Understanding Regressions with Observations Collected at High Frequency over Long Span*

Yoosoon Chang
Department of Economics
Indiana University

Ye Lu
School of Economics
University of Sydney

Joon Y. Park
Department of Economics
Indiana University

January 30, 2025

Abstract

In this paper, we analyze regressions with observations collected at small time intervals over a long period of time. For the formal asymptotic analysis, we assume that samples are obtained from continuous time stochastic processes, and let the sampling interval δ shrink down to zero and the sample span T increase up to infinity. In this setup, we show that the standard Wald statistic diverges to infinity and the regression becomes spurious as long as $\delta \to 0$ sufficiently fast relative to $T \to \infty$. Such a phenomenon is indeed what is frequently observed in practice for the type of regressions considered in the paper. In contrast, our asymptotic theory predicts that the spuriousness disappears if we use the robust version of the Wald test with an appropriate long-run variance estimate. This is supported, strongly and unambiguously, by our empirical illustration using the regression of long-term on short-term interest rates.

JEL Classification: C13, C22

Keywords and phrases: high frequency regression, spurious regression, continuous-time model, asymptotics, long-run variance estimation

*We are very grateful to the editor Stéphane Bonhomme and two anonymous referees for their valuable comments. Earlier versions of this paper have been circulated since 2014. We would also like to thank Torben Andersen, Donald Andrews, Federico Bandi, Jihyun Kim, Hwagyun Hagen Kim, Barbara Rossi and Jae Sim for helpful discussions, and to the participants at 2013 Princeton-QUT-SMU Conference on Measuring Risk (Princeton), 2013 MEG Meeting (Bloomington), 2014 SETA (Taipei), 2014 Conference on Econometrics for Macroeconomics and Finance (Hitotsubashi), 2015 Financial Econometrics Conference (Toulouse), 2015 Workshop on Development in Time Series Econometrics (Cambridge), 2015 Frontiers of Theoretical Econometrics (Konstanz), and 2015 World Congress of Econometric Society (Montreal), and the seminar attendants at University of Washington, Michigan, NYU, Yale, Bank of Portugal, Carlos III University of Madrid, UPF, and CEMFI for useful discussions.

1 Introduction

A great number of economic and financial time series are now collected and made available at high frequencies, and naturally many empirical researchers find it difficult to decide at what frequency they collect the samples to estimate and test their models. Naturally we may think that we should use all available observations, since neglecting any available observations means a loss in information. Nevertheless, this is not the usual practice in applied empirical research. In most cases, samples used in practical applications are obtained at a frequency lower than the maximum frequency available. For instance, many time series models in financial economics are fitted using monthly observations, when their daily samples or even intra-day samples are available at no extra costs. Some researchers seem to believe, rather vaguely, that high frequency observations include excessive noise or erratic volatilities, and that they do not bring in any significant amount of marginal information. Others keep silent on this issue, and seem to simply choose the sampling frequency that yields sensible results.

In the paper, we formally analyze the effect of sampling frequency on the standard tests for a class of high frequency regressions.[1] For our analysis, we consider the standard regression model for continuous time stochastic processes, and assume that the regression is fitted by discrete time observations collected from the underlying processes at sampling interval $\delta > 0$ over sample span T. Under this set-up, we establish our asymptotics by letting $\delta \to 0$ and $T \to \infty$ jointly,[2] and use them to analyze regressions with observations collected at high frequency over long span. Both stationary and nonstationary continuous time regression models are analyzed. The former is the continuous time version of the standard stationary time series regression, whereas the latter is a continuous time analogue of the cointegrating regression model. Our asymptotics are derived under general conditions accommodating a large class of stationary and nonstationary regression models, and therefore, they are widely applicable in practical applications.

One of the main findings from our analysis is that both types of regressions eventually become spurious as the sampling frequency increases.[3] Even under the correct null hy-

[1]As will be explained later, we only consider the class of high frequency regressions consisting of 'stock' variables in the paper. We have other classes of high frequency regressions relying on 'flow' variables, for which our analysis here does not apply.

[2]It is necessary to let $\delta \to$ to analyze high frequency regressions. In our asymptotics, we also need to let $T \to \infty$, since otherwise the tests considered in the paper for high frequency regressions all become inconsistent.

[3]The problems using high frequency observations analyzed in our paper disappear if the sampling frequency goes, say, well beyond monthly for a wide range of economic variables that we have had a chance to analyze. However, using low frequency observations, when observations are available at higher frequencies, necessarily incurs some information loss. We therefore recommend to use high frequency observations, if available, with a methodology which is high-frequency compatible. Our paper provides some of such

pothesis, the standard test statistics, such as the t-ratios and Wald statistics, increase up to infinity as the sampling interval decreases down to zero.[4] Therefore, they would always lead us to reject the correct null hypothesis if the sampling interval is sufficiently small. This is completely analogous to the conventional spurious regression in econometrics, which was first studied through simulations by Granger and Newbold (1974) and studied theoretically later by Phillips (1986). The spuriousness in the conventional spurious regression is due to the presence of a unit root in the regression error that generates strong serial dependence. The same problem arises in the regressions we consider. The regression error from a continuous time process becomes strongly dependent as the sampling interval decreases, which yields the same type of spuriousness in the conventional spurious regression.

In time series regressions, we often use the robust version of standard tests with a long-run variance estimator in place of the usual variance estimator, to allow for the presence of serial dependence. Our asymptotic theory shows that the robust test may or may not diverge to infinity under the correct null hypothesis, depending upon how we choose the bandwidth parameter in our long-run variance estimator used in the test.[5] If it is chosen appropriately, the test becomes valid at high frequency and the spuriousness at high frequency disappears. However, with a conventional choice of bandwidth made in the discrete time setup, the test diverges as the sampling interval decreases like in the case of the standard tests, resulting in the spuriousness at high frequency. The use of data-dependent bandwidth choice also has a critical effect on the validity of robust tests. For instance, the test becomes generally valid if the bandwidth selection is made using the procedure by Andrews (1991), while the spuriousness at high frequency arises if the method proposed by Newey and West (1994) is employed. In all actual regressions we consider in the paper, the tests behave exactly as predicted by our asymptotic theory.

It seems possible to find a discrete time model, which yields a particular feature of our asymptotics based on the continuous time regression. For instance, we may use a near-unit root model to demonstrate how and why a test fails to perform properly in the presence of strong persistence, similarly as we observe in our analysis. This possibility is explored by Müller (2005) for the test of stationarity. However, the near-unit root model cannot generate any other features of high frequency regressions that our framework and analysis

methodologies.

[4]This, of course, implies that the usual asymptotics with fixed δ completely fails to provide a valid approximation to the actual distribution of the t-ratio or Wald statistic in the class of high frequency regressions considered in the paper.

[5]The basic asymptotics of HAC estimation in continuous time is developed in Lu and Park (2019), which was written concurrently with this paper. This paper extends them to allow for data-dependent bandwidth selection procedures, and also establishes general sufficient conditions for the discretization errors being negligible asymptotically, which are widely valid for both stationary and nonstationary continuous time regressions.

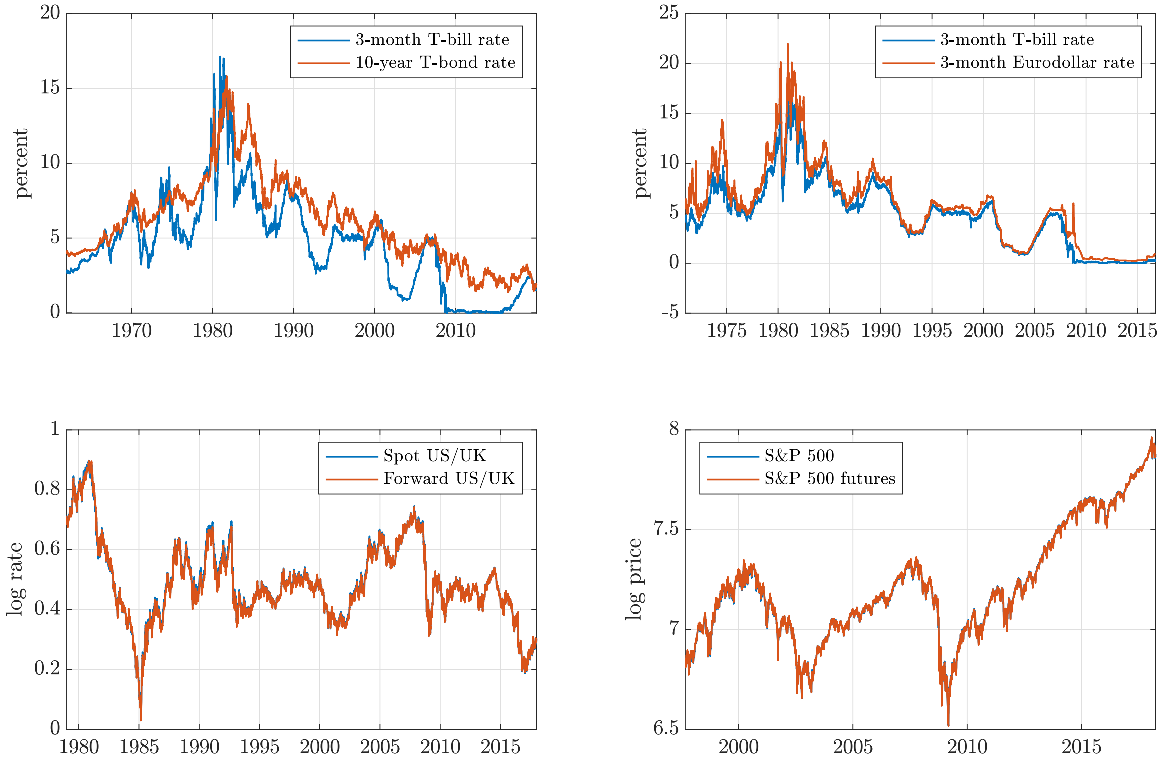
reveal. A majority of previous works, which use continuous time models to study discrete time observations, rely on the limit approximation by a continuous time process of a discrete time series model varying with the sample size n like the near-unit root model, see, e.g., Perron (1991a,b). Our framework is totally different: we assume that discrete samples are collected from a given continuous time model. In our analysis, we fix a continuous time model, and study the characteristics of discrete samples as we change the sampling interval δ and the sample span T.

The rest of the paper is organized as follows. Section 2 explains the background and motivation of our analysis in the paper. In particular, we provide some illustrative examples that are analyzed throughout the paper to show the practical relevancy of our asymptotic theory. Section 3 introduces the regression models, the setup for our asymptotics and some preliminaries. The spuriousness of the high frequency regressions are derived and investigated in Section 4. In particular, we establish under fairly general conditions that the coefficient in the first order autoregression of the regression error converges to unity as the sampling interval decreases. Section 5 presents the limit theory for the robust versions of the Wald test statistic defined with long-run variance estimators. We also demonstrate that the bandwidth selection is important, and that the modified tests may or may not yield spurious results depending upon the bandwidth choice. An empirical illustration with the regression of long-term on short-term interest rates is provided in Section 6, and the simulations reported in Section 7 demonstrate the practical relevance of our asymptotic theory. Section 8 concludes the paper, and Appendices provide mathematical proofs, additional simulation results and additional figures.

2 Background and Motivation

It is widely observed that test results are critically dependent upon the choice of sampling frequency in many time series regressions. To illustrate more explicitly the dependency on the sampling interval of test results, we consider a simple bivariate regression of (y_i) on (x_i) written as $y_i = \beta_0 + \beta_1 x_i + u_i$, where β_0 and β_1 are respectively the intercept and slope parameters and (u_i) are the regression errors. For (y_i) and (x_i), we consider the following four pairs:

Figure 1: Data Plots for Models I–IV

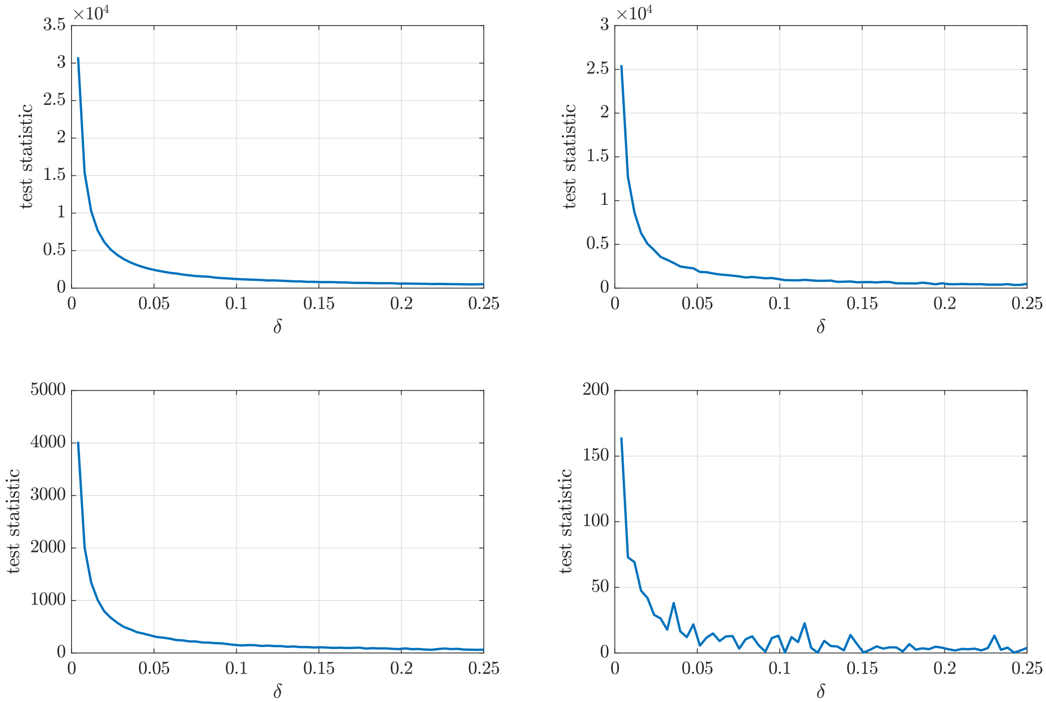


Notes: Presented are daily sample paths of the regressand y and the regressor x used for the empirical illustrations of Models I–IV. Top-left panel presents the 10-year T-bond rate and 3-month T-bill rate from January 2, 1962 to December 31, 2019. Top-right panel presents the 3-month Eurodollar rate and 3-month T-bill rate from January 4, 1971 to October 7, 2016. Bottom-left panel presents the log US/UK 3-month forward exchange rate and log US/UK spot exchange rate from January 2, 1979 to December 29, 2017. Bottom-right panel presents the log S&P 500 Index future and log S&P 500 Index from September 10, 1997 to March 29, 2018.

Model	(y_i)	(x_i)
I	10-year T-bond rates	3-month T-bill rates
II	3-month Eurodollar rates	3-month T-bill rates
III	log US/UK exchange rates forward	log US/UK exchange rates spot
IV	log S&P 500 index futures	log S&P 500 index

We consider Models I/II primarily as stationary regressions, and Models III/IV as cointegrating regressions. However, whether or not Models I/II and Models III/IV are truly specified respectively as stationary and cointegrating regressions is not important. Our asymptotics accommodate both types of regressions.

Figure 2: Wald Tests in Models I-IV



Notes: Presented are the values of the standard Wald statistics, which are defined explicitly in (3.2), for the null hypothesis $\mathrm{H}_0 : \beta_0 = 0$ and $\beta_1 = 1$ in Models I-IV (Model I in the upper-left, Model II in the upper-right, Model III in the lower-left, and Model IV in the lower-right panel), obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).

The plots for (y_i) and (x_i) in Models I-IV are given in Figure 1.[6] In all models, possibly except for Model I, two series (y_i) and (x_i) move very closely with each other. Therefore, the most natural hypothesis to be tested appears to be $\mathrm{H}_0 : \beta_0 = 0$ and $\beta_1 = 1$. The hypothesis may or may not hold. In particular, the null hypothesis $\beta_0 = 0$ does not necessarily hold when there are differences in the term or liquidity premium as well as the general risk premium between two assets represented by (y_i) and (x_i). However, in the paper, we do not intend to provide any answers to whether or not the hypothesis should hold in any of the models we consider here. Instead, we simply analyze the dependence of test results on the sampling frequency.

In Figure 2, we present the values of the standard Wald test relying on 5% chi-square

[6]The S&P index futures used here and elsewhere in the paper are the log prices of the E-mini S&P 500 contracts expiring in the front month (the nearest delivery month) downloaded from Investing.com. The E-mini S&P 500 contract represents one-fifth of the value of the standard-size futures contract, and it is traded electronically on the Chicago Mercantile Exchange (CME). The E-mini contract has been traded much more actively and widely than the standard contract, and it has become the primary futures trading vehicle for the S&P 500 index since the inception its trading on September 9, 1997.

critical values in Models I-IV against various sampling intervals from one quarter with $\delta = 1/4$ to one day with $\delta = 1/252$ in yearly unit. As discussed, the values of the Wald test change dramatically as the sampling interval δ varies. In particular, they tend to increase very rapidly as δ gets smaller and becomes near zero. The rate of increase in the values of the Wald test as δ approaches zero varies across different models. However, it is common to all models that the values of the Wald test start to increase sharply as the sampling interval becomes approximately one month or shorter, and eventually explode at daily frequency. Consequently, the Wald tests in all models unambiguously and unanimously reject the null hypothesis. In contrast, it appears that the Wald tests yield some mixed results for the same hypothesis as the sampling interval moves away from a neighborhood of zero and further increases.[7]

The dependency of the test results on the sampling frequency is of course extremely undesirable, since in most cases the hypothesis of interest is not specific to sampling frequency and we expect it to hold for all samples collected at any sampling interval. Subsequently, we consider a continuous time regression model and build up an appropriate framework to analyze this dependency of the test results on the sampling frequency. We find that what we observe here as the common feature of the Wald tests is not an anomaly. From our asymptotic analysis relying on $\delta \to 0$ as well as $T \to \infty$, it actually becomes clear that the test is expected to diverge up to infinity with probability one as δ decreases down to zero. Roughly, this happens since the serial correlation at any finite lag of discrete samples converges to unity as the sampling frequency increases if the samples are taken from continuous time stochastic processes. We may allow for the presence of jumps, if the jump activity is regular and there are only a finite number of jumps in any time interval.

3 The Model, Setup and Preliminaries

Consider the standard regression model

$$y_i = x_i'\beta + u_i \tag{3.1}$$

for $i = 1, \ldots, n$, where (y_i) and (x_i) are respectively the regressand and regressor, β is the regression coefficient and (u_i) are the regression errors. Though it is possible to analyze more general regressions, the simple model we consider here is sufficient to illustrate the

[7]We expect that the Wald tests relying on chi-square critical values are subject to nonnegligible size distortions, due mainly to the presence of persistency in the regressors and regression errors of our models. Nevertheless, we will not pay attention to this and other related problems of the Wald tests because, as discussed, the focus of this paper is to analyze their frequency dependence, not their performance at any particular frequency.

main issue dealt with in the paper. Throughout, we denote by $\hat{\beta}$ the OLS estimator of β. The general linear hypothesis on β, formulated typically as $R\beta = r$ with known matrix R and vector r of conformable dimensions, is often tested using the Wald statistic defined by

$$F(\hat{\beta}) = (R\hat{\beta} - r)' \left[R \left(\sum_{i=1}^{n} x_i x_i' \right)^{-1} R' \right]^{-1} (R\hat{\beta} - r)/\hat{\sigma}^2, \tag{3.2}$$

where $\hat{\sigma}^2$ is the usual estimator for the error variance obtained from the OLS residuals $(\hat{u}_i)$. In the presence of serial correlation in (u_i), the Wald statistic introduced in (3.2) is in general not applicable. Therefore, in this case, modified versions of the Wald statistic such as

$$G(\hat{\beta}) = (R\hat{\beta} - r)' \left[R \left(\sum_{i=1}^{n} x_i x_i' \right)^{-1} R' \right]^{-1} (R\hat{\beta} - r)/\hat{\omega}^2, \tag{3.3}$$

where $\hat{\omega}^2$ is a consistent estimator for the long-run variance of (u_i) based on $(\hat{u}_i)$, or

$$H(\hat{\beta}) = (R\hat{\beta} - r)' \left[R \left(\sum_{i=1}^{n} x_i x_i' \right)^{-1} n\hat{\Omega} \left(\sum_{i=1}^{n} x_i x_i' \right)^{-1} R' \right]^{-1} (R\hat{\beta} - r), \tag{3.4}$$

where $\hat{\Omega}$ is a consistent estimator for the long-run variance of $(x_i u_i)$ based on $(x_i \hat{u}_i)$. We consider two different types of regressions given by (3.1): stationary type regression and cointegration type regression. The test based on (3.4) is generally more appropriate for the stationary type regression, whereas only the test based on (3.3) is sensible for the cointegration type regression.[8]

We analyze regression (3.1), when (y_i) and (x_i) are high frequency observations.[9] For the subsequent analysis, we let (y_i) and (x_i) be samples collected at discrete time intervals from the underlying continuous time processes denoted respectively by $Y = (Y_t)$ and $X = (X_t)$, i.e., we let

$$y_i = Y_{i\delta} \quad \text{and} \quad x_i = X_{i\delta}$$

for $i = 1, \ldots, n$ be discrete samples from the continuous time processes Y and X over time $[0, T]$ collected at the sampling interval with length $\delta > 0$, where $T = n\delta$. Our asymptotics will be obtained by letting $\delta \to 0$ and $T \to \infty$ jointly. Under our setup, the regression model

[8] Note that the long-run variance of $(x_i u_i)$ does not exist if (x_i) is nonstationary.

[9] In the paper, high frequency observations are defined to be samples collected at sampling intervals which are small relative to their time span. For instance, five years of daily observations are considered to be high frequency observations. Our empirical analysis in the paper uses daily observations for fifty-eight years.

introduced in (3.1) can be analyzed using the corresponding continuous time regression

$$Y_t = X_t'\beta + U_t \tag{3.5}$$

for $0 \leq t \leq T$, where Y and X are the regressand and regressor processes, and $U = (U_t)$ is the error process, from which (u_i) are defined similarly as (y_i) and (x_i) are defined from Y and X respectively.

We use the asymptotics relying on $\delta \to 0$ and $T \to \infty$, since it is most appropriate for our study which investigates the effect of increasing sampling frequency on the behaviors of the estimators and test statistics for the continuous time regression model (3.5). It is necessary to set $\delta \to 0$, since we want to analyze high frequency regressions. Moreover, we need to set $T \to \infty$ for our study, since it is required to have consistency for the estimators and test statistics considered in the paper. For fixed T, all of them are inconsistent. Finally, we require $\delta \to 0$ faster relative to $T \to \infty$ in our asymptotics to ensure that the discrete time regression (3.1) meaningfully approximates the continuous time regression (3.5) under the situation where the sample span expands.

The underlying continuous time model in (3.5), which we rewrite as $Y_t = \alpha + \beta X_t + U_t$ in its simplest form for our discussions below, is introduced to study the type of high frequency regressions we consider in Section 2. There are other continuous time models that are more suitable to analyze different types of regressions relying on high frequency observations. Most notably, we may consider a continuous time regression given by

$$dY_t = (\alpha + \beta X_t)dt + dU_t, \tag{3.6}$$

where $dU = (dU_t)$ is a martingale differential error, which is often specified further as $dU_t = \sigma_t dW_t$ for $t \geq 0$ with instantaneous volatility $\sigma = (\sigma_t)$ and Brownian motion $W = (W_t)$. This type of regression is used in Choi, Jacewitz, and Park (2016) to investigate stock return predictability. Moreover, we may also look at a continuous time regression of the form

$$dY_t = \alpha dt + \beta dX_t + dU_t, \tag{3.7}$$

where dU is defined as in (3.6). See, e.g., Chang, Choi, Kim, and Park (2016), which uses regression in (3.7) to evaluate factor pricing models.

In continuous time regressions, the differentials dY and dX should be used to represent *flow* variables like stock or portfolio returns over an infinitesimal interval, whereas the levels Y and X are more appropriate to specify *stock* variables such as stock prices, interest rates and predictive ratios.[10][11] Also note that we need the dt term to include a constant term or to relate any stock variable X to a flow variable dY. In the paper, we only consider the continuous time regression in *levels*, which is useful to analyze the high frequency regression modeling a relationship between a 'stock' variable and other 'stock' covariates. Therefore, we focus on the continuous time regression (3.5), and will not consider other types of continuous time regressions such as (3.6) and (3.7). The asymptotics of these continuous time regressions are quite distinct from our asymptotics in the paper.

For any stochastic process $Z = (Z_t)$ appearing in the paper, we assume that $Z = Z^c + Z^d$, where Z^c is the continuous part and Z^d the jump part defined as $Z_t^d = \sum_{0 \leq s \leq t} \Delta Z_s$ with $\Delta Z_t = Z_t - Z_{t-}$.

Assumption A. Let Z be any element in U^2, XX' or XU. We have

$$\sum_{0 \leq t \leq T} \mathbb{E}|\Delta Z_t| = O(T).$$

Moreover, if we define $\Delta_{\delta,T}(Z) = \sup_{0 \leq s,t \leq T, |t-s| \leq \delta} |Z_t^c - Z_s^c|$, then

$$\max\left(\delta, \frac{\delta}{T} \sup_{0 \leq t \leq T} |Z_t|\right) = O_p\Big(\Delta_{\delta,T}(Z)\Big)$$

as $\delta \to 0$ and $T \to \infty$.

The conditions in Assumption A are very mild and they are satisfied by virtually all stochastic processes used in both theoretical and empirical applications. The first condition is met, for instance, for all processes with compound Poisson type jumps as long as their sizes are bounded in L^1 and their intensity is proportional to T. The second condition holds trivially for a wide class of stochastic processes. If T is fixed, $\Delta_{\delta,T}(Z)$ represents the usual modulus of continuity of the stochastic process Z. On the other hand, we let $T \to \infty$ in our setup and therefore it may be regarded as the global modulus of continuity. Typically, we have

$$\Delta_{\delta,T}(Z) = \delta^{1/2-\epsilon} \lambda_T \tag{3.8}$$

for some $\epsilon \geq 0$ and a nonrandom sequence (λ_T) of T that is bounded away from zero and, as an example, the condition is clearly satisfied if $\sup_{0 \leq t \leq T} |Z_t| = O_p(T)$,[12] regardless of

[10] Here we follow the usual convention and classify a variable as a *flow* variable if it is defined over an interval of time, and as a *stock* variable if it is defined at a specific time.

[11] The regressand Y in both (3.6) and (3.7) is therefore a flow variable, whereas the regressand X is a stock variable and a flow variable in (3.6) and (3.7), respectively.

[12] Note that we have $\sup_{0 \leq t \leq T} |Z_t| = O_p(\sqrt{T})$ if Z is Brownian motion. Therefore, the condition here is

how we set δ and T as long as $\delta \to 0$ and $T \to \infty$.

Note that the second condition holds in any case as long as $\delta \to 0$ fast enough relative to $T \to \infty$. This is a necessary condition to guarantee that discretization errors become negligible asymptotically and our asymptotics are determined solely by those of the underlying continuous time processes. If, for instance, we set $\delta \to 0$ slowly relative to $T \to \infty$, our asymptotics in the paper will not be applicable. Sufficient conditions for the applicability of our asymptotics are dependent on the underlying model and statistical procedure we use, and they will be introduced later.

The following lemma allows us to approximate the sample moments in discrete time by the corresponding sample moments in continuous time. Here and elsewhere in the paper, we use $\|\cdot\|$ to denote the Euclidian norm for a vector or a matrix.

Lemma 3.1. *Let Assumption A hold. If we define* $Z = U^2, XX'$ *or* XU *and* $z_i = Z_{i\delta}$ *for* $i = 1, \ldots, n$*, we have*

$$\frac{1}{n}\sum_{i=1}^{n} z_i = \frac{1}{T}\int_0^T Z_t dt + O_p\Big(\Delta_{\delta,T}(\|Z\|)\Big)$$

for all small δ *and large* T*.*

In our subsequent analysis, we impose a set of sufficient conditions to ensure the asymptotic negligibility of the approximation error $\Delta_{\delta,T}(\|Z\|)$, for $Z = U^2, XX'$ and XU, so that we may approximate all relevant sample moments by their continuous analog without affecting their asymptotics. Once the approximations are made, the rest of our asymptotics rely entirely on the asymptotics of moments in continuous time. This will be introduced below.

Assumption B. $T^{-1}\int_0^T U_t^2 dt \to_p \sigma^2$ for some $\sigma^2 > 0$ as $T \to \infty$.

Needless to say, Assumption B holds for a wide variety of asymptotically stationary stochastic processes.

As discussed, we consider two different types of regressions. Below we introduce assumptions for each of these regressions. We denote by $D[0,1]$ the space of cadlag functions endowed with the usual Skorohod topology.

Assumption C1. We assume that
(a) $T^{-1}\int_0^T X_t X_t' dt \to_p M$ as $T \to \infty$ for some nonrandom matrix $M > 0$, and
(b) we have

$$T^{-1/2}\int_0^T X_t U_t dt \to_d \mathbb{N}(0, \Pi)$$

very mild and expected to be satisfied even by some explosive processes.

as $T \to \infty$, where $\Pi = \lim_{T\to\infty} T^{-1}\mathbb{E}\left(\int_0^T X_t U_t dt\right)\left(\int_0^T X_t U_t dt\right)' > 0$, which is assumed to exist.

Assumption C2. We assume that
(a) for X^T defined as

$$X_t^T = \Lambda_T^{-1} X_{Tt}$$

on $[0,1]$ with an appropriate nonsingular normalizing sequence (Λ_T) of matrices, we have $X^T \to_d X^\circ$ in the product space of $D[0,1]$ as $T \to \infty$ with linearly independent limit process X°, and
(b) if we define U^T on $[0,1]$ as

$$U_t^T = T^{-1/2}\int_0^{Tt} U_s ds,$$

then $U^T \to_d U^\circ$ in $D[0,1]$ jointly with $X^T \to_d X^\circ$ in the product space of $D[0,1]$ as $T \to \infty$, where U° is Brownian motion with variance $\pi^2 = \lim_{T\to\infty} T^{-1}\mathbb{E}\left(\int_0^T U_t dt\right)^2 > 0$, which is assumed to exist.

Assumptions C1 and C2 are introduced for the stationary regression and the cointegrating regression, respectively. They are not stringent and expected to hold for a wide class of regressions in continuous time. Assumption C1 is the continuous time analog of the standard assumptions for the stationary regression in discrete time: Parts (a) and (b) require continuous time law of large numbers and central limit theory, respectively. The former holds for all stationary continuous time process X with finite second moment, and the latter holds for stationary processes under suitable mixing and moment conditions as shown in, e.g., Rozanov (1960), similarly as for discrete time series. Assumption C2 is also the continuous time analog of the standard assumptions for the cointegrating regression in discrete time. Part (a) is satisfied for general null recurrent diffusions and jump diffusions, as shown by Jeong and Park (2011), Jeong and Park (2014) and Kim and Park (2017). Part (b) is the continuous time version of the invariance principle, which holds for a class of functions of continuous-time stationary ergodic Markov process, for example, as shown in Theorem 2.1 of Bhattacharya (1982).[13] Note that we assume in Assumption C1 that X and U are uncorrelated as in the standard regression model. However, in Assumption C2, we allow X and U to be dependent with each other in an arbitrary manner.

In parallel with Assumptions C1 and C2, respectively for the stationary and cointegrating regressions, we introduce Assumptions D1 and D2 below.

[13]If $U^T = (U_t^T)$ is defined as $U_t^T = T^{-1/2}\sum_{i=1}^{[Tt]} u_i$ for a discrete time series $(u_i)_{i=1}^T$, this assumption reduces to the usual invariance principle employed routinely for the cointegrating regression in discrete time.

Assumption D1. $\Delta_{\delta,T}(U^2), \Delta_{\delta,T}(\|XX'\|) \to_p 0$ and $\sqrt{T}\Delta_{\delta,T}(\|XU\|) \to_p 0$ as $\delta \to 0$ and $T \to \infty$.

Assumption D2. $\Delta_{\delta,T}(U^2), \|\Lambda_T\|^2\Delta_{\delta,T}(\|XX'\|) \to_p 0$ and $\sqrt{T}\|\Lambda_T\|\Delta_{\delta,T}(\|XU\|) \to_p 0$ as $\delta \to 0$ and $T \to \infty$.

The conditions in Assumptions D1 and D2 hold if $\delta \to 0$ fast enough relative to $T \to \infty$. If the sample paths of U and X are bounded and differentiable with bounded derivatives, then $\Delta_{\delta,T}(\|Z\|) = O_p(\delta)$ for $Z = U^2, XX'$ and XU and $\|\Lambda_T\| = O_p(1)$. In this case, it suffices to have $\delta = o(T^{-1/2})$. Generally, we require $\delta \to 0$ faster relative to $T \to \infty$ if the underlying processes U and X are more volatile locally and more explosive globally. If they behave all like Brownian motion, for instance, then we may easily deduce that $\Delta_{\delta,T}(\|Z\|) = O_p\left(\delta^{1/2-\epsilon}T^{1/2+\epsilon}\right)$ with an arbitrarily small $\epsilon > 0$ for $Z = U^2, XX'$ and XU from, e.g., Kanaya, Kim, and Park (2018). Therefore, the conditions in Assumption D1 are met if we set $\delta = O\left(T^{-2-\epsilon}\right)$ for any $\epsilon > 0$ arbitrarily small. To satisfy the conditions in Assumption D2, we should have $\delta \to 0$ faster, but so much unless X is overly explosive. If $\|\Lambda_T\| = O(\sqrt{T})$ as in the case of X being given by Brownian motion, we only need to require $\delta = O\left(T^{-3-\epsilon}\right)$ for $\epsilon > 0$ arbitrarily small. Therefore, our conditions here are expected to hold for a large class of continuous time models.

In our asymptotic analysis, we let $\delta \to 0$ and $T \to \infty$ jointly, under either Assumption D1 or D2. Our asymptotics are joint, not sequential, in δ and T. We allow $\delta \to 0$ and $T \to \infty$ jointly, as long as δ and T satisfy an appropriate condition specified in Assumption D1 or D2. In both of these assumptions, we require $\delta \to 0$ sufficiently fast relative to $T \to \infty$. It is therefore expected that our joint asymptotics yield the same results as the sequential asymptotics relying on $\delta \to 0$ followed by $T \to \infty$.

4 Spuriousness of Regression at High Frequency

In this section, we establish the asymptotics of the OLS estimator $\hat{\beta}$ of β in regression (3.1) and analyze the asymptotic behaviors of the standard Wald test based on the test statistic $F(\hat{\beta})$ in (3.2) under the null hypothesis $\mathrm{H}_0 : R\beta = r$. In the subsequent development of our asymptotic theory for the cointegrating regression in continuous time, we assume that there exist an invertible sequence of matrices $\Lambda_T^\bullet$ such that

$$\Lambda_T^{\bullet\prime} R \sim_p R\Lambda_T',$$

where $M \sim_p N$ implies $M = N\big(1 + o_p(1)\big)$. This assumption would be satisfied if (i) rows of R have no asymptotically redundant restrictions and (ii) columns of R have no asymptotically disappearing restrictions.[14] In our general setup, each restriction represented by a row of R is defined meaningfully when it involves only a subvector of β whose OLS estimators converge at a common rate. Here we require that R is defined to be valid asymptotically.

Theorem 4.1. *Assume $R\beta = r$ and let Assumptions A and B hold.*
(a) Under Assumption C1, we have

$$\sqrt{T}(\hat{\beta} - \beta) \to_d N$$
$$\delta F(\hat{\beta}) \to_d N'R'\left(RM^{-1}R'\right)^{-1} RN/\sigma^2,$$

where $N =_d \mathbb{N}\left(0, M^{-1}\Pi M^{-1}\right)$, as $\delta \to 0$ and $T \to \infty$ satisfying Assumption D1.
(b) Under Assumptions C2, we have

$$\sqrt{T}\Lambda_T'(\hat{\beta} - \beta) \to_d P$$
$$\delta F(\hat{\beta}) \to_d P'R'\left(RQ^{-1}R'\right)^{-1} RP/\sigma^2$$

where $P = \left(\int_0^1 X_t^\circ X_t^{\circ\prime} dt\right)^{-1} \int_0^1 X_t^\circ dU_t^\circ$ and $Q = \int_0^1 X_t^\circ X_t^{\circ\prime} dt$, as $\delta \to 0$ and $T \to \infty$ satisfying Assumption D2.

For both stationary and cointegration type regressions, the OLS estimator $\hat{\beta}$ is generally consistent for β under our asymptotics relying on $\delta \to 0$ sufficiently fast relative to $T \to \infty$. It is crucial that we have $T \to \infty$ for the consistency of $\hat{\beta}$. If, for instance, T is fixed, $\delta \to 0$ alone is not sufficient for its consistency. On the other hand, for both stationary and cointegration type regressions, we have $F(\hat{\beta}) \to_p \infty$ as $\delta \to 0$ and $T \to \infty$. This implies that the Wald test always leads us to reject the null hypothesis when it is correct, and the asymptotic size becomes unity, as $\delta \to 0$ and $T \to \infty$. The regressions therefore become spurious in this sense.

It is easy to see why this happens. Suppose that the law of large numbers and the central limit theorem hold for U, as we assume in Assumption C1 or C2. Moreover, we let Assumption A hold for U, and set $\Delta_{\delta,T}(U) \to_p 0$ or more strongly $\sqrt{T}\Delta_{\delta,T}(U) \to_p 0$ if needed as $\delta \to 0$ and $T \to \infty$. We may easily deduce that

$$\frac{1}{n}\sum_{i=1}^{n} u_i = \frac{1}{T}\int_0^T U_t dt + o_p(1) \to_p 0$$

as $n \to \infty$ (with $\delta \to 0$ and $T \to \infty$), and therefore, the law of large numbers holds for (u_i).

[14] If R is defined appropriately, we may simply set $\Lambda_T^{\bullet\prime} R = R\Lambda_T'$ as assumed in our proofs.

However, we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} u_i = \frac{1}{\sqrt{\delta}}\left[\frac{1}{\sqrt{T}}\int_0^T U_t dt + o_p(1)\right] \to_p \infty$$

as $n \to \infty$ (with $\delta \to 0$ and $T \to \infty$), and consequently, the central limit theory fails to hold for (u_i). In fact, in our setup, (u_i) becomes strongly dependent as $\delta \to 0$, since the correlation between u_i and u_{i-j} for any i and j becomes unity as $\delta \to 0$. Therefore, it is well expected that the central limit theory does not hold for (u_i).

Our results here are very much analogous to those from the conventional spurious regression, which was first investigated through simulations by Granger and Newbold (1974) and explored later analytically by Phillips (1986). As is now well known, the regression of two independent random walks, or more generally, integrated time series with no cointegration, yields spurious results, and the Wald statistic for testing no long-run relationship diverges to infinity, implying falsely the presence of cointegration. Granger and Newbold (1974) originally suggest that this is due to the existence of strong serial dependence in the regression error. On the other hand, we show in the paper that an authentic relationship in stationary time series or the presence of cointegration among nonstationary time series is always rejected if the test is based on the Wald statistic relying on observations collected at high frequencies. Our spurious regression here is therefore in contrast with the conventional spurious regression. True relationship is rejected and tested to be false in the former, while false relationship is rejected and tested to be true in the latter. However, our regression and the conventional spurious regression have the same reason why they yield nonsensical results: They both have regression errors that are strongly dependent, and the central limit theory does not hold for them.

To further analyze the serial dependency in (u_i), we consider the AR(1) specification[15]

$$u_i = \rho u_{i-1} + \varepsilon_i, \tag{4.1}$$

and introduce some additional assumptions in

Assumption E. (a) We let U^c, the continuous part of U, be a semimartingale given by $U^c = A + M$, where A and M are respectively the bounded variation and martingale components of U^c satisfying

$$\sup_{0\le s,t\le T}\frac{|A_t - A_s|}{|t-s|} = O_p(p_T) \quad \text{and} \quad \sup_{0\le s,t\le T}\frac{\big|[M]_t - [M]_s\big|}{|t-s|} = O_p(q_T),$$

[15]We use an AR(1) specification for (u_i) here not as a true model but as a fitted model. Clearly, AR(1) processes cannot be defined meaningfully in a continuous time framework.

and $p_T \Delta_{\delta,T}(U) \to_p 0$ and $(q_T/\sqrt{T})\Delta_{\delta,T}(U) \to_p 0$ with $\delta = O_p(\Delta^2_{\delta,T}(U))$ as $\delta \to 0$ and $T \to \infty$. (b) Moreover, we assume that $\sum_{0 \le t \le T} \mathbb{E}(\Delta U_t)^4 = O(T)$ and $T^{-1}[U]_T \to_p \tau^2$ for some $\tau^2 > 0$ as $T \to \infty$.

The conditions introduced in Assumption E are mild and expected to hold for a wide class of asymptotically stationary error processes. In Part (a), we require that both the bounded variation component and the quadratic variation of the martingale component of the continuous part of the error process U be Lipschitz continuous and δ be small enough to allow their Lipschitz constants to increase with T. On the other hand, Part (b) holds, for instance, if the number of jumps increases at T-rate and jump size has finite fourth moment, and if the instantaneous variance of U is asymptotically stationary. Note that $[U]_T = [U^c]_T + \sum_{0 \le t \le T}(\Delta U_t)^2$ and $[U^c]_T = [M]_T$.

The asymptotics for the estimated AR coefficient $\tilde{\rho}$ of ρ in (4.1) are given as

Lemma 4.2. *Under Assumption E, we have*

$$\tilde{\rho} = 1 - \frac{\tau^2}{2\sigma^2}\delta + o_p(\delta)$$

as $\delta \to 0$ and $T \to \infty$.

It follows immediately from Lemma 4.2 that

$$\tilde{\rho} \to_p 1$$

particularly as $\delta \to 0$. Therefore, (u_i) becomes strongly dependent, and the regression becomes spurious as the sampling interval δ approaches 0. Our regression is completely analogous in this regard to the conventional spurious regression, except that we let $\delta \to 0$ in contrast with the conventional spurious regression requiring $n \to \infty$.[16] Therefore, the results in Theorem 4.1 may well be expected. Though we let $T \to \infty$, as well as $\delta \to 0$, to get a more explicit limit of $\tilde{\rho}$ as in Lemma 4.2, the condition $T \to \infty$ is not essential for the spuriousness in regression (3.1). This is clear from our proof of Lemma 4.2. If T is assumed to be fixed and set at $T = 1$ without loss of generality, we have $\delta = 1/n$ and (u_i) asymptotically reduces to a near unit root process with AR coefficient $\rho = 1 - c/n$ with $c = \tau^2/2\sigma^2$.

Of course, it is also possible to formulate and analyze the classical spurious regression in continuous time. If the underlying regression error process U is indeed nonstationary

[16] Clearly, our regression is not directly comparable to the conventional spurious regression. Unlike the classical spurious regression having an error process that is nonstationary and has a stochastic trend, we set the error process U to be essentially stationary in our model.

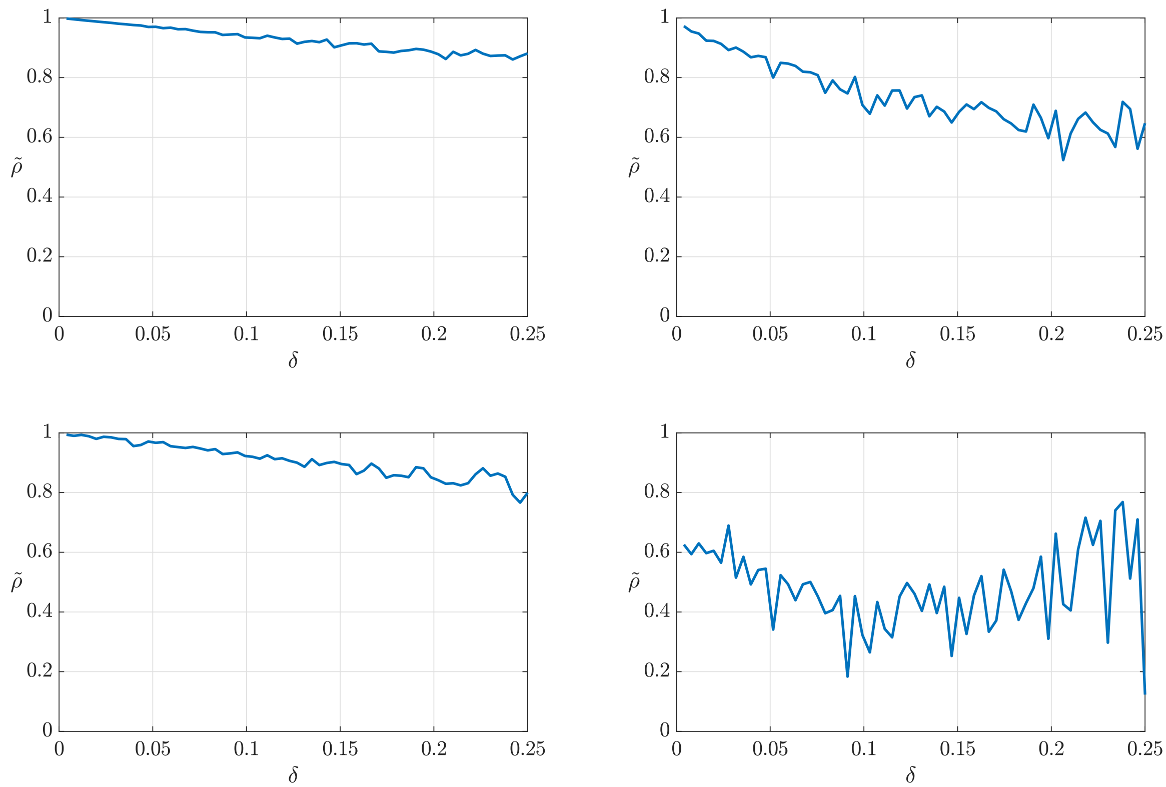
and has a stochastic trend, contrarily to our assumption, we have $T^{-1}\int_0^T U_t^2 dt \to_p \infty$ as $T\to\infty$. Therefore, we expect that our regression becomes spurious as long as T is large even if δ is not small. In the paper, however, we assume $T^{-1}\int_0^T U_t^2 dt \to_p \sigma^2$ as $T\to\infty$, and let $\delta\to 0$ to analyze the spuriousness generated by high frequency observations. In our setup, the regression error (u_i) becomes strongly persistent and the regression becomes spurious, simply because we collect samples too frequently.

The speed at which $\tilde{\rho}$ diverges away from the unity as δ increases depends on the ratio τ^2/σ^2. Roughly, τ^2 measures the mean local variation, while σ^2 represents the mean global variation of the error process U. Therefore, we may refer to it as the *local-to-global variation ratio*. The larger the value of the ratio is, the slower $\tilde{\rho}$ converges to the unity as $\delta\to 0$. Note that the ratio becomes large if, in particular, either the existence of excessive volatility makes local variation large, or the presence of strong mean reversion makes global variation small. If U is the Ornstein-Uhlenbeck process given by $dU_t = -\kappa U_t dt + \upsilon dW_t$, we have $\tau^2 = \upsilon^2$ and $\sigma^2 = \upsilon^2/2\kappa$. Therefore, the ratio becomes $\tau^2/\sigma^2 = 2\kappa$, which becomes larger if we have stronger mean reversion. The local-to-global variation ratios of U estimated using the fitted residuals of Models I-IV are 1.08, 14.30, 3.16 and 169.9, respectively. The ratio is particularly large for Model IV, which explains why its estimated residual AR coefficient is not quite close to the unity at the highest frequency, daily.

The actual estimates of the autoregressive coefficients for the fitted residuals from Models I-IV are plotted in Figure 3 against various values of the sampling interval. It is clearly seen that the estimates of the autoregressive coefficients tend to increase as the sampling interval shrinks. In particular, except for Model IV, the estimates approach unity as the sampling interval decreases. This is exactly what we expect from Lemma 4.2. Model IV is rather exceptional. For Model IV, the estimated autoregressive coefficients do not show any monotonous increasing trend, unlike all other models. In fact, Lemma 4.2 does not seem to apply for Model IV, which alludes that Assumption E does not hold for Model IV. This is perhaps due to more irregular and frequent jump activities present in stock prices than are allowed in Assumption E. Of course, not all types of jumps are permitted in our paper, though our assumptions on jumps are fairly general and weak. In particular, we assume that the jump intensity is proportional to time span, following most of the existing literature, which excludes the possibility of having jumps with intensity varying with sampling frequency.

Needless to say, all our analysis for (u_i) applies also to any linear combination of the vector time series $(x_i u_i)$, $(c' x_i u_i)$ for an arbitrary nonrandom vector c, say, if we assume the vector process $c'XU$ satisfies the same conditions as those we impose on U above in Assumption E.

Figure 3: Estimated Residual AR Coefficients in Models I-IV



Notes: Presented are estimated autoregressive coefficients in the first order autoregression of the residuals from regressions of Models I-IV (Model I in the upper-left, Model II in the upper-right, Model III in the lower-left, and Model IV in the lower-right panel), obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).

5 Asymptotics of Robust Wald Tests

In this section, we develop the asymptotics of the robust Wald tests based on the test statistics $G(\hat{\beta})$ and $H(\hat{\beta})$ in (3.3) and (3.4), which will be referred to as the G and H-tests for short. These robust test statistics involve the long-run variance estimators of (u_i) and $(x_i u_i)$, instead of the usual variance estimators used for the standard Wald test, to take into account the presence of serial correlation in (u_i) and $(x_i u_i)$. Our asymptotics in this section show that, in general, the spuriousness in regression at high frequency we observe and analyze in the previous sections is expected to appear also in the G and H-tests, as well as in the Wald test. Therefore, it is not exclusively due to neglecting the presence of serial correlation in (u_i) and $(x_i u_i)$, which necessarily emerges in high frequency regression. The spuriousness disappears only when we properly estimate the long-run variances of (u_i) and $(x_i u_i)$ using bandwidths chosen suitably. In what follows, we write $M \approx N$ to signify that M and N are asymptotically equivalent, and $M \sim_p N$ to denote more specifically that $M = N\bigl(1 + o_p(1)\bigr)$.

Under the null hypothesis, we may expect that

$$G(\hat\beta) \approx P_T'R'\left[RQ_T^{-1}R'\right]^{-1}RP_T/(\delta\hat\omega^2)$$
$$H(\hat\beta) \approx P_T'R'\left[RQ_T^{-1}(\delta\hat\Omega)Q_T^{-1}R'\right]^{-1}RP_T \tag{5.1}$$

with

$$P_T = \left(\int_0^T X_tX_t'dt\right)^{-1}\int_0^T X_tU_tdt, \quad Q_T = \int_0^T X_tX_t'dt,$$

if δ is sufficiently small relative to T. Therefore, it is clearly seen that the long-run variance estimators, $\hat\omega^2$ and $\hat\Omega$ respectively of (u_i) and (x_iu_i), hold very important roles in the asymptotics of G and H-tests.

To analyze the long-run variance estimators of (u_i) and (x_iu_i), we set $v_i = u_i$ or x_iu_i and let $v_i = V_{i\delta}$, $i = 1, \ldots, n$, with $V = U$ or XU, and assume for expositional simplicity that V is one-dimensional. The commonly used long-run variance estimator π_n^2 of (v_i) is given by

$$\pi_n^2 = \sum_{|j|\le n} K\left(\frac{j}{b_n}\right)\gamma_n(j), \tag{5.2}$$

where K is the kernel function, $\gamma_n(j) = n^{-1}\sum_i v_iv_{i-j}$ is the sample autocovariance function of (v_i) and b_n is the bandwidth parameter.[17] If δ is fixed and (v_i) has a well defined long-run variance π^2, then we would expect $\pi_n^2 \to_p \pi^2$ as long as $b_n \to \infty$ and $b_n/n \to 0$ as $n \to \infty$.[18] If we set $\delta \to 0$ as in our setup, however, π_n^2 behaves much differently in asymptotics.

The type of long-run variance estimator in (5.2) defined with high frequency observations is analyzed in Lu and Park (2019). Following them, we relate it to the long-run variance estimator ϖ_T^2 of the underlying continuous time process V, which is given by

$$\varpi_T^2 = \int_{|s|\le T} K\left(\frac{s}{B_T}\right)\Gamma_T(s)ds, \tag{5.3}$$

where K is the kernel function, $\Gamma_T(s) = T^{-1}\int_0^T V_tV_{t-s}dt$ is the sample autocovariance function, and B_T is the bandwidth parameter. Under very general regularity conditions,

[17] Of course, (u_i) is not observable and should be estimated in practice. However, this is unimportant and it is ignored here.

[18] In the paper, we only consider inferences based on consistent long-run variance estimators. Therefore, in particular, we do not allow the bandwidth parameter to be set as $b_n = bn$ for some constant $b > 0$, which yields the so-called fixed-b asymptotics. See, e.g., Sun (2014) for some recent developments on the subject.

they show that

$$\varpi_T^2 \to_p \varpi^2 \tag{5.4}$$

as $T \to \infty$, where ϖ^2 is the long-run variance of V, if and only if $B_T \to \infty$ and $B_T/T \to 0$ as $T \to \infty$.

It is easy to see that

$$\gamma_n(j) = \frac{1}{n}\sum_{i=1}^{n} v_i v_{i-j} = \frac{1}{T}\sum_{i=1}^{n} \delta V_{i\delta} V_{(i-j)\delta} \approx \frac{1}{T}\int_0^T V_t V_{t-j\delta} dt = \Gamma_T(j\delta), \tag{5.5}$$

and therefore, we have

$$\delta\pi_n^2 \approx \sum_{|j|\leq n} \delta K\left(\frac{j\delta}{b_n\delta}\right)\Gamma_T(j\delta) \approx \int_{|s|\leq T} K\left(\frac{s}{B_{n,\delta}}\right)\Gamma_T(s)ds, \tag{5.6}$$

where we let $B_{n,\delta} = b_n\delta$ be the continuous time bandwidth corresponding to the discrete time bandwidth b_n. [19] Consequently, we may readily deduce from (5.3), (5.4) and (5.6) that, under suitable regularity conditions,

$$\delta\pi_n^2 \to_p \varpi^2, \tag{5.7}$$

if and only if

$$B_{n,\delta} \to \infty \quad \text{and} \quad B_{n,\delta}/T \to 0 \tag{5.8}$$

as $\delta \to 0$ and $T \to \infty$.

If a discrete time bandwidth b_n yields the corresponding continuous time bandwidth $B_{n,\delta} = b_n\delta$ satisfying conditions in (5.8), we say that it is *high-frequency compatible*. Of the two high-frequency compatibility conditions in (5.8), only the first one $B_{n,\delta} = b_n\delta \to \infty$ matters, since the second one $B_{n,\delta}/T = b_n/n \to 0$ is required for any discrete time bandwidth b_n as well. For the high-frequency compatibility of a discrete time bandwidth b_n, we should have $b_n \to \infty$ faster than $\delta \to 0$. It is easy and straightforward to find a high-frequency compatible bandwidth b_n. For instance, if we set $b_n = cn^a/\delta^{1-a}$ with some $c > 0$ and $0 < a < 1$, then $B_{n,\delta} = b_n\delta = cT^a \to \infty$ as long as $T \to \infty$, and therefore, b_n becomes high-frequency compatible. This bandwidth choice will be referred to as the rule of thumb in continuous time (CRT).

However, the usual discrete time bandwidth given by $b_n = cn^a$ with some $c > 0$ and $0 < a < 1$, the scheme that we will simply refer to as the rule of thumb (RT), is *not* high-frequency compatible, since $B_{n,\delta} = b_n\delta = c\delta^{1-a}T^a \to 0$ if $\delta \to 0$ fast enough relative to

[19] We refer to Lu and Park (2019, p.263) for more details and formal justifications of our approximations in (5.5) and (5.6).

$T \to \infty$. In this case, we have

$$\frac{\delta \pi_n^2}{B_{n,\delta}} \approx \frac{1}{B_{n,\delta}} \int_{|s| \leq T} K\left(\frac{s}{B_{n,\delta}}\right) \Gamma_T(s) ds \to_p \Gamma(0) \int K(x) dx,$$

where $\Gamma(s)$ is the autocovariance function of V, which implies, in particular, that

$$\delta \pi_n^2 \to_p 0 \tag{5.9}$$

as $\delta \to 0$ and $T \to \infty$.[20]

Now it is clear that the asymptotics of the G and H-tests rely critically on whether or not the bandwidths used in their long-run variance estimators are high-frequency compatible. We expect them to have well defined limit distributions if they are based on the long-run variances estimated with high-frequency compatible bandwidths, for which (5.7) holds. However, if the long-run variance estimators rely on the bandwidths that are high-frequency incompatible and (5.9) holds, then we have

$$G(\hat{\beta}) \to_p \infty \quad \text{and} \quad H(\hat{\beta}) \to_p \infty$$

even under the null hypothesis, as $\delta \to 0$ and $T \to \infty$, exactly as in the case of the standard Wald test.

In discrete time framework, the bandwidth parameter b_n is often set to be data-dependent following Andrews (1991) and Newey and West (1994), which we simply refer to as the Andrews bandwidth and Newey-West bandwidth, respectively. Specifically, both Andrews and Newey-West bandwidths are based on the infeasible optimal bandwidth given by

$$b_n^* = \left(\frac{r \kappa_r^2 \theta_r^2}{\int K(x)^2 dx} n \right)^{1/(2r+1)}, \tag{5.10}$$

where r is the characteristic exponent of kernel function K, $\kappa_r = \lim_{x \to 0} (1 - K(x))/|x|^r$ and $\theta_r = \sum_j |j|^r \gamma(j) / \sum_j \gamma(j)$ with $\gamma(j)$ denoting the autocovariance function of (v_i). Note that θ_r is unknown and should be estimated. Andrews (1991) assumes that (v_i) is AR(1), in which case we have

$$\theta_1^2 = \frac{4\rho^2}{(1-\rho)^2(1+\rho)^2}, \qquad \theta_2^2 = \frac{4\rho^2}{(1-\rho)^4} \tag{5.11}$$

[20] If we set $b_n = bn$ for some constant $b > 0$ as in the fixed-b asymptotics in discrete time, then it follows that $B_{n,\delta} = b_n \delta = bT$. Indeed, it is not difficult to see that, in our asymptotic setting, using a fixed-smoothing parameter at high frequency regression yields the fixed-b asymptotics in continuous time. Therefore, the fixed-b approach provides self-normalization. We are grateful to a referee for pointing this out.

for $r = 1, 2$. He suggests to run an AR(1) regression for (v_i) and use the OLS estimate $\tilde{\rho}$ of the autoregressive coefficient ρ to obtain an estimate $\tilde{\theta}_r$ of θ_r in (5.11) for $r = 1, 2$.[21] On the other hand, Newey and West (1994) propose to nonparametrically estimate θ_r by $\hat{\theta}_r = \sum_{|j| \leq a_n} |j|^r \gamma_n(j) / \sum_{|j| \leq a_n} \gamma_n(j)$ with $a_n = cn^p$ for some constants c and $0 < p < 1$.

Lemma 5.1. *Let $V = U$ and XU, respectively for the cointegrating regression and the stationary regression, satisfy Assumption E, and assume $\sup_{0 \leq t \leq \infty} \mathbb{E}|V_t|^2 < \infty$. The Andrews bandwidth is high-frequency compatible, while the Newey-West bandwidth is not. For the latter, we have $B_{n,\delta} \sim_p \delta^{2r(1-p)/(2r+1)} T^{(2pr+1)/(2r+1)}$ as $\delta \to 0$ and $T \to \infty$.*

Obviously, the Newey-West bandwidth is not high-frequency compatible because $B_{n,\delta} \to 0$ if $\delta = o\left(T^{-(2pr+1)/(2r(1-p))}\right)$, which holds whenever $\delta \to 0$ fast enough relative to $T \to \infty$. The G and H-tests with the Newey-West bandwidth becomes invalid at high frequency. They are expected to be generally valid if the Andrews bandwidth is used.

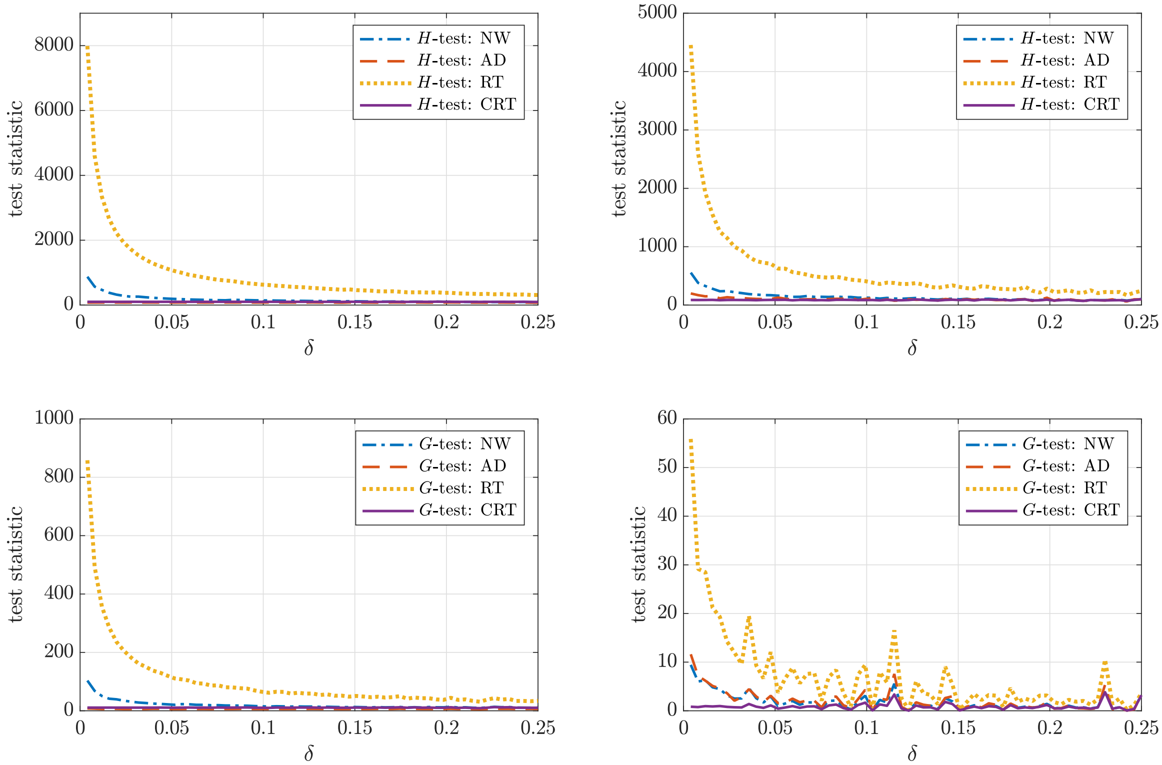
The actual values of the G and H-tests in Models I-IV introduced in Section 2 are presented in Figure 4 for various sampling intervals δ ranging from 1/252 to 1/4, which correspond to the daily and quarterly frequencies. We use the H-test for Models I/II, and the G-test for Models III/IV. Of course, this implies that we interpret Models I/II as the stationary regression, and Models III/IV as the cointegrating regression. We consider four schemes for the bandwidth choice: the rule of thumb (RT), the rule of thumb in continuous time (CRT), Andrews (AD) and Newey-West (NW).[22] As shown, CRT and AD are high-frequency compatible, whereas RT and NW are not. Our asymptotics show that the G and H-tests are valid only when high-frequency compatible bandwidths are used. If high-frequency incompatible bandwidths are used, the G and H-tests are expected to diverge even if the null hypothesis is true.

When we use high-frequency compatible CRT, the G and H-tests yield values that are not sensitive to the sampling frequency. They remain stable more or less across the entire range of the sampling intervals we consider. The G and H-tests with AD, which is also high-frequency compatible in our setup, behave as expected only in Models I-III. In Model IV, they become unreliable at high frequency. This is mainly because the estimated autoregressive coefficient for Model IV does not converge to unity, as demonstrated in Figure 3. In sharp contrast, the values of the G and H-tests become heavily dependent upon the sampling frequency if high-frequency incompatible RT and NW are used. In this case, the overall pattern of frequency dependence of the G and H-tests is exactly identical to

[21] Here we only consider the univariate version of bandwidth in Andrews (1991) to make our discussions simple and focused. However, in our simulation and application, we use its multivariate version with an identity weighting matrix.

[22] Here and elsewhere in the paper, we use the Parzen kernel. All our results are not sensitive to the choice of kernel function.

Figure 4: Robust Wald Tests in Models I-IV



Notes: Presented are the values of the H and G-statistics for the null hypothesis $\mathrm{H}_0 : \beta_0 = 0$ and $\beta_1 = 1$ in Models I/II and III/IV, respectively, obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency). The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT).

that of the standard Wald test we explore in Section 2. The test values change dramatically as the sampling frequency varies. They increase rapidly as the sampling interval becomes smaller than one month, and explode as the sampling interval approaches one day. This shows in particular that the spuriousness we observe from the standard Wald test cannot be simply dealt with by using its robust version that allows for the presence of serial correlation.

In the rest of the section, we introduce a set of regularity conditions and formally establish the asymptotics for the G and H-tests. To ensure that the G and H-tests have well defined limit null distributions, we need to assume

Assumption F. Let $V = U$ and XU, respectively for the cointegrating regression and the stationary regression, satisfy Assumptions 2.2(b), 2.3 and 4.1, and K satisfy Assumptions 2.1, 2.2(a) and 4.2 in Lu and Park (2019). Moreover, we assume that b_n used in $\hat{\omega}^2$ and $\hat{\Omega}$ is high-frequency compatible, and $b_n \delta \Delta_{\delta,T}(V) \to 0$ as $\delta \to 0$ and $T \to \infty$.

Assumption F specifies the regularity conditions on the processes U and XU, and the kernel function K used in the long-run variance estimation. See Lu and Park (2019) for more discussions on these conditions.

The limit null distributions of the G and H-tests are presented below. We let q be the number of restrictions, and χ_q^2 denote the chi-square distribution with q degrees of freedom.

Theorem 5.2. *Assume $R\beta = r$ and let Assumptions A and F hold.*
(a) Under Assumptions C1 and D1, we have

$$H(\hat{\beta}) \to_d \chi_q^2$$

as $\delta \to 0$ and $T \to \infty$.
(b) Under Assumptions C2 and D2, we have

$$G(\hat{\beta}) \to_d P^{*\prime} R' \left(RQ^{-1}R'\right)^{-1} RP^*$$

as $\delta \to 0$ and $T \to \infty$, where $P^* = \left(\int_0^1 X_t^\circ X_t^{\circ\prime} dt\right)^{-1} \int_0^1 X_t^\circ dU_t^*$ with standard Brownian motion U^* defined as $\pi U^* = U^\circ$ and $Q = \int_0^1 X_t^\circ X_t^{\circ\prime} dt$ using the notations in Theorem 4.1.

Both G and H-tests have well defined limit null distributions respectively for general stationary and nonstationary regressions, if in particular we use high-frequency compatible bandwidths for their long-run variance estimates. The H-test has the standard chi-square limit null distribution for stationary regressions. On the other hand, the limit null distribution of the G-test is generally nonnormal and nonstandard.[23] If, however, the limit processes X° and U° are independent, then its limit null distribution reduces to chi-square distribution.

6 An Empirical Illustration

For a long time, economists expect a stable relationship between short-term and long-term interest rates, especially since Fama (1984) discusses the expectation hypothesis of the term structure of interest rates and its implications, and multiple subsequent studies on the dynamics of short and long interest rates. Campbell and Shiller (1991) provide further insights on the expectation hypothesis by examining yield spread and interest rate movements, Cochrane (2001) by exploring exchange rate dynamics, and Ang and Piazzesi (2003) by analyzing term structure dynamics with macroeconomic and latent variables.

[23]In this case, we may use subsampling to obtain the critical values of the G-test as in, e.g., Jiang, Lu, and Park (2020, Section 4.2).

Motivated by this well-expected stable relationship between short-term and long-term rates, we empirically investigate the extent to which the interest rates of securities with time varying maturities move together using the tests introduced in the previous section.

The consistent co-movement of long and short rates, should we establish this as a stylized fact, would have important monetary policy implications. Such a finding would imply that the U.S. Federal Reserve System (Fed) could effectively control long rates simply by manipulating the policy rate through its traditional open market operations.[24] However, the notion that central banks have full control over long-term rates through their policy actions has been challenged by what is known as *Greenspan conundrum*, a puzzling observation made by the former Fed Chairman Allan Greenspan in 2005. Greenspan noted that, despite the Fed raising short-term policy rates, long-term rates remained relatively low, contrary to what traditional monetary policy would predict. During the period leading up to the conundrum, long-term rates did not rise as much as anticipated, creating a disconnect between short and long-term interest rates.

The Greenspan conundrum highlighted the complexities of interest rate dynamics and raised questions about the effectiveness of conventional monetary policy tools and the transmission mechanism between short-term and long-term interest rates. It led to further research and analysis on the relationship between short and long rates and discussions on the Greenspan conundrum and its implications.[25] Several explanations put forward to account for the unexpected behavior of long-term interest rates in the Greenspan conundrum.[26] We empirically examine the extent to which the Fed effectively managed long-term interest rates through manipulating short-term policy rates prior to the Greenspan conundrum.

We also observed during the 2008-9 global financial crisis that efficacy of the Fed's conventional policy to control long-term rates by manipulating short rates was widely questioned. The Fed's efforts to counter the negative shocks associated with the financial market meltdown were ineffective since nominal rates had fallen quite dramatically in the decade preceding the crisis, due in large part to persistently low inflation expectations (Bernanke, 2020). In an environment with the policy rate at or near the zero lower bound (ZLB), the

[24]Interest rate targeting had indeed been the central conventional monetary policy tool employed by the Fed over the past four decades to execute its dual mandate, as reflected in the testimony of Fed Chairman Alan Greenspan (1987-2006) to Congress.

[25]Ben Bernanke, who succeeded Greenspan as Chairman of the Federal Reserve, has discussed the Greenspan conundrum in various speeches and writings during his tenure, 2006-2014, Rudebusch (2006) discusses the bond yield "conundrum" from a macro-finance perspective, Gürkaynak, Sack, and Wright (2007) discuss the conundrum in their study of the dynamics of the term structure of interest rates and its relationship with monetary policy actions, and Thornton (2018) on Greenspan's conundrum and the Fed's ability to affect long-term yields.

[26]They include global factors, such as increased capital flows from foreign investors seeking higher yields, and changes in investor expectations, improvements in productivity, and increased demand for long-term bonds from pension funds and insurance.

Fed was compelled to consider implementing a set of non-conventional policy tools,[27] and they actually implemented many of these tools during and after the 2008-9 financial crisis. With these non-conventional monetary policies, the Fed hoped to directly affect long rates in spite of its inability to further lower nominal short-term rates.[28]

Long rates, of course, may change independently of the Fed due to portfolio reallocation strategies by investors that reflect state specific risks and expected return forecasts by asset class over the course of the business cycle. Moreover, the dynamics of the short-term and long-term rates have been clearly decoupled by the ZLB. Hence, we may no longer find a stable relationship between long-term and short-term interest rates for the periods during and following the ZLB period. We therefore investigate whether our hypothesis holds true also in the context of the 2008-9 global financial crisis and ZLB.

Our analysis yields consistent results for both pre and post-Greenspan conundrum and financial crisis periods. We present the results pertaining to the pre and post-Greenspan conundrum period in the main text, and provide the results for pre and post-financial crisis period in Appendix C.

We conduct our hypothesis testing in a simple stationary continuous time regression of long-term interest rate Y_t on short-term interest rate X_t

$$Y_t = \alpha + \beta X_t + U_t \tag{6.1}$$

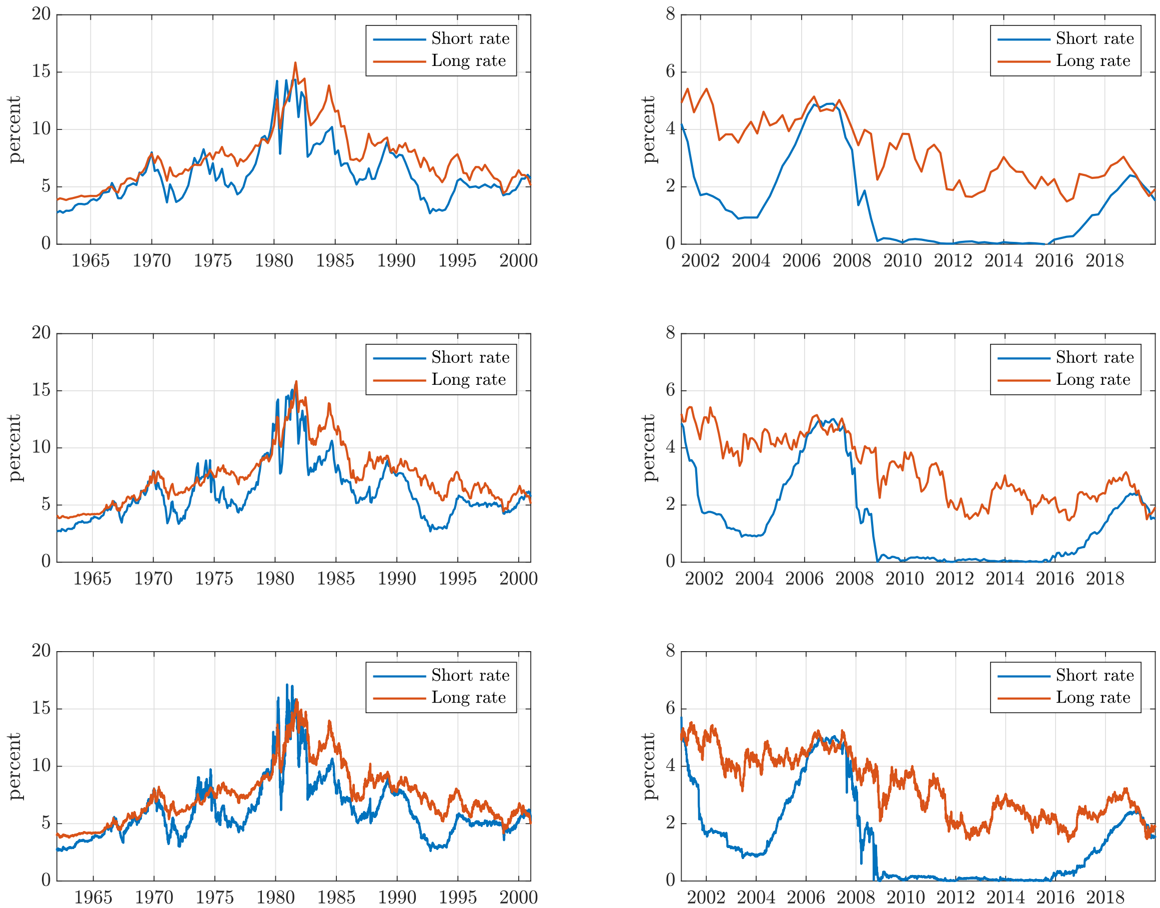
using observations $(Y_{i\delta})$ and $(X_{i\delta})$, $i = 1, \ldots, n$, collected at discrete time intervals of length δ, corresponding to quarterly, monthly and daily frequencies, where α represents the historic average of term premium, β the parameter revealing the existence of the aforementioned comovement of long-term and short-term interest rates, and U_t the regression error. In our simple empirical model above, we assume that the term premium α is constant. Of course, it would be more realistic to assume time-varying term premium since the presence of the latter has been reported by many previous studies, including the work by Ang and Piazzesi (2003).[29] Allowing for time-varying term premium in a realistic way, however, would involve some complicated nonparametric estimation. Due to this consideration, we assume that the intercept α in our simple empirical model (6.1) is constant, and interpret it as historic average term premium. Accordingly, when $\alpha = 0$, we interpret that there is no term premium on average. When the parameter β is unity, the long and short rates

[27] Non-conventional policy tools include most notably quantitative easing (QE), forward guidance, large scale asset purchases (LSAP), repurchase agreement (REPO), reverse REPO, negative interest rates and yield curve control.

[28] Interest rates hit the zero lower bound in September 2008. The three-month secondary market rate was 0.03% on September 17, 2008, and the Fed made its first purchase of financial assets in November 2008.

[29] Ang and Piazzesi (2003) find that macroeconomic factors reflecting broader economic conditions are crucial in explaining the movements of bond yields, which in turn suggests presence of time-varying term.

Figure 5: Short and Long Rates Before and After the Observed Greenspan Conundrum at Quarterly, Monthly and Daily Frequencies



Notes: The panels on the left present short (3-month T-bill) and long (10-year T-bond) rates before the observed Greenspan conundrum, 1962-2000, and those on the right plot the same series after the observed Greenspan conundrum, 2001-2019, and the top, middle and bottom panels present short and long rates at quarterly, monthly and daily frequencies, respectively.

increase or decrease in the same direction and by the same amount, implying that long-term interest rates move in tandem with short-term interest rates. As long as $\beta = 1$, we may still perfectly predict the movement of the long rate by that of the short rate conditional on the term premium. Therefore non-rejection of $\beta = 1$ may provide an empirical justification for Fed's efforts to influence long rates through its usual open market operations.

We use the interest rate on the ten-year U.S. Treasury bond for the long-term interest rate ($Y_{i\delta}$), and the three-month U.S. Treasury bill secondary market rate for the short-term rate ($X_{i\delta}$). Both data series are obtained at quarterly, monthly and daily frequencies from the Federal Reserve Economic Data (FRED) of St. Louis Fed. The two interest rates at quarterly, monthly and daily frequencies are plotted in Figure 5 where the panels on the left and right are respectively for two sub-samples, 1962-2000 and 2001-2019, signifying the periods before and after the observed Greenspan conundrum, which is defined as before and after the end of the year 2000. We start from 1962 when the ten-year rate became available at daily frequency.

We test if $\alpha = 0$ and $\beta = 1$ both separately and jointly under the null hypotheses $\mathrm{H}_0^\alpha : \alpha = 0$, $\mathrm{H}_0^\beta : \beta = 1$, and $\mathrm{H}_0^{\alpha,\beta} : (\alpha, \beta) = (0, 1)$ for the two subsamples using the four robust H-tests, AD, CRT, NW and RT, as well as the non-robust Wald test introduced in the previous section. All tests are valid asymptotically for low frequency data. For regressions with observations collected at high frequency, however, only two robust H-tests, AD and CRT, are valid according to our theoretical results and simulation evidence. The other two robust tests, NW and RT, are valid only in low frequency regressions. For easy reference, we call the AD and CRT as high-frequency-valid tests and NW and RT as high-frequency-invalid tests.

To empirically demonstrate this point, we test our hypotheses using both low frequency (quarterly) and high frequency (daily) data. Monthly data are also considered for comparison, as they can be considered as either low or high frequency depending on the sample size. Recall that our asymptotics are applicable only when $\delta \to 0$ sufficiently fast relative to $T \to \infty$. High-frequency-invalid tests may still produce reasonable results at a monthly frequency if their divergence rates are not too rapid. As shown in our simulations, the high-frequency-invalid NW test diverges more slowly as the sampling interval decreases compared to the high-frequency-invalid RT test.

The test results from the quarterly, monthly and daily stationary regressions are presented in Table 1, Table 2 and Table 3, respectively. We also conduct the same hypothesis testing for *cointegrating* regressions using G-tests, and the results are reported in Table 4, Table 5 and Table 6, respectively, for samples at quarterly, monthly and daily frequencies, in Appendix B. The results from the pre- and post-Greenspan conundrum samples are presented in the upper and lower panels in each table. The test values along with the 1% and 5% critical values for the five tests considered are reported. For the robust tests on H_0^β for pre-crisis samples, the p-values are also provided. The test values and the associated p-values are marked with one star (two stars) when the null hypothesis is rejected at 5% (1%) significance level. Non-rejection of a null hypothesis at 5% significance level provide stronger support for the null than the non-rejection at 1% level.[30]

In what follows, we regard only rejection at the 1% level as significant. This is because we have a persistent covariate in our regression, and therefore, the actual rejection probabilities

[30]Though we do not report to save space, the results obtained using the data from the entire sample period 1962-2019 are qualitatively similar to those obtained using the post-crisis data.

Table 1: Testing Results from Quarterly Regressions

	Nonrobust test	Robust H-test			
Pre-Greenspan conundrum sample (1962–2000)					
		AD	CRT	NW	RT
H_0^{α}	72.09**	8.98**	12.13**	12.22**	36.64**
H_0^{β}	10.51**	1.30	1.60	1.62	4.50*
		[0.2538]	[0.2053]	[0.2037]	[0.0339]
$H_0^{\alpha,\beta}$	220.73**	26.57**	38.37**	38.74**	132.07**
Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^{α}	412.09**	66.66**	112.40**	114.59**	325.24**
H_0^{β}	61.30**	49.65**	42.86**	43.16**	98.73**
$H_0^{\alpha,\beta}$	478.06**	82.84**	115.50**	117.47**	325.24**

Notes: Presented are test statistics (and p-values in square brackets) of the quarterly stationary regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT), among which only AD and CRT are high-frequency compatible in theory. The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

for all our tests are expected to be larger than their nominal sizes. It is indeed well known that the actual rejection probabilities of tests in time series regressions with persistent covariates are generally larger, often considerably if their persistency is strong, than their nominal values. See more discussions on this problem and relevant simulation results in Section 7.

Most interestingly, we find strong support for the hypothesis $\beta = 1$ in the pre-Greenspan conundrum sample at all three data frequencies only when the high-frequency-valid tests are used. In the post-Greenspan conundrum sample, however, it is decisively rejected by all tests across all data frequencies. This is as expected and consistent with the persistently low long rates despite rising short-term policy rates during the Greenspan conundrum observed in the early 2000s. Our results therefore provide empirical support for using the short rate to control long rates within the conventional monetary policy regime that prevailed until the Greenspan conundrum observed in the early 2000s.

We also find that term premium does exist in both subsamples at all data frequencies. The no term premium hypothesis $\alpha = 0$ is clearly rejected by all five tests regardless of whether it is tested separately under $H_0^{\alpha} : \alpha = 0$ or jointly under $H_0^{\alpha,\beta} : (\alpha, \beta) = (0, 1)$ during both sample periods in the regressions with observations collected at all three data

Table 2: Testing Results from Monthly Regressions

Pre-Greenspan conundrum sample (1962–2000)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^{α}	233.44**	9.30**	13.71**	22.21**	90.44**
H_0^{β}	39.80**	1.68	2.14	3.32	12.61**
		[0.1954]	[0.1432]	[0.0684]	[0.0004]
$H_0^{\alpha,\beta}$	656.63**	23.44**	38.07**	64.41**	307.98**

Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^{α}	1266.72**	56.29**	101.52**	162.79**	718.86**
H_0^{β}	198.96**	42.80**	42.82**	59.26**	231.82**
$H_0^{\alpha,\beta}$	1454.55**	63.85**	102.93**	163.04**	719.86**

Notes: Presented are test statistics(and p-values in square brackets) of the monthly stationary regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT), among which only AD and CRT are high-frequency compatible in theory. The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

frequencies.

The behavior of all five tests across varying frequencies in both sample periods aligns well with our asymptotic theory and simulation evidence. As presented in Tables 1–3, and consistent with our theoretical predictions, the high-frequency-invalid tests, NW and RT, show a clear tendency to diverge as the sampling frequency increases from quarterly to daily for both sample periods. In the pre-Greenspan conundrum sample, where the null hypothesis $\beta = 1$ is supported by high-frequency-valid tests across all frequencies, the high-frequency-invalid NW and RT test statistics exhibit substantial variability with sampling frequency. In the quarterly regression, both tests support the null hypothesis $\beta = 1$. However, in the monthly regression, the RT test no longer supports this null hypothesis, whereas the NW test, which diverges more slowly than the RT test, continues to do so. In the high-frequency daily regression, neither the NW nor the RT test supports the null hypothesis.

These findings underscore the importance of using high-frequency-valid tests to ensure correct inferences. Without such tests, researchers have no choice but to use lower-frequency data, even when high-frequency data are available. This may result in dependence of test results on sampling frequency and significantly reduce test power.[31]

[31] As discussed in Section 7, using data sampled at higher frequencies substantially improves the stability of

Table 3: Testing Results from Daily Regressions

Pre-Greenspan conundrum sample (1962–2000)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^α	5648.93**	10.52**	15.61**	128.17**	997.49**
H_0^β	1174.12**	2.32	2.93	21.14**	162.33**
		[0.1280]	[0.0868]	[0.0000]	[0.0000]
$H_0^{\alpha,\beta}$	13930.71**	22.74**	38.64**	422.53**	3531.82**

Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^α	28263.04**	57.69**	104.53**	1109.89**	8251.69**
H_0^β	4459.63**	45.04**	45.40**	370.35**	2706.39**
$H_0^{\alpha,\beta}$	32382.96**	65.57**	105.86**	1111.55**	8270.83**

Notes: Presented are test statistics (and p-values in square brackets) of the dailyly stationary regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT), among which only AD and CRT are high-frequency compatible in theory. The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^α and H_0^β are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

7 Simulation

Our simulation shows that the asymptotics developed in the paper are relevant and useful to analyze regressions using high-frequency observations. In our simulation, we consider two types of regressions, the stationary and cointegration regressions, based on the continuous time regression (6.1) for $0 \le t \le T$, where $X = (X_t)$ and $U = (U_t)$ are given by

$$dX_t = -\kappa_x X_t dt + \sigma_x dW_t^x \quad \text{and} \quad dU_t = -\kappa_u U_t dt + \sigma_u dW_t^u,$$

where $W^x = (W_t^x)$ and $W^u = (W_t^u)$ are two independent Brownian motions. We let $\kappa_u > 0$ for both types of regressions, and let $\kappa_x > 0$ and $\kappa_x = 0$ for the stationary and cointegrating regressions, respectively.

For the stationary regression, we set the parameter values $(\kappa_x, \sigma_x) = (0.1020, 1.5513)$ and $(\kappa_u, \sigma_u) = (6.9011, 2.7565)$, which are obtained from X and $\hat{U}$ in Model II fitted to our simulation model. Both X and U are therefore specified as stationary Ornstein-Uhlenbeck

test results and enhances test power, particularly when the underlying error process is not overly persistent. See Figure 8 and Table 13. These findings suggest that high-frequency data offer useful information that can make tests more stable and powerful.

(OU) processes in our stationary regression. For the cointegrating regression, the parameter values are given as $(\kappa_x, \sigma_x) = (0, 0.0998)$ and $(\kappa_u, \sigma_u) = (1.5718, 0.0097)$, which are identical to the estimates from X and $\hat{U}$ in Model III fitted with restriction $\kappa_x = 0$ to our simulation model. In our cointegrating regression, X therefore becomes a Brownian motion, while U is a stationary OU process. We consider the test of the null hypothesis $\mathrm{H}_0 : \alpha = 0$ and $\beta = 1$ using the H and G-tests respectively for the stationary and cointegrating regressions.

In the simulation, the exact transition densities of Brownian motion and OU process are used to generate daily sample paths of X and U over $T = 30$ and 50 years with 3,000 iterations. We then collect discrete samples at varying frequencies ranging from daily to quarterly levels, which correspond to $\delta = 1/252$ and $1/4$, respectively. The long-run variance estimates in the test statistics are obtained using Parzen kernel and the four different bandwidth choices, which are introduced earlier in Section 5 and referred to as RT, CRT, AD and NW, respectively. As discussed, CRT and AD are high-frequency compatible, while RT and NW are not.

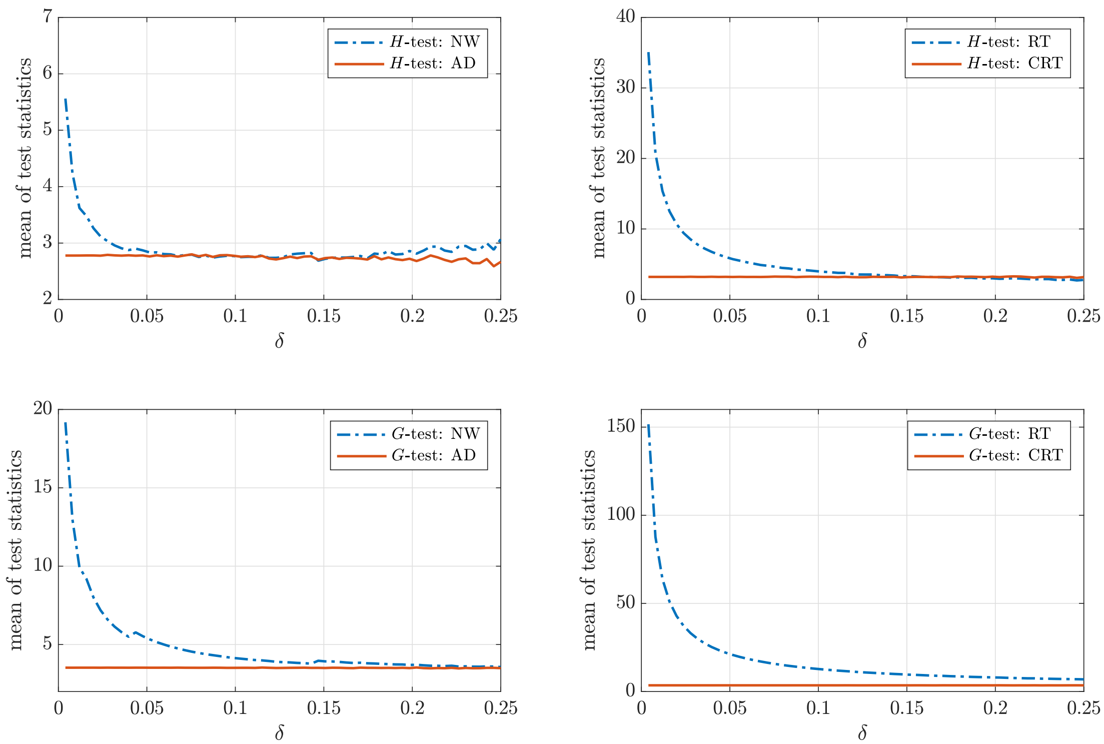
Figure 6 presents the means of the H-test statistics for the stationary regression and the G-test statistics for the cointegrating regression computed under the null hypothesis, relying on 30 years of simulated data sampled at frequencies varying from daily to quarterly frequencies corresponding to sampling intervals from $\delta = 1/252$ to $\delta = 1/4$, respectively.[32] We can see that, for both H and G-tests, the behaviors of the simulated means of the test statistics are critically dependent upon the high frequency compatibility of the bandwidth used in estimating the required long-run variance. If high-frequency compatible bandwidths (AD, CRT) are used, both tests are insensitive to the sampling intervals and they yield stable test values across all different sampling frequencies. On the other hand, the tests constructed with high-frequency incompatible bandwidths (NW, RT) are extremely unreliable and they provide test values changing radically as sampling intervals and frequencies vary. In particular, the averages of their test values increase very rapidly as the sampling frequency exceeds the monthly level or as the sampling interval δ decreases to less than $1/12 \approx 0.08$ and approaches the daily level, exactly as we observe for the H and G-tests applied to the actual data in Figure 4.

Furthermore, we compute the rejection probabilities of the H and G-tests for the stationary and cointegrating regressions under the null hypothesis using the 5% chi-square critical value for $T = 30$ and 50. They are presented in Figure 7 in Appendix D.[33] If high-

[32] The simulated means of the test statistics for $T = 50$ show the same patterns as those for $T = 30$ reported in Figure 6 here.

[33] We present, in Figure 7 in Appendix D, the simulated rejection probabilities for both $T = 30$ and 50 rather than just for $T = 30$, as we did for presenting simulated means of test statistics in Figure 6. We present the results for two different time spans in Figure 7 to illustrate how the size distortion of high-frequency-compatible tests is reduced when time span T increases from 30 to 50 years.

Figure 6: Simulated Means of Robust Wald Test Statistics



Notes: Presented are simulated means of test statistics for the null hypothesis $H_0 : (\alpha, \beta) = (0, 1)$ of the H-test (upper panels) for the stationary regression and the G-test (lower panels) for the stationary regression, over 3,000 simulation iterations. The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT). The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency) over a sample span of $T = 30$ years.

frequency incompatible bandwidths NW and RT are used, not only the means but also the rejection probabilities of the tests become highly sensitive to the sampling frequency and increase rapidly as the sampling interval decreases down below one month, or when δ takes values below $1/12 \approx 0.08$.[34] In particular, increasing time span T from 30 to 50 years does not alleviate this problem. In contrast, the rejection probabilities of the tests with high-frequency compatible bandwidths AD and CRT are remarkably stable across all sampling frequencies, and they tend closer to the nominal size of the tests as T increases from 30 to 50 years.

Figure 7 shows that both the H and G-tests yield substantial size distortions, even when we use high-frequency compatible bandwidths AD and CRT with samples over an

[34] As shown in the lower panels of Figure 7, the G-test with bandwidth RT yields large size distortion even at relatively low frequencies for the simulated cointegrating regression model, and this is the case for both $T = 30$ and 50 years.

extended time span of 50 years. These distortions are attributed to the presence of persistent covariate X and persistent error process U in the stationary regression model and cointegrating regression model, respectively, used for our simulations. We use such models in our simulations, since they are more relevant to our empirical illustrations.[35] The size distortions of tests in time series regressions, caused generally by either a persistent covariate or error term, are already well documented in the existing literature. See, for example, Müller (2014) and the references cited there. We will not analyze this problem further in any detail, since it goes beyond the scope of our paper. However, we take this problem seriously, and consider only rejection at the 1% level as significant in our empirical illustrations in Section 6.

We also find clear evidence that using high-frequency data is beneficial for our tests when high-frequency compatible bandwidths AD and CRT are used. As shown in the upper panels of Figure 8 in Appendix D, the simulated means of test statistics become significantly less volatile across sampling frequencies with higher-frequency data. Moreover, the lower panels of Figure 8 show the instability of test results at each sampling interval δ, measured by the percentage of different test results arising from a one-day change in sampling frequency. From the results there we can clearly see that using higher-frequency data significantly reduces the instability of test results caused by small and insignificant changes in sampling frequency.

Furthermore, we find that higher-frequency data can improve test power. According to Table 13 in Appendix D, the power of the H-test for stationary regression is substantially increased with higher-frequency data, especially under a more local alternative. However, the G-test used in cointegrating regression shows only minor improvements. Further simulations, not reported here due to space limitations, indicate that the effect of higher-frequency data on test power depends on the persistence level of the error process in the underlying continuous-time regression: the less persistent the error process, the greater the potential improvement in power from using higher-frequency data.

Lastly, to see more clearly the effect of our double asymptotics relying on $\delta \to 0$ and $T \to \infty$ jointly, we set $\delta = (T/3)^{-2}$ so that T changes simultaneously along with δ. Under this setup, we simulate the means of the H and G-tests for the stationary and cointegrating regressions under the null hypothesis, which is reported in Figure 9 in Appendix D. The overall pattern of the frequency dependency remains the same as what we observe in Figure 6, which is obtained by changing only δ with T fixed. The only notable difference is that the simulated means of the tests with high-frequency compatible bandwidths, CRT

[35] The size distortions reported here decrease rapidly as we use more stationary models in our simulations. These simulation results, however, are not reported, since they are well expected and not directly useful for our empirical illustrations.

and AD, now show downward trends as the sampling interval gets smaller. As discussed, no such trends appear when T is fixed. Therefore, we may conclude that the trends here are generated not directly by varying δ but by T changing with δ. The downward trends emerging under our double asymptotics just indicate that the tests have positive biases in our simulation model when T is small. The simulated means of the tests with high-frequency incompatible bandwidths RT and NW behave similarly, regardless of T being fixed or varying with δ.

We conduct some additional simulations based on the models that can capture more realistic features of the economic and financial series used in our empirical illustrations presented in the previous section. Details of the additional simulation models are described and the results obtained from these models are reported in Appendix E.

8 Conclusion

The Wald test is widely used to test for restrictions in regressions. However, the test is extremely sensitive to the sampling frequency, and it is very likely that the test result depends on the frequency of the samples we use for the test. This, of course, is highly undesirable, since in most cases the sampling frequency has no bearing on the hypothesis to be tested. The dependence of the Wald test on the sampling frequency is manifested vividly in many time series regressions. In particular, when observations are used at high frequency, the standard Wald test almost explodes and we are always led to reject the null hypothesis no matter whether it is true or not. The problem, however, appears to have long been either overlooked or neglected in the literature. Certainly, it has now become increasingly more important as more economic and financial time series are collected and made available at high frequencies.

In the paper, we develop a new continuous time framework and develop relevant asymptotics, relying on sampling interval δ and time span T jointly, to analyze various versions of the Wald test in regressions with high frequency observations over long sample span. Our framework accommodates both stationary and cointegrating regressions, and our asymptotics hold under mild regularity conditions. According to our asymptotic theory, the standard Wald test is expected to diverge if $\delta \to 0$ fast enough relative to $T \to \infty$. This is exactly what we observe in practice. The robust Wald tests, which use the long-run variance estimates in place of the usual variance estimates, are expected to behave similarly unless the bandwidths for their long-run variance estimates are appropriately chosen to be high-frequency compatible. The high frequency compatibility of a bandwidth ensures that the resulting long-run variance estimate properly captures the clear and present serial correlation at high frequency. Only when high-frequency compatible bandwidths are used do the robust Wald tests become valid and behave as expected under the null hypothesis.

References

Aït-Sahalia, Y., and R. Kimmel (2007): "Maximum likelihood estimation of stochastic volatility models," *Journal of Financial Economics*, 83(2), 413–452.

Andrews, D. W. K. (1991): "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation," *Econometrica*, 59, 817–858.

Ang, A., and M. Piazzesi (2003): "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50, 745–787.

Bernanke, B. S. (2020): "The New Tools of Monetary Policy," *American Economic Review*, 110, 943–983.

Bhattacharya, R. N. (1982): "On the functional central limit theorem and the law of the iterated logarithm for Markov processes," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 60(2), 185–201.

Campbell, J. Y., and R. J. Shiller (1991): "Yield Spreads and Interest Rate Movements: A Bird's Eye View," *Review of Economic Studies*, 58, 495–514.

Chang, Y., Y. Choi, H. Kim, and J. Y. Park (2016): "Evaluating Factor Pricing Models Using High Frequency Panels," *Quantitative Economics*, 7, 889–933.

Choi, Y., S. Jacewitz, and J. Y. Park (2016): "A Reexamination of Stock Return Predictability," *Journal of Econometrics*, 192, 168–189.

Cochrane, J. H. (2001): "Expectations and Exchange Rate Dynamics," *Journal of Political Economy*, 109, 663–696.

Fama, E. F. (1984): "The Information in the Term Structure," *Journal of Financial Economic*, 13, 509–528.

Granger, C. W. J., and P. Newbold (1974): "Spurious Regressions in Econometrics," *Journal of Econometrics*, 2, 111–120.

Gürkaynak, R. S., B. P. Sack, and J. H. Wright (2007): "The U.S. Treasury Yield Curve: 1961 to the Present," *Journal of Monetary Economics*, 54, 2291–2304.

Heston, S. L. (1993): "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies*, 6(2), 327–343.

Jeong, M., and J. Y. Park (2011): "Asymptotic Theory of Maximum Likelihood Estimator for Diffusion Model," Working Paper, Indiana University.

——— (2014): "Asymptotic Theory of Maximum Likelihood Estimator for Jump Diffusion Model," Working Paper, Indiana University.

Jiang, B., Y. Lu, and J. Y. Park (2020): "Testing for Stationoarity at High Frequency," *Journal of Econometrics*, 215, 341–374.

Kanaya, S., J. Kim, and J. Y. Park (2018): "On the Modulus of Continuity of Brownian Motion," Working Paper, Indiana University.

Kim, J., and J. Y. Park (2017): "Asymptotics for Recurrent Diffusions with Application to High Frequency Regression," *Journal of Econometrics*, 196, 37–54.

Lu, Y., and J. Y. Park (2019): "Estimation of Longrun Variance of Continuous Time Stochastic Process Using Discrete Sample," *Journal of Econometrics*, 210, 236–267.

Müller, U. K. (2005): "Size and Power of Tests of Stationarity in Highly Autocorrelated Time Series," *Journal of Econometrics*, 128, 195–213.

Müller, U. K. (2014): "HAC corrections for strongly autocorrelated time series," *Journal of Business & Economic Statistics*, 32(3), 311–322.

Newey, W. K., and K. D. West (1994): "Automatic Lag Selection in Covariance Matrix Estimation," *Review of Economic Studies*, 61, 631–653.

Perron, P. (1991a): "A Continuous-Time Approximation to the Stationary First-Order Autoregressive Model," *Econometric Theory*, 7, 236–252.

——— (1991b): "A Continuous-Time Approximation to the Unstable First-Order Autoregressive Model: The Case Without an Intercept," *Econometrica*, 59, 211–236.

Phillips, P. C. B. (1986): "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, 33, 311–340.

Rozanov, Y. A. (1960): "A central limit theorem for additive random functions," *Theory of Probability & Its Applications*, 5(2), 221–223.

Rudebusch, G. D. (2006): "The Bond Yield "Conundrum" from a Macro-Finance Perspective," Federal Reserve Bank of San Francisco Working Paper No. 2006-06.

Sun, Y. (2014): "Let's Fix It: Fixed-b Asymptotics Versus Small-b Asymptotics in Heteroskedasticity and Autocorrelation Robust Inference," *Journal of Econometrics*, 178, 659–677.

Thornton, D. L. (2018): "Greenspan's Conundrum and the Fed's Ability to Affect Long-Term Yields," *Journal of Money, Credit and Banking*, 50, 513–543.

Appendices

A Mathematical Proofs

Proof of Lemma 3.1 We may assume without loss of generality that Z is univariate by looking at each component separately. Note that

$$\begin{aligned}\frac{1}{T}\int_0^T Z_t dt &= \frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} Z_t dt \\ \frac{1}{n}\sum_{i=1}^n z_i &= \frac{1}{T}\sum_{i=1}^n \delta Z_{(i-1)\delta} + \frac{\delta}{T}(Z_T - Z_0),\end{aligned}$$

from which it follows that

$$\frac{1}{T}\int_0^T Z_t dt - \frac{1}{n}\sum_{i=1}^n z_i = \frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (Z_t - Z_{(i-1)\delta})dt + O_p\left(\frac{\delta}{T}\sup_{0\le t\le T}|Z_t|\right).$$

However, we have

$$|Z_t - Z_{(i-1)\delta}| \le |Z_t^c - Z_{(i-1)\delta}^c| + \sum_{(i-1)\delta < s \le t} \Delta Z_t$$

for all $i = 1, \ldots, n$ and t such that $(i-1)\delta < t \le i\delta$. Consequently, we have

$$\left|\frac{1}{n}\sum_{i=1}^n z_i - \frac{1}{T}\int_0^T Z_t dt\right| \le \frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} |Z_t - Z_{(i-1)\delta}|dt + O_p\left(\frac{\delta}{T}\sup_{0\le t\le T}|Z_t|\right)$$

and

$$\begin{aligned}\frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} |Z_t - Z_{(i-1)\delta}|dt &\le \frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta}\left(|Z_t^c - Z_{(i-1)\delta}^c| + \sum_{(i-1)\delta<s\le t}|\Delta Z_t|\right)dt \\ &\le \frac{1}{T}\sum_{i=1}^n \int_{(i-1)\delta}^{i\delta}\left(|Z_t^c - Z_{(i-1)\delta}^c| + \sum_{(i-1)\delta<s\le i\delta}|\Delta Z_t|\right)dt \\ &\le \left(\sup_{|t-s|\le\delta}|Z_t^c - Z_s^c|\right) + \frac{\delta}{T}\sum_{0<t\le T}|\Delta Z_t| \\ &= O_p\Big(\Delta_{\delta,T}(Z)\Big) + O_p(\delta),\end{aligned}$$

and we may deduce the stated result immediately. □

Proof of Theorem 4.1 Under Assumptions A and B, we have

$$\frac{1}{n}\sum_{i=1}^{n} u_i^2 = \frac{1}{T}\int_0^T U_t^2 dt + o_p(1) \to_p \sigma^2$$

as $\delta \to 0$ and $T \to \infty$ with $\Delta_{\delta,T}(U) = o(1)$ as in Assumption D1 or D2.

For the proof of part (a), we write

$$\sqrt{T}(\hat{\beta} - \beta) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} \frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i, \tag{A.1}$$

and note that, under Assumptions A and C1, we have

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_i' = \frac{1}{T}\int_0^T X_t X_t' dt + o_p(1) \to_p M > 0$$
$$\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i = \frac{1}{\sqrt{T}}\int_0^T X_t U_t dt + o_p(1) \to_d \mathbb{N}(0, \Pi)$$

as $\delta \to 0$ and $T \to \infty$ satisfying Assumption D1, and that

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n} u_i^2 - \frac{1}{T}\left(\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n} u_i x_i'\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1}\left(\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n} x_i u_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} u_i^2 + O_p(T^{-1}).\end{aligned}$$

Therefore, the stated results follow immediately.

The proof of part (b) is analogous. We write

$$\sqrt{T}\Lambda_T'(\hat{\beta} - \beta) = \left(\frac{1}{n}\sum_{i=1}^{n} \Lambda_T^{-1} x_i x_i' \Lambda_T^{-1\prime}\right)^{-1} \frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n} \Lambda_T^{-1} x_i u_i, \tag{A.2}$$

and note that, under Assumptions A and C2, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} \Lambda_T^{-1} x_i x_i' \Lambda_T^{-1\prime} &= \frac{1}{T}\int_0^T \Lambda_T^{-1} X_t X_t' \Lambda^{-1\prime} dt + o_p(1) \\ &= \int_0^1 X_t^T X_t^{T\prime} dt + o_p(1) \to_d \int_0^1 X_t^\circ X_t^{\circ\prime} dt := Q \end{aligned} \tag{A.3}$$

and

$$\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n}\Lambda_T^{-1}x_iu_i = \frac{1}{\sqrt{T}}\int_0^T \Lambda_T^{-1}X_tU_tdt + o_p(1)$$
$$= \int_0^1 X_t^T dU_t^T + o_p(1) \to_d \int_0^1 X_t^\circ dU_t^\circ$$

as $\delta \to 0$ and $T \to \infty$ satisfying Assumption D2. Therefore, it follows that

$$\sqrt{T}\Lambda_T'(\hat{\beta}-\beta) \to_d \left(\int_0^1 X_t^\circ X_t^{\circ\prime}dt\right)^{-1}\int_0^1 X_t^\circ dU_t^\circ := P. \tag{A.4}$$

To derive the asymptotic null distribution of $F(\hat{\beta})$, we first note that, by (A.3),

$$nR\left(\sum_{i=1}^{n}x_ix_i'\right)^{-1}R' \sim_p R\Lambda_T^{-1\prime}Q^{-1}\Lambda_T^{-1}R'. \tag{A.5}$$

Moreover, based on (A.4), we have

$$\sqrt{T}(R\hat{\beta}-r) = \sqrt{T}R(\hat{\beta}-\beta) = R\Lambda_T^{-1\prime}\sqrt{T}\Lambda_T'(\hat{\beta}-\beta) \sim_p R\Lambda_T^{-1\prime}P. \tag{A.6}$$

Lastly, the denominator of $F(\hat{\beta})$ satisfies

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}u_i^2 - \frac{1}{T}\left(\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n}u_ix_i'\Lambda_T^{-1\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\Lambda_T^{-1}x_ix_i'\Lambda_T^{-1\prime}\right)^{-1}\left(\frac{\sqrt{\delta}}{\sqrt{n}}\sum_{i=1}^{n}\Lambda_T^{-1}x_iu_i\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}u_i^2 + O_p(T^{-1}) \to_p \sigma^2. \tag{A.7}$$

Then it follows from (A.5), (A.6) and $\Lambda_T^{\bullet\prime}R = R\Lambda_T'$ that

$$n\Lambda_T^{\bullet\prime}R\left(\sum_{i=1}^{n}x_ix_i'\right)^{-1}R'\Lambda_T^{\bullet} \to_d RQ^{-1}R', \tag{A.8}$$
$$\sqrt{T}\Lambda_T^{\bullet\prime}(R\hat{\beta}-r) \to_d RP, \tag{A.9}$$

from which and (A.7) we can deduce that

$$\delta F(\hat{\beta}) = \sqrt{T}(R\hat{\beta}-r)'\Lambda_T^{\bullet}\left[n\Lambda_T^{\bullet\prime}R\left(\sum_{i=1}^{n}x_ix_i'\right)^{-1}R'\Lambda_T^{\bullet}\right]^{-1}\sqrt{T}\Lambda_T^{\bullet\prime}(R\hat{\beta}-r)/\hat{\sigma}^2$$
$$\to_d P'R'\left(RQ^{-1}R'\right)^{-1}RP/\sigma^2,$$

as to be shown. □

Proof of Lemma 4.2 Write

$$\tilde{\rho} - 1 = \frac{\sum_{i=1}^{n} u_{i-1}(u_i - u_{i-1})}{\sum_{i=1}^{n} u_{i-1}^2}. \tag{A.10}$$

As shown earlier, we have $n^{-1}\sum_{i=1}^{n} u_{i-1}^2 \to_p \sigma^2$. Moreover, note that

$$u_{i-1} = \frac{1}{2}\left[(u_i + u_{i-1}) - (u_i - u_{i-1})\right],$$

and therefore, we may deduce that

$$\begin{aligned}\sum_{i=1}^{n} u_{i-1}(u_i - u_{i-1}) &= \frac{1}{2}\left[\sum_{i=1}^{n}(u_i^2 - u_{i-1}^2) - \sum_{i=1}^{n}(u_i - u_{i-1})^2\right] \\ &= \frac{1}{2}\left(u_n^2 - u_0^2\right) - \frac{1}{2}\sum_{i=1}^{n}(u_i - u_{i-1})^2,\end{aligned}$$

which will be further analyzed subsequently.

We have

$$\begin{aligned}(U_{i\delta} - U_{(i-1)\delta})^2 = 2\int_{(i-1)\delta}^{i\delta}(U_{t-} - U_{(i-1)\delta})dU_t^c + \Big([U^c]_{i\delta} - [U^c]_{(i-1)\delta}\Big) \\ + \sum_{(i-1)\delta < t \leq i\delta} \Delta(U_t - U_{(i-1)\delta})^2,\end{aligned} \tag{A.11}$$

and

$$\begin{aligned}\Delta(U_t - U_{(i-1)\delta})^2 &= (U_t - U_{(i-1)\delta})^2 - (U_{t-} - U_{(i-1)\delta})^2 \\ &= (U_t - U_{t-})(U_t + U_{t-} - 2U_{(i-1)\delta}) \\ &= (U_t - U_{t-})\Big[(U_t - U_{t-}) + 2(U_{t-} - U_{(i-1)\delta})\Big] \\ &= 2(U_{t-} - U_{(i-1)\delta})\Delta U_t + (\Delta U_t)^2\end{aligned} \tag{A.12}$$

for $i = 1, \ldots, n$. Therefore, it follows from (A.11) and (A.12) that

$$\sum_{i=1}^{n}(U_{i\delta-} - U_{(i-1)\delta})^2 = [U]_T + 2Z_T, \tag{A.13}$$

where $Z = Z^c + Z^d$ with

$$Z_T^c = \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_{t-} - U_{(i-1)\delta}) dU_t^c$$
$$Z_T^d = \sum_{i=1}^n \sum_{(i-1)\delta < t \leq i\delta} (U_{t-} - U_{(i-1)\delta}) \Delta U_t.$$

Note that

$$[U]_T = \sum_{i=1}^n \Big([U^c]_{i\delta} - [U^c]_{(i-1)\delta} \Big) + \sum_{0<t\leq T} (\Delta U_t)^2$$

for any n and δ such that $T = n\delta$. In what follows, we use

$$U_{t-} - U_{(i-1)\delta} = (U_t^c - U_{(i-1)\delta}^c) + \sum_{(i-1)\delta < s < t} \Delta U_s, \tag{A.14}$$

which holds for t, $(i-1)\delta < t \leq i\delta$, and all $i = 1, \ldots, n$.

To consider Z^c, we write

$$Z^c = Z^a + Z^b, \tag{A.15}$$

where

$$\begin{aligned} Z_T^a &= \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_{t-} - U_{(i-1)\delta}) dA_t \\ &= \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_t^c - U_{(i-1)\delta}^c) dA_t + \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} \left(\sum_{(i-1)\delta < s < t} \Delta U_s \right) dA_t \end{aligned}$$

and

$$\begin{aligned} Z_T^b &= \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_{t-} - U_{(i-1)\delta}) dB_t \\ &= \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} (U_t^c - U_{(i-1)\delta}^c) dB_t + \sum_{i=1}^n \int_{(i-1)\delta}^{i\delta} \left(\sum_{(i-1)\delta < s < t} \Delta U_s \right) dB_t, \end{aligned}$$

which we analyze subsequently. For Z_a, we have

$$\left|\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t^c - U_{(i-1)\delta}^c)dA_t\right| \leq \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}|U_t^c - U_{(i-1)\delta}^c||dA_t|$$
$$\leq p_T\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}|U_t^c - U_{(i-1)\delta}^c|dt$$
$$\leq p_T\Delta_{\delta,T}(U)\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}dt = O_p\Big(p_T T\Delta_{\delta,T}(U)\Big)$$

and

$$\left|\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)dA_t\right| \leq \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(\sum_{(i-1)\delta<s<t}|\Delta U_s|\right)|dA_t|$$
$$\leq p_T\sum_{i=1}^{n}\left(\sum_{(i-1)\delta<t\leq i\delta}|\Delta U_t|\right)\int_{(i-1)\delta}^{i\delta}dt$$
$$= p_T\delta\sum_{0<t\leq T}|\Delta U_t| = O_p(p_T T\delta),$$

from which it follows that

$$Z_T^a = O_p(p_T T\Delta_{\delta,T}(U)) + O_p(p_T T\delta) = O_p(p_T T\Delta_{\delta,T}(U)), \tag{A.16}$$

since $\delta = O(\Delta_{\delta,T}(U))$.

For Z^b, it suffices to look at its quadratic variation, since it can be embedded into a continuous martingale. However, we have

$$\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t^c - U_{(i-1)\delta}^c)^2 d[B]_t \leq q_T\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}(U_t^c - U_{(i-1)\delta}^c)^2 dt$$
$$\leq q_T\Delta_{\delta,T}^2(U)\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}dt = O_p\left(q_T T\Delta_{\delta,T}^2(U)\right).$$

Furthermore, it follows that

$$\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)^2 d[B]_t \leq q_T\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)^2 dt,$$

and that

$$\mathbb{E}\left[\sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)^2 dt\right] = \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\mathbb{E}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)^2 dt$$
$$= \sum_{i=1}^{n}\int_{(i-1)\delta}^{i\delta}\left[\sum_{(i-1)\delta<s<t}\mathbb{E}(\Delta U_s)^2\right] dt$$
$$\leq \sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\mathbb{E}(\Delta U_t)^2\int_{(i-1)\delta}^{i\delta} dt$$
$$= \delta\sum_{0<t\leq T}\mathbb{E}(\Delta U_t)^2 = O(\delta T).$$

Therefore, we may deduce that

$$Z_T^b = O_p\left(\sqrt{q_T T}\Delta_{\delta,T}(U)\right) + O_p\left(\sqrt{q_T T\delta}\right) = O_p\left(\sqrt{q_T T}\Delta_{\delta,T}(U)\right), \tag{A.17}$$

since $\sqrt{\delta} = O(\Delta_{\delta,T}(U))$. The order of Z^c may now be easily obtained as

$$Z_T^c = O_p\left(p_T T\Delta_{\delta,T}(U)\right) + O_p\left(\sqrt{q_T T}\Delta_{\delta,T}(U)\right)$$
$$= T\left[O_p\left(p_T\Delta_{\delta,T}(U)\right) + O_p\left(\sqrt{q_T/T}\Delta_{\delta,T}(U)\right)\right] \tag{A.18}$$

from (A.16) and (A.17).

To analyze Z^d, we let

$$Z_T^d = \sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}(U_{t-} - U_{(i-1)\delta})\Delta U_t$$
$$= \sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}(U_t^c - U_{(i-1)\delta}^c)\Delta U_t + \sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)\Delta U_t$$

We have

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}(U_t^c-U_{(i-1)\delta}^c)\Delta U_t\right]^2\\
&=\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\mathbb{E}(U_t^c-U_{(i-1)\delta}^c)^2\mathbb{E}(\Delta U_t)^2\\
&\leq\left[\max_{1\leq i\leq n}\sup_{(i-1)\delta<t\leq i\delta}\mathbb{E}(U_t^c-U_{(i-1)\delta}^c)^2\right]\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\mathbb{E}(\Delta U_t)^2\\
&=\left[\max_{1\leq i\leq n}\sup_{(i-1)\delta<t\leq i\delta}\mathbb{E}(U_t^c-U_{(i-1)\delta}^c)^2\right]\sum_{0<t\leq T}\mathbb{E}(\Delta U_t)^2=O\left(T\Delta_{\delta,T}^2(U)\right). \qquad \text{(A.19)}
\end{aligned}
$$

Moreover, we may easily deduce that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)\Delta U_t\right]^2\\
&=\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\mathbb{E}\left(\sum_{(i-1)\delta<s<t}\Delta U_s\right)^2\mathbb{E}(\Delta U_t)^2\\
&=\sum_{i=1}^{n}\sum_{(i-1)\delta<t\leq i\delta}\left[\sum_{(i-1)\delta<s<t}\mathbb{E}(\Delta U_s)^2\right]\mathbb{E}(\Delta U_t)^2\\
&\leq\sum_{i=1}^{n}\sum_{(i-1)\delta<s,t\leq i\delta}\mathbb{E}(\Delta U_s)^2\mathbb{E}(\Delta U_t)^2\\
&\leq\sum_{i=1}^{n}\sum_{(i-1)\delta<s,t\leq i\delta}\mathbb{E}(\Delta U_t)^4=\sum_{0<t\leq T}\mathbb{E}(\Delta U_t)^4=O(T). \qquad \text{(A.20)}
\end{aligned}
$$

Therefore, it follows from (A.19) and (A.20) that

$$
Z_T^d=O_p\left(\sqrt{T}\Delta_{\delta,T}(U)\right)+O_p(\sqrt{T})=O_p(\sqrt{T})=o_p(T) \qquad \text{(A.21)}
$$

as $\delta\to 0$ and $T\to\infty$.

Now we have, due in particular to (A.18) and (A.21),

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}u_{i-1}(u_i-u_{i-1}) &= \frac{1}{2n}(u_n^2-u_0^2)-\frac{1}{2n}\sum_{i=1}^{n}(u_i-u_{i-1})^2\\ &= \delta\left[\frac{1}{2T}(U_T^2-U_0^2)-\frac{1}{2T}\sum_{i=1}^{n}(U_{i\delta}-U_{(i-1)\delta})^2\right]\\ &= -\delta\left(\frac{1}{2T}[U]_T+o_p(1)\right)=-\frac{\delta\tau^2}{2}+o_p(\delta)\end{aligned}$$

as $\delta\to 0$ and $T\to\infty$. Consequently, the stated result follows immediately from (A.10), and the proof is complete. □

Proof of Lemma 5.1 For the Andrews bandwidth, we may readily deduce from Lemma 4.2 that

$$\tilde{\theta}_1^2=\frac{\tau^4}{\delta^2\sigma^4}+o_p(\delta^{-2}),\qquad \tilde{\theta}_2^2=\frac{4\tau^8}{\delta^4\sigma^8}+o_p(\delta^{-4}),$$

and in particular $\tilde{\theta}_r^2=O_p(\delta^{-2r})$ for $r=1,2$ as $\delta\to 0$ and $T\to\infty$ satisfying Assumption E. With these choices of θ_r^2, $r=1,2$, the feasible optimal bandwidth $\tilde{b}_n^*$ in (5.10) becomes

$$\tilde{b}_n^*=\left(\frac{\tau^4\pi_1^2}{\sigma^4\int K(x)^2dx}\,n\right)^{1/3}\delta^{-2/3}\Big(1+o_p(1)\Big)\quad\text{or}\quad\left(\frac{8\tau^8\pi_2^2}{\sigma^8\int K(x)^2dx}\,n\right)^{1/5}\delta^{-4/5}\Big(1+o_p(1)\Big),$$

and therefore,

$$B_{n,\delta}=\tilde{b}_n^*\delta=\left(\frac{\tau^4\pi_1^2}{\sigma^4\int K(x)^2dx}\,T\right)^{1/3}(1+o_p(1))\quad\text{or}\quad\left(\frac{8\tau^8\pi_2^2}{\sigma^8\int K(x)^2dx}\,T\right)^{1/5}(1+o_p(1))$$

depending upon whether $r=1$ or 2, as $\delta\to 0$ and $T\to\infty$ satisfying Assumption E. Consequently, we have $B_{n,\delta}\to_p\infty$ and $B_{n,\delta}/T\to_p 0$, which implies that the Andrews bandwidth is high-frequency compatible.

For the Newey-West bandwidth, note that for $r=1,2$,

$$\delta^r\hat{\theta}_r=\frac{\delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j)}{\delta\sum_{|j|\le a_n}\gamma_n(j)},\tag{A.22}$$

and we will show in the following that as $\delta\to 0$ and $T\to\infty$,

$$\delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j)\sim_p\int_{|s|\le A_{n,\delta}}|s|^r\Gamma_T(s)ds\tag{A.23}$$

for $r=0,1,2$, where $A_{n,\delta}=a_n\delta$. To see this, we write

$$\int_{|s|\le A_{n,\delta}}|s|^r\Gamma_T(s)ds-\delta^{1+r}\sum_{|j|\le a_n}|j|^r\gamma_n(j)=P_{n,\delta}+Q_{n,\delta}+R_{n,\delta},$$

where $P_{n,\delta}, Q_{n,\delta}$ and $R_{n,\delta}$ are the same as those in the proof of Proposition 4.2 in Lu and Park (2019) with A_T replaced by $A_{n,\delta}$, and the analyses of their stochastic orders are also analogous. In particular, given $\sup_{0\leq t\leq\infty}\mathbb{E}|V_t|^2 < \infty$, Lemma A.2 in Lu and Park (2019) holds, and we have $P_{n,\delta} = O(\Delta_{\delta,T}(V)A_{n,\delta}^{1+r}), Q_{n,\delta} = O_p(\Delta_{\delta,T}(V)A_{n,\delta}^{1+r} + \delta A_{n,\delta}^r)$ and $R_{n,\delta} = O(\delta A_{n,\delta}^r)$. Therefore,

$$\int_{|s|\leq A_{n,\delta}} |s|^r \Gamma_T(s)ds - \delta^{1+r} \sum_{|j|\leq a_n} |j|^r \gamma_n(j) = O(\Delta_{\delta,T} A_{n,\delta}^{1+r} + \delta A_{n,\delta}^r). \tag{A.24}$$

Note that $A_{n,\delta} = cn^p\delta = c\delta^{1-p}T^p \to 0$ as $\delta \to 0$ sufficiently fast relative to $T \to \infty$, so we have

$$A_{n,\delta}^{-(1+r)} \int_{|s|\leq A_{n,\delta}} |s|^r \Gamma_T(s)ds = \int_{|s|\leq 1} |s|^r \left(\frac{1}{T}\int_0^T V_t V_{t-sA_{n,\delta}}dt\right) ds \to_p \frac{2\sigma^2}{1+r}$$

as $\delta \to 0$ and $T \to \infty$. This implies $A_{n,\delta}^{-(1+r)}\left[\delta^{1+r}\sum_{|j|\leq a_n}|j|^r\gamma_n(j) - \int_{|s|\leq A_{n,\delta}}|s|^r\Gamma_T(s)ds\right] = O(\Delta_{\delta,T}(V) + \delta/A_{n,\delta}) = O(\Delta_{\delta,T}(V) + \delta^p T^{-p}) = o_p(1)$ which proves (A.23). Now we can readily deduce from (A.22), (A.23) and (A.24) that

$$A_{n,\delta}^{-r}\delta^r\hat{\theta}_r \sim_p A_{n,\delta}^{-r}\frac{\int_{|s|\leq A_{n,\delta}}|s|^r\Gamma_T(s)ds}{\int_{|s|\leq A_{n,\delta}}\Gamma_T(s)ds} = \frac{A_{n,\delta}^{-(1+r)}\int_{|s|\leq A_{n,\delta}}|s|^r\Gamma_T(s)ds}{A_{n,\delta}^{-1}\int_{|s|\leq A_{n,\delta}}\Gamma_T(s)ds} \to_p \frac{1}{1+r},$$

and therefore for $B_{n,\delta} = \hat{b}_n^*\delta = \left(r\kappa_r\hat{\theta}_r^2 n/\int K(x)^2dx\right)^{1/(2r+1)}\delta$, we have

$$A_{n,\delta}^{-2r/(2r+1)}T^{-1/(2r+1)}B_{n,\delta} \to_p \left(\frac{r\pi_r^2}{(1+r)^2\int K(x)^2dx}\right)^{1/(2r+1)}$$

from which it follows that

$$\delta^{-2r(1-p)/(2r+1)}T^{-(2pr+1)/(2r+1)}B_{n,\delta} \to_p \left(\frac{rc^{2r}\pi_r^2}{(1+r)^2\int K(x)^2dx}\right)^{1/(2r+1)}$$

given $A_{n,\delta} = c\delta^{1-p}T^p$. Therefore, $B_{n,\delta} \to_p 0$ if $\delta \to 0$ fast enough relative to $T \to \infty$ so that $\delta = o\left(T^{-(2pr+1)/(2r(1-p))}\right)$, and hence Newey-West bandwidth is not high-frequency compatible. □

Proof of Theorem 5.2 To show Part (a), we follow the proof in part (a) of Theorem 4.1 and note that $\delta\hat{\Omega} \to_p \Pi$ under Assumption F, due to Theorem 4.1 in Lu and Park (2019). Then we can show that, under $H_0 : R\beta = r$,

$$H(\hat{\beta}) = \sqrt{T}(\hat{\beta}-\beta)'R'\left[R\left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}\delta\hat{\Omega}\left(\frac{1}{n}\sum_{i=1}^n x_i x_i'\right)^{-1}R'\right]^{-1}R\sqrt{T}(\hat{\beta}-\beta),$$

where

$$R\sqrt{T}(\hat{\beta}-\beta) \to_d RM^{-1}\mathbb{N}(0,\Pi) =_d \mathbb{N}(0, RM^{-1}\Pi M^{-1}R'),$$
$$R\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} \delta\hat{\Omega} \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} R' \to_p RM^{-1}\Pi M^{-1}R',$$

as $\delta \to 0$ and $T \to \infty$ under Assumption A, C1 and D1. Therefore, it follows immediately that $H(\hat{\beta}) \to_p \chi_q^2$ as $\delta \to 0$ and $T \to \infty$.

The proof of Part (b) follows from that of Part (b) of Theorem 4.1, as the only difference between the test statistics $G(\hat{\beta})$ and $F(\hat{\beta})$ is in the denominator. Note that the denominator of $G(\hat{\beta})$ satisfies $\delta\hat{\omega}^2 \to_p \pi^2$ where π^2 is the LRV of U_t defined in Assumption C2. It then follows from the definition of $G(\hat{\beta})$ and (A.8)–(A.9) that

$$\begin{aligned} G(\hat{\beta}) &= \sqrt{T}(R\hat{\beta}-r)'\Lambda_T^{\bullet}\left[n\Lambda_T^{\bullet\prime}R\left(\sum_{i=1}^{n} x_i x_i'\right)^{-1} R'\Lambda_T^{\bullet}\right]^{-1}\sqrt{T}\Lambda_T^{\bullet\prime}(R\hat{\beta}-r)/(\delta\hat{\omega}^2) \\ &\to_d P^{*\prime}R'\left(RQ^{-1}R'\right)^{-1}RP^{*} \end{aligned}$$

where $P^* = \left(\int_0^1 X_t^\circ X_t^{\circ\prime}dt\right)^{-1}\int_0^1 X_t^\circ dU_t^*$ where $U^* = U^\circ/\pi$. This complete the proof of the theorem. □

B Additional Empirical Results: Cointegrating Regressions

In Tables 4, 5, and 6, we present the testing results for three null hypotheses $\mathrm{H}_0^\alpha : \alpha = 0$, $\mathrm{H}_0^\beta : \beta = 1$ and $\mathrm{H}_0^{\alpha,\beta} : (\alpha, \beta) = (0, 1)$ for quarterly, monthly and daily *cointegrating* regressions using G-tests, for the pre and post-Greenspan conundrum period, which are defined as before and after the end of year 2000. Overall, these results for the cointegrating regression are consistent with those for the stationary regression presented in Tables 1, 2 and 3 in Section 6 of the main text. They are also consistent with the stationary and cointegrating regression results obtained from the sub-samples before and after the global financial crisis that are presented below in Tables 7–12 in Appendix C.

C Testing with Pre & Post-Financial Crisis Samples

C.1 Testing in Stationary Regressions

In Tables 7, 8, and 9, we present the testing results for three null hypotheses $\mathrm{H}_0^\alpha : \alpha = 0$, $\mathrm{H}_0^\beta : \beta = 1$ and $\mathrm{H}_0^{\alpha,\beta} : (\alpha, \beta) = (0, 1)$ for quarterly, monthly and daily *stationary* regressions using H-tests, for two sub-samples before and after the 2008-9 global financial crisis, which are defined as before and after the end of year 2007. We find that the results presented here are consistent with the results for the stationary regressions before and after the Greenspan conundrum period presented in Tables 1, 2 and 3 in Section 6 of the main text.

Table 4: Testing Results from Quarterly Cointegrating Regressions

Pre-Greenspan conundrum sample (1962–2000)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	72.09**	7.28**	11.79**	11.86**	38.65**
H_0^{β}	10.51**	1.06	1.72	1.73	5.63*
		[0.3031]	[0.1899]	[0.1887]	[0.0176]
$H_0^{\alpha,\beta}$	220.73**	22.28**	36.11**	36.30**	118.34**
Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	412.09**	41.14**	77.60**	79.20**	250.63**
H_0^{β}	61.30**	6.12**	11.54**	11.78**	37.28**
$H_0^{\alpha,\beta}$	478.06**	47.73**	90.02**	91.88**	290.75**

Notes: Presented are test statistics (and p-values in square brackets) of the quarterly cointegrating regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 5: Testing Results from Monthly Cointegrating Regressions

Pre-Greenspan conundrum sample (1962–2000)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	233.44**	7.22**	12.65**	20.47**	94.64**
H_0^{β}	39.80**	1.23	2.16	3.49	16.13**
		[0.2674]	[0.1420]	[0.0618]	[0.0001]
$H_0^{\alpha,\beta}$	656.63**	20.30**	35.57**	57.56**	266.20**
Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	1266.72**	38.95**	76.52**	126.58**	588.74**
H_0^{β}	198.96**	6.12**	12.02**	19.88**	92.47**
$H_0^{\alpha,\beta}$	1454.55**	44.72**	87.87**	145.35**	676.04**

Notes: Presented are test statistics (and p-values in square brackets) of the monthly cointegrating regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 6: Testing Results from Daily Cointegrating Regressions

Pre-Greenspan conundrum sample (1962–2000)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^α	5648.93**	7.45**	14.36**	143.63**	1193.82**
H_0^β	1174.12**	1.55	2.99	29.85**	248.13**
		[0.2133]	[0.0840]	[0.0000]	[0.0000]
$H_0^{\alpha,\beta}$	13930.71**	18.38**	35.42**	354.21**	2944.06**

Post-Greenspan conundrum sample (2001–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^α	28263.04**	40.77**	80.19**	922.12**	6896.29**
H_0^β	4459.63**	6.43**	12.65**	145.50**	1088.17**
$H_0^{\alpha,\beta}$	32382.96**	46.72**	91.88**	1056.54**	7901.56**

Notes: Presented are test statistics (and p-values in square brackets) of the daily cointegrating regressions for two sub-samples before and after January 1, 2001. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^α and H_0^β are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 7: Testing Results from Quarterly Stationary Regressions

Pre-financial crisis sample (1962–2007)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^α	128.38**	19.20**	24.02**	24.18**	69.57**
H_0^β	21.65**	2.87	3.44	3.46	9.52**
		[0.0902]	[0.0636]	[0.0628]	[0.0020]
$H_0^{\alpha,\beta}$	297.21**	39.38**	54.10**	54.53**	176.87**

Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^α	436.36**	119.82**	123.10**	131.21**	316.13**
H_0^β	53.15**	37.53**	25.44**	24.83**	43.63**
$H_0^{\alpha,\beta}$	482.17**	122.92**	126.36**	136.38**	336.74**

Notes: Presented are test statistics (and p-values in square brackets) of the daily stationary regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^α and H_0^β are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 8: Testing Results from Monthly Stationary Regressions

	Nonrobust test	Robust H-test			
Pre-financial crisis sample (1962–2007)					
		AD	CRT	NW	RT
H_0^{α}	400.59**	20.63**	25.91**	40.42**	166.58**
H_0^{β}	75.87**	3.50	4.22*	6.45*	25.02**
		[0.0613]	[0.0398]	[0.0111]	[0.0000]
$H_0^{\alpha,\beta}$	875.82**	37.03**	53.12**	86.31**	407.71**
Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^{α}	1286.44**	107.53**	114.46**	196.73**	720.30**
H_0^{β}	144.75**	36.20**	21.42**	25.35**	82.45**
$H_0^{\alpha,\beta}$	1438.94**	119.52**	116.55**	207.81**	772.63**

Notes: Notes: Presented are test statistics (and p-values in square brackets) of the daily stationary regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 9: Testing Results from Daily Stationary Regressions

	Nonrobust test	Robust H-test			
Pre-financial crisis sample (1962–2007)					
		AD	CRT	NW	RT
H_0^{α}	9459.58**	23.10**	29.45**	216.85**	1865.35**
H_0^{β}	2068.05**	4.56*	5.44*	36.30**	306.98**
		[0.0328]	[0.0197]	[0.0000]	[0.0000]
$H_0^{\alpha,\beta}$	18887.81**	37.37**	55.54**	517.52**	4758.46**
Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust H-test			
		AD	CRT	NW	RT
H_0^{α}	28495.64**	109.42**	116.82**	1105.07**	8112.10**
H_0^{β}	3196.55**	34.35**	21.34**	127.96**	915.51**
$H_0^{\alpha,\beta}$	31731.91**	119.98**	119.28**	1185.72**	8770.33**

Notes: Presented are test statistics (and p-values in square brackets) of the daily stationary regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 10: Testing Results from Quarterly Cointegrating Regressions

Pre-financial crisis sample (1962–2007)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	128.38**	13.24**	20.44**	20.56**	66.04**
H_0^{β}	21.65**	2.23	3.45	3.47	11.14**
		[0.1351]	[0.0634]	[0.0626]	[0.0008]
$H_0^{\alpha,\beta}$	297.21**	30.65**	47.32**	47.60**	152.89**

Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	436.36**	91.47**	116.02**	123.05**	300.83**
H_0^{β}	53.15**	11.14**	14.13**	14.99**	36.64**
$H_0^{\alpha,\beta}$	482.17**	101.07**	128.20**	135.97**	332.41**

Notes: Presented are test statistics (and p-values in square brackets) of the daily cointegrating regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

C.2 Testing in Cointegrating Regressions

In Tables 10, 11 and 12, we present the testing results for three null hypotheses $H_0^{\alpha} : \alpha = 0$, $H_0^{\beta} : \beta = 1$ and $H_0^{\alpha,\beta} : (\alpha, \beta) = (0, 1)$ for quarterly, monthly and daily *cointegrating* regressions using G-tests for two sub-samples before and after the 2008-9 financial crisis. These results complement the results presented in Tables 7, 8 and 9 in Section C.1 for stationary regressions with H-tests. We find that the results for the cointegrating regressions shown in this appendix are consistent with the results for the stationary regressions obtained with the same sub-samples presented earlier in Section C.1.

D Additional Simulation Figures and Tables

In this section, we present three additional figures, Figure 7, Figure 8, Figure 9, and an additional table, Table 13, to illustrate the simulation results reported in Section 7.

Figure 7 shows the simulated rejection probabilities of the 5% H and G-tests respectively for the stationary and cointegrating regressions under the null hypothesis $H_0 : (\alpha, \beta) = (0, 1)$ across varying sampling frequencies, while the time span is fixed at $T = 30$ and $T = 50$. Figure 8 presents, in the upper panels, the simulated means of H and G-test statistics, and, in the lower panels, the percentages of unstable test results caused by a one-day change in sampling frequency, using high-frequency compatible bandwidths AD and CRT. The upper panels highlight the reduced volatility of the mean test statistics at high frequencies,

Table 11: Testing Results from Monthly Cointegrating Regressions

Pre-financial crisis sample (1962–2007)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	400.59**	12.69**	21.03**	33.56**	156.75**
H_0^{β}	75.87**	2.40	3.98*	6.36*	29.69**
		[0.1211]	[0.0460]	[0.0117]	[0.0000]
$H_0^{\alpha,\beta}$	875.82**	27.74**	45.99**	73.38**	342.71**
Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	1286.44**	71.44**	101.85**	178.62**	660.48**
H_0^{β}	144.75**	8.04**	11.46**	20.10**	74.32**
$H_0^{\alpha,\beta}$	1438.94**	79.91**	113.92**	199.80**	738.78**

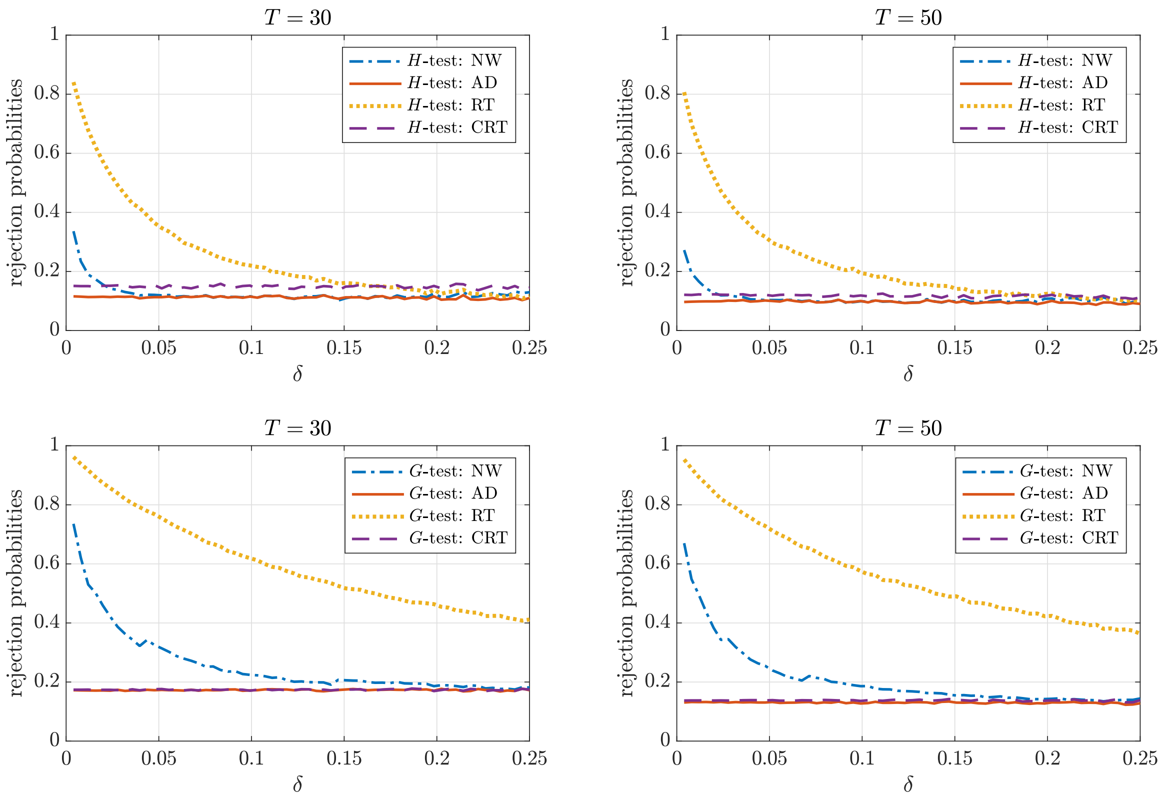
Notes: Presented are test statistics (and p-values in square brackets) of the daily cointegrating regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Table 12: Testing Results from Daily Cointegrating Regressions

Pre-financial crisis sample (1962–2007)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	9459.58**	12.78**	23.31**	210.65**	1933.62**
H_0^{β}	2068.05**	2.79	5.10*	46.05**	422.73**
		[0.0947]	[0.0240]	[0.0000]	[0.0000]
$H_0^{\alpha,\beta}$	18887.81**	25.51**	46.55**	420.59**	3860.83**
Post-financial crisis sample (2008–2019)					
	Nonrobust test	Robust G-test			
		AD	CRT	NW	RT
H_0^{α}	28495.64**	73.69**	105.38**	1026.72**	7626.55**
H_0^{β}	3196.55**	8.27**	11.82**	115.17**	855.52**
$H_0^{\alpha,\beta}$	31731.91**	82.05**	117.35**	1143.33**	8492.70**

Notes: Presented are test statistics (and p-values in square brackets) of the daily cointegrating regressions for two sub-samples before and after January 1, 2008. The bandwidth choices are given by Andrews (AD), continuous-time rule of thumb (CRT), Newey-West (NW) and discrete-time rule of thumb (RT). The critical values of the 5% and 1% level χ^2 tests for the single hypotheses H_0^{α} and H_0^{β} are, respectively, 3.84 and 6.63. The critical values of the 5% and 1% level χ^2 test for the joint hypothesis $H_0^{\alpha,\beta}$ are, respectively, 5.99 and 9.21. Rejection at the 5% level is signified by $*$ and at the 1% level signified by $**$.

Figure 7: Simulated Rejection Probabilities of Robust Wald Tests



Notes: Presented are the simulated rejection probabilities of the H-test (upper panels) for the stationary regression and the G-test (lower panels) for the cointegrating regression with $T = 30$ and $T = 50$ under the null hypothesis $\mathrm{H}_0 : (\alpha, \beta) = (0, 1)$ using the 5% chi-square critical value 5.99. The simulation models are presented in Section 7, and the results are based on 3,000 simulation iterations. The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT). The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).

which was not visible in Figure 6 when plotted with test statistics based on high-frequency incompatible bandwidths NW and RT. Figure 9 shows the simulated means of the H and G-test statistics for the same null hypothesis where the sampling interval δ varies with the time span T at the rate given by $\delta = (T/3)^{-2}$ to demonstrate the effect of our double asymptotics under which $\delta \to 0$ as $T \to \infty$.

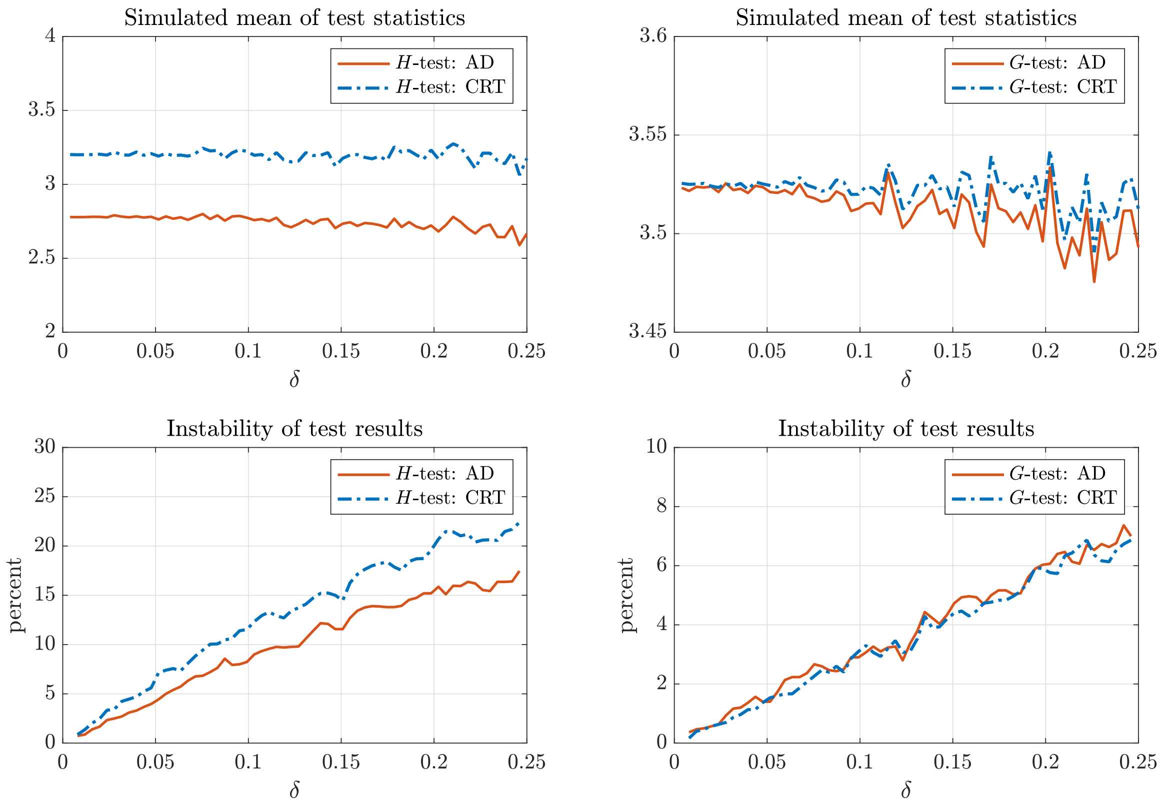
Table 13 presents the simulated size-adjusted power (adjusted to 5%) of the H-test and G-test using high-frequency compatible bandwidths AD and CRT, for the null hypothesis $\mathrm{H}_0 : \beta = 1$. Observations are sampled at daily, monthly, and quarterly frequencies over a time span of $T = 50$ years. Improvements in size-adjusted power from quarterly to monthly and daily frequencies are calculated as percentages and shown in brackets.

Table 13: Simulated Size-Adjusted Power

H-test		Daily	Monthly	Quarterly
AD	$\beta_0 = 0.95$	0.6362	0.6198	0.5460
		(16.52%)	(13.52%)	–
	$\beta_0 = 0.90$	0.9726	0.9712	0.9440
		(3.03%)	(2.88%)	–
CRT	$\beta_0 = 0.95$	0.6164	0.5914	0.5252
		(17.36%)	(12.60%)	–
	$\beta_0 = 0.90$	0.9682	0.9642	0.9352
		(3.53%)	(3.10%)	–
G-test		Daily	Monthly	Quarterly
AD	$\beta_0 = 0.990$	0.7472	0.7446	0.7326
		(1.99%)	(1.64%)	–
	$\beta_0 = 0.985$	0.9218	0.9210	0.9164
		(0.59%)	(0.50%)	–
CRT	$\beta_0 = 0.990$	0.7590	0.7562	0.7492
		(1.31%)	(0.93%)	–
	$\beta_0 = 0.985$	0.9288	0.9280	0.9232
		(0.61%)	(0.52%)	–

Notes: The table presents the simulated size-adjusted power of the H-test for stationary regression and the G-test for cointegrating regression, with observations sampled at daily, monthly and quarterly frequencies and a time span of $T = 50$ years. The null hypothesis for the test is $\mathrm{H}_0 : \beta = 1$, and the true values of β in simulations are given by β_0 in the table. Critical values are adjusted to ensure that the size of all tests is 5%. Percentages in brackets represent the improvement in size-adjusted power when increasing the sampling frequency from quarterly to monthly and daily levels. The bandwidth choices used are high-frequency compatible bandwidths, Andrews (AD) and continuous-time rule of thumb (CRT). Details of the simulation models are provided in Section 7.

Figure 8: Simulated Means of Robust Wald Test Statistics and Instability of Test Results Caused by One-Day Change in Sampling Frequency

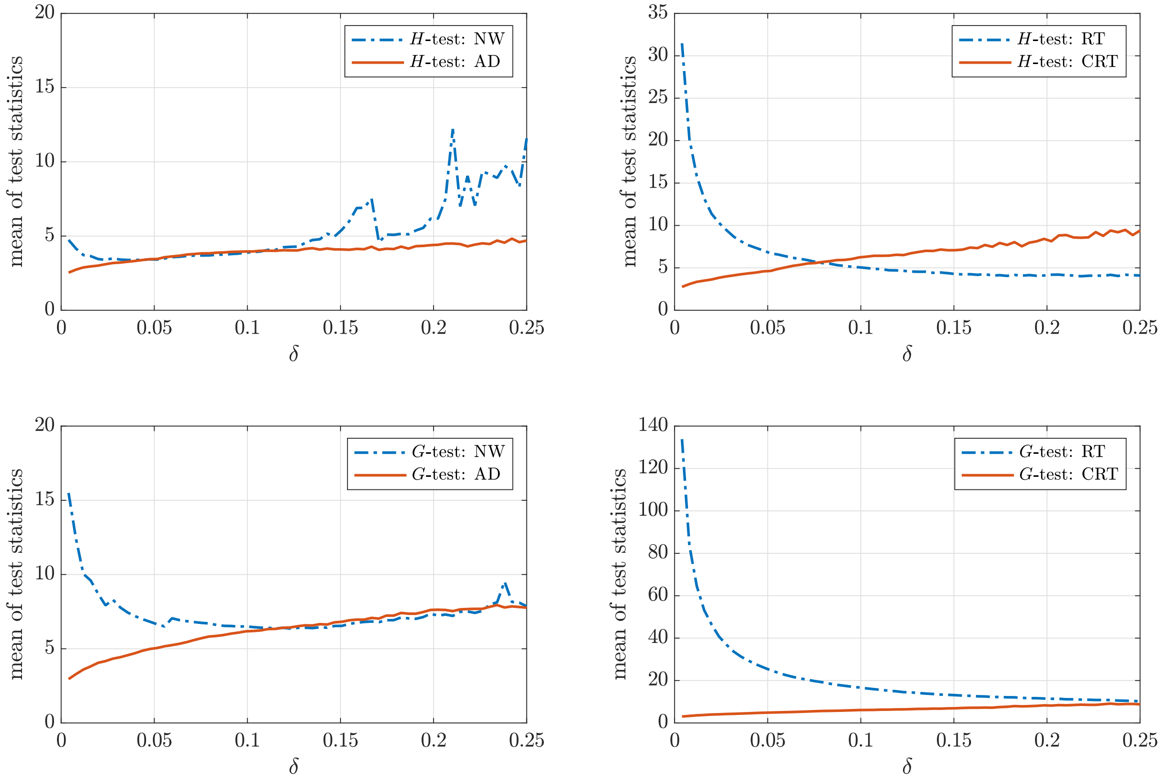


Notes: The upper panels present the simulated means of the H and G-statistics for testing $\mathrm{H}_0 : (\alpha, \beta) = (0, 1)$ in stationary and cointegrating regressions, respectively, when high-frequency compatible bandwidths AD and CRT are applied. The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency) over a sample span of $T = 30$ years. The lower panels show the instability of test results at each sampling interval δ, measured by the percentage of unstable test results (using 5% critical value) arising from a one-day change in sampling frequency. An unstable test result at sampling interval δ is identified when the test result differs from that obtained using data sampled at one day higher or lower frequency. Results are based on 3,000 simulation iterations, with simulation models detailed in Section 7.

E Additional Simulation Results

In this section, we present some additional simulation results based on the models that can capture more realistic features of the economic and financial series used in our empirical illustrations. In particular, we introduce stochastic volatilities in $X = (X_t)$ and $U = (U_t)$, and allow them to interact with each other. Again, we consider two types of regressions, the stationary and cointegrating regressions, based on the continuous time regression (6.1), and test the null hypothesis $\mathrm{H}_0 : \alpha = 0$ and $\beta = 1$ using our H and G-tests, respectively. Apart from using more realistic models, all the other simulation setups are the same as in Section 7.

Figure 9: Simulated Means of Robust Wald Test Statistics Under Joint Asymptotics



Notes: Presented are the simulated means of the H-statistics (upper panels) for the stationary regression and the G-statistics (lower panels) for the cointegrating regression computed under $\mathrm{H}_0 : (\alpha, \beta) = (0, 1)$, based on 3,000 simulation iterations. The simulation models are presented in Section 7. The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency) over a sample span of $T = 3\delta^{-1/2}$. The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT).

For the stationary regression, we let

$$dX_t = \kappa_x(\mu_x - X_t)dt + \sigma_x\sqrt{X_t}dW_{1t}$$
$$dU_t = -\kappa_u U_t dt + \sigma_u dW_{2t},$$

where $W_1 = (W_{1t})$ and $W_2 = (W_{2t})$ are two standard Browian motions with $d[W_1, W_2] = \upsilon dt$, $-1 \leq \upsilon \leq 1$, and set their parameter values to satisfy stationarity conditions so that X and U are, respectively, stationary Feller's square root process and Ornstein-Uhlenckbeck process.[36] Compared with the simulation model in Section 7, where X has constant volatility and is independent of U, the model here is more realistic as it allows time varying

[36] In particular, this requires $2\kappa_x\mu_x \geq \sigma_x^2$, which ensures the process X is bounded above from zero which is suitable for modelling the interest rates in normal times. This condition is imposed for all Feller's square root processes used in our simulations.

volatility in X as well as interaction of X and U. The parameters in the model are set to be $(\kappa_x, \mu_x, \sigma_x) = (0.106, 5.495, 0.600)$, $(\kappa_u, \sigma_u) = (9.420, 3.108)$ and $\upsilon = -0.241$, which are obtained by fitting the empirical daily data of Model II to the model.[37] The parameters $\kappa_x, \mu_x, \sigma_x, \kappa_u$ and σ_u are estimated by the maximum likelihood method, and the estimate for υ is obtained by the sample analogue principle using $\upsilon = [X,U]_T \Big/ \left(\sigma_x \sigma_u \int_0^T X_t dt\right)$, where $[X,U]_T$ denotes the quadratic variation of X and U in $[0,T]$.

The simulated means of the test statistics for $T = 30$ are presented in the upper panels of Figure 10 and the simulated rejection probabilities of the 5% test for both $T = 30$ and $T = 50$ are presented in the upper panels of Figure 11; they are the counterparts of the upper panels in Figures 6 and 7, respectively. The simulation results are again in line with our theory: the test statistics employing the NW and RT bandwidths diverge as sampling frequency increases, while the test statistics employing the AD and CRT bandwidths behave much more stably across different sampling frequencies. Our theory is therefore fully demonstrated even when more realistic features are added to the simple baseline models. For more realistic models we consider here, the actual rejection probabilities are all significantly above the nominal 5% level for all bandwidth choices. The overall levels of the actual rejection probabilities appear to be generally higher for realistic models than for simple models, although comparisons across the tests with different bandwidth choices cannot be clearly made in any meaningful manners.

We also simulate a more sophisticated cointegrating continuous-time regression model than the one we considered in Section 7, motivated by our empirical Model IV introduced in Section 2 which involves a regression of log S&P 500 futures price on log S&P 500 index price. In particular, we assume the covariate X follows the Heston (1993) process, as the latter is a commonly used model for log stock index price process. More specifically, following equation (35) in Aït-Sahalia and Kimmel (2007), we assume that

$$d\begin{bmatrix} X_t \\ V_t^x \end{bmatrix} = \begin{bmatrix} r - d + (\lambda(1-\upsilon^2) - 1/2)V_t^x \\ \kappa_x(\mu_x - V_t^x) \end{bmatrix} dt + \begin{bmatrix} \sqrt{(1-\upsilon^2)V_t^x} & \upsilon\sqrt{V_t^x} \\ 0 & \sigma_x\sqrt{V_t^x} \end{bmatrix} d\begin{bmatrix} W_{1t}^x \\ W_{2t}^x \end{bmatrix},$$

where V^x is the spot variance process of X, r is the risk-free interest rate, d is the dividend yield of the stock index, and $W^x = (W_1^x, W_2^x)$ is standard bivariate Brownian motion. For simplicity, we let r and d be constant. In the model, the spot variance process V^x follows a Feller's square root process which is assumed to be stationary. Moreover, υ, as the correlation of the shocks to the stock price and the shocks to the volatility, is interpreted as the leverage effect and assumed to take value between -1 and 0. In our simulations, we set the values of r and d to be 2% and 1.5% respectively, which are obtained from the historical average of the 3-month T-bill rate and the S&P 500 dividend yield. Moreover, we set $\kappa_x = 3, \mu_x = 0.10, \sigma_x = 0.25, \upsilon = -0.8$ and $\lambda = 4$, which are used by Aït-Sahalia and Kimmel (2007) in their simulations. Lastly, each simulated sample path is initialized at $V_0^x = \mu_x$, its unconditional mean level, and $X_0 = \ln(100)$.

We also use a stochastic volatility model for the error process U in the cointegrating

[37] The 3-month T-bill rates hit the zero lower bound on some days after 2007. To ensure sensible estimation results, we exclude the post-2007 period of the data to fit the simulation model.

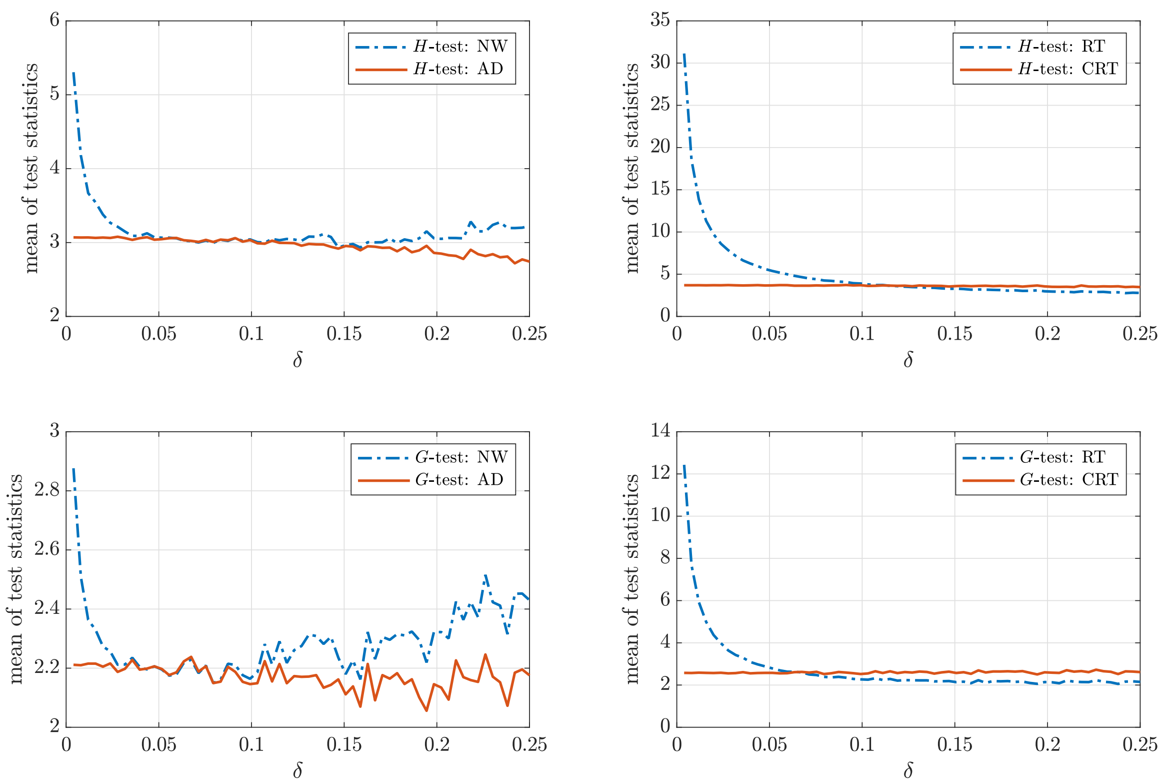
regression. In particular, we specify U jointly with its spot variance process V^u as

$$d\begin{bmatrix} U_t \\ V_t^u \end{bmatrix} = \begin{bmatrix} -\kappa_u U_t \\ \kappa_v(\mu_v - V_t^u) \end{bmatrix} dt + \begin{bmatrix} \sqrt{V_t^u} & 0 \\ 0 & \sigma_v\sqrt{V_t^u} \end{bmatrix} d\begin{bmatrix} W_{1t}^u \\ W_{2t}^u \end{bmatrix},$$

where $W^u = (W_1^u, W_2^u)$ is standard bivariate Brownian motion. Note that U is specified as a zero-mean stationary process whose volatility is generated by the Feller's square root process. In our simulations, we set $\kappa_u = 20, \kappa_v = 2, \mu_v = 0.0002$ and $\sigma_v = 0.008$. Each simulated sample path is at $V_0^u = \mu_v$, its unconditional mean level, and $U_0 = 0$. The initial 500 observations are discarded to remove any potential initialization effects.

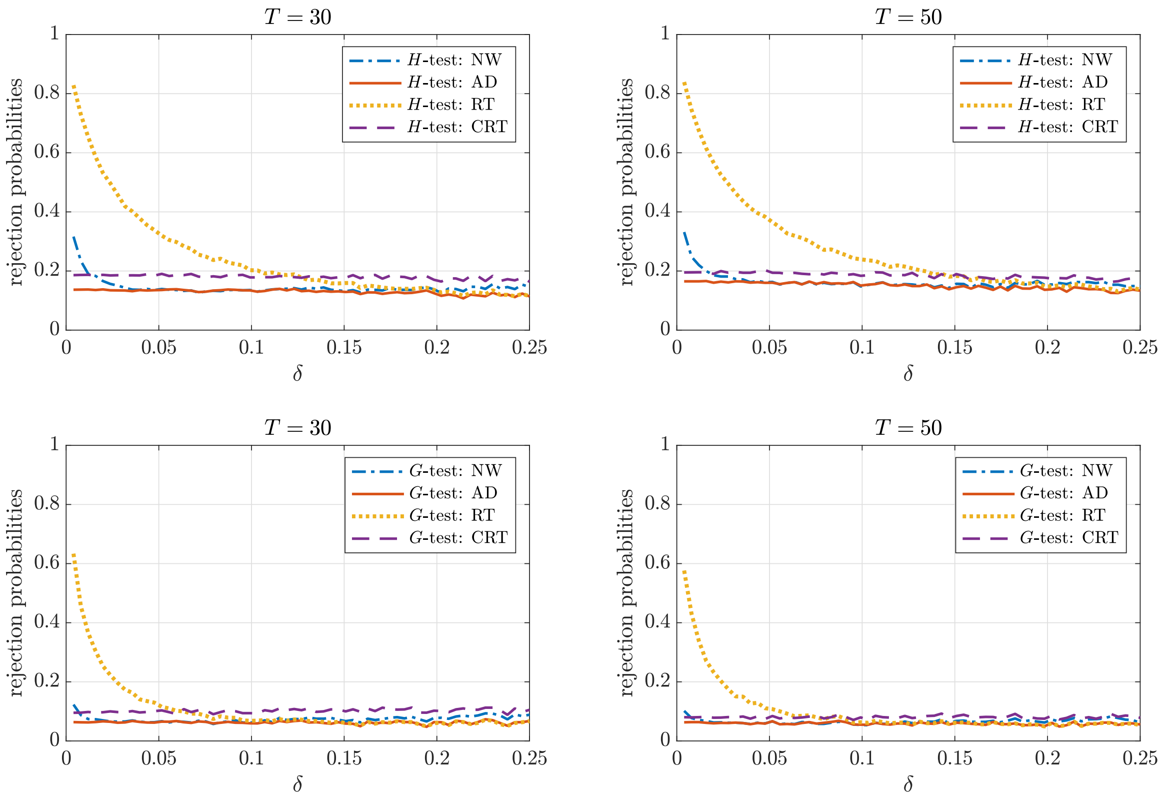
The simulated means of the test statistics for $T = 30$ are presented in the lower panels of Figure 10, and the simulated rejection probabilities of the 5% test for both $T = 30$ and $T = 50$ are presented in the lower panels of Figure 11; they are the counterparts of the lower panels in Figures 6 and 7, respectively. Once again, our simulation results are consistent with our theory. They clearly demonstrate that the tests employing the NW and RT bandwidths tend to diverge as sampling frequency increases, whereas the tests employing the AD and CRT bandwidths do not show such a tendency. Such contrast in the limit behaviors of the tests is much more conspicuous between the tests with the RT and CRT bandwidths, than between the tests with the NW and AD bandwidths. We do not attempt to further interpret the details of our simulation results here. There are many aspects of our simulation models, including their parameter values, that could affect the performances of the tests with different choices of bandwidths in different ways, and we do not believe that we can unravel the individual effect of any one of those aspects on the test with a particular choice of bandwidth.

Figure 10: Simulated Means of Robust Wald Test Statistics (Additional Simulation)



Notes: Presented are the simulated means of the test statistics for the null hypothesis $\mathrm{H}_0 : (\alpha, \beta) = (0, 1)$ of the H-test (upper panels) for the stationary regression and the G-test (lower panels) for the cointegrating regression, over 3,000 simulation iterations. The simulation models are presented in Appendix E. The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT). The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency) over a sample span of $T = 30$ years.

Figure 11: Simulated Rejection Probabilities of Robust Wald Tests (Additional Simulation)



Notes: Presented are the simulated rejection probabilities of the H-test (upper panels) for the stationary regression and the G-test (lower panels) for the cointegrating regression with $T = 30$ and $T = 50$ under $\mathrm{H}_0 : (\alpha, \beta) = (0, 1)$ using the 5% chi-square critical value 5.99. The simulation models are presented in Appendix E, and the results are based on 3,000 simulation iterations. The bandwidth choices are given by Newey-West (NW), Andrews (AD), discrete-time rule of thumb (RT), and continuous-time rule of thumb (CRT). The test statistics are obtained using observations collected at frequencies ranging from $\delta = 1/252$ (daily frequency) to $\delta = 1/4$ (quarterly frequency).