big(\underset{m\times m}{V_2'C_2}\big)\underset{m\times m}{\Omega_{\nu\nu}}. \tag{S36}$$

Post-multiplying (S36) by $\underset{m\times m}{(V_2'C_2)'}$ and using (S35), then

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \Big(\underset{m\times p}{V_2'}\,\underset{p\times \ell}{C_1} - \underset{m\times p}{V_2'}\,\underset{p\times n}{V_1}\,\underset{n\times \ell}{N} - \underset{m\times \ell}{\widetilde{M}}\Big)\underset{\ell\times m}{\Omega_{\varepsilon\nu}}\big(\underset{m\times m}{V_2'C_2}\big)' + \underset{m\times m}{\big(V_2'C_2\big)}\,\underset{m\times m}{\Omega_{\nu\nu}}\,\big(\underset{m\times m}{V_2'C_2}\big)'. \tag{S37}$$

Therefore, equation (S37) defines the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Together with equation (S34), we show that this equation defines the one-to-one mapping between the parametrization in Lubik and Schorfheide $\{\Theta, \Theta^{\text{LS}}\}$ and the parametrization in Bianchi–Nicolò $\{\Theta, \Theta^{\text{BN}}\}$.

Appendix B: Additional results for Galí (2021)

Table SI reports the posterior distribution of the model parameters for each of the three specifications that are possible when adopting our method to solve for the model of Galí (2021) under two degrees of indeterminacy. For each specification, the log-posterior mode is −33.1 and the table shows that the estimates are equivalent up to a transforma-

Table SI. Posterior distribution of model parameters under 2-degree indeterminacy.

	$\{\nu_1=\nu_\pi, \nu_2=\nu_y\}$		$\{\nu_1=\nu_\pi, \nu_2=\nu_b\}$		$\{\nu_1=\nu_y, \nu_2=\nu_b\}$	
	Mean	90% prob. int.	Mean	90% prob. int.	Mean	90% prob. int.
$100(\lambda_l^{-1}-1)$	0.028	[0.018, 0.038]	0.031	[0.018, 0.043]	0.030	[0.012, 0.042]
κ	0.042	[0.034, 0.050]	0.039	[0.031, 0.047]	0.039	[0.031, 0.047]
g	0.48	[0.46, 0.50]	0.49	[0.46, 0.51]	0.48	[0.46, 0.51]
π^*	0.89	[0.44, 1.36]	0.90	[0.41, 1.36]	0.89	[0.40, 1.34]
ϕ_π	0.29	[0.14, 0.43]	0.34	[0.15, 0.56]	0.26	[0.12, 0.39]
ϕ_q	0.02	[0.01, 0.04]	0.04	[0.01, 0.07]	0.03	[0.01, 0.04]
ρ_i	0.50	[0.27, 0.74]	0.50	[0.25, 0.77]	0.48	[0.23, 0.73]
σ_q	0.97	[0.45, 1.51]	1.21	[0.45, 2.06]	1.01	[0.46, 1.58]
σ_s	0.11	[0.09, 0.12]	0.11	[0.09, 0.14]	0.11	[0.09, 0.13]
σ_i	0.11	[0.09, 0.14]	0.12	[0.09, 0.15]	0.12	[0.09, 0.14]
ρ_q	0.71	[0.55, 0.87]	0.74	[0.57, 0.91]	0.72	[0.56, 0.87]
ρ_s	0.89	[0.84, 0.94]	0.88	[0.81, 0.95]	0.88	[0.83, 0.94]
σ_{ν_1}	0.30	[0.25, 0.34]	0.30	[0.25, 0.34]	0.70	[0.61, 0.79]
σ_{ν_2}	11.14	[5.48, 16.55]	0.71	[0.61, 0.80]	10.10	[4.83, 15.39]
$\varphi_{\nu_1,i}$	−0.54	[−0.75, −0.31]	−0.59	[−0.80, −0.39]	−0.38	[−0.65, −0.11]
$\varphi_{\nu_1,q}$	0.18	[−0.27, 0.60]	0.21	[−0.23, 0.62]	0.11	[−0.43, 0.66]
$\varphi_{\nu_1,s}$	0.58	[0.46, 0.72]	0.54	[0.38, −0.69]	−0.54	[−0.70, −0.40]
$\varphi_{\nu_2,i}$	−0.72	[−0.88, −0.54]	−0.39	[−0.71, −0.10]	−0.76	[−0.93, −0.61]
$\varphi_{\nu_2,q}$	−0.06	[−0.42, 0.29]	0.05	[−0.44, 0.56]	−0.08	[−0.46, 0.26]
$\varphi_{\nu_2,s}$	−0.51	[−0.71, −0.32]	−0.55	[−0.72, −0.38]	−0.45	[−0.65, −0.25]
φ_{ν_1,ν_2}	0.28	[0.03, 0.54]	0.23	[0.05, 0.41]	0.63	[0.44, 0.81]

TABLE SII. Raftery–Lewis diagnostics for each parameter chain in Galí (2021).

Variable	Thin	Burn	Total (N)	Nmin	I-stat
$100(\lambda_l^{-1}-1)$	1	2	345	322	1.071
κ	1	2	304	322	0.944
g	1	2	331	322	1.028
π^*	1	3	293	322	0.910
ϕ_π	1	2	304	322	0.944
ϕ_q	1	2	317	322	0.984
ρ_i	1	3	361	322	1.121
σ_q	1	2	331	322	1.028
σ_s	1	3	376	322	1.168
σ_i	1	2	345	322	1.071
ρ_q	1	3	376	322	1.168
ρ_s	1	2	331	322	1.028
σ_{ν_π}	1	3	340	322	1.056
σ_{ν_y}	1	2	345	322	1.071
$\varphi_{\nu_\pi,i}$	1	3	392	322	1.217
$\varphi_{\nu_\pi,q}$	1	3	376	322	1.168
$\varphi_{\nu_\pi,s}$	1	4	410	322	1.273
$\varphi_{\nu_y,i}$	1	3	376	322	1.168
$\varphi_{\nu_y,q}$	1	4	410	322	1.273
$\varphi_{\nu_y,s}$	1	3	385	322	1.196
φ_{ν_π,ν_y}	1	13	985	322	3.059

Note: The table reports the Raftery–Lewis diagnostics for each parameter chain. We consider the $5th$ quantile, $q=0.05$, with an accuracy $r=0.02$ and a probability $s=0.9$ of obtaining an estimate in the interval $(q-r, q+r)$. The diagnostics reports the suggested number of burn-in iterations ("Burn"), the suggested number of iterations ("Total (N)"), the suggested minimum number of iterations based on zero autocorrelation ("Nmin") and the dependence factor ("I-stat"). The dependence factor is computed as I-stat = (Burn + Total)/Nmin, and interpreted as the proportional increase in the number of iterations attributable to autocorrelation.

tion of the correlations between the exogenous shocks and the forecast errors included in the auxiliary process.[3]

Table SII reports the Raftery–Lewis diagnostics for each parameter chain of the model of Galí (2021).

[3]To obtain the estimates in Table SI, we use a uniform distribution over the interval $(0, 20)$ for the standard deviation of the sunspot shocks. This guarantees that the posterior distribution of the sunspot shock ν_b is not at the boundary of the range specified for the uniform prior.

Appendix C: Additional results for Lubik and Schorfheide (2004)

Table SIII. Raftery–Lewis diagnostics for each parameter chain in Lubik and Schorfheide (2004).

	Mixture					Random walk				
	Thin	Burn	Total (N)	Nmin	I-stat	Thin	Burn	Total (N)	Nmin	I-stat
ψ_π	2	5	2981	1286	2.318	4	14	5790	1286	4.502
ψ_y	1	2	1299	1286	1.010	1	1	1287	1286	1.000
ρ_R	1	3	1545	1286	1.201	1	3	1432	1286	1.113
τ	1	3	1564	1286	1.216	1	3	1538	1286	1.196
κ	2	7	3037	1286	2.362	1	4	1654	1286	1.286
ρ_g	2	7	2967	1286	2.307	4	15	6287	1286	4.889
ρ_z	1	2	1332	1286	1.036	1	2	1321	1286	1.027
r^*	1	3	1468	1286	1.141	1	3	1474	1286	1.146
π^*	1	3	1500	1286	1.166	3	11	4136	1286	3.216
σ_R	1	2	1366	1286	1.062	1	3	1526	1286	1.187
σ_g	1	2	1372	1286	1.067	1	3	1384	1286	1.076
σ_z	1	2	1304	1286	1.014	1	1	1285	1286	0.999
ρ_{gz}	2	7	3051	1286	2.372	2	8	2972	1286	2.311
σ_η	2	6	2758	1286	2.145	1	3	1390	1286	1.081

Note: The table reports the Raftery–Lewis diagnostics for each parameter chain using the hybrid ("Mixture") and the random walk algorithm ("Random walk"). We consider the $5th$ quantile, $q = 0.05$, with an accuracy $r = 0.01$ and a probability $s = 0.9$ of obtaining an estimate in the interval $(q-r, q+r)$. The diagnostics reports the suggested number of burn-in iterations ("Burn"), the suggested number of iterations ("Total (N)"), the suggested minimum number of iterations based on zero autocorrelation ("Nmin") and the dependence factor ("I-stat"). The dependence factor is computed as I-stat = (Burn + Total)/Nmin, and interpreted as the proportional increase in the number of iterations attributable to autocorrelation.

Appendix D: Suggestions on practical implementation

In this section, we consider the case that a researcher estimates a LRE model using Bayesian techniques and a conventional Metropolis–Hastings algorithm in Dynare. Let us consider the following LRE model:

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{S38}$$

with a maximum degree of indeterminacy denoted by m.[4] As explained in detail in Section 3, the proposed methodology appends to the original LRE model the following system of m equations:

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \tag{S39}$$

where Φ is a diagonal matrix whose entries are $\{1/\alpha_1, .., 1/\alpha_m\}$. Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the resulting augmented LRE model can be written as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{S40}$$

[4] We refer the reader to Section 3 for definitions and notation.

Auxiliary autoregressive parameters As a first step, we discuss how to handle the vector of additional autoregressive parameters, $\{\alpha_i\}_{i=1}^m$, introduced under our methodology. We can distinguish three cases:

1. When the threshold for the different regions of determinacy is known analytically, then α_i can be expressed as a function of the other parameters. In this case, there is no need to specify a prior on α_i and the prior probability of (in)determinacy is given by the prior on the parameter vector θ.

2. When the threshold is unknown and the researcher writes her own code, she can start with all the roots inside the unit circle for α at each draw of θ and then flip the appropriate number of elements in the vector α. This case coincides with the approach that we adopt in Section 5 to estimate the model of Galí (2021) for which there is no need to specify a prior on α_i and the prior probability of indeterminacy depends on the prior on the parameter vector θ. In other words, in this case α_i is treated as an unknown transformation of the structural parameters that guarantees that a solution, if it exists, can be found independently of the degree of indeterminacy.

3. When the threshold is unknown and the researcher wants to use standard estimation packages such as Dynare, there are two options. First, the researcher estimates the model separately for different degrees of indeterminacy. This is the simplest approach and we describe it more in detail below. Second, the researcher estimates the model over the whole parameter space. In this case, the researcher can complement the Dynare codes with a function that pins down the degrees of indeterminacy. This can be done writing a function that, starting with all $\{\alpha_i\}_{i=1}^m$ inside the unit circle, solves the model and keeps flipping $\{\alpha_i\}_{i=1}^m$ until the augmented state space returns determinacy. In this case, the $\{\alpha_i\}_{i=1}^m$ are still treated as a transformation of the structural parameters of the model. Alternatively, the researcher can decide to treat $\{\alpha_i\}_{i=1}^m$ as additional parameters. In this case, the researcher should choose priors that are symmetric with respect to the various determinacy regions and orthogonal with respect to the priors on the other parameters. The researcher could use a uniform distribution over the interval [0.5, 1.5] or any symmetric interval around 1 as a prior distribution. The assumptions that the priors are symmetric around 1 and orthogonal with respect to the structural parameters imply that the a priori probabilities of the different determinacy regions only depend on the priors on the structural parameters of the model. The posterior distribution of the parameters is not affected by treating $\{\alpha_i\}_{i=1}^m$ as additional parameters. However, the priors on $\{\alpha_i\}_{i=1}^m$ would have an impact on the marginal data density computed by Dynare. The marginal data density can be corrected ex post by taking into account that uniform priors on $\{\alpha_i\}_{i=1}^m$ simply rescale the joint prior on the model parameters. Alternatively, a researcher could implement a simple modification of the code used to compute Geweke (1999)'s modified harmonic mean estimator to remove the impact of the priors on α_i. For example, $\{\alpha_i\}_{i=1}^m$ could be weighted using their own prior when computing the modified harmonic mean estimator.

Priors for the correlations between the sunspot and fundamental shocks In Section 2.2, we discuss in detail the economic rationale for how to construct a baseline solution using our methodology. Therefore, it seems natural to center the prior distribution for the correlations on zero, the value associated with the "baseline solution." However, as carefully explained in Section 2.2, it is important to stress that under the baseline solution, the choice of which forecast errors to include in the auxiliary processes matters for the solution when the correlations are restricted to zero. At the same time, as explained in Section 3, a set of correlations under the representation that includes a given subset of nonfundamental shocks has a unique mapping to *different* values of the correlations in the representation with another subset of nonfundamental disturbances, and vice versa. Therefore, in order for the alternative representations to deliver the same fit to the data, a researcher has to leave the *correlations unrestricted*. One simple option is to set a uniform prior distributions over the interval $(-1, 1)$ for the correlations of the sunspot shocks. As shown for the estimation of the model of Galí (2021) in Section 5, this approach guarantees that the fit of the model does not depend on which nonfundamental shock is included in the auxiliary processes. Of course, if a researcher has reasons to believe that one baseline solution is more meaningful than the other, she can choose the priors accordingly.

Lubik and Schorfheide (2004) centered the prior distribution for the additional parameters introduced in their representation to values that minimize the distance between the impulse responses of the model under indeterminacy and determinacy evaluated at the boundary of the region of determinacy. While our approach and intuition differ, our theoretical results show the equivalence between the two representations in Section 3. Therefore, the priors for the correlations between sunspot shocks and fundamental shocks could also be specified in a way to replicate the approach of Lubik and Schorfheide (2004). Specifically, we could center the prior on the auxiliary parameters as in Lubik and Schorfheide (2004) and then map those values into correlations in our approach that would deliver the same fit of the model to the data. However, our suggestion to choose a flat prior such as a uniform distribution considers a priori the mapped parameterization suggested by Lubik and Schorfheide (2004) as equally likely with respect to the continuum of indeterminate equilibria.

Model comparison A researcher might be interested in comparing the fit of the model under determinacy and under indeterminacy. Note that, while under indeterminacy the volatility of the sunspot shocks and their correlations with the exogenous shocks are estimated, those parameters *should* be restricted to zero (or any other constant) under determinacy. Model comparison can then be conducted by using standard techniques, such as the harmonic mean estimator proposed by Geweke (1999).

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