Supplement to
Estimating Macroeconomic Models of Financial Crises: An Endogenous Regime-Switching Approach*

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This supplementary appendix provides technical details and additional results.

S.1 Additional Model Details

S.1.1 Stationary Equilibrium Conditions and the Steady State

In the stationary equilibrium, all variables are scaled by Z_{t-1}. So, for example, $\tilde{C}_t = C_t/Z_{t-1}$. The the exceptions are $\tilde{K}_{t-1} = K_{t-1}/Z_{t-1}$, $\tilde{B}_{t-1} = B_{t-1}/Z_{t-1}$, $\tilde{\mu}_t = \mu_t Z_{t-1}^{\rho}$ and $\tilde{\lambda}_t = \lambda_t Z_{t-1}^{\rho}$.

The full list of equilibrium conditions is as follows. The first-order conditions from the household-firm (5 equations)

$$d_t \left(\tilde{C}_t - \frac{H_t^{\omega}}{\omega} \right)^{-\rho} = \tilde{\mu}_t$$

$$(1-\alpha-\eta) A_t \tilde{Z}_t^{\alpha} \tilde{K}_{t-1}^{\eta} H_t^{\alpha} \tilde{V}_t^{-\alpha-\eta} = P_t \left(1 + \phi r_t + \frac{\tilde{\lambda}_t}{\tilde{\mu}_t} \phi (1+r_t) \right)$$

$$\alpha A_t \tilde{Z}_t^{\alpha} \tilde{K}_{t-1}^{\eta} H_t^{\alpha-1} \tilde{V}_t^{1-\alpha-\eta} = \phi \tilde{W}_t \left(r_t + \frac{\tilde{\lambda}_t}{\tilde{\mu}_t} (1+r_t) \right) + H_t^{\omega-1}$$

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$$\tilde{\mu}_t = \tilde{\lambda}_t + \beta\left(1+r_t\right)\mathbb{E}_t\tilde{\mu}_{t+1}\tilde{Z}_t^{-\rho}$$

$$\beta\mathbb{E}_t\tilde{\mu}_{t+1}\tilde{Z}_t^{-\rho}\begin{pmatrix} \eta A_{t+1}\tilde{Z}_{t+1}^{\alpha}\tilde{K}_t^{\eta-1}H_{t+1}^{\alpha}\tilde{V}_{t+1}^{1-\alpha-\eta} \\ +1-\delta \\ +\iota\Lambda_k\left(\frac{\tilde{K}_{t+1}\tilde{Z}_{t+1}-\Lambda_k\tilde{K}_t}{\tilde{K}_t}\right) \\ +\frac{\iota}{2}\left(\frac{\tilde{K}_{t+1}\tilde{Z}_{t+1}-\Lambda_k\tilde{K}_t}{\tilde{K}_t}\right)^2 \end{pmatrix} = \tilde{\mu}_t\left(1+\iota\left(\frac{\tilde{K}_t\tilde{Z}_t-\Lambda_k K_{t-1}}{K_{t-1}}\right)\right) - \tilde{\lambda}_t\kappa q_t$$

market price equations (2 equations)

$$\tilde{W}_t = H_t^{\omega-1}$$

$$q_t = 1+\iota\left(\frac{\tilde{K}_t\tilde{Z}_t-\Lambda_k\tilde{K}_{t-1}}{\tilde{K}_{t-1}}\right)$$

budget constraints (2 equations)

$$\tilde{C}_t+\tilde{I}_t+e_t = A_t\tilde{Z}_t^{\alpha}\tilde{K}_{t-1}^{\eta}H_t^{\alpha}\tilde{V}_t^{1-\alpha-\eta} - P_t\tilde{V}_t - \phi r_t\left(\tilde{W}_tH_t+P_t\tilde{V}_t\right) - \frac{1}{(1+r_t)}\tilde{B}_t\tilde{Z}_t + \tilde{B}_{t-1}$$

$$\tilde{K}_t\tilde{Z}_t = \left(1-\delta\right)\tilde{K}_{t-1}+\tilde{I}_t-\frac{\iota}{2}\left(\frac{\tilde{K}_t\tilde{Z}_t-\Lambda_k\tilde{K}_{t-1}}{\tilde{K}_{t-1}}\right)^2\tilde{K}_{t-1}$$

the debt cushion definition and regime switching slackness condition (2 equations)

$$\tilde{B}_t^* = \frac{1}{(1+r_t)}\tilde{B}_t\tilde{Z}_t - \phi\left(1+r_t\right)\left(\tilde{W}_tH_t+P_t\tilde{V}_t\right)+\kappa_tq_t\tilde{K}_t\tilde{Z}_t$$

$$\varphi\left(s_t\right)\tilde{B}_{ss}^* + \nu\left(s_t\right)\left(\tilde{B}_t^*-\tilde{B}_{ss}^*\right) = \left(1-\varphi\left(s_t\right)\right)\tilde{\lambda}_{ss} + \left(1-\nu\left(s_t\right)\right)\left(\tilde{\lambda}_t-\tilde{\lambda}_{ss}\right)$$

some definitions (2 equations)

$$\tilde{Y}_t = A_t\tilde{Z}_t^{\alpha}\tilde{K}_{t-1}^{\eta}H_t^{\alpha}\tilde{V}_t^{1-\alpha-\eta} - P_t\tilde{V}_t$$

$$\Phi_t^{ca} = \frac{\tilde{B}_t\tilde{Z}_t-\tilde{B}_{t-1}}{\tilde{Y}_t}$$

an interest rate and debt premium equation (1 equation)

$$r_t = r_t^* + \psi\left(\exp\left(\bar{b}-\tilde{B}_t\right)-1\right)$$

and exogenous processes (6 equations)

$$\log A_t = (1-\rho_a)\log \bar{A} + \rho_a \log A_{t-1} + \sigma_a \varepsilon_{a,t}$$

$$\log \tilde{Z}_t = (1-\rho_z)\log \bar{Z} + \rho_z \log \tilde{Z}_{t-1} + \sigma_z \varepsilon_{z,t}$$

$$\log P_t = (1-\rho_p)\log \bar{P} + \rho_g \log P_{t-1} + \sigma_p \varepsilon_{p,t}$$

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t}$$

$$\log e_t = (1-\rho_e)\log \bar{e} + \rho_e \log e_{t-1} + \sigma_e \varepsilon_{e,t}$$

$$r_t^* = (1-\rho_r)\,\bar{r} + \rho_r r_{t-1}^* + \sigma_r \varepsilon_{r,t}$$

In total, there are 20 equations, the following 20 unknowns

$$\left\{\tilde{C}_t, H_t, \tilde{K}_t, \tilde{I}_t, \tilde{V}_t, \tilde{B}_t, \tilde{B}_t^*, q_t, \tilde{W}_t, \tilde{Y}_t, \Phi_t^{ca}, \tilde{\mu}_t, \tilde{\lambda}_t, A_t, \tilde{Z}_t, P_t, d_t, e_t, r_t\right\},$$

and 6 shocks.

$$\{\varepsilon_{a,t}, \varepsilon_{z,t}, \varepsilon_{u,t}, \varepsilon_{g,t}, \varepsilon_{r,t}, \varepsilon_{\kappa\, t}\}.$$

S.1.2 Steady State Solution

Here, we provide the derivations for Step 1 of Appendix B.2. First, note that we can directly solve for part of the steady state:

$$A_{ss} = A^*, Z_{ss} = Z^*, P_{ss} = P^*, d_{ss} = 1, e_{ss} = e^*, r_{ss}^* = \bar{r}, q_{ss} = 1.$$

Suppose now that we know $\tilde{B}_{ss}$, then

$$r_{ss} = r_{ss}^* + \psi\left(\exp\left(\bar{b} - \tilde{B}_{ss}\right) - 1\right).$$

Next, by using

$$\frac{\tilde{\lambda}_{ss}}{\tilde{\mu}_{ss}} = 1 - \beta\,(1+r_{ss})\,\tilde{Z}_{ss}^{-\rho},$$

we can define

$$\Omega_v = \frac{A_{ss}\tilde{Z}_{ss}^{\alpha}\tilde{K}_{ss}^{\eta}H_{ss}^{\alpha}\tilde{V}_{ss}^{1-\alpha-\eta}}{\tilde{V}_{ss}P_{ss}} = \frac{1+\phi r_{ss} + \phi\,(1+r_{ss})\left(1-\beta\,(1+r_{ss})\,\tilde{Z}_{ss}^{-\rho}\right)}{(1-\alpha-\eta)}$$

$$\Omega_h = \frac{A_{ss}\tilde{Z}_{ss}^{\alpha}\tilde{K}_{ss}^{\eta}H_{ss}^{\alpha}\tilde{V}_{ss}^{1-\alpha-\eta}}{\tilde{W}_{ss}H_{ss}} = \frac{1+\phi\left(r_{ss}+(1+r_{ss})\left(1-\beta(1+r_{ss})\tilde{Z}_{ss}^{-\rho}\right)\right)}{\alpha}$$

$$\Omega_k = \frac{A_{ss}\tilde{Z}_{ss}^{\alpha}\tilde{K}_{ss}^{\eta}H_{ss}^{\alpha}\tilde{V}_{ss}^{1-\alpha-\eta}}{K_{ss}} = \frac{\frac{1-\kappa\left(1-\beta(1+r_{ss})\tilde{Z}_{ss}^{-\rho}\right)}{\beta\tilde{Z}_{ss}^{-\rho}}-1+\delta}{\eta}$$

and then solve for

$$\tilde{W}_{ss} = \left(\frac{A_{ss}\tilde{Z}_{ss}^{\alpha}}{\Omega_k^{\eta}\Omega_h^{\alpha}\left(\Omega_v P_{ss}\right)^{1-\alpha-\eta}}\right)^{\frac{1}{\alpha}}$$

$$H_{ss} = \tilde{W}_{ss}^{\frac{1}{\omega-1}}$$

$$\tilde{V}_{ss} = \frac{\Omega_h}{\Omega_v P_{ss}}\tilde{W}_{ss}H_{ss}$$

$$K_{ss} = \frac{\Omega_h}{\Omega_k}\tilde{W}_{ss}H_{ss}$$

$$\tilde{I}_{ss} = \left(\tilde{Z}_{ss}-1+\delta\right)\tilde{K}_{ss}$$

$$\tilde{Y}_{ss} = A_{ss}\tilde{Z}_{ss}^{\alpha}\tilde{K}_{ss}^{\eta}H_{ss}^{\alpha}\tilde{V}_{ss}^{1-\alpha-\eta} - P_{ss}\tilde{V}_{ss}$$

$$\Phi_{ss}^{ca} = \left(\tilde{Z}_{ss}-1\right)\frac{\tilde{B}_{ss}}{\tilde{Y}_{ss}}$$

$$\tilde{B}_{ss}^{*} = \frac{1}{(1+r_{ss})}\tilde{B}_{ss}\tilde{Z}_{ss} - \phi(1+r_{ss})\left(\tilde{W}_{ss}H_{ss}+P_{ss}\tilde{V}_{ss}\right) + \kappa q_{ss}\tilde{K}_{ss}\tilde{Z}_{ss}$$

$$\tilde{C}_{ss} = Y_{ss} - \phi r_{ss}\left(\tilde{W}_{ss}H_{ss}+P_{ss}\tilde{V}_{ss}\right) - \frac{1}{(1+r_{ss})}\tilde{B}_{ss}\tilde{Z}_{ss} + \tilde{B}_{ss} - e_{ss} - I_{ss}$$

$$\tilde{\mu}_{ss} = d_{ss}\left(\tilde{C}_{ss}-\frac{H_{ss}^{\omega}}{\omega}\right)^{-\rho}.$$

Finally, $\tilde{B}_{ss}$ solves

$$\bar{\varphi}\tilde{B}_{ss}^{*} = (1-\bar{\varphi})\tilde{\lambda}_{ss}.$$

S.2 Bayesian Estimation Procedure

This appendix outlines the Bayesian estimation procedure that we use.

S.2.1 State Space

For likelihood estimation, the state space representation is

$$\mathcal{X}_t = \mathcal{H}_{s_t}\left(\mathcal{X}_{t-1}, \varepsilon_t\right) \tag{S1}$$

$$\mathcal{Y}_t = \mathcal{G}_{s_t}\left(\mathcal{X}_t, \mathcal{U}_t\right), \tag{S2}$$

where $\mathcal{X}_t$ denotes the state, $\mathcal{Y}_t$ denotes the observation, ε_t denotes the structural shocks, and $\mathcal{U}_t$ denotes the observation errors.

Recall that the second-order approximation takes the form

$$\mathbf{x}_t \approx \mathbf{x}_{ss} + H_{s_t}^{(1)} S_t + \frac{1}{2} H_{s_t}^{(2)} \left(S_t \otimes S_t\right) \tag{S3}$$

$$\mathbf{y}_t \approx \mathbf{y}_{ss} + G_{s_t}^{(1)} S_t + \frac{1}{2} G_{s_t}^{(2)} \left(S_t \otimes S_t\right), \tag{S4}$$

where $S_t = \left[\begin{array}{ccc} (\mathbf{x}_{t-1} - \mathbf{x}_{ss})' & \varepsilon_t' & 1 \end{array}\right]'$. Therefore, we can define the state variables as

$$\mathcal{X}_t = \left[\begin{array}{ccccc} \mathbf{x}_t' & \mathbf{x}_{t-1}' & \mathbf{y}_t' & \mathbf{y}_{t-1}' & \epsilon_t \end{array}\right]'. \tag{S5}$$

The nonlinear transition equations,

$$\mathcal{X}_t = \mathcal{H}_{s_t}\left(\mathcal{X}_{t-1}, \varepsilon_t\right), \tag{S6}$$

can be represented as

$$\left[\begin{array}{c} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \varepsilon_t \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_{ss} + H_{s_t}^{(1)} S_t + \frac{1}{2} H_{s_t}^{(2)} \left(S_t \otimes S_t\right) \\ \mathbf{x}_{t-1} \\ \mathbf{y}_{ss} + G_{s_t}^{(1)} S_t + \frac{1}{2} G_{s_t}^{(2)} \left(S_t \otimes S_t\right) \\ \mathbf{y}_{t-1} \\ \varepsilon_t \end{array}\right]. \tag{S7}$$

The observables are $\Delta \log Y_t$, $\Delta \log C_t$, $\Delta \log I_t$, r_t^{obs}, Φ_c^{ca}, $\Delta \log P_t$. Matching these variables with the model's counterparts, we have:

$$\Delta \log Y_t = \Delta \log \left(\tilde{Y}_t Z_{t-1}\right) = \Delta \log \tilde{Y}_t + \Delta \log Z_{t-1} = \log \tilde{Y}_t - \log \tilde{Y}_{t-1} + \log \tilde{Z}_{t-1}. \tag{S8}$$

Similarly, consumption growth is:

$$\Delta \log C_t = \log \tilde{C}_t - \log \tilde{C}_{t-1} + \log \tilde{Z}_{t-1}, \tag{S9}$$

while investment growth is

$$\Delta \log I_t = \log \bar{I}_t - \log \bar{I}_{t-1} + \log \tilde{Z}_{t-1}. \tag{S10}$$

For the real interest rate, we observe r_t plus the external financing premium, so we have:

$$r_t^{obs} = r_t + EFPD_t. \tag{S11}$$

Last, Φ_t^{ca} is a model-defined variable, and

$$\Delta \log P_t = \log P_t - \log P_{t-1}. \tag{S12}$$

Therefore, the observation equation

$$\mathcal{Y}_t = \mathcal{G}_{s_t}(\mathcal{X}_t, \mathcal{U}_t) \tag{S13}$$

is given by

$$\begin{bmatrix} \Delta \log Y_t \\ \Delta \log C_t \\ \Delta \log I_t \\ r_t \\ \Delta B_t / Y_t \\ \Delta \log P_t \end{bmatrix} = D \begin{bmatrix} \mathbf{x}_t \\ \mathbf{x}_{t-1} \\ \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \varepsilon_t \end{bmatrix} + \mathcal{U}_t \tag{S14}$$

where D denotes a selection matrix.

S.2.2 Filtering

To filter the likelihood, we use the Unscented Kalman Filter (UKF) (Julier and Uhlmann, 1999). The UKF calculates the state mean and covariance by propagating deterministically chosen sigma-points through the nonlinear functions. The transformed points are then used to calculate the mean and covariance matrix. As Julier and Uhlmann (1999) note, the critical assumption to apply the UKF is that the prediction density and the filtering density are both Gaussian. The filtering and smoothing procedure largely follow Binning and Maih (2015), so here we just outline the procedure.

The filter starts by combining the state vector and exogenous disturbances into a single vector, $\mathcal{X}^a_{t-1} = [\mathcal{X}_{t-1}, \epsilon_t]'$, with the following mean and covariance matrix conditional on $Y_{1:t-1}$ and regime s_{t-1}:

$$\mathcal{X}^a_{t-1}(s_{t-1}) = \begin{bmatrix} \mathcal{X}_{t-1|t-1}(s_{t-1}) \\ 0_\epsilon \end{bmatrix} \tag{S15}$$

$$P^a_{t-1}(s_{t-1}) = \begin{bmatrix} P^x_{t-1|t-1}(s_{t-1}) & 0 \\ 0 & I \end{bmatrix}. \tag{S16}$$

The sigma-points $\mathcal{X}^a_{i,t-1}(s_{t-1})$ that consist of the sigma-points for state variables $\mathcal{X}^x_{i,t-1}(s_{t-1})$ and the sigma-points for exogenous shocks $\mathcal{X}^\varepsilon_{i,t-1}(s_{t-1})$ are chosen as follows:

$$\mathcal{X}^a_{0,t-1}(s_{t-1}) = \mathcal{X}^a_{t-1}(s_{t-1}) \tag{S17}$$

$$\mathcal{X}^a_{0,t-1}(s_{t-1}) = \mathcal{X}^a_{t-1}(s_{t-1}) \tag{S18}$$

$$\mathcal{X}^a_{i,t-1}(s_{t-1}) = \mathcal{X}^a_{t-1}(s_{t-1}) + (h\sqrt{P^a_{t-1}(s_{t-1})})_i \text{ for } i = 1 \dots L \tag{S19}$$

$$\mathcal{X}^a_{i,t-1}(s_{t-1}) = \mathcal{X}^a_{t-1}(s_{t-1}) - (h\sqrt{P^a_{t-1}(s_{t-1})})_{i-L} \text{ for } i = L+1 \dots 2L, \tag{S20}$$

where $h = \sqrt{3}$ and L denotes the number of state variables and exogenous shocks. The weights for the sigma-points are given by:

$$w_0 = \frac{h - L}{2h} \tag{S21}$$

$$w_i = \frac{1}{2h} \text{ for } i = 1 \dots 2L \tag{S22}$$

The sigma-points and the assigned weights are then used to calculate the expected mean and covariance by propagating sigma-points through transition equations and taking a weighted average:

$$\mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) = H_{s_t}(\mathcal{X}^x_{i,t-1}(s_{t-1}), \mathcal{X}^\varepsilon_{i,t-1}(s_{t-1})) \tag{S23}$$

$$\mathcal{X}_{t|t-1}(s_{t-1}, s_t) = \sum_{i=0}^{2L} w_i \mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) \tag{S24}$$

$$P^x_{t|t-1}(s_{t-1}, s_t) = \sum_{i=0}^{2L} w_i \tilde{\mathcal{X}}_i \tilde{\mathcal{X}}_i^T \tag{S25}$$

$$\mathcal{Y}_{t|t-1}(s_{t-1}, s_t) = D\mathcal{X}_{t|t-1}(s_{t-1}, s_t) \tag{S26}$$

where $\tilde{\mathcal{X}}_i = \mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) - \mathcal{X}_{t|t-1}(s_{t-1}, s_t)$. From these conditions, we obtain the Gaussian approximation predictive density $p(\mathcal{X}_t|\mathcal{Y}_{1:t-1}, s_{t-1}, s_t) = N(\mathcal{X}_{t|t-1}(s_{t-1}, s_t), P^x_{t|t-1}(s_{t-1}, s_t))$. The predictive density is then updated using the Kalman filter as follows:

$$P^y_{t|t-1}(s_{t-1}, s_t) = DP^x_{t|t-1}(s_{t-1}, s_t)D^T + R \tag{S27}$$

$$P^{xy}_{t|t-1}(s_{t-1}, s_t) = P^x_{t|t-1}(s_{t-1}, s_t)D^T \tag{S28}$$

$$K_t(s_{t-1}, s_t) = P^{xy}_{t|t-1}(s_{t-1}, s_t)(P^y_{t|t-1}(s_{t-1}, s_t))^{-1} \tag{S29}$$

$$\mathcal{X}_{t|t}(s_{t-1}, s_t) = \mathcal{X}_{t|t-1}(s_{t-1}, s_t) + K_t(s_{t-1}, s_t)(\mathcal{Y}_t - \mathcal{Y}_{t|t-1}(s_{t-1}, s_t)) \tag{S30}$$

$$P^x_{t|t}(s_{t-1}, s_t) = P^x_{t|t-1}(s_{t-1}, s_t) - K_t(s_{t-1}, s_t)P^y_{t|t-1}(s_{t-1}, s_t)K^T_t(s_{t-1}, s_t). \tag{S31}$$

This updating step gives $p(\mathcal{X}_t|\mathcal{Y}_{1:t}, s_{t-1}, s_t) = N(\mathcal{X}_{t|t}(s_{t-1}, s_t), P^x_{t|t}(s_{t-1}, s_t))$. As a by-product of the filter, we also obtain the density of $\mathcal{Y}_t$ conditional on $\mathcal{Y}_{1:t-1}$, s_t, and s_{t-1}

$$p(\mathcal{Y}_t|\mathcal{Y}_{1:t-1}, s_{t-1}, s_t; \theta) = N(\mathcal{Y}_{t|t-1}(s_{t-1}, s_t), P^y_{t|t-1}(s_{t-1}, s_t)). \tag{S32}$$

Since the UKF with regime switching creates a large number of nodes at each iteration where the filtered mean and covariance matrix need to be evaluated, we implement the following collapsing procedure suggested by Kim and Nelson (1999):

$$\mathcal{X}_{t|t}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:t})}\Big\{\sum_{i=1}^{M} \Pr(s_{t-1} = i, s_t = j|\mathcal{Y}_{1:t})\mathcal{X}_{t|t}(s_{t-1} = i, s_t = j)\Big\} \tag{S33}$$

$$\begin{aligned} P^x_{t|t}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:t})}\Big\{\sum_{i=1}^{M} \Pr(s_{t-1} = i, s_t = j|\mathcal{Y}_{1:t})[P^x_{t|t}(s_{t-1} = i, s_t = j) \\ +(\mathcal{X}_{t|t}(s_t = j) - \mathcal{X}_{t|t}(s_{t-1} = i, s_t = j))(\mathcal{X}_{t|t}(s_t = j) - \mathcal{X}_{t|t}(s_{t-1} = i, s_t = j))^T]\Big\}, \end{aligned} \tag{S34}$$

where $\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t})$ and $\Pr(s_t|\mathcal{Y}_{1:t})$ are obtained from the following Hamilton filter

$$\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1}) = \Pr(s_t|s_{t-1}) \Pr(s_{t-1}|\mathcal{Y}_{1:t-1}) \tag{S35}$$

$$\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t}) = \frac{p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1}) \Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1})}{\sum_{s_t}\sum_{s_{t-1}} p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1}) \Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1})} \tag{S36}$$

$$\Pr(s_t|\mathcal{Y}_{1:t}) = \sum_{s_{t-1}} \Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t}). \tag{S37}$$

The resulting conditional marginal likelihood is

$$p(\mathcal{Y}_t|\mathcal{Y}_{1:t-1};\theta) = \sum_{s_t}\sum_{s_{t-1}} p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1}) \Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1}). \tag{S38}$$

S.2.3 Smoothing

Once we evaluated the likelihood of the data and performed the filtering using the UKF for $t = 1, \ldots, T$, we can also obtain $\Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T})$, $\Pr(s_t|\mathcal{Y}_{1:T})$, $x_{t|T}(s_t, s_T)$, and $P^x_{t|T}(s_t, s_T)$ as follows:

$$\begin{aligned}
\Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T}) &= \frac{\Pr(s_{t+1}|\mathcal{Y}_{1:T}) \Pr(s_t|\mathcal{Y}_{1:t}) \Pr(s_{t+1}|s_t)}{\Pr(s_{t+1}|\mathcal{Y}_{1:t})} && \text{(S39)}\\
\Pr(s_t|\mathcal{Y}_{1:T}) &= \sum_{s_{t+1}} \Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T}) && \text{(S40)}\\
\mathcal{X}_{t|T}(s_t, s_{t+1}) &= \mathcal{X}_{t|t}(s_t) + \tilde{K}_t(s_t, s_{t+1})(\mathcal{X}_{t+1|T}(s_{t+1}) - \mathcal{X}_{t+1|t}(s_t, s_{t+1})) && \text{(S41)}\\
P^x_{t|T}(s_t, s_{t+1}) &= P^x_{t|t}(s_t) - \tilde{K}_t(s_t, s_{t+1})(P^x_{t+1|T}(s_{t+1}) - P^x_{t+1|T}(s_t, s_{t+1}))\tilde{K}_t(s_t, s_{t+1})^T.
\end{aligned}$$

Given the above smoothing algorithm, we implement another collapsing procedure similar to that in the filtering step:

$$\mathcal{X}_{t|T}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:T})}\left\{\sum_{j=1}^{M} \Pr(s_t = i, s_{t+1} = j|\mathcal{Y}_{1:T})\mathcal{X}_{t|T}(s_t = i, s_{t+1} = j)\right\}, \tag{S42}$$

$$\begin{aligned}
P^x_{t|T}(s_t &= j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:T})}\left\{\sum_{j=1}^{M} \Pr(s_t = i, s_{t+1} = j|\mathcal{Y}_{1:T})[P^x_{t|T}(s_t = i, s_{t+1} = j)\right.\\
+(\mathcal{X}_{t|T}(s_t &= j) - \mathcal{X}_{t|T}(s_t = i, s_{t+1} = j))(\mathcal{X}_{t|T}(s_t = j) - \mathcal{X}_{t|T}(s_t = i, s_{t+1} = j))^T]\Big\}. && \text{(S43)}
\end{aligned}$$

S.3 Calibrated Parameters

To calibrate the parameters that we do not estimate, we largely follow Mendoza (2010), targeting the same moments, but adapting the computations to our model specification.

We start by using the steady state of the model in which there is no working capital constraint and the borrowing constraint does not bind. That is, $\phi = 0$ and $\bar{\varphi} = 0$. The latter implies $\lambda_{ss} = 0$. We follow Mendoza and set $\rho = 2$. From the data, we pin down $\bar{Z} = 1.006$. Next, we set

$$\beta (1 + r_{ss}) \bar{Z}^{-\rho} = 1,$$

and define the following ratios

$$\Omega_v = \frac{1}{1 - \alpha - \eta}, \Omega_h = \frac{1}{\alpha}, \Omega_k = \frac{1}{\eta} \left(\frac{1}{\beta \bar{Z}^{-\rho}} - 1 + \delta \right).$$

Therefore, the factor payment ratios are

$$\frac{\tilde{V}_{ss} P_{ss}}{Y_{ss} + P_{ss} \tilde{V}_{ss}} = \frac{1}{\Omega_v} = 1 - \alpha - \eta$$

$$\frac{\tilde{W}_{ss} H_{ss}}{\tilde{Y}_{ss}} = \frac{1}{\Omega_h \left(1 - \frac{1}{\Omega_v} \right)} = \frac{\alpha}{\alpha + \eta}$$

$$\frac{\left(\frac{1}{\beta \tilde{Z}_{ss}^{-\rho}} - 1 + \delta \right) \tilde{K}_{ss}}{\tilde{Y}_{ss}} = \frac{\eta}{\alpha + \eta},$$

and from Mendoza's calibration, we obtain

$$\begin{bmatrix} 1 - \alpha - \eta = 0.102 \\ \frac{\alpha}{\alpha + \eta} = 0.66 \end{bmatrix} . \Longrightarrow \begin{bmatrix} \alpha = 0.59268 \\ \eta = 0.30532 \end{bmatrix}.$$

Setting the depreciation rate at Mendoza's annual value of 8.8 percent, we also obtain

$$(1 - \delta)^4 = 1 - 0.088 \Longrightarrow \delta = 0.022766.$$

Mendoza's annual capital-to-gross output ratio of 1.758. So, in our quarterly model, this ratio, denoted Ω_k^{-1}, is given by

$$\Omega_k^{-1} = \left(\frac{1}{\eta} \left(\frac{1}{\beta \bar{Z}^{-\rho}} - 1 + \delta \right) \right)^{-1} = 4 * 1.758 \Longrightarrow \beta = 0.99156,$$

implying the following annualized real interest rate of

$$(1 + r_{ss})^4 = \left(\frac{1}{\beta} \bar{Z}^{\rho} \right)^4 = 1.0852,$$

which matches nearly exactly Mendoza's number.

All other calibrated parameters are taken directly from Mendoza, with the caveat that we divide the mean technology A^*, reported in Mendoza and Villalvazo (2020) by four to account for the quarterly frequency of the model.

S.4 Data Appendix

National accounts are from the National Statistic Office. The data series used in the analysis merge two sets of official statistics by updating the level of the national accounts based on 1993 constant prices with the quarterly rate of growth of the accounts based on 2008 constant prices. The merging is necessary as the deflators to splice the accounts in levels were not available at the time of the last download of the data (May 2017). The two sets of national accounts overlap from 1993:Q1 to 2006:Q4. Over this period, the difference in annual rate of growth is less than 0.01 percent in absolute value for GDP, less than 0.05 percent for consumption, less than 2 percent for investment, and less than 1 and 3 percent for imports and exports, respectively. The correlations between the series are more than 0.9 for all series except investment, which is 0.84, pointing to possibly larger measurement errors in this variable. The differences are smaller, the closer to the end of the sample period. For this reason, we choose to update the 1993 accounts rather than backdate the 2008 ones.

The specific sources of the data are as follows:

- 1980:Q1-2006:Q4 (Labeled 1993 accounts)—Supply and demand of goods and services. Original Series (not seasonally adjusted). Constant prices, annual 1993 = 100. We obtained these from the Central Bank of Mexico (Gabriel, 2008).
- 2006:Q1-2016:Q4 (Labeled 2008 accounts)—Supply and demand of goods and services. Original series (not seasonally adjusted). Constant prices, annual 2008 = 100 (Oferta y demanda de bienes y servicios. Series originales. A precios constantes 2008). Available from `http://www3.inegi.org.mx/sistemas/tabuladosbasicos/tabdirecto.aspx`.

The data are not seasonally adjusted and show a strong seasonal pattern. To seasonally adjust all series (assumed to be I(1) processes), we adjust the log-difference using the X-12 procedure with the additive option in Eviews. We then use the log of the first observation of the raw series (not seasonally adjusted) and cumulate the seasonally adjusted log-difference. The net exports to GDP series is calculated as real exports minus real imports divided by real GDP.

The current account as a percentage of GDP is from the balance of payment statistics, obtained from the OECD Economic Outlook Database (Series MEX.CBGDPR.Q, OECD-EO-MEX-CBGDPR-Q).

As a proxy for the relative price of intermediate goods, entered as observable in estimation, we use a measure of Mexico's terms of trade obtained from Banco de México (PPI Producer and International Trade Price Indexes, series SP12753).

Mexico's country interest rate is calculated following Uribe and Yue (2006) as

$$r_t = r_t^* + spread_t \tag{S44}$$

where r^* is the US real interest rate, and *spread* is a proxy for Mexico's country risk or sovereign spread. We compute r^* as 3-month Treasury Constant Maturity Rate adjusted for ex post CPI (annualized) quarterly inflation, using period average data. The source of these data is FRED. For the country spread, as customary, we use the Mexico's component of the JP Morgan EMBI.

Unfortunately, the EMBI spread is available only starting from 1993. In order to estimate the country spread before 1993, we rely on empirical modeling of the relation between the *domestic* real interest rates and country risk at the Banco de Mexico (Aportela Rodriguez et al., 2001) that estimates a close and stable relation between a measure of the domestic real interest rate and the EMBI spread over the period over which both these variables are available. The only quarterly interest series available we are aware off going back to 1980 is a three-month nominal short-term rate obtained from Banco de Mexico (Average monthly yield on 90-days Cetes, series SF3338).[S1] So we estimate a relationship between this nominal interest rate, i_t, and the EMBI during the period over which the EMBI is observable, adjusting for inflation, π_t, which was an important source of nominal interest rate variation in the 1980s, and then invert it. Specifically, we posit the following simplified version of the model that (Aportela Rodriguez et al., 2001) estimate:

$$i_t = \alpha_0 + \alpha_1 \pi_t + \alpha_2 EMBI_t. \tag{S45}$$

We then solve the fitted equation for the country risk component of the domestic real interest rate, which we denote as $E\hat{M}BI_t$. The estimated regression is (t-statistics in paren-

[S1] There are three missing monthly observations in this series: August and September 1986 and November 1988. We fill these gaps using July 1986 for 1986Q3 and the average of October and December 1988 for 1988:Q4.

Table S.1: Parameter Estimates for Alternative Models

Par.	Description	Prior	Endogenous SV	Endogenous No SV	Exogenous SV	Exogenous No SV	No Constraint SV	No Constraint No SV
ι	Capital Adj.	N(10,5)	5.7692	10.5143	5.3602	9.6687	5.7838	11.4643
ϕ	Working Cap.	U(0,1)	0.7689	0.7836	0.7739	0.6951	0.8519	0.7076
$\bar{r}$	Int Rate Mean	N(0.0177,0.005)	0.0062	0.0143	0.0083	0.0071	0.0068	0.0119
κ^*	Leverage	U(0,1)	0.1818	0.1871	0.2024	0.1679	–	–
ψ_r	Elasticity	U(-10,10)	–	–	–	–	0.0081	0.0072
ρ_a	TFP	B(0.6,0.2)	0.9816	0.9265	0.9816	0.9227	0.9771	0.8777
ρ_z	TFP Growth	B(0.6,0.2)	0.8112	0.8633	0.8253	0.8968	0.8477	0.8571
ρ_p	Imp Price	B(0.6,0.2)	0.9780	0.8943	0.9782	0.9115	0.9772	0.9007
ρ_r	Int Rate	B(0.6,0.2)	0.9562	0.8926	0.9587	0.8807	0.9795	0.8985
ρ_e	Expend	B(0.6,0.2)	0.8786	0.8810	0.8629	0.7774	0.8574	0.9442
ρ_d	Pref	B(0.6,0.2)	0.8824	0.9135	0.8828	0.9018	0.8729	0.9042
$\sigma_a(L)$	TFP	IG(0.005,0.01)	0.0046	0.0124	0.0066	0.0108	0.0052	0.0220
$\sigma_a(H)$		IG(0.005,0.01)	0.0120	–	0.0085	–	0.0116	–
$\sigma_z(L)$	TFP Growth	IG(0.005,0.01)	0.0025	0.0047	0.0018	0.011	0.002	0.0106
$\sigma_z(H)$		IG(0.005,0.01)	0.0101	–	0.0124	–	0.0195	–
$\sigma_p(L)$	Imp Price	IG(0.05,0.01)	0.0270	0.0462	0.0294	0.0467	0.0336	0.0501
$\sigma_p(H)$		IG(0.05,0.01)	0.0630	–	0.063	–	0.0633	–
$\sigma_r(L)$	Int Rate	IG(0.01,0.025)	0.0019	0.0040	0.0017	0.0040	0.0024	0.0080
$\sigma_r(H)$		IG(0.01,0.025)	0.0069	–	0.0063	–	0.0588	–
$\sigma_e(L)$	Expend	IG(0.5,0.5)	0.1599	0.2957	0.1452	0.2775	0.1472	0.3168
$\sigma_e(H)$		IG(0.5,0.5)	0.3843	–	0.3914	–	0.1622	–
$\sigma_d(L)$	Pref	IG(0.05,0.01)	0.0484	0.0360	0.0469	–	0.0574	0.0561
$\sigma_d(H)$		IG(0.05,0.01)	0.0601	–	0.0791	0.0508	0.0841	–
$P_{0,1}$	Enter Binding	U(0,1)	–	–	0.0635	0.0380	–	–
$P_{1,0}$	Exit Binding	U(0,1)	–	–	0.4370	0.5089	–	–
$\log \gamma_{0,1}$	Enter Binding	U(-20,20)	2.0650	2.3015	–	–	–	–
$\log \gamma_{1,1}$	Exit Binding	U(-20,20)	4.9252	2.9832	–	–	–	–
$P\sigma_l$	Stay Low Vol	B(0.975,0.025)	0.9582	–	0.9579	–	0.9598	–
$P\sigma_h$	Stay High Vol	B(0.975,0.025)	0.9492	–	0.9372	–	0.9387	–

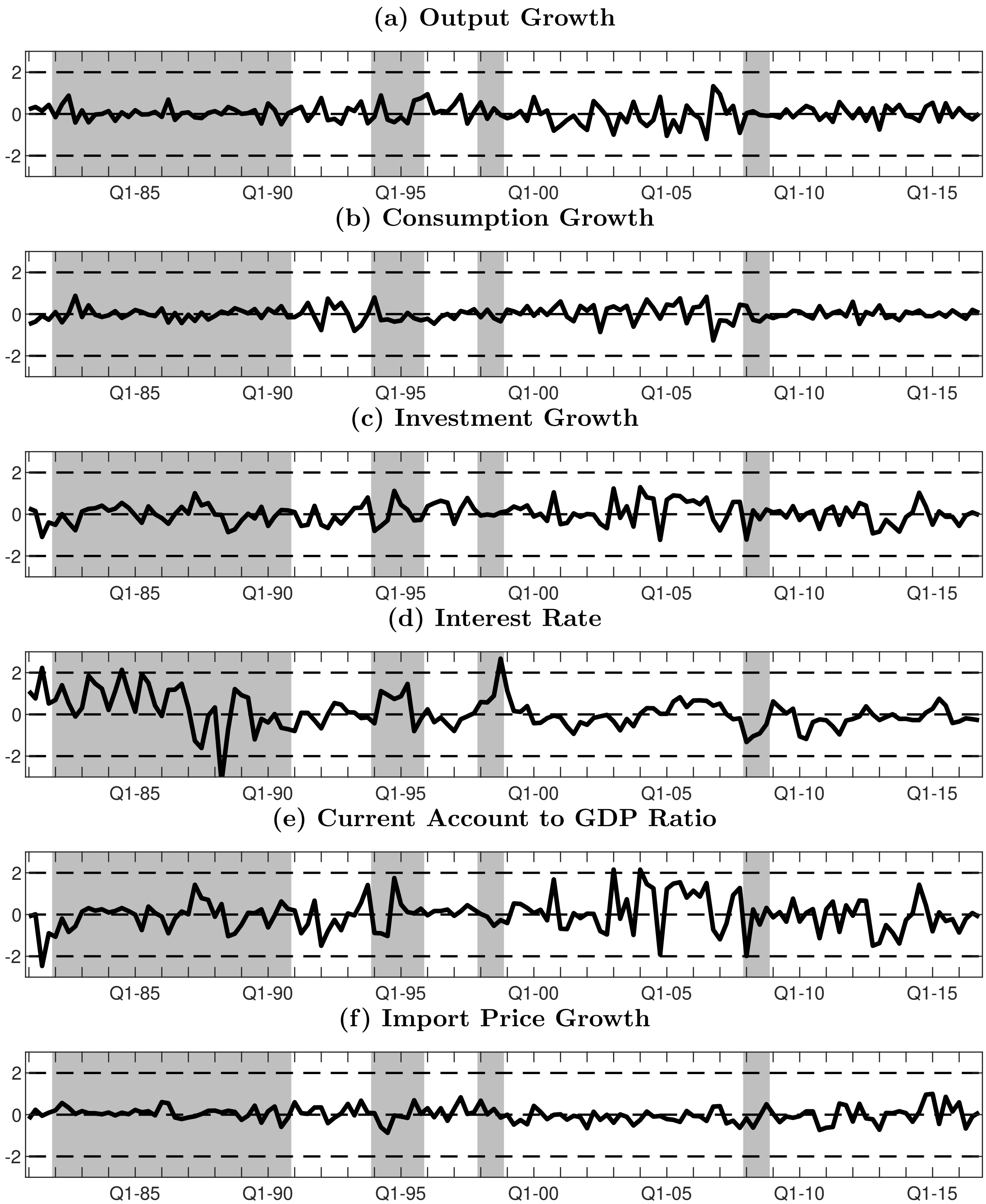
theses and $R^2 = 0.883$):

$$\hat{i_t} = \underset{(-0.42)}{-0.00346} + \underset{(4.46)}{0.397}\pi_t + \underset{(7.37)}{2.770}E\hat{MB}I_t. \tag{S46}$$

S.5 Additional Estimation Results

In this appendix, we report additional empirical results, including on measurement errors and estimated parameters under alternative model specifications. The appendix also gives details on how we estimate the model under the alternative specifications considered.

Figure S.1: Measurement Errors



Note: The figure plots measurement errors between the model estimates and the data show in Figure 3. Light gray areas indicate periods of currency or external debt crisis as identified in Reinhart and Rogoff (2009).

S.5.1 Measurement Errors

Figure S.1 plots the implied measurement error from Figure 3, in standard deviation units. It shows that measurement errors for output, consumption, investment, and import price growth are all relatively small. The interest rate and the current account-to-GDP ratio have larger errors, but these are generally within two standard deviations.

S.5.2 Alternative Model Estimates

We compare our baseline model estimates to a version with exogenous regime switching and one without the constraint. All versions are estimated with and without stochastic volatility, and have identical priors for parameters that are common across models. Table S.1 shows the posterior modes. For the exogenous switching model, we assume the transitions between the non-binding and binding regimes are not logistic functions but constant probabilities:

$$\mathbb{P}^c = \begin{bmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{bmatrix} = \begin{bmatrix} 1 - P_{0,1} & P_{0,1} \\ P_{1,0} & 1 - P_{1,0} \end{bmatrix} \tag{S47}$$

where $P_{i,j} = \Pr\left(s_{t+1} = j | s_t = i\right)$, which are estimated. For the model with no constraint, we eliminate the regime with a binding constraint. To keep the model stationary, we assume a debt-elastic interest rate rule

$$r_t = r_t^* + \psi\left(\exp\left(\bar{B} - B_t\right) - 1\right), \tag{S48}$$

where r_t^* follows the exogenous process above and the ψ is estimated.

S.6 Second Order Derivations

Second order derivatives are 3-dimensional arrays. In particular, $F_{s,xx}\left(x,0,0\right)$ is $n \times n_x \times n_x$, $F_{s,x\varepsilon}\left(x,0,0\right)$ is $n \times n_x \times n_\varepsilon$, $F_{s,x\chi}\left(x,0,0\right)$ is $n \times n_x \times 1$, $F_{s,\varepsilon\varepsilon}\left(x,0,0\right)$ is $n \times n_\varepsilon \times n_\varepsilon$, $F_{s,\varepsilon\chi}\left(x,0,0\right)$ is $n \times n_\varepsilon \times 1$, and $F_{s,\chi\chi}\left(x,0,0\right)$ is $n \times 1 \times 1$. In addition, $f_{y_{t+1}y_{t+1}}$ is $n \times n_y \times n_y$, $f_{y_{t+1}x_t}$ is $n \times n_y \times n_x$, etc. Denote $\left[f_{y_{t+1}y_{t+1}}\right]_{jk}^{i}$ as the element (i,j,k) of $f_{y_{t+1}y_{t+1}}$, meaning $\left[f_{y_{t+1}y_{t+1}}\right]_{jk}^{i} = f_{y_{j,t+1}y_{k,t+1}}^{(i)}$. In this subsection, the derivatives will use the fact that $f = 0$ from the outset for notational simplicity.

Derivative for x, x

The derivative with respect to (x_{t-1}, x_{t-1}) is a $n \times n_x \times n_x$ array. The (a,b,c)-element is given by

$$
\begin{aligned}
&[F_{s,xx}]^a_{b,c} = \\
&\sum_{s'} P_{s,s'} \left(\begin{array}{c}
\sum_i f^a_{y_{i,t}} g^i_{s,x_b x_c} + \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_j x_i} h^j_{s,x_b} h^i_{s,x_c} \\
+ \sum_i f^a_{x_{i,t}} h^i_{s,x_b x_c} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,x_b x_c} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{k,t+1} y_{j,t+1}} \sum_i g^j_{s' x_i} h^i_{s,x_c} + \sum_j f^a_{y_{k,t+1} y_{j,t}} g^j_{s,x_c} \\ + \sum_j f^a_{y_{k,t+1}, x_{j,t}} h^j_{s,x_c} + f^a_{y_{k,t+1} x_{c,t-1}} \end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,x_b} \right) \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} y_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,x_c} + \sum_j f^a_{y_{k,t} y_{j,t}} g^j_{s,x_c} \\ + \sum_j f^a_{y_{k,t} x_{j,t}} h^j_{s,x_c} + f^a_{y_{k,t} x_{c,t-1}} \end{array} \right) g^k_{s,x_b} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} x_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,x_c} + \sum_j f^a_{y_{j,t} x_{k,t}} g^j_{s,x_c} \\ + \sum_j f^a_{x_{j,t} x_{k,t}} h^j_{s,x_c} + f^a_{x_{k,t} x_{c,t-1}} \end{array} \right) h^k_{s,x_b} \\
+ \sum_j f^a_{y_{j,t+1} x_{b,t-1}} \sum_i g^j_{s',x_i} h^i_{s,x_c} + \sum_i f^a_{y_{i,t} x_{b,t-1}} g^i_{s,x_c} \\
+ \sum_i f^a_{x_{i,t} x_{b,t-1}} h^i_{s,x_c} + f^a_{x_{b,t-1} x_{c,t-1}}
\end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,x_b} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,x_c} + \sum_j f_{y_{j,t}} g^j_{s,x_c} + \sum_j f_{x_{j,t}} h^j_{s,x_c} + f_{x_{c,t-1}} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,x_c} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,x_b} + \sum_j f_{y_{j,t}} g^j_{s,x_b} + \sum_j f_{x_{j,t}} h^j_{s,x_b} + f_{x_{b,t-1}} \right)
\end{aligned}
$$

Note that the last two lines show dependence on the endogenous probabilities. This is a linear system that can be expressed as

$$
\sum_{s'} \underbrace{\left[\begin{array}{ccc} m^{h_s}_{a,b,c} & m^{g_s}_{a,b,c} & m^{g_{s'}}_{a,b,c} \end{array} \right]}_{m_{a,b,c}} \left[\begin{array}{c} vec\left(h_{s,xx}\right) \\ vec\left(g_{s,xx}\right) \\ vec\left(g_{s',xx}\right) \end{array} \right] + \sum_{s'} n_{a,b,c} = 0
$$

where $vec\left(h_{s,xx}\right) = \left[\begin{array}{ccc} vec\left(h^i_{a,1}\right)' & \cdots & vec\left(h^i_{a,n_x}\right)' \end{array} \right]'$, etc. Stacking these equations for $a =$

$1, \ldots, n$, and $b, c = 1, \ldots, n_x$:

$$\sum_{s'} \underbrace{\begin{bmatrix} m^{h_s}_{1,1,1} & m^{g_s}_{1,1,1} & m^{g_{s'}}_{1,1,1} \\ m^{h_s}_{1,1,2} & m^{g_s}_{1,1,2} & m^{g_{s'}}_{1,1,2} \\ \vdots & \vdots & \vdots \\ m^{h_s}_{1,1,n_x} & m^{g_s}_{1,1,n_x} & m^{g_{s'}}_{1,1,n_x} \\ m^{h_s}_{1,2,1} & m^{g_s}_{1,2,1} & m^{g_{s'}}_{1,2,1} \\ \vdots & \vdots & \vdots \\ m^{h_s}_{1,n_x,n_x} & m^{g_s}_{1,n_x,n_x} & m^{g_{s'}}_{1,n_x,n_x} \\ m^{h_s}_{2,1,1} & m^{g_s}_{2,1,1} & m^{g_{s'}}_{2,1,1} \\ \vdots & \vdots & \vdots \\ m^{h_s}_{n,n_x,n_x} & m^{g_s}_{n,n_x,n_x} & m^{g_{s'}}_{n,n_x,n_x} \end{bmatrix}}_{m_{xx}(s,s') = \left[m^h_{xx}(s,s') \; m^g_{xx}(s,s') \; m^{g'}_{xx}(s,s') \right]} \begin{bmatrix} vec\left(h_{s,xx}\right) \\ vec\left(g_{s,xx}\right) \\ vec\left(g_{s',xx}\right) \end{bmatrix} + \sum_{s'} \underbrace{\begin{bmatrix} n_{1,1,1} \\ n_{1,1,2} \\ \vdots \\ n_{1,1,n_x} \\ n_{1,2,1} \\ \vdots \\ n_{1,n_x,n_x} \\ n_{2,1,1} \\ \vdots \\ n_{n,n_x,n_x} \end{bmatrix}}_{n_{xx}(s,s')} = 0.$$

Then, we can stack these for s:

$$\underbrace{\begin{bmatrix} \sum_{s'} m^h_{xx}\left(1, s'\right) & \sum_{s'} m^g_{xx}\left(1, s'\right) & \cdots & 0 & m^{g'}_{xx}\left(1, n_s\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m^{g'}_{xx}\left(n_s, 1\right) & \cdots & \sum_{s'} m^h_{xx}\left(n_s, s'\right) & \sum_{s'} m^g_{xx}\left(n_s, s'\right) \end{bmatrix}}_{M_{xx}} \begin{bmatrix} vec\left(h_{1,xx}\right) \\ vec\left(g_{1,xx}\right) \\ \vdots \\ vec\left(h_{n_s,xx}\right) \\ vec\left(g_{n_s,xx}\right) \end{bmatrix} + \underbrace{\begin{bmatrix} \sum_{s'} n_{xx}\left(1, s'\right) \\ \vdots \\ \sum_{s'} n_{xx}\left(n_s, s'\right) \end{bmatrix}}_{N_{xx}} = 0$$

which is then solved by matrix inversion.

Derivative for x, ε

The derivative with respect to (x_{t-1}, ε_t) is a $n \times n_x \times n_\varepsilon$ array. The (a, b, c)-element is given by

$$
\begin{aligned}
&[F_{s,x\varepsilon}]^a_{bc} = \\
&\sum_{s'} P_{s,s'} \left(\begin{array}{c}
\sum_i f^a_{y_{i,t}} g^i_{s,x_b x_c} + \sum_i f^a_{x_{i,t}} h^i_{s,x_b \varepsilon_c} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,x_b \varepsilon_c} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{k,t+1} y_{j,t+1}} \sum_i g^j_{s' x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{k,t+1} y_{j,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{y_{k,t+1}, x_{j,t}} h^j_{s,\varepsilon_c} + f^a_{y_{k,t+1} \varepsilon_{c,t}} \end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,x_b} \right) \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} y_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{k,t} y_{j,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{y_{k,t} x_{j,t}} h^j_{s,\varepsilon_c} + f^a_{y_{k,t} \varepsilon_{c,t}} \end{array} \right) g^k_{s,x_b} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} x_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{j,t} x_{k,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{x_{j,t} x_{k,t}} h^j_{s,\varepsilon_c} + f^a_{x_{k,t} \varepsilon_{c,t}} \end{array} \right) h^k_{s,x_b} \\
+ \sum_j f^a_{y_{j,t+1} x_{b,t-1}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_i f^a_{y_{i,t} x_{b,t-1}} g^i_{s,\varepsilon_c} \\
+ \sum_i f^a_{x_{i,t} x_{b,t-1}} h^i_{s,\varepsilon_c} + f^a_{x_{b,t-1} \varepsilon_{c,t}} \\
+ \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_j x_i} h^j_{s,x_b} h^i_{s,\varepsilon_c}
\end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,x_b} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f_{y_{j,t}} g^j_{s,\varepsilon_c} + \sum_j f_{x_{j,t}} h^j_{s,\varepsilon_c} + f_{\varepsilon_{c,t}} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\varepsilon_c} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,x_b} + \sum_j f_{y_{j,t}} g^j_{s,x_b} + \sum_j f_{x_{j,t}} h^j_{s,x_b} + f_{x_{b,t-1}} \right)
\end{aligned}
$$

Note that the last two lines show dependence on the endogenous probabilities. This is a linear system that can be expressed as

$$
\sum_{s'} \underbrace{\left[\begin{array}{cc} m^{h_s}_{a,b,c} & m^{g_s}_{a,b,c} \end{array} \right]}_{m_{a,b,c}} \left[\begin{array}{c} vec\left(h_{s,x\varepsilon}\right) \\ vec\left(g_{s,x\varepsilon}\right) \end{array} \right] + \sum_{s'} n_{a,b,c} = 0
$$

Stacking these equations for $a = 1, \ldots, n$, $b = 1, \ldots, n_x$, and $c = 1, \ldots, n_\varepsilon$:

$$\sum_{s'} \underbrace{\begin{bmatrix} m^{h_s}_{1,1,1} & m^{g_s}_{1,1,1} \\ m^{h_s}_{1,1,2} & m^{g_s}_{1,1,2} \\ \vdots & \vdots \\ m^{h_s}_{1,1,n_x} & m^{g_s}_{1,1,n_x} \\ m^{h_s}_{1,2,1} & m^{g_s}_{1,2,1} \\ \vdots & \vdots \\ m^{h_s}_{1,n_\varepsilon,n_x} & m^{g_s}_{1,n_\varepsilon,n_x} \\ m^{h_s}_{2,1,1} & m^{g_s}_{2,1,1} \\ \vdots & \vdots \\ m^{h_s}_{n,n_\varepsilon,n_x} & m^{g_s}_{n,n_\varepsilon,n_x} \end{bmatrix}}_{m_{x\varepsilon}(s,s') = \left[m^h_{x\varepsilon}(s,s') \; m^g_{x\varepsilon}(s,s') \right]} \begin{bmatrix} vec\left(h_{s,x\varepsilon}\right) \\ vec\left(g_{s,x\varepsilon}\right) \end{bmatrix} + \sum_{s'} \underbrace{\begin{bmatrix} n_{1,1,1} \\ n_{1,1,2} \\ \vdots \\ n_{1,1,n_x} \\ n_{1,2,1} \\ \vdots \\ n_{1,n_\varepsilon,n_x} \\ n_{2,1,1} \\ \vdots \\ n_{n,n_\varepsilon,n_x} \end{bmatrix}}_{n_{x\varepsilon}(s,s')} = 0.$$

Then, we can stack these for s:

$$\underbrace{\begin{bmatrix} \sum_{s'} m^h_{x\varepsilon}\left(1, s'\right) & \sum_{s'} m^g_{x\varepsilon}\left(1, s'\right) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{s'} m^h_{x\varepsilon}\left(n_s, s'\right) & \sum_{s'} m^g_{x\varepsilon}\left(n_s, s'\right) \end{bmatrix}}_{M_{x\varepsilon}} \begin{bmatrix} vec\left(h_{1,x\varepsilon}\right) \\ vec\left(g_{1,x\varepsilon}\right) \\ \vdots \\ vec\left(h_{n_s,x\varepsilon}\right) \\ vec\left(g_{n_s,x\varepsilon}\right) \end{bmatrix} + \underbrace{\begin{bmatrix} \sum_{s'} n_{x\varepsilon}\left(1, s'\right) \\ \vdots \\ \sum_{s'} n_{x\varepsilon}\left(n_s, s'\right) \end{bmatrix}}_{N_{x\varepsilon}} = 0$$

which is then solved by matrix inversion.

Derivative for x, χ

The derivative with respect to (x_{t-1}, χ) is a $n \times n_x$ matrix. The (a,b)-element is given by

$$
\begin{aligned}
&[F_{s,x\chi}]^a_b = \\
&\sum_{s'} P_{s,s'} \left(\begin{array}{c}
\sum_i f^a_{y_{i,t}} g^i_{s,x_b\chi} + \sum_k f^a_{y_{k,t+1}} \sum_i g^k_{s',x_i\chi} h^i_{s,x_b} \\
+ \sum_i f^a_{x_{i,t}} h^i_{s,x_b\chi} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,x_b\chi} \\
+ \sum_k \left(\begin{array}{c}
\sum_j f^a_{y_{k,t+1} y_{j,t+1}} \left(\sum_i g^j_{s' x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) \\
+ \sum_j f^a_{y_{k,t+1} y_{j,t}} g^j_{s,\chi} + \sum_j f^a_{y_{k,t+1}, x_{j,t}} h^j_{s,\chi} \\
+ \sum_j f^a_{y_{k,t+1} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t+1} \theta_{j,t}} \hat{\theta}_{j,t}
\end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,x_b} \right) \\
+ \sum_k \left(\begin{array}{c}
\sum_j f^a_{y_{j,t+1} y_{k,t}} \sum_i \left(g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{k,t} y_{j,t}} g^j_{s,\chi} \\
+ \sum_j f^a_{y_{k,t} x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{y_{k,t} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t} \theta_{j,t}} \hat{\theta}_{j,t}
\end{array} \right) g^k_{s,x_b} \\
+ \sum_k \left(\begin{array}{c}
\sum_j f^a_{y_{j,t+1} x_{k,t}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{j,t} x_{k,t}} g^j_{s,\chi} \\
+ \sum_j f^a_{x_{j,t} x_{k,t}} h^j_{s,\chi} + \sum_j f^a_{x_{k,t} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{x_{k,t} \theta_{j,t}} \hat{\theta}_{j,t}
\end{array} \right) h^k_{s,x_b} \\
+ \sum_j f^a_{y_{j,t+1} x_{b,t-1}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_i f^a_{y_{i,t} x_{b,t-1}} g^i_{s,\chi} \\
+ \sum_i f^a_{x_{i,t} x_{b,t-1}} h^i_{s,\chi} + \sum_j f^a_{x_{b,t-1} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{x_{b,t-1} \theta_{j,t}} \hat{\theta}_{j,t} \\
+ \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_j x_i} h^j_{s,x_b} h^i_{s,\chi}
\end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,x_b} \right) \left(\begin{array}{c}
\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,\chi} + \sum_j f_{y_{j,t}} g^j_{s,\chi} \\
+ \sum_j f_{x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\theta_{j,t}} \hat{\theta}_{j,t}
\end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\chi} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,x_b} + \sum_j f_{y_{j,t}} g^j_{s,x_b} + \sum_j f_{x_{j,t}} h^j_{s,x_b} + f_{x_{b,t-1}} \right)
\end{aligned}
$$

This is a linear system that can be expressed as

$$
\sum_{s'} \underbrace{\left[\begin{array}{ccc} m^{h_s}_{a,b} & m^{g_s}_{a,b} & m^{g_{s'}}_{a,b} \end{array} \right]}_{m_{a,b}} \left[\begin{array}{c} vec\left(h_{s,x\chi}\right) \\ vec\left(g_{s,x\chi}\right) \\ vec\left(g_{s',x\chi}\right) \end{array} \right] + \sum_{s'} n_{a,b} = 0
$$

Stacking these equations for $a = 1, \ldots, n$, and $b = 1, \ldots, n_x$:

$$
\sum_{s'} \underbrace{\left[\begin{array}{ccc}
m^{h_s}_{1,1} & m^{g_s}_{1,1} & m^{g_{s'}}_{1,1} \\
m^{h_s}_{1,2} & m^{g_s}_{1,2} & m^{g_{s'}}_{1,2} \\
\vdots & \vdots & \vdots \\
m^{h_s}_{1,n_x} & m^{g_s}_{1,n_x} & m^{g_{s'}}_{1,n_x} \\
m^{h_s}_{2,1} & m^{g_s}_{2,1} & m^{g_{s'}}_{2,1} \\
\vdots & \vdots & \vdots \\
m^{h_s}_{n,n_x} & m^{g_s}_{n,n_x} & m^{g_{s'}}_{n,,n_x}
\end{array} \right]}_{m_{x\chi}(s,s') = \left[m^h_{x\chi}(s,s') \; m^g_{x\chi}(s,s') \; m^{g'}_{x\chi}(s,s') \right]} \left[\begin{array}{c} vec\left(h_{s,x\chi}\right) \\ vec\left(g_{s,x\chi}\right) \\ vec\left(g_{s',x\chi}\right) \end{array} \right] + \sum_{s'} \underbrace{\left[\begin{array}{c} n_{1,1} \\ n_{1,2} \\ \vdots \\ n_{1,n_x} \\ n_{2,1} \\ \vdots \\ n_{n,n_x} \end{array} \right]}_{n_{x\chi}(s,s')} = 0.
$$

Then, we can stack these for s:

$$\underbrace{\begin{bmatrix} \sum_{s'} m^h_{x\chi}(1,s') & \sum_{s'} m^g_{x\chi}(1,s') & \cdots & 0 & m^{g'}_{x\chi}(1,n_s) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m^{g'}_{x\chi}(n_s,1) & \cdots & \sum_{s'} m^h_{x\chi}(n_s,s') & \sum_{s'} m^g_{x\chi}(n_s,s') \end{bmatrix}}_{M_{x\chi}} \begin{bmatrix} vec(h_{1,x\chi}) \\ vec(g_{1,x\chi}) \\ \vdots \\ vec(h_{n_s,x\chi}) \\ vec(g_{n_s,x\chi}) \end{bmatrix} + \underbrace{\begin{bmatrix} \sum_{s'} n_{x\chi}(1,s') \\ \vdots \\ \sum_{s'} n_{x\chi}(n_s,s') \end{bmatrix}}_{N_{x\chi}} = 0$$

which is solved by matrix inversion.

Derivative for ε, ε

The derivative with respect to $(\varepsilon_t, \varepsilon_t)$ is a $n \times n_\varepsilon \times n_\varepsilon$ array. The (a,b,c)-element is given by

$$[F_{s,\varepsilon\varepsilon}]^a_{bc} =$$
$$\sum_{s'} P_{s,s'} \left(\begin{array}{c} \sum_i f^a_{y_{i,t}} g^i_{s,\varepsilon_b\varepsilon_c} + \sum_i f^a_{x_{i,t}} h^i_{s,\varepsilon_b\varepsilon_c} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,\varepsilon_b\varepsilon_c} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{k,t+1}y_{j,t+1}} \sum_i g^j_{s'x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{k,t+1}y_{j,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{y_{k,t+1},x_{j,t}} h^j_{s,\varepsilon_c} + f^a_{y_{k,t+1}\varepsilon_{c,t}} \end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,\varepsilon_b} \right) \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1}y_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{k,t}y_{j,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{y_{k,t}x_{j,t}} h^j_{s,\varepsilon_c} + f^a_{y_{k,t}\varepsilon_c} \end{array} \right) g^k_{s,\varepsilon_b} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1}x_{k,t}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f^a_{y_{j,t}x_{k,t}} g^j_{s,\varepsilon_c} \\ + \sum_j f^a_{x_{j,t}x_{k,t}} h^j_{s,\varepsilon_c} + f^a_{x_{k,t}\varepsilon_{c,t}} \end{array} \right) h^k_{s,\varepsilon_b} \\ + \sum_j f^a_{y_{j,t+1}\varepsilon_{b,t}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_i f^a_{y_{i,t}\varepsilon_{b,t}} g^i_{s,\varepsilon_c} + \sum_i f^a_{x_{i,t}x_{b,t-1}} h^i_{s,\varepsilon_c} + f^a_{\varepsilon_{b,t}\varepsilon_{c,t}} \\ + \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_jx_i} h^j_{s,\varepsilon_b} h^i_{s,\varepsilon_c} \end{array} \right)$$
$$+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\varepsilon_b} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_c} + \sum_j f_{y_{j,t}} g^j_{s,\varepsilon_c} + \sum_j f_{x_{j,t}} h^j_{s,\varepsilon_c} + f_{\varepsilon_{c,t}} \right)$$
$$+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\varepsilon_c} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_b} + \sum_j f_{y_{j,t}} g^j_{s,\varepsilon_b} + \sum_j f_{x_{j,t}} h^j_{s,x_b} + f_{\varepsilon_{b,t}} \right)$$

This is a linear system that can be expressed as

$$\sum_{s'} \underbrace{\left[\begin{array}{ccc} m_{a,b,c}^{h_s} & m_{a,b,c}^{g_s} & m_{a,b,c}^{g_{s'}} \end{array}\right]}_{m_{a,b,c}} \left[\begin{array}{c} vec\left(h_{s,\varepsilon\varepsilon}\right) \\ vec\left(g_{s,\varepsilon\varepsilon}\right) \\ vec\left(g_{s',\varepsilon\varepsilon}\right) \end{array}\right] + \sum_{s'} n_{a,b,c} = 0$$

Stacking these equations for $a = 1, \ldots, n$, and $b, c = 1, \ldots, n_x$:

$$\sum_{s'} \underbrace{\left[\begin{array}{cc} m_{1,1,1}^{h_s} & m_{1,1,1}^{g_s} \\ m_{1,1,2}^{h_s} & m_{1,1,2}^{g_s} \\ \vdots & \vdots \\ m_{1,1,n_\varepsilon}^{h_s} & m_{1,1,n_\varepsilon}^{g_s} \\ m_{1,2,1}^{h_s} & m_{1,2,1}^{g_s} \\ \vdots & \vdots \\ m_{1,n_\varepsilon,n_\varepsilon}^{h_s} & m_{1,n_\varepsilon,n_\varepsilon}^{g_s} \\ m_{2,1,1}^{h_s} & m_{2,1,1}^{g_s} \\ \vdots & \vdots \\ m_{n,n_\varepsilon,n_\varepsilon}^{h_s} & m_{n,n_\varepsilon,n_\varepsilon}^{g_s} \end{array}\right]}_{m_{\varepsilon\varepsilon}(s,s') = \left[m_{\varepsilon\varepsilon}^{h}(s,s')\ m_{\varepsilon\varepsilon}^{g}(s,s')\right]} \left[\begin{array}{c} vec\left(h_{s,\varepsilon\varepsilon}\right) \\ vec\left(g_{s,\varepsilon\varepsilon}\right) \end{array}\right] + \sum_{s'} \underbrace{\left[\begin{array}{c} n_{1,1,1} \\ n_{1,1,2} \\ \vdots \\ n_{1,1,n_\varepsilon} \\ n_{1,2,1} \\ \vdots \\ n_{1,n_\varepsilon,n_\varepsilon} \\ n_{2,1,1} \\ \vdots \\ n_{n,n_\varepsilon,n_\varepsilon} \end{array}\right]}_{n_{\varepsilon\varepsilon}(s,s')} = 0.$$

Then, we can stack these for s:

$$\underbrace{\left[\begin{array}{ccccc} \sum_{s'} m_{\varepsilon\varepsilon}^{h}\left(1,s'\right) & \sum_{s'} m_{\varepsilon\varepsilon}^{g}\left(1,s'\right) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{s'} m_{\varepsilon\varepsilon}^{h}\left(n_s,s'\right) & \sum_{s'} m_{\varepsilon\varepsilon}^{g}\left(n_s,s'\right) \end{array}\right]}_{M_{\varepsilon\varepsilon}} \left[\begin{array}{c} vec\left(h_{1,\varepsilon\varepsilon}\right) \\ vec\left(g_{1,\varepsilon\varepsilon}\right) \\ \vdots \\ vec\left(h_{n_s,\varepsilon\varepsilon}\right) \\ vec\left(g_{n_s,\varepsilon\varepsilon}\right) \end{array}\right] + \underbrace{\left[\begin{array}{c} \sum_{s'} n_{\varepsilon\varepsilon}\left(1,s'\right) \\ \vdots \\ \sum_{s'} n_{\varepsilon\varepsilon}\left(n_s,s'\right) \end{array}\right]}_{N_{\varepsilon\varepsilon}} = 0$$

which is solved by matrix inversion.

Derivative for ε, χ

The derivative with respect to (ε_t, χ) is a $n \times n_\varepsilon$ matrix. The (a, b)-element is given by

$$
\begin{aligned}
&[F_{s,\varepsilon\chi}]^a_b = \\
&\sum_{s'} P_{s,s'} \left(\begin{array}{c}
\sum_i f^a_{y_{i,t}} g^i_{s,\varepsilon_b\chi} + \sum_i f^a_{x_{i,t}} h^i_{s,\varepsilon_b\chi} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,\varepsilon_b\chi} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{k,t+1} y_{j,t+1}} \left(\sum_i g^j_{s' x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) \\ + \sum_j f^a_{y_{k,t+1} y_{j,t}} g^j_{s,\chi} + \sum_j f^a_{y_{k,t+1}, x_{j,t}} h^j_{s,\chi} \\ + \sum_j f^a_{y_{k,t+1} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t+1} \theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,\varepsilon_b} \right) \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} y_{k,t}} \sum_i \left(g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{k,t} y_{j,t}} g^j_{s,\chi} \\ + \sum_j f^a_{y_{k,t} x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{y_{k,t} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t} \theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) g^k_{s,\varepsilon_b} \\
+ \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} x_{k,t}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{j,t} x_{k,t}} g^j_{s,\chi} \\ + \sum_j f^a_{x_{j,t} x_{k,t}} h^j_{s,\chi} + \sum_j f^a_{x_{k,t} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{x_{k,t} \theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) h^k_{s,\varepsilon_b} \\
+ \sum_j f^a_{y_{j,t+1} \varepsilon_{b,t}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_i f^a_{y_{i,t} \varepsilon_{b,t}} g^i_{s,\chi} \\
+ \sum_i f^a_{x_{i,t} \varepsilon_{b,t}} h^i_{s,\chi} + \sum_j f^a_{\varepsilon_{b,t} \theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\varepsilon_{b,t} \theta_{j,t}} \hat{\theta}_{j,t} \\
+ \sum_k f^a_{y_{k,t+1}} \sum_i g^k_{s',x_i\chi} h^i_{s,\varepsilon_b} + \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_j x_i} h^j_{s,\varepsilon_b} h^i_{s,\chi}
\end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\varepsilon_b} \right) \left(\begin{array}{c} \sum_j f_{y_{j,t+1}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f_{y_{j,t}} g^j_{s,\chi} \\ + \sum_j f_{x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) \\
&+ \sum_{s'} \left(\sum_j P_{s,s',y_{j,t}} g^j_{s,\chi} \right) \left(\sum_j f_{y_{j,t+1}} \sum_i g^j_{s',x_i} h^i_{s,\varepsilon_b} + \sum_j f_{y_{j,t}} g^j_{s,\varepsilon_b} + \sum_j f_{x_{j,t}} h^j_{s,\varepsilon_b} + f_{\varepsilon_{b,t}} \right)
\end{aligned}
$$

This is a linear system that can be expressed as

$$
\sum_{s'} \underbrace{\left[\begin{array}{ccc} m^{h_s}_{a,b,c} & m^{g_s}_{a,b,c} & m^{g_{s'}}_{a,b,c} \end{array} \right]}_{m_{a,b,c}} \left[\begin{array}{c} vec\left(h_{s,\varepsilon\chi}\right) \\ vec\left(g_{s,\varepsilon\chi}\right) \\ vec\left(g_{s',\varepsilon\chi}\right) \end{array} \right] + \sum_{s'} n_{a,b,c} = 0
$$

Stacking these equations for $a = 1, \ldots, n$, and $b = 1, \ldots, n_\varepsilon$:

$$
\sum_{s'} \underbrace{\left[\begin{array}{cc} m^{h_s}_{1,1} & m^{g_s}_{1,1} \\ m^{h_s}_{1,2} & m^{g_s}_{1,2} \\ \vdots & \vdots \\ m^{h_s}_{1,n_\varepsilon} & m^{g_s}_{1,n_\varepsilon} \\ m^{h_s}_{2,1} & m^{g_s}_{2,1} \\ \vdots & \vdots \\ m^{h_s}_{n,n_\varepsilon} & m^{g_s}_{n,n_\varepsilon} \end{array} \right]}_{m_{\varepsilon\chi}(s,s') = \left[m^h_{\varepsilon\chi}(s,s') \; m^g_{\varepsilon\chi}(s,s') \right]} \left[\begin{array}{c} vec\left(h_{s,\varepsilon\chi}\right) \\ vec\left(g_{s,\varepsilon\chi}\right) \end{array} \right] + \sum_{s'} \underbrace{\left[\begin{array}{c} n_{1,1} \\ n_{1,2} \\ \vdots \\ n_{1,n_\varepsilon} \\ n_{2,1} \\ \vdots \\ n_{n,n_x} \end{array} \right]}_{n_{\varepsilon\chi}(s,s')} = 0.
$$

Then, we can stack these for s:

$$\underbrace{\begin{bmatrix} \sum_{s'} m^h_{\varepsilon\chi}(1,s') & \sum_{s'} m^g_{\varepsilon\chi}(1,s') & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{s'} m^h_{\varepsilon\chi}(n_s,s') & \sum_{s'} m^g_{\varepsilon\chi}(n_s,s') \end{bmatrix}}_{M_{\varepsilon\chi}} \begin{bmatrix} vec\left(h_{1,\varepsilon\chi}\right) \\ vec\left(g_{1,\varepsilon\chi}\right) \\ \vdots \\ vec\left(h_{n_s,\varepsilon\chi}\right) \\ vec\left(g_{n_s,\varepsilon\chi}\right) \end{bmatrix} + \underbrace{\begin{bmatrix} \sum_{s'} n_{\varepsilon\chi}(1,s') \\ \vdots \\ \sum_{s'} n_{\varepsilon\chi}(n_s,s') \end{bmatrix}}_{N_{\varepsilon\chi}} = 0$$

Derivative for χ, χ

The derivative with respect to (χ,χ) is a n vector. The (a)-element is given by

$$[F_{s,\chi\chi}]^a =$$

$$= \sum_{s'} P_{s,s'} \left(\begin{array}{c} \sum_i f^a_{y_{i,t}} g^i_{s,\chi\chi} + \sum_i f^a_{x_{i,t}} h^i_{s,\chi\chi} + \sum_k f^a_{y_{k,t+1}} g^k_{s',\chi\chi} + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',x_j} h^j_{s,\chi\chi} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{k,t+1} y_{j,t+1}} \left(\sum_i g^j_{s' x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) \\ + \sum_j f^a_{y_{k,t+1} y_{j,t}} g^j_{s,\chi} + \sum_j f^a_{y_{k,t+1}, x_{j,t}} h^j_{s,\chi} \\ + \sum_j f^a_{y_{k,t+1}\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t+1}\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) \left(\sum_i g^k_{s',x_i} h^i_{s,\chi} + g^k_{s',\chi} \right) \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} y_{k,t}} \sum_i \left(g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{k,t} y_{j,t}} g^j_{s,\chi} \\ + \sum_j f^a_{y_{k,t} x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{y_{k,t}\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{y_{k,t}\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) g^k_{s,\chi} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1} x_{k,t}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{j,t} x_{k,t}} g^j_{s,\chi} \\ + \sum_j f^a_{x_{j,t} x_{k,t}} h^j_{s,\chi} + \sum_j f^a_{x_{k,t}\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{x_{k,t}\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) h^k_{s,\chi} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1}\theta_{k,t+1}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{j,t}\theta_{k,t+1}} g^j_{s,\chi} \\ + \sum_j f^a_{x_{j,t}\theta_{k,t+1}} h^j_{s,\chi} + \sum_j f^a_{\theta_{k,t+1}\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\theta_{k,t+1}\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) \hat{\theta}_{k,s'} \\ + \sum_k \left(\begin{array}{c} \sum_j f^a_{y_{j,t+1}\theta_{k,t}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f^a_{y_{j,t}\theta_{k,t}} g^j_{s,\chi} \\ + \sum_j f^a_{x_{j,t}\theta_{k,t}} h^j_{s,\chi} + \sum_j f^a_{\theta_{k,t}\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\theta_{k,t}\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right) \hat{\theta}_{k,s} \\ + \sum_k f^a_{y_{k,t+1}} \sum_j \sum_i g^k_{s',x_j x_i} h^j_{s,\chi} h^i_{s,\chi} + 2 \sum_k f^a_{y_{k,t+1}} \sum_i g^k_{s',x_i \chi} h^i_{s,\chi} \\ + \sum_k f^a_{y_{k,t+1}} \sum_j g^k_{s',\varepsilon_j \varepsilon_j} + 2 \sum_k \sum_j f^a_{y_{k,t+1}\varepsilon_{j,t+1}} g^k_{s',\varepsilon_j} + \sum_k f^a_{\varepsilon_{k,t+1}\varepsilon_{k,t+1}} \\ + \sum_k \sum_j f^a_{y_{j,t+1} y_{k,t+1}} \sum_i g^j_{s',\varepsilon_i} g^k_{s',\varepsilon_i} \end{array} \right)$$

$$+ \sum_{s'} 2 \left(P_{s,s',y_{j,t}} g^j_{s,\chi} \right) \left(\begin{array}{c} \sum_j f_{y_{j,t+1}} \left(\sum_i g^j_{s',x_i} h^i_{s,\chi} + g^j_{s',\chi} \right) + \sum_j f_{y_{j,t}} g^j_{s,\chi} \\ + \sum_j f_{x_{j,t}} h^j_{s,\chi} + \sum_j f^a_{\theta_{j,t+1}} \hat{\theta}_{j,t+1} + \sum_j f^a_{\theta_{j,t}} \hat{\theta}_{j,t} \end{array} \right)$$

This is a linear system that can be expressed as

$$\sum_{s'} \underbrace{\begin{bmatrix} m^{h_s}_{a,b,c} & m^{g_s}_{a,b,c} & m^{g_{s'}}_{a,b,c} \end{bmatrix}}_{m_{a,b,c}} \begin{bmatrix} vec\left(h_{s,\chi\chi}\right) \\ vec\left(g_{s,\chi\chi}\right) \\ vec\left(g_{s',\chi\chi}\right) \end{bmatrix} + \sum_{s'} n_{a,b,c} = 0$$

Stacking these equations for $a = 1, \ldots, n$:

$$\sum_{s'} \underbrace{\begin{bmatrix} m^{h_s}_1 & m^{g_s}_1 & m^{g_{s'}}_1 \\ m^{h_s}_2 & m^{g_s}_2 & m^{g_{s'}}_2 \\ \vdots & \vdots & \vdots \\ m^{h_s}_n & m^{g_s}_n & m^{g_{s'}}_n \end{bmatrix}}_{m_{\chi\chi}(s,s')=\left[m^h_{\chi\chi}(s,s')\ m^g_{\chi\chi}(s,s')\ m^{g'}_{\chi\chi}(s,s')\right]} \begin{bmatrix} vec\left(h_{s,\chi\chi}\right) \\ vec\left(g_{s,\chi\chi}\right) \\ vec\left(g_{s',\chi\chi}\right) \end{bmatrix} + \sum_{s'} \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{bmatrix}}_{n_{\chi\chi}(s,s')} = 0.$$

Then, we can stack these for s:

$$\underbrace{\begin{bmatrix} \sum_{s'} m^h_{\chi\chi}\left(1,s'\right) & \sum_{s'} m^g_{\chi\chi}\left(1,s'\right) & \cdots & 0 & m^{g'}_{\chi\chi}\left(1,n_s\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m^{g'}_{\chi\chi}\left(n_s,1\right) & \cdots & \sum_{s'} m^h_{\chi\chi}\left(n_s,s'\right) & \sum_{s'} m^g_{\chi\chi}\left(n_s,s'\right) \end{bmatrix}}_{M_{xx}} \begin{bmatrix} vec\left(h_{1,\chi\chi}\right) \\ vec\left(g_{1,\chi\chi}\right) \\ \vdots \\ vec\left(h_{n_s,\chi\chi}\right) \\ vec\left(g_{n_s,\chi\chi}\right) \end{bmatrix} + \underbrace{\begin{bmatrix} \sum_{s'} n_{\chi\chi}\left(1,s'\right) \\ \vdots \\ \sum_{s'} n_{\chi\chi}\left(n_s,s'\right) \end{bmatrix}}_{N_{\chi\chi}} = 0$$

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