Supplementary Material

Supplement to "Identification of time and risk preferences in buy price auctions": Appendix

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This supplementary appendix contains proofs for some results in the paper and additional discussion of identification of utility functions from certainty equivalents.

Appendix SA: Proofs of selected results from the main text

Proof of Proposition 2

Consider the term

$$e^{-\gamma}U(v-r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}F_V^n(v)E_n\big[U\big(v-\max\{r,Y\}\big)|Y\le v\big]. \tag{SA.1}$$

Note that

$$\begin{aligned}
&F_V^n(v)E_n\big[U\big(v-\max(r,Y)\big)|Y\le v\big]\\
&\quad=\int_0^v U\big(v-\max\{r,y\}\big)nF_V^{n-1}(y)f_V(y)\,dy\\
&\quad=\int_0^r U(v-r)nF_V^{n-1}(y)f_V(y)\,dy+\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)\,dy\\
&\quad=U(v-r)F_V^n(r)+\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)\,dy.
\end{aligned}$$

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So we can write (SA.1) as

$$
\begin{aligned}
& e^{-\gamma}U(v-r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}U(v-r)F_V^n(r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)\,dy \\
&= e^{-\gamma}U(v-r)\left[1+\sum_{n=1}^{\infty}\frac{\gamma^n F_V^n(r)}{n!}\right]+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)\,dy \\
&= e^{-\gamma}U(v-r)\exp\bigl(\gamma F_V(r)\bigr)+e^{-\gamma}\sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)\,dy \\
&= U(v-r)\exp\bigl[\gamma F_V(r)-\gamma\bigr]+e^{-\gamma}\int_r^v U(v-y)f_V(y)\left[\sum_{n=1}^{\infty}\frac{\gamma^n nF_V^{n-1}(y)}{n!}\right]dy,
\end{aligned}
$$

where the last equality follows from the dominated convergence theorem. We also have

$$
\sum_{n=1}^{\infty}\frac{\gamma^n nF_V^{n-1}(y)}{n!}=\sum_{n=1}^{\infty}\frac{\gamma\gamma^{n-1}F_V^{n-1}(y)}{(n-1)!}=\gamma\exp\bigl(\gamma F_V(y)\bigr),
$$

so

$$
U^R(v,t)=\delta(T-t)\left\{U(v-r)\exp\bigl[\gamma F(r)-\gamma\bigr]+\int_r^v U(v-y)\exp\bigl(\gamma F_V(y)-\gamma\bigr)\gamma f_V(y)\,dy\right\}.
$$

The other parts of the proposition are straightforward to verify.

Proof of Proposition 4

We show properties of the inverse cutoff function defined by

$$
\begin{aligned}
p(c,r,\tau,t) &= c-U^{-1}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r)+\int_r^c U(c-y)h(y,\tau,t)\,dy\right)\right) \\
&= c-M(c,r,\tau,t)
\end{aligned}
$$

over the support $r\in[0,\infty)$, $c\in[r,\infty)$, $\tau\in(0,\infty)$, and $t\in(0,\infty)$.

We start by deriving some useful properties of $U^{-1\prime}(x)$ and $U^{-1\prime\prime}(x)$ given Assumption 1. Starting with the identity

$$
z=U^{-1}\bigl(U(z)\bigr),
$$

differentiate w.r.t. z to get

$$
1=U^{-1\prime}\bigl(U(z)\bigr)U'(z).
$$

Evaluating this expression at $z=U^{-1}(x)$, we obtain

$$
U^{-1\prime}(x)=\frac{1}{U'\bigl(U^{-1}(x)\bigr)}=\bigl(U'\bigl(U^{-1}(x)\bigr)\bigr)^{-1}.
$$

Differentiating this results in

$$\begin{aligned} U^{-1\prime\prime}(x) &= -\big(U'\big(U^{-1}(x)\big)\big)^{-2} U''\big(U^{-1}(x)\big) U^{-1\prime}(x) \\ &= -\big(U'\big(U^{-1}(x)\big)\big)^{-2} U''\big(U^{-1}(x)\big)\big(U'\big(U^{-1}(x)\big)\big)^{-1} \\ &= -\big(U'\big(U^{-1}(x)\big)\big)^{-3} U''\big(U^{-1}(x)\big). \end{aligned}$$

Given our assumptions on $U(x)$, these results imply the following statements:

(i) We have $U^{-1\prime}(0) = 1$.

(ii) We have $U^{-1\prime\prime}(0) = -U''(0)$.

(iii) We have that $U^{-1\prime}(\cdot) > 0$ and is bounded away from 0 and ∞.

(iv) We have that $U^{-1\prime\prime}(\cdot) \geq 0$ and is bounded away from ∞.

With these results in hand, consider the statements in the proposition one by one. First, $p_c(c, r, \tau, t) > 0$ because, by Assumption 3, the derivative of M w.r.t. its first argument is strictly less than 1.

Second, $p_c(c, r, \tau, t) < 1$, since

$$\begin{aligned} & p_c(c, r, \tau, t) \\ &\quad = 1 - U^{-1\prime}\bigg(\delta(\tau)\bigg(\alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t)\, dy\bigg)\bigg) \\ &\quad\quad \cdot \delta(\tau)\bigg(\alpha(r, \tau, t) U'(c-r) + \int_r^c U'(c-y) h(y, \tau, t)\, dy\bigg), \end{aligned}$$

and because under our assumptions, $U^{-1\prime}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, $\alpha(y, \tau, t) > 0$, and $h(y, \tau, t) > 0$ for $y > r$.

Third, $p_r(c, r, \tau, t) > 0$, since

$$\begin{aligned} & p_r(c, r, \tau, t) \\ &\quad = -U^{-1\prime}\bigg(\delta(\tau)\bigg(\alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t)\, dy\bigg)\bigg) \\ &\quad\quad \cdot \delta(\tau)\bigg(\frac{\partial \alpha(r, \tau, t)}{\partial r} U(c-r) - \alpha(r, \tau, t) U'(c-r) + U(c-r) h(r, \tau, t)\bigg) \\ &\quad = U^{-1\prime}\bigg(\delta(\tau)\bigg(\alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t)\, dy\bigg)\bigg) \\ &\quad\quad \cdot \delta(\tau) \alpha(r, \tau, t) U'(c-r). \end{aligned}$$

The second line follows since $\frac{\partial \alpha(r,\tau,t)}{\partial r} = h(r, \tau, t)$, and the term is strictly positive since under our assumptions, $U^{-1\prime}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, and $\alpha(y, \tau, t) > 0$.

Fourth, $p_\tau(c,r,\tau,t) \geq 0$, since

$$p_\tau(c,r,\tau,t) = -U^{-1\prime}\bigg(\delta(\tau)\bigg(\alpha(r,\tau,t)U(c-r) + \int_r^c U(c-y)h(y,\tau,t)\,dy\bigg)\bigg) \cdot \left[\begin{array}{c} \delta'(\tau)\Big(\alpha(r,\tau,t)U(c-r) + \int_r^c U(c-y)h(y,\tau,t)\,dy\Big) \\ + \delta(\tau)\dfrac{\partial\Big(\alpha(r,\tau,t)U(c-r)+\int_r^c U(c-y)h(y,\tau,t)\,dy\Big)}{\partial\tau} \end{array}\right].$$

The first term in the square brackets is weakly negative since Assumption 1 implies $\delta'(\cdot) < 0$, $\alpha(\cdot,\cdot,\cdot) > 0$, $h(\cdot,\cdot,\cdot) > 0$, and $U(\cdot) \geq 0$. The second term in the square brackets is weakly negative since $\delta(\tau) > 0$ and the derivative of the expected utility from rejecting the BP w.r.t. τ is weakly negative (since the distribution of the highest competitor valuation is stochastically increasing in the length of the bidding phase τ; this derivative is zero when $c = r$). Since $U^{-1\prime}(x) > 0$, this implies $p_\tau(c,r,\tau,t) \geq 0$.

Fifth, $p(c,r,\tau,t) = r$ iff $c = r$, because

$$\begin{aligned} p(c,c,\tau,t) &= c - U^{-1}\bigg(\delta(\tau)\bigg(\alpha(c,\tau,t)U(c-c) + \int_c^c U(c-y)h(y,\tau,t)\,dy\bigg)\bigg) \\ &= c - U^{-1}(0) \\ &= c = r. \end{aligned}$$

The "only if" follows because $p_c(c,r,\tau,t) > 0$ and because $p(c,r,\tau,t)$ is only defined for $c \geq r$.

Sixth, $p(c,r,\tau,t) \geq r$ from a similar argument, since $p(c,r,\tau,t) = r$ when $c = r$ and $p_c(c,r,\tau,t) > 0$.

Seventh, $p(c,r,\tau,t) \leq c$, since

$$p(c,r,\tau,t) = c - U^{-1}\bigg(\delta(\tau)\bigg(\alpha(r,\tau,t)U(c-r) + \int_r^c U(c-y)h(y,\tau,t)\,dy\bigg)\bigg)$$

and $U^{-1}(\cdot) \geq 0$.

Next, $p_c(z,z,\tau,t) = 1 - \delta(\tau)\alpha(z,\tau,t)$, since

$$\begin{aligned} &p_c(z,z,\tau,t) \\ &\quad = 1 - U^{-1\prime}\bigg(\delta(\tau)\bigg(\alpha(z,\tau,t)U(z-z) + \int_z^z U(z-y)h(y,\tau,t)\,dy\bigg)\bigg) \\ &\qquad \cdot \delta(\tau)\bigg(\alpha(z,\tau,t)U'(z-z) + \int_z^z U'(z-y)h(y,\tau,t)\,dy\bigg) \\ &\quad = 1 - U^{-1\prime}(0)\delta(\tau)\alpha(z,\tau,t)U'(0) \\ &\quad = 1 - \delta(\tau)\alpha(z,\tau,t). \end{aligned}$$

Finally, $p_r(z,z,\tau,t)=\delta(\tau)\alpha(z,\tau,t)$, since

$$\begin{aligned}
p_r(z,z,\tau,t) &= -U^{-1\prime}\Big(\delta(\tau)\Big(\alpha(z,\tau,t)U(z-z)+\int_z^z U(z-y)h(y,\tau,t)\,dy\Big)\Big)\\
&\quad\cdot\delta(\tau)\Big(\frac{\partial\alpha(z,\tau,t)}{\partial z}U(z-z)-\alpha(z,\tau,t)U'(z-z)+U(z-z)h(z,\tau,t)\Big)\\
&= U^{-1\prime}\Big(\delta(\tau)\Big(\alpha(z,\tau,t)U(z-z)+\int_z^z U(z-y)h(y,\tau,t)\,dy\Big)\Big)\\
&\quad\cdot\delta(\tau)\alpha(z,\tau,t)U'(z-z)\\
&= U^{-1\prime}(0)\delta(\tau)\alpha(z,\tau,t)U'(0)\\
&= \delta(\tau)\alpha(z,\tau,t),
\end{aligned}$$

where the last line follows because $U^{-1\prime}(0)=U'(0)=1$.

Next, we consider the second derivatives of the inverse cutoff function w.r.t. c and r, that is, $p_{cc}(c,r)$, $p_{rr}(c,r)$, and $p_{cr}(c,r)$. We drop the τ and t arguments for compactness.

For $p_{cc}(c,r)$, we have

$$\begin{aligned}
p_c(c,r) = 1 &- U^{-1\prime}\Big(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\Big)\\
&\cdot\delta\Big[\alpha(r)U'(c-r)+\int_r^c U'(c-y)h(y)\,dy\Big],
\end{aligned}$$

so

$$\begin{aligned}
p_{cc}(c,r) = &-U^{-1\prime\prime}\Big(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\Big)\\
&\cdot\delta^2\Big[\alpha(r)U'(c-r)+\int_r^c U'(c-y)h(y)\,dy\Big]^2\\
&- U^{-1\prime}\Big(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\Big)\\
&\cdot\delta\Big[\alpha(r)U''(c-r)+\int_r^c U''(c-y)h(y)\,dy+h(c)\Big].
\end{aligned}$$

Under our assumptions, all these terms are bounded away from ∞ and $-\infty$, so $p_{cc}(c,r)$ is bounded away from ∞ and $-\infty$. Moreover, if we evaluate this expression at $c=r=z$, we get

$$\begin{aligned}
p_{cc}(z,z) = &-U^{-1\prime\prime}\Big(\delta\Big(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\Big)\Big)\\
&\cdot\delta^2\Big[\alpha(z)U'(z-z)+\int_z^z U'(z-y)h(y)\,dy\Big]^2\\
&- U^{-1\prime}\Big(\delta\Big(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\Big)\Big)
\end{aligned}$$

$$\cdot\,\delta\Big[\alpha(z)U''(z-z)+\int_z^z U''(z-y)h(y)\,dy+h(z)\Big]$$
$$=-U^{-1\prime\prime}(0)\delta^2\alpha(z)^2-U^{-1\prime}(0)\delta\big[\alpha(z)U''(0)+h(z)\big]$$
$$=-U''(0)\delta\alpha(z)\big(1-\delta\alpha(z)\big)-\delta h(z).$$

For $p_{rr}(c,r)$, we have

$$p_r(c,r)=U^{-1\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)\delta\alpha(r)U'(c-r),$$

so

$$p_{rr}(c,r)=-U^{-1\prime\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)\delta^2\alpha(r)^2U'(c-r)^2$$
$$+U^{-1\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)$$
$$\cdot\,\delta\big[\alpha'(r)U'(c-r)-\alpha(r)U''(c-r)\big].$$

Again, under our assumptions, all the terms in this expression are bounded away from ∞ and $-\infty$, so $p_{rr}(c,r)$ is bounded away from ∞ and $-\infty$. If we evaluate this expression at $c=r=z$, we get

$$p_{rr}(z,z)=-U^{-1\prime\prime}\bigg(\delta\Big(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\Big)\bigg)\delta^2\alpha(z)^2U'(z-z)^2$$
$$+U^{-1\prime}\bigg(\delta\Big(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\Big)\bigg)$$
$$\cdot\,\delta\big[\alpha'(z)U'(z-z)-\alpha(z)U''(z-z)\big]$$
$$=-U^{-1\prime\prime}(0)\delta^2\alpha(z)^2U'(0)^2+U^{-1\prime}(0)\delta\big[\alpha'(z)U'(0)-\alpha(z)U''(0)\big]$$
$$=U''(0)\delta^2\alpha(z)^2+\delta\big[\alpha'(z)-\alpha(z)U''(0)\big]$$
$$=-U''(0)\delta\alpha(z)\big(1-\delta\alpha(z)\big)+\delta\alpha'(z).$$

For $p_{rc}(c,r)=p_{cr}(c,r)$, we have

$$p_r(c,r)=U^{-1\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)\delta\alpha(r)U'(c-r),$$

so

$$p_{rc}(c,r)=U^{-1\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)\delta\alpha(r)U''(c-r)$$
$$+U^{-1\prime\prime}\bigg(\delta\Big(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)\,dy\Big)\bigg)\delta\alpha(r)U'(c-r)$$
$$\cdot\bigg[\delta\Big(\alpha(r)U'(c-r)+\int_r^c U'(c-y)h(y)\,dy\Big)\bigg].$$

Again, all the terms are bounded away from ∞ and $-\infty$, so $p_{rc}(c,r)$ is bounded away from ∞ and $-\infty$. Evaluated at $c=r=z$, we get

$$\begin{aligned}
p_{rc}(z,z) &= U^{-1\prime}\left(\delta\left(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\right)\right)\delta\alpha(z)U''(z-z)\\
&\quad + U^{-1\prime\prime}\left(\delta\left(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)\,dy\right)\right)\delta\alpha(z)U'(z-z)\\
&\quad \cdot\left[\delta\left(\alpha(z)U'(z-z)+\int_z^z U'(z-y)h(y)\,dy\right)\right]\\
&= U^{-1\prime}(0)\delta\alpha(z)U''(0)+U^{-1\prime\prime}(0)\delta\alpha(z)U'(0)\delta\alpha(z)U'(0)\\
&= \delta\alpha(z)U''(0)-U''(0)\delta\alpha(z)\delta\alpha(z)\\
&= U''(0)\delta\alpha(z)\big(1-\delta\alpha(z)\big).
\end{aligned}$$

Proof of Proposition 8

Since the hazard rate of the first action (accept or reject the BP) is observed in the data and satisfies

$$\theta(t_1|p,r,\tau_0)=\lambda(t_1)\big(1-F_V(r)\big), \tag{SA.2}$$

it is clear that $\lambda(t_1)(1-F_V(r))$ is identified on $r\in[\underline{r},\overline{r}]$ and $t_1\in[0,\overline{T})$.

We next show that this implies that $\alpha(r,\tau_0,t_1)$ is identified on $r\in[\underline{r},\overline{r}]$ and $t_1\in[0,T-\tau_0)$. By definition

$$\alpha(r,\tau_0,t_1)=\exp\big(\gamma F_V(r)-\gamma\big),$$

where

$$\gamma=\int_t^{t+\tau_0}\lambda(s)\,ds.$$

Therefore

$$\begin{aligned}
\alpha(r,\tau_0,t_1) &= \exp\left(-\big(1-F_V(r)\big)\int_t^{t+\tau_0}\lambda(s)\,ds\right)\\
&= \exp\left(-\int_t^{t+\tau_0}\lambda(s)\big(1-F_V(r)\big)\,ds\right).
\end{aligned}$$

Since $\lambda(t_1)(1-F_V(r))$ is identified on $r\in[\underline{r},\overline{r}]$ and $t_1\in[0,T)$, this implies that $\alpha(r,\tau_0,t_1)$ is identified on $r\in[\underline{r},\overline{r}]$ and $t_1\in[0,T-\tau_0)$.

Next we show that $h(y,\tau_0,t_1)$ is identified on $y\in[\underline{r},\overline{r}]$ and $t_1\in[0,T-\tau_0)$. Again, by definition

$$\begin{aligned}
h(y,\tau_0,t_1) &= \exp\big(\gamma F_V(y)-\gamma\big)\gamma f_V(y)\\
&= \alpha(r,\tau_0,t_1)\int_t^{t+\tau_0}\lambda(s)f_V(y)\,ds.
\end{aligned}$$

Since $\lambda(t_1)(1-F_V(y))$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$, its derivative $-\lambda(t_1) f_V(y)$ is also identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$. This implies that $\int_t^{t+\tau_0} \lambda(s) f_V(y)\, ds$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T-\tau_0)$. Therefore, $h(y, \tau_0, t)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T-\tau_0)$.

Next, we consider identification of $c(p, r, \tau_0, t_1)$. From Section 3.3, we know that

$$\Pr(B = 1 | p, r, \tau_0, t_1) = \frac{1 - F_V\big(c(p, r, \tau_0, t_1)\big)}{1 - F_V(r)},$$

where $\Pr(B = 1 | p, r, \tau_0, t_1)$ is observed on the support $r \in [\underline{r}, \overline{r}]$, $p \in [r, \overline{p}]$, and $t_1 \in [0, T)$ (at τ_0). Therefore,

$$\begin{aligned}
\Pr(B = 1 | p, r, \tau_0, t_1) &= \frac{\lambda(t_1)\big(1 - F_V\big(c(p, r, \tau_0, t_1)\big)\big)}{\lambda(t_1)\big(1 - F_V(r)\big)} \\
&= \frac{\lambda(t_1)\big(1 - F_V\big(c(p, r, \tau_0, t_1)\big)\big)}{\theta(t_1 | p, r, \tau_0)},
\end{aligned}$$

and therefore $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$ is identified on the same support. Note that this term is the hazard rate of the BP being accepted.

Since we have already identified $\lambda(t_1)(1 - F_V(r))$ on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$, this implies that

$$c(p, r, \tau_0, t_1) = z,$$

where z satisfies

$$\lambda(t_1)\big(1 - F_V\big(c(p, r, \tau_0, t_1)\big)\big) = \lambda(t_1)\big(1 - F_V(z)\big). \tag{SA.3}$$

Intuitively, this says that the cutoff at (p, r, τ_0, t_1) is equal to the hypothetical reserve price that would imply that the hazard rate of the first action is equal $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$.

It remains to be verified that we can identify the z that satisfies (SA.3). Note that the right-hand side (r.h.s.) of (SA.3) is strictly decreasing in z. Since $c(p, r, \tau_0, t_1) \geq \underline{r}$, the left-hand side (l.h.s.) is less than or equal to the r.h.s. at $z = \underline{r}$. Hence, we want to increase z above $\underline{r}$ to satisfy (SA.3). The problem is that we only observe the r.h.s. for $z \in [\underline{r}, \overline{r}]$. However, as long as $c(p, r, \tau_0, t_1) \leq \overline{r}$, we can find a $z \in [\underline{r}, \overline{r}]$ that satisfies (SA.3). This implies that $c(p, r, \tau_0, t_1)$ is identified on the set (p, r, t_1) such that $c(p, r, \tau_0, t_1) \leq \overline{r}$.[1] This immediately implies that the inverse cutoff function $p(c, r, \tau_0, t_1)$ is identified on the set $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \overline{r}]$.

Thus, we have shown that the following statements hold:

(i) The function $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$.

(ii) The function $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$.

[1] This set exists. To show this, consider a situation where $r = \underline{r}$ and $p = \underline{r} + \varepsilon$ for some arbitrarily small ε. For small enough ε, $c(p, r, \tau_0, t_1)$ will be below $\overline{r}$ (since c is continuous and $c(\overline{r}, \overline{r}, \tau_0, t_1) = \overline{r}$). Obviously the size of this set will depend on the range $[\underline{r}, \overline{r}]$.

(iii) The function $p(c, r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \overline{r}]$.

Recall that our integral equation

$$U\big(c - p(c, r, \tau_0, t_1)\big) = \delta(\tau_0)\left(\alpha(r, \tau_0, t_1)U(c - r) + \int_r^c U(c - y)h(y, \tau_0, t_1)\, dy\right) \tag{SA.4}$$

can be reduced to

$$U''(c - r) = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)} U'(c - r), \tag{SA.5}$$

where

$$\Phi(c, r, \tau_0, t_1) = \alpha(r, \tau_0, t_1)\left[\frac{\big(1 - p_c(c, r, \tau_0, t_1)\big)}{p_r(c, r, \tau_0, t_1)} - 1\right].$$

Identification of $\alpha(r, \tau_0, t_1)$, $h(y, \tau_0, t_1)$, and $p(c, r, \tau_0, t_1)$ implies that we can identify $\frac{\Phi_r(c,r,\tau_0,t_1)+h(r,\tau_0,t_1)}{\Phi(c,r,\tau_0,t_1)}$ on $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \overline{r}]$. Hence, by arguments similar to Proposition 3, equation (SA.5) identifies $U(\cdot)$ on $[0, \underline{r} - \overline{r}]$. By the same arguments as in Section 3.4, $\delta(\cdot)$ is identified at τ_0.

Proof of Proposition 9

Assumption 8 further restricts the support of p to $[p_0 - \varepsilon, p_0 + \varepsilon]$. We also assume that p_0 is such that there exists a $r^* \in (\underline{r}, \overline{r})$ and a t_1^* such that $c(p_0, r^*, \tau_0, t_1^*) \in (\underline{r}, \overline{r})$. By the same arguments as in the proof of Proposition 8, we know that the following statements hold:

(i) The function $\alpha(r, \tau, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T} - \tau_0)$.

(ii) The function $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T} - \tau_0)$.

By the same arguments as above (and the condition that $c \in (\underline{r}, \overline{r})$), one can see that $c(p, r, \tau_0, t_1)$ will be identified for $p \in (p_0 - \varepsilon, p_0 + \varepsilon)$, $r \in (r - \eta, r + \eta)$, $t_1 = t_1^*$, and $\tau = \tau_0$ for η sufficiently small. Therefore, the inverse cutoff function $p(c, r, \tau_0, t_1)$ will be identified at $t_1 = t_1^*$, and $\tau = \tau_0$ in a ball centered at $(c(p_0, r^*, \tau_0, t_1^*), r^*)$. This implies that $p_r(c, r, \tau_0, t_1)$ and $p_c(c, r, \tau_0, t_1)$ are identified over that same region, as are $\Phi(c, r, \tau_0, t_1)$ and $\Phi_r(c, r, \tau_0, t_1)$. We have

$$\frac{U''(c - r)}{U'(c - r)} = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)}. \tag{SA.6}$$

Hence, the Arrow–Pratt measure of risk aversion $\frac{U''}{U'}$ is identified at the point $c(p_0, r^*, \tau_0, t_1^*) - r^*$. Again, by the same arguments as Section 3.4, $\delta(\cdot)$ is identified at τ_0.

Appendix SB: Proof that $U''' \leq 0$ is a sufficient condition for Assumption 3

We have

$$M(v, r, \tau, t) = U^{-1}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)\, dy\right)\right)$$

so

$$
\begin{aligned}
M_v(v, r, \tau, t) &= U^{-1\prime}\Big(\delta(\tau)\Big(\alpha(r, \tau, t)U(v-r) + \int_r^v U(v-y)h(y, \tau, t)\,dy\Big)\Big) \\
&\quad \cdot \delta(\tau)\Big(\alpha(r, \tau, t)U'(v-r) + \int_r^v U'(v-y)h(y, \tau, t)\,dy\Big) \\
&= \frac{\delta(\tau)\Big(\alpha(r, \tau, t)U'(v-r) + \int_r^v U'(v-y)h(y, \tau, t)\,dy\Big)}{U'\Big(U^{-1}\Big(\delta(\tau)\Big(\alpha(r, \tau, t)U(v-r) + \int_r^v U(v-y)h(y, \tau, t)\,dy\Big)\Big)\Big)} \\
&< \Big(\delta(\tau)\Big(\alpha(\cdot)U'(v-r) + \int_r^v U'(v-y)h(\cdot)\,dy \\
&\quad + \Big(1 - \alpha(\cdot) - \int_r^v h(\cdot)\,dy\Big)U'(0)\Big)\Big) \\
&\quad \Big/\Big(U'\Big(U^{-1}\Big(\delta(\tau)\Big(\alpha(\cdot)U(v-r) + \int_r^v U(v-y)h(\cdot)\,dy \\
&\quad + \Big(1 - \alpha(\cdot) - \int_r^v h(\cdot)\,dy\Big)U(0)\Big)\Big)\Big)\Big).
\end{aligned}
$$

The strict inequality holds because of our normalizations that $U(0) = 0$ and $U'(0) = 1$, and because $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t)\,dy > 0$ for any finite v.

Therefore, we have

$$
M_v(v, r, \tau, t) < \frac{\delta(\tau)EU'(x)}{U'\big(U^{-1}\big(\delta(\tau)EU(x)\big)\big)},
$$

where the random variable x has a mixed discrete-continuous distribution, taking the value 0 with probability $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t)\,dy$ and the value $v - r$ with probability $\alpha(r, \tau, t)$, and having density $h(y, \tau, t)$ over the interval $(0, v - r)$. Because $U'' \le 0$ and $\delta(\tau) < 1$, Jensen's inequality implies that $\delta(\tau)EU(x) < U(Ex)$. Therefore,

$$
\begin{aligned}
M_v(v, r, \tau, t) &< \frac{\delta(\tau)EU'(x)}{U'\big(U^{-1}\big(U(Ex)\big)\big)} \\
&= \frac{\delta(\tau)EU'(x)}{U'(Ex)}.
\end{aligned}
$$

Since $U''' \le 0$, Jensen's inequality implies $EU'(x) \le U'(Ex)$. Hence,

$$
M_v(v, r, \tau, t) < \delta(\tau) < 1.
$$

Appendix SC: Identification of utility functions from certainty equivalents

Suppose that U is a utility function defined on $\mathcal{X} \subset \mathbb{R}$ and that $\mathcal{F}$ is a collection of distributions with supports contained in $\mathcal{X}$. This generates a certainty equivalent functional

(also called a quasilinear mean)

$$m(F) = U^{-1}\Big(\int U(x)\,dF(x)\Big), \quad F \in \mathcal{F}.$$

Now suppose that we are given a collection of lotteries $\mathcal{F}$ and a quasilinear mean functional m. If $\mathcal{F}$ is sufficiently rich, it is plausible that the utility function U is uniquely determined (up to affine transformations) by m. We show that this is true even for a well chosen one-dimensional family of lotteries.

Our example is adapted from the proof of Theorem 83 in Hardy, Littlewood, and Polya (1952). Let $\mathcal{X} = [a, b]$ and consider the collection of lotteries $\mathcal{F} = \{F_t(x), t \in [0, 1]\}$, where the F_t are mixtures of point masses at the endpoints a and b:

$$F_t(x) = (1-t)\delta_a(x) + t\delta_b(x).$$

Note that

$$m(F_0) = m(\delta_a) = a,$$
$$m(F_1) = m(\delta_b) = b,$$

and since m is continuous and strictly increasing, $m(F_t)$ takes every value in $[a, b]$.

Suppose that there is another function V satisfying

$$m(F) = V^{-1}\Big(\int V(x)\,dF(x)\Big), \quad F \in \mathcal{F}.$$

Let

$$\tilde{x}(t) = m(F_t) = U^{-1}\big[(1-t)U(a) + tU(b)\big] = V^{-1}\big[(1-t)V(a) + tV(b)\big].$$

We have

$$U\big(\tilde{x}(t)\big) = (1-t)U(a) + tU(b),$$

and we can solve for t and $(1-t)$:

$$t = \frac{U\big(\tilde{x}(t)\big) - U(a)}{U(b) - U(a)},$$
$$(1-t) = \frac{U(b) - U\big(\tilde{x}(t)\big)}{U(b) - U(a)}.$$

Now

$$V\big(\tilde{x}(t)\big) = (1-t)V(a) + tV(b)$$
$$= \frac{U(b) - U\big(\tilde{x}(t)\big)}{U(b) - U(a)} \cdot V(a) + \frac{U\big(\tilde{x}(t)\big) - U(a)}{U(b) - U(a)} \cdot V(b).$$

This is a linear (in fact, affine) function of $U(\tilde{x}(t))$, so we can write

$$V\big(\tilde{x}(t)\big) = \alpha + \beta U\big(\tilde{x}(t)\big),$$

where α and β do not depend on t and $\beta > 0$. Since this holds for all $t \in [0, 1]$, we have

$$V(x) = \alpha + \beta U(x) \quad \forall x \in [a, b].$$

Thus V must be an affine transformation of U.

References

Hardy, G., J. E. Littlewood, and G. Polya (1952), *Inequalities*, second edition. Cambridge University Press. [11]

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