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# Average crossing time: An alternative characterization of mean aversion and reversion

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This study compares and contrasts the multiple characterizations of mean reversion in financial time series as regards the restrictions they imply. This is accomplished by translating them into statements about an alternative measure, the "Average Crossing Time" or ACT. We argue that the ACT measure, per se, provides not only a useful benchmark for the degree of mean reversion/aversion, but also an intuitive, and easily quantified sense of one time series being "more strongly mean-reverting/averting" than another. We conclude our discussion by deriving the ACT measure for a wide class of stochastic processes and detailing its statistical characteristics. Our analysis is principally undertaken within a class of well-understood production based asset pricing models.

Keywords. Mean aversion, mean reversion, average crossing time, time series, asset pricing.

JEL classification. C13, C53, E3, E44, E47, G1, G12.

## 1. Introduction

There has been a long-standing debate in the asset pricing literature as to whether time series of equity and bonds returns are "mean reverting" or "mean averting." While the conventional wisdom is that the former returns are mean reverting and the latter mean averting, the issue is by no means settled.[1] The debate has also diffused to the pre-

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[1]There is a plethora of empirical studies on mean reversion in stock prices at various time horizons. See, for example, papers by Summers (1986), Campbell and Mankiw (1987), Fama and French (1988), Lo and MacKinlay (1988), and Poterba and Summers (1988). Others, most notably, Kim, Nelson, and Startz (1991), and Richardson and Stock (1989) have challenged some of their conclusions. See also the conflicting perspectives in, for example, Lewellen (2004), Torous, Valkanov, and Yau (2004), and Campbell and Yogo (2006) versus Goyal and Welch (2003), Welch and Goyal (2008), and Bossaerts and Hillion (1999). Other important work includes Cochrane (2011), Kim and Nelson (1998), Bessembinder et al. (1995), and Daniel (2001). Zakamulin (2016) provided an excellent summary of this literature and explores the evidence for mean reversion and predictability over periods exceeding 10 years.


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dictability literature. Predictability studies implicitly assume that the predicting variables (dividend-price ratios, earnings-price ratios) follow stationary processes that *revert to some unspecified normal value* (the mean of the process).[2] Intuitively, a stationary stochastic process illustrates this property: above average values of the process must regularly be followed by below average values and vice versa. Is a "mean averting" process then one for which this property is absent? If so, could such a "mean averting" process be "stationary"? What does the mean reversion/aversion distinction ultimately signify? This paper offers one answer to this question.

In particular, we compare and contrast the multiple characterizations of mean reversion in financial time series as regards the restrictions they imply. This is accomplished by translating them into statements about an alternative measure, the "Average Crossing Time" or *ACT*. We argue that the *ACT* measure, per se, provides not only a useful benchmark for the degree of mean reversion/aversion, but also an intuitive, and easily quantified, sense of one time series being "more strongly mean-reverting/averting" than another. We conclude our discussion by deriving the *ACT* measure for a wide class of stochastic processes and detailing its statistical characteristics. While our analysis is principally undertaken within a family of well-understood production-based asset-pricing models, we expand the discussion to include arbitrary AR-1 and random walk processes.

As the name suggests, the *ACT* measures the average number of periods in the evolution of a discrete time stochastic process for which the process is strictly above or below its mean value. Broadly speaking, mean reverting processes are shown to have a relatively shorter average crossing time as compared to mean averting ones. Using the *ACT* measure, we also explore the antecedent probability structures behind traditional notions of mean reversion and aversion. Our analysis is both analytical and computational, with the latter consisting of wide-ranging numerical simulations of the aforementioned dynamic production-based macro-finance models where the time series of endogenous security prices and returns are stationary by construction. Using the *ACT* measure, we then explore the extent to which these model-generated series satisfy the various characterizations.

Negative (positive) autocorrelation in financial return series, for example, is sometimes cited as an identifying characteristic of mean reversion (aversion) and we adopt it as our benchmark property. In the case of the simple baseline equilibrium model we explore, this identifying characteristic classifies the stationary time series characterizing equity returns as "mean reverting" yet the *stationary time series characterizing bond returns as mean averting*, a result consistent with conventional understanding. Mean reversion, as defined by the benchmark property, is thus not unique to the phenomenon that "above average values of stochastic process must regularly be followed by below average values and vice-versa" *since mean averting time series are equally consistent with*

[2]As Campbell and Shiller (2005) note: "It seems reasonable to suspect that prices are not likely ever to drift too far from their normal levels relative to indicators of fundamental value, ... *when stock prices are very high relative to these indicators, then prices will eventually fall in the future to bring the ratios back to more normal historical levels*."

*this characterization*. We later show that realistically parameterized versions of the baseline model generically display benchmark mean aversion in *all* financial time series, whether equity or debt, a result that perhaps diminishes the usefulness of the benchmark characterization itself.

An outline of the paper is as follows: In Section 2, we identify four prominent definitions of mean reversion and mean aversion found in the literature, and partially characterize their interrelationships. In Section 3, these definitions are applied to the analysis of a simple baseline dynamic macroeconomic model.[3] The *ACT* characterization of "mean reversion" is introduced in Section 4. In Section 5, we add additional features to the baseline model and study the resulting implications for the strength of mean reversion/aversion in model-generated equity returns, bond returns, and equity premium data. Section 6 explores the statistical properties of the *ACT* measure per se, including the computation of standard errors, a consideration of its small sample properties and an exploration of its relationship to the sample impulse response function of the underlying stochastic process. We also compute the *ACT* for familiar time series processes such as an AR-1 and a random walk. These extensions are undertaken within a wider discussion of the usefulness of the *ACT* measure for distinguishing stationary stochastic processes from nonstationary ones. Section 7 (detailed in Appendix A which may be found in the Replication file (Donaldson and Mehra (2021)) further generalizes the baseline model and explores how the generalizations influence the *ACTs* of its financial time series. Section 8 relates these concepts to the data by computing empirical autocorrelations and *ACTs* for prominent financial return series while Section 9 concludes.

All proofs of Propositions are in the online Technical Appendix in the Replication file (Donaldson and Mehra (2021)).

## 2. Mean reversion

The empirical finance literature proposes multiple characterizations of "mean reversion." In the discussion below, we examine four specific characterizations and explore their interrelationships. They are as follows, expressed in terms of an arbitrary *stationary* stochastic process $\{\tilde{x}_t\}$.[4]

A stationary stochastic process $\{\tilde{x}_t\}$ is said to be mean reverting if and only if:

[3]The primary intellectual antecedents of the present study are Basu and Vinod (1994), Cecchetti, Lam, and Mark (1990), Guvenen (2009), and Lansing (2015). In a Lucas (1978) style exchange model where dividends follow a Markov switching regime (see Hamilton (1989)), Cecchetti, Lam, and Mark (1990) are able to replicate the observed patterns of mean reversion measures at various horizons. Guvenen (2009) explored asset pricing in a model where firm owners and workers have differential access to securities markets: firm owners trade both equity and default free bonds while workers are limited to bond trading. Lansing (2015) explored the asset pricing consequences of variation in factor shares. Both report slight negative correlation in equity returns based on data, and as equilibrium outcomes of their models. We note that the analysis in Basu and Vinod (1994) explores some of the same issues and motivates the present study. None of these studies, however, explores the fundamental information being conveyed by the mean reversion/aversion distinction, but rather focus on identifying this property in model generated return series.

[4]In all that follows a $\sim$ above a variable denotes a stochastic process while its absence indicates a particular realization of the process.

I. $$\operatorname{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < 0; \tag{1}$$

II. $$\frac{\operatorname{var}(\tilde{x}_{t+1} + \cdots + \tilde{x}_{t+J})}{J \operatorname{var}(\tilde{x}_{t+1})} < 1, \quad \text{for any } J \geq 2. \tag{2}$$

Property I is cited by Guvenen (2009) and Lansing (2015). An early proponent of Property II is Summers (1986). Property II is also used in Poterba and Summers (1988) and Mukherji (2011) for their discussions of mean reversion in stock price and rate of return series.[5]

The relationship between Properties I and II is captured in Proposition 2.1.

Proposition 2.1. *Let* $\{\tilde{x}_t\}$ *be a stationary stochastic process with an ergodic probability distribution. With respect to that distribution, statistical Properties* I *and* II *detailed above are related according to*

(a) II $\Rightarrow$ I

(b) *If*

$$\left|\operatorname{cov}(\tilde{x}_t, \tilde{x}_{t+1})\right| > \left|\sum_{s=2}^{J-1} \operatorname{cov}(\tilde{x}_t, \tilde{x}_{t+s})\right| \quad \textit{for all } J, \tag{3}$$

*then* I $\Rightarrow$ II.[6]

Condition (3) captures the idea that a process displays strong comovement in adjacent elements but that the effect diminishes very rapidly for series elements increasingly in the future.

The significance of Proposition 2.1, simple as it is, lies in Part (a): if a time series generated by a particular model fails to satisfy Property I, it will fail to satisfy Property II as well. In this sense, Property II is more restrictive than Property I. For this reason, we focus on assessing how restrictive Property I actually is, knowing that Property II is even more so. In fact, they will both be so strikingly restrictive that it is not surprising to find that they are not robustly satisfied in data.

Work by Campbell and Mankiw (1987), Cochrane (1988), and Campbell (2018) proposes a weaker version of Property II: rather than requiring the variance ratio to be less

[5]Property II stands in specific contrast to the analogous property of a random walk where $\operatorname{var}(\tilde{x}_{t-1} + \cdots + \tilde{x}_{t+J}) = J \operatorname{var}(\tilde{x}_{t+1})$.

[6]For mean aversion, the connections of Proposition 2.1 are reversed. Using the notation of Proposition 2.1, they are:

(a) I $\Rightarrow$ II

(b)

$$\text{If } \left|\operatorname{cov}(\tilde{x}_t, \tilde{x}_{t+1})\right| < \left|\sum_{s=2}^{J-1} \operatorname{cov}(\tilde{x}_t, \tilde{x}_{t+s})\right| \quad \text{for all } J \text{ then II } \Rightarrow \text{ I.} \tag{3a}$$

than one at all horizons, they admit the possibility of positive autocorrelations at short horizons together with negative autocorrelations at longer horizons, leading to a variance ratio that is eventually less than one. Specifically, these authors first define $V(J)$ as

$$V(J) = \frac{\text{var}(\tilde{x}_{t+1} + \cdots + \tilde{x}_{t+J})}{J\,\text{var}(\tilde{x}_{t+1})},$$

for any stationary stochastic process $\{\tilde{x}_t\}$. They then define a stochastic process to be mean reverting if and only if

II L. $\lim_{J\to\infty} V(J) < 1$

We choose to denote this Property as II L since it is a limiting version of Property II.

III. For any time integers $0 \le r < s < t < u$,

$$\text{cov}(\tilde{x}_s - \tilde{x}_r, \tilde{x}_u - \tilde{x}_t) < 0. \tag{4}$$

Property III, to our knowledge first proposed in Exley, Mehta, and Smith (2004), is a comment about sequential *changes* in the values of the stochastic process $\{\tilde{x}_t\}$ rather than a statement about the statistical properties of the values themselves. Interpreting $\{\tilde{x}_t\}$ as $\{\tilde{p}_t^e\}$, the price of equity capital at time $t$, Property III suggests that increases in the price of equity over a particular interval of time will generally be followed by declines in the price in future time intervals. As such, it represents a sense of mean reversion different from Properties I and II. In all of the characterizations we consider, if the identifying inequality is reversed, the series is said to be mean averting.

Property III can be guaranteed if certain sufficient conditions are satisfied.

For any time integers $h > 0$, $k > 0$, define $v(h-k)$ as

$$v(h-k) \equiv \text{var}(\tilde{x}_h - \tilde{x}_k).$$

This allows a simple statement of the following proposition.

PROPOSITION 2.2. *If* $v(\,)$ *is concave, then* $\{\tilde{x}_t\}$ *is mean reverting by Property* III.[7]

PROOF. See Exley, Mehta, and Smith (2004). □

If condition (4) were not satisfied then increases in the value of the series would, on average, be followed by further increases and declines by further declines. Accordingly, the range of possible evolutions of a series would tend to fan out with the variance of the difference in series values growing with the difference in time indices, as in a random walk. Concavity of $v(\,)$ precludes this effect: increases in $\tilde{x}_t$ must be followed, on average, by eventual decreases and vice versa, a weak sense of mean reversion. Property III is rarely employed in the empirical finance literature, and, as we demonstrate later, neither implies nor is it implied by either Property I or II. We will thus largely focus on Properties I and II since they are the most frequently cited:

[7] If $v(\,)$ is convex, then $\{\tilde{x}_t\}$ is mean averting.

IV. $$E_t[\tilde{x}_{t+1} - x_t] = \kappa[x_t - \bar{x}], \tag{5}$$

where $E_t$ is the period $t$ conditional expectations operator, $\bar{x}$ is the unconditional mean of the series $\{\tilde{x}_t\}$ and $\kappa < 0$ a constant.[8]

Property IV identifies a mean reverting process as one that is always being "pulled towards its mean": if, at some $t$, the process is above its mean ($x_t > \bar{x}$), the expected change next period should be negative and vice versa, with the "strength" of the pull-back determined by $\kappa$ and the extent of the current deviation $[x_t - \bar{x}]$. Of all the characterizations of mean reversion that we consider, this is perhaps the most intuitively appealing. Other authors identify mean reversion with conditional or unconditional variance compression, or simply the absence of random walks. Koijen, Rodriguez, and Sbuelz (2009) explored optimal portfolio allocations when equity index returns display momentum at short horizons and Property IV mean reversion at long horizons. While we leave these important generalizations to future analysis, they may all be accommodated by the perspective to follow.

As noted in the Introduction, the notion of mean reversion is one of the above average realizations of the series being regularly followed by below-average values and vice versa. Which of the above properties is consistent with this intuition? Are the properties consistent with one another? To what extent do they refine the basic concept of stationarity?[9] Since the context of these questions is usually one of equilibrium economic and financial time series, we choose to address them first within the framework of a simple stochastic general equilibrium macroeconomic model, although the analysis applies to any stochastic process that can be well approximated as a Markov chain. Following Cochrane (2011), the implicit context is one of quarterly frequency.

[8]This Ornstein–Uhlenbeck style characterization is emphasized in Bekaert and Hodrick (2017), for example. It is also expressed as $E_t[\ell n(\tilde{x}_{t+1}) - \ell n(x_t)] = \kappa[\ell n(x_t) - \bar{x}]$; other similar representations may be found in the literature, for example, Marques (2004).

[9]As regards these questions, we can get an indication of the answer by looking at the simplest "canonical mean reverting process," an AR(1):

$$\tilde{x}_{t+1} = \rho x_t + \tilde{\varepsilon}_{t+1}, \quad 0 < \rho < 1, \qquad \tilde{\varepsilon}_t \sim N(0, \sigma^2_{\tilde{\varepsilon}}) \quad \text{for all } t.$$

For this process, $\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) = \rho\sigma^2_{x_t}$, which is mean *averting* by Property I. Furthermore, since

$$\text{var}\left(\frac{\tilde{x}_t + \tilde{x}_{t+1}}{2}\right) = \frac{1+\rho}{1-\rho^2}\sigma^2_{\tilde{\varepsilon}} > \frac{\sigma^2_{\tilde{\varepsilon}}}{1-\rho^2} = \text{var}(\tilde{x}_t)$$

it is mean averting by Property II, confirming Proposition 2.1. As for Property III,

$$\text{cov}(\tilde{x}_s - \tilde{x}_r, \tilde{x}_u - \tilde{x}_t) = \text{cov}\left(\sum_{j=r+1}^{s} \tilde{\varepsilon}_j \rho^{j-s}, \sum_{j=r+1}^{s} \tilde{\varepsilon}_j \rho^{u-s}\right) = 0, \quad \text{for all } r < s < t < u;$$

hence the process is also not Property III mean reverting. Lastly,

$$E_t(\tilde{x}_{t+1} - x_t) = E_t\big((\rho - 1)x_t + \tilde{\varepsilon}_{t+1}\big) = (\rho - 1)(x_t - \bar{x}), \quad \text{since } \bar{x} = 0.$$

An AR(1) process is set up structurally to satisfy Property IV.

## 3. Modeling perspective and the baseline model

We first focus on a simple representative agent neoclassical stochastic macroeconomic model with "planning" representation:

$$\begin{aligned}
&\max E\left(\sum_{t=0}^{\infty}\beta^t u(\tilde{c}_t, 1-\tilde{n}_t)\right)\\
&\text{s.t.}\quad c_t + i_t \le y_t = f(k_t, n_t)\lambda_t,\\
&\qquad k_{t+1} = (1-\Omega)k_t + i_t, \quad k_0 \text{ given}, 0 \le n_t \le 1,\\
&\qquad \tilde{\lambda}_{t+1} \sim G(\tilde{\lambda}_{t+1}; \lambda_t).
\end{aligned} \tag{6}$$

Adopting the customary notation, $u(c_t, 1-n_t)$ represents the representative agent's period utility function defined over his period $t$ consumption $c_t$ and leisure, $(1-n_t)$, where $n_t$ is labor supplied, $f(k_t, n_t)\lambda_t$ denotes the representative firm's CRS production function of capital stock $k_t$ and labor supplied with $\{\tilde{\lambda}_t\}$ the stochastic total factor productivity shock. The probability distribution function for $\{\tilde{\lambda}_{t+1}\}$ conditional on $\lambda_t$ is denoted $G(\tilde{\lambda}_{t+1}; \lambda_t)$ and is assumed to be known to the representative agent.[10] Lastly, $\beta$ denotes the representative agent's subjective time discount factor and $\Omega$ the economy's period depreciation rate.

Model (6) has been extensively studied in the literature. Our interest, however, is the decentralized interpretation,[11] which allows us to model an implied financial market where risk free debt and equity are competitively traded.[12] Under this interpretation, the period $t$ dividend satisfies

$$d_t = f(k_t, n_t)\lambda_t - w_t n_t - i_t \tag{7}$$

while the ex-dividend aggregate equity price, $p_t^e = p^e(k_t, \lambda_t)$, is identified with next period's capital stock:

$$p_t^e = k_{t+1}.\text{[13]} \tag{8}$$

In (7), $w_t$ denotes the competitive wage rate, which, in equilibrium, satisfies

$$w_t = f_2(k_t, n_t)\lambda_t.$$

Accordingly,

$$1 + \tilde{r}_{t+1}^e = \frac{\tilde{p}_{t+1}^e + \tilde{d}_{t+1}}{p_t^e} = f_1(k_{t+1}, n_{t+1})\tilde{\lambda}_{t+1} + (1-\Omega) \quad \text{(by CRS)}, \tag{9}$$

[10] The productivity disturbance $\{\tilde{\lambda}_t\}$ will typically be of the form $\tilde{\lambda}_t = e^{\tilde{x}_t}$ where $\{\tilde{x}_t\}$ is an AR(1) process.

[11] See Prescott and Mehra (1980), Brock (1982), or Donaldson and Mehra (1984).

[12] As such, the financial market can be regarded as "complete."

[13] In a related study, Lansing (2015) referred to this dividend expression as the "macroeconomic dividend." With this identification, the dividend is assumed to be exclusively financed out of capital's income share.

where $\tilde{r}^e_{t+1} = f_1(k_{t+1}, n_{t+1})\tilde{\lambda}_{t+1} - \Omega$ denotes the net return on unlevered equity from the "end of period $t$" to the "end of period $t+1$."

The period price, $p^b_t$, of a risk-free bond paying one unit of consumption in period $t+1$, irrespective of the realized state, is

$$p^b_t = p^b(k_t, \lambda_t) = \beta \int \frac{u_1(\tilde{c}_{t+1}, 1 - \tilde{n}_{t+1})}{u_1(c_t, 1 - n_t)} dG(\tilde{\lambda}_{t+1}; \lambda_t) \tag{10}$$

with the risk-free rate $r^b_t = r^b(k_{t-1}, \lambda_{t-1})$ satisfying $(1 + r^b_{t+1}) = 1/p^b_t$. Accordingly, the equity premium is defined by $r^p_t = r^e_t - r^b_t$. Recalling footnote 4, a tilde above a quantity indicates that it is to be interpreted as random; an absent tilde indicates a particular realization of the random quantity.

### 3.1 *The baseline model*

We first restrict problem (6) by requiring that

$$u(c_t) = \ell n(c_t), \qquad y_t = f(k_t, n_t)\lambda_t = k^\alpha_t \lambda_t, \quad n_t \equiv 1 \quad \text{and} \quad \Omega = 1 \tag{11}$$

with $\{\tilde{\lambda}_t\}$ a strictly positive i.i.d. stochastic process. (12)

Optimal policy functions assume the form[14]

$$c_t = c(k_t, \lambda_t) = (1 - \alpha\beta) y_t \quad \text{and} \tag{13}$$

$$i_t = k_{t+1} = \alpha\beta y_t = \alpha\beta k^\alpha_t \lambda_t.^{15} \tag{14}$$

Accordingly,

$$p^e_t = p(k_t, \lambda_t) = k_{t+1} = \alpha\beta k^\alpha_t \lambda_t, \tag{15}$$

$$p^b_t = p^b_t(k_t, \lambda_t) = \big(\beta E\big(\lambda^{-1}_t\big)/(\alpha\beta)^\alpha\big) k^{\alpha(1-\alpha)}_t \lambda^{1-\alpha}_t, \tag{16}$$

$$d_t = \alpha k^\alpha_t \lambda_t - \alpha\beta k^\alpha_t \lambda_t = \alpha(1-\beta) k^\alpha_t \lambda_t, \tag{17}$$

$$r^e_t = f_1(k_t)\lambda_t = \alpha k^{\alpha-1}_t \lambda_t. \tag{18}$$

As a specialization of Problem (6), there exists an ergodic probability distribution for $\{(\tilde{k}_t, \tilde{\lambda}_t)\}$ that captures the Baseline model's long run stochastic behavior; the same can thus be said for $\{\tilde{p}^e_t\}$, $\{\tilde{p}^b_t\}$, $\{\tilde{d}_t\}$, $\{\tilde{r}^e_t\}$, $\{\tilde{r}^b_t\}$, and $\{\tilde{r}^p_t\}$.[16]

We first explore this model with reference to Property I.

PROPOSITION 3.1. *In model* (6), *specialized as per identifications* (11) *and* (12), *the equity price* $\{\tilde{p}^e_t\}$, *the default-free bond price* $\{\tilde{p}^b_t\}$, *and the dividend series* $\{\tilde{d}_t\}$ *are all mean averting by Property* I.[17]

[14] See Donaldson and Mehra (1984) and Mehra (1984).

[15] By recursive substitution, $k_t = [(\alpha\beta)^{1+\alpha+\alpha^2+\cdots+\alpha^{t-1}}] k_0^{\alpha^t} \prod_{s=0}^{t-1} \lambda_s^{\alpha^{t-1-s}}$.

[16] In order to provide a closed form expression for $r^b_t$, the productivity shock must be further specialized. See Section 4.

[17] The part of Proposition 3.1 dealing specifically with $\{\tilde{p}^e_t\}$ was first presented in Basu and Vinod (1994) and Basu and Samanta (2001) in a slightly less general setting. We extend their explorations with a different goal in mind.

This result confirms that the notions of Property I mean reversion and stationarity (the existence of a long run ergodic probability distribution on capital stock—also the equity price—to which the economy converges) are not equivalent, and that the distinction arises in the simplest equilibrium macroeconomic models.

Corollary 3.1. *Dividends, equity prices and default-free debt prices are mean averting under Property* II.

Furthermore, these observations are generic in the sense expressed in the following result.

Proposition 3.2. *Consider any equilibrium model of the general form* (6) *for which the equilibrium investment function* $i(k_t, \lambda_t)$ *is continuous and increasing in both its arguments. Suppose also that the period t price of equity and the period* $t+1$ *level of the capital stock coincide* (*no costs of adjustment*). *Then, under both Properties* I *and* II $\{\tilde{p}_t^e\}$ *will be mean averting. If* $p_t^b = h(k_t, \lambda_t)$, *where* $h(\cdot)$ *is continuous and increasing in both its arguments, then* $\{\tilde{p}_t^b\}$ *will be mean averting as well.*

All of the macroeconomic models to be considered in this paper satisfy the conditions of the above proposition; hence, the conclusion to Proposition 3.2 applies quite generally.

We next examine mean reversion (Property I) in the equity and bond return series for this model.

Proposition 3.3. *For Model* (6), *specialized by* (11) *and* (12):

(a) $\text{corr}(\tilde{r}_t^e, \tilde{r}_{t+1}^e) \leq 0$; *equity returns are mean reverting by Property* I.

(b) $\text{corr}(\tilde{r}_t^b, \tilde{r}_{t+1}^b) > 0$; *that is, bond returns are mean averting by Property* I.

It is clear from the Proof of Proposition 3.3 that concavity of the production function $(\alpha - 1) < 0$ plays the key role in inducing Property I mean reversion in equity returns, a fact first observed in Basu and Vinod (1994). Risk-free returns, however, are Property I mean averting.

Taken together, Propositions 3.1 and 3.3 remind us that mean reversion in equity returns need not imply mean reversion in equity prices (at least by the criterion of Property I).[18] That mean reversion in returns is compatible with mean aversion in prices is already well known (Spierdijk and Bikker (2012)). To find this compatibility in the simplest possible dynamic equilibrium context is, however, somewhat surprising. Without further specialization of the productivity process,[19] it is difficult to derive fully general results for Property III. Consider Proposition 3.4.

[18] Indeed, in this simple model both equity prices and dividends are mean averting, yet equity returns are nevertheless mean reverting.

[19] To show, for example, that $\{\tilde{p}_t^e\}$ is mean averting, it is sufficient to show $\text{var}\{\tilde{p}_t^e - \tilde{p}_t^s\}$ is convex as a function of $t - s$, where $p_t^e = k_{t+1} = [(\alpha\beta)^{1+\alpha+\alpha^2+\cdots+\alpha^{t-1}}]k_0^{\alpha^t} \prod_{j=0}^{t-1} \lambda_j^{\alpha^{t-1-j}}$. Variances of products of random variables are complex quantities.

Proposition 3.4. *Consider model* (6) *specialized as per* (11) *with a productivity shock of the form* $\{e^{\tilde{\lambda}_t}\}$, *where* $\tilde{\lambda}_t = \tilde{\varepsilon}_t$, $\{\tilde{\varepsilon}_t\}$ *i.i.d.* $N(0, \sigma_\varepsilon^2)$.[20] *Then,*

(a) *The equity price series and dividend series are mean averting by Property* III;

(b) *The return on equity is mean reverting by Property* III.

The proof of Proposition 3.4 reveals that concavity in production ($\alpha < 1$) is, once again, key in generating Property III mean reversion in equity returns. These results are entirely consistent with those obtained for our earlier analysis of Properties I and II. Nevertheless, the fact that Property III does not distinguish among AR(1) processes for various $\rho$ tends to disqualify it as a discriminating mean reversion characterization.

For the baseline model, Property IV is not satisfied; there exists no single $\kappa$ for which condition (5) holds. It is "weakly" mean reverting, however, in the sense that if $\tilde{\varepsilon}_t \equiv 0$ for all $t$, then $k_t = p_t^e < \bar{p}_t^e = \bar{k}$ implies $p_{t+1}^e > p_t^e$ and vice versa if $p_t^e > \bar{p}_t^e$ (a "bar" above a variable denotes its mean value).

If the notion of mean reversion is intended to capture the property that above average values of a stochastic process must regularly be followed by below average values, then all the series considered thus far, $\{\tilde{p}_t^e\}$, $\{\tilde{d}_t\}$, $\{\tilde{r}_t^e\}$, $\{\tilde{p}_t^b\}$, and $\{\tilde{r}_t^b\}$ qualify: each follows a stationary stochastic process that converges to a unique, irreducible ergodic set. Yet, as Propositions 3.1–3.3 make clear,

(i) mean aversion in a time series (by Properties I–III) does not imply non-stationary, and (ii) stationary of a series does not guarantee mean reversion by any of the Properties I–IV.

Properties I–IV thus appear to represent restrictive distinctions relative to the basic intuitive sense of a mean reverting series. A useful characterization of mean reversion should also allow the easy comparison of one time series being more highly mean-reverting than another. Our intuition suggests that a more highly mean reverting series should cross its mean more often; that is, with greater "frequency."[21] Crossing the mean with greater frequency must in turn imply less persistence as regards the series being exclusively either in states above or below its mean. In the next section, we develop this notion of persistence and relate it to the Property I characterization of mean reversion presented earlier.

## 4. An alternative metric

In view of the preceding discussion, we propose "Average Crossing Time" (*ACT*) as a simple measure of persistence. Intuitively, a larger *ACT* roughly corresponds to less frequent crossings of the mean, in turn suggesting weaker mean reversion. We offer the simple *ACT* measure because not only does it help to evaluate Properties I–IV, but also because it provides a natural, intuitive sense of one series being more weakly mean reverting than another.

[20] This productivity shock process is typically used in the business-cycle literature.

[21] A crossing of the mean from above at time $t$ would signify that $x_t \geq \bar{x}$, yet $x_{t+1} < \bar{x}$, and analogously for crossing the mean from below.

Table 1. Baseline model: correlations and ACTs[(i)].

| $\alpha = 0.36$, $\beta = 0.99$, $\Omega = 1$, $\tilde{\lambda}_t = e^{\tilde{\varepsilon}_t}$, $\{\tilde{\varepsilon}_t\}$ i.i.d., $\sigma_\varepsilon = 0.00712$[(ii)] | | | | |
|---|---|---|---|---|
| $\text{corr}(\tilde{p}^e_t, \tilde{p}^e_{t+1})$ | $\text{corr}(\tilde{p}^b_t, \tilde{p}^b_{t+1})$ | $\text{corr}(\tilde{r}^e_t, \tilde{r}^e_{t+1})$ | $\text{corr}(\tilde{r}^b_t, \tilde{r}^b_{t+1})$ | $\text{corr}(\tilde{r}^p_t, \tilde{r}^p_{t+1})$ |
| 0.36 | 0.36 | −0.32 | 0.36 | −0.21 |
| $ACT\{\tilde{p}^e_t\}$ | $ACT\{\tilde{p}^b_t\}$ | $ACT\{\tilde{r}^e_t\}$ | $ACT\{\tilde{r}^b_t\}$ | $ACT\{\tilde{r}^p_t\}$ |
| 2.61 | 2.61 | 1.66 | 2.61 | 1.76 |

(i) These ACT computations (averages) include the period of crossing. See Footnote 23 ahead for the justification behind this convention. Statistics are computed on the basis of 1000 independently constructed series each of length 10,000. In all cases, the initial value in the series corresponds to the steady state capital stock level.

(ii) This choice of shock standard deviation is common in the business cycle literature. We retain this value throughout the paper.

Definition. A discrete-time stochastic process's average crossing time ($ACT$) is the average number of time periods before the process transitions from above its unconditional mean to below its unconditional mean or vice versa.

Subject to certain modest refinements, the $ACT$ can be computed by dividing the length of the series by the number of crossings of the mean observed over its duration.[22] Under this concept, an economic time series is defined to be *mean reverting if and only if its ACT is finite*.[23,24,25] To gain some intuition for the $ACT$-autocorrelation relationship, we computed these quantities for the financial time series generated by the Baseline model. Table 1 presents the results.

Two observations stand out. First, the ordering (smallest to largest) of correlations and $ACT$s is the same, subject to rounding and numerical approximations: a more positive autocorrelation is associated with a larger $ACT$, which implies less frequent "crossings." Second, the single negatively autocorrelated series, $\{\tilde{r}^e_t\}$, is the only one for which the $ACT$ is less than two. In the remainder of this section, we explore the generality of these observations.

First, note that the state variables for any Dynamic Stochastic General Equilibrium (DSGE) model follow a Markov process for an appropriately defined state space, and thus can be well approximated by a Markov chain of sufficiently high dimension. The same is true for all the endogenous return series arising in equilibrium. Accordingly, we focus our attention on chain representations. Furthermore, when appropriately constructed, a specific two state Markov chains turns out to be all that is necessary for the $ACT$ computation.

[22] Marques (2004) introduced the "frequency of transition" concept as a measure of persistence. Roughly speaking, his frequency of transition measure is the reciprocal of ACT. He notes that a white noise process has unconditional $E(\gamma) = 0.5$, corresponding to the ACT computation of 2 as demonstrated in Proposition 4.2 to follow.

[23] This characterization of mean reversion is made more precise in Section 6.

[24] Dividing the length of the time series by the number of crossings only leads to a precise ACT measurement if the series is of great length. See Section 6 and, in particular, the notes to Figure 11(a) where alternative methods of calculation are compared.

[25] Model (6), specialized by restrictions (11) is level stationary so that all the relevant series will have well-defined means.

To see this, consider an $N$ state irreducible Markov chain $\{\tilde{\gamma}_t\}$ with states indexed by $\gamma_i, \gamma_j, i, j : 1, 2, \ldots, N$, transition probabilities $\phi_{ij}, i = 1, 2, \ldots, N, j = 1, 2, \ldots, N$ and ergodic probabilities $\pi_i : i = 1, 2, \ldots, N$, $\pi_i > 0\ \forall i$. We will subsequently interpret these $\{\tilde{\gamma}_t\}$ as equilibrium return or price measurements. Let $T$ denote the chain's transition probability matrix with entries $\phi_{ij}$. Without much loss of generality, we assume there exists a state $\hat{j}$ such that

$$\gamma_{\hat{j}+1} < E(\tilde{\gamma}_t) < \gamma_{\hat{j}},$$

a restriction that allows the unambiguous definition of the sets $\gamma^A$ and $\gamma^B$, where

$$\gamma^A = \{\gamma_j : \gamma_j > E(\tilde{\gamma}_t)\}, \qquad \gamma^B = \{\gamma_j : \gamma_j < E(\tilde{\gamma}_t)\} \quad \text{and} \quad \gamma^A \cap \gamma^B = \emptyset.$$[26]

We construct a derivative (two state) Markov chain $\{\tilde{\gamma}_t^{AB}\}$ on the sets $\gamma^A$ and $\gamma^B$ by defining its $\gamma^A$, $\gamma^B$ transition probabilities as follows:

$$\phi_{AA} = \operatorname{Prob}(\tilde{\gamma}_{t+1} \in \gamma^A | \gamma_t \in \gamma^A) = \sum_{j=1}^{\hat{j}} \left( \left( \frac{\pi_j}{\sum_{\ell=1}^{\hat{j}} \pi_\ell} \right) \left( \sum_{k=1}^{\hat{j}} \phi_{jk} \right) \right),$$

$$\phi_{AB} = \operatorname{Prob}(\tilde{\gamma}_{t+1} \in \gamma^B | \gamma_t \in \gamma^A) = \sum_{j=1}^{\hat{j}} \left( \left( \frac{\pi_j}{\sum_{\ell=1}^{\hat{j}} \pi_\ell} \right) \left( \sum_{k=\hat{j}+1}^{N} \phi_{jk} \right) \right),$$

$$\phi_{BA} = \operatorname{Prob}(\tilde{\gamma}_{t+1} \in \gamma^A | \gamma_t \in \gamma^B) = \sum_{j=\hat{j}+1}^{N} \left( \left( \frac{\pi_j}{\sum_{\ell=\hat{j}+1}^{N} \pi_\ell} \right) \left( \sum_{k=1}^{\hat{j}} \phi_{jk} \right) \right) \quad \text{and}$$

$$\phi_{BB} = \operatorname{Prob}(\tilde{\gamma}_{t+1} \in \gamma^B | \gamma_t \in \gamma^B) = \sum_{j=\hat{j}+1}^{N} \left( \left( \frac{\pi_j}{\sum_{\ell=\hat{j}+1}^{N} \pi_\ell} \right) \left( \sum_{k=\hat{j}+1}^{N} \phi_{jk} \right) \right).$$

[26] If, for some $j^*$, $\gamma_{j^*} = E(\tilde{\gamma}_t)$, then $\gamma_{j^*}$ is in neither $\gamma^A$ nor $\gamma^B$ which distorts the relationship of the original transition matrix, which includes $\gamma_{j^*}$, to the "aggregated" process $\{\tilde{\gamma}_t^{AB}\}$.

Let $T^{AB}$ denote the transition probability matrix with the above entries,

$$T^{AB}: \begin{matrix} & \gamma^A & \gamma^B \\ \gamma^A & \phi_{AA} & \phi_{AB} \\ \gamma^B & \phi_{BA} & \phi_{BB} \end{matrix},$$

and denote the long run ergodic probabilities governing the relative frequency of observing, respectively, elements of $\gamma^A$ and $\gamma^B$ by $\pi_A$ and $\pi_B$ where

$$\pi_A = \sum_{\gamma_j \in \gamma^A} \pi_j, \quad \text{and} \quad \pi_B = \sum_{\gamma_j \in \gamma^B} \pi_j.$$

For any discrete time Markov process $\{\tilde{\eta}_t\}$, we define the average crossing time *from above* as the average number of periods the process assumes values above its mean, inclusive of the first period it "crosses," that is, assumes a value below its mean. We denote this quantity by $ACT^A\{\tilde{\eta}_t\}$. The average crossing time *from below* is defined analogously and is denoted by $ACT^B\{\tilde{\eta}_t\}$. In the case of the two state Markov chain $\{\tilde{\gamma}_t^{AB}\}$, these quantities are easily computed as follows:

$$\begin{aligned} ACT^A\{\tilde{\gamma}_t^{AB}\} &= \sum_{n=1}^{\infty} n \operatorname{Prob}\big(\tilde{\gamma}_{t+n}^{AB} \in \gamma^B | \gamma_{t+j}^{AB} \in \gamma^A, j = 0, 1, 2, \dots, n-1\big) \\ &= \sum_{n=1}^{\infty} n \big(\phi^{AA}\big)^{n-1}\big(1-\phi^{AA}\big) \\ &= \frac{1-\phi^{AA}}{\phi^{AA}} \frac{\phi^{AA}}{\big(1-\phi^{AA}\big)^2} = \frac{1}{1-\phi^{AA}}. \end{aligned}$$ [27]

Similarly,

$$ACT^B\{\tilde{\gamma}_t^{AB}\} = \sum_{n=1}^{\infty} n \operatorname{Prob}\big(\tilde{\gamma}_{t+n}^{AB} \in \gamma^A | \gamma_{t+j}^{AB} \in \gamma^B, j = 0, 1, 2, \dots, n-1\big) = \frac{1}{1-\phi^{BB}}.$$

With these quantities in mind, we focus exclusively on the "set chain" $\{\tilde{\gamma}_t^{AB}\}$ rather than its antecedent, $\{\tilde{\gamma}_t\}$. The justification for this choice takes the form of a small proposition.

[27] One could also define the average crossing time from $\gamma^A$ to $\gamma^B$ by the average time the process remains in state $\gamma^A$, *not* including the period of crossing. Identify this quantity as $\widehat{ACT}^A$ where

$$\widehat{ACT}^A = \sum_{n=1}^{\infty} n \operatorname{Prob}\big(\tilde{\gamma}_{t+n}^{AB} \in \gamma^A | \tilde{\gamma}_{t+n+1}^{AB} \in \gamma^B \text{ and } \gamma_{t+j}^{AB} \in \gamma^A, j = 0, 1, 2, \dots, n-1\big) = \frac{\phi^{AA}}{\big(1-\phi^{AA}\big)}.$$

Clearly, $ACT^A = \widehat{ACT}^A + 1$, as the respective formulae confirm. We choose to work with $ACT^A$ rather than $\widehat{ACT}^A$ and the analogous $ACT^B$ rather than $\widehat{ACT}^B$ as to do so proves to be algebraically simpler.

PROPOSITION 4.1. *For any irreducible Markov chain* $\{\tilde{\gamma}_t\}$

$$ACT^A\{\tilde{\gamma}_t\} = ACT^A\{\tilde{\gamma}_t^{AB}\} \quad \textit{and} \quad ACT^B\{\tilde{\gamma}_t\} = ACT^B\{\tilde{\gamma}_t^{AB}\}.$$

*Accordingly,* $ACT\{\tilde{\gamma}_t\} = ACT\{\tilde{\gamma}_t^{AB}\}$.

Proposition 4.1 simply claims that the $ACT^A$ and $ACT^B$ values for the original chain and its derived "set chain" are identical. Hence, we shift our focus to the latter. We do not claim, however, that $\text{corr}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) = \text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB})$, or that any other statistical properties beyond $ACT^A$ and $ACT^B$ are the same for both series.

We next initiate a characterization of $\{\tilde{\gamma}_t^{AB}\}$. Its properties are listed below. All the calculations are entirely straightforward and are provided in the Technical Appendix in the Replication file (Donaldson and Mehra (2021)).

A.

$$\pi^A = \frac{1-\phi^{BB}}{2-(\phi^{AA}+\phi^{BB})} \quad \text{and} \quad \pi^B = \frac{1-\phi^{AA}}{2-(\phi^{AA}+\phi^{BB})},$$

where $\pi^A$, $\pi^B$ represent, respectively, the ergodic probabilities of the process being in set $\gamma^A$ or $\gamma^B$.

B. As noted earlier, $ACT^A = \frac{1}{1-\phi^{AA}}$ and $ACT^B = \frac{1}{1-\phi^{BB}}$. Accordingly, "Average Crossing Time," *ACT*, satisfies

$$\begin{aligned} ACT_{\{\tilde{\gamma}_t^{AB}\}} &= \pi^A ACT^A_{\{\tilde{\gamma}_t^{AB}\}} + \pi^B ACT^B_{\{\tilde{\gamma}_t^{AB}\}} \\ &= \frac{1}{2-(\phi^{AA}+\phi^{BB})}\left(\frac{1-\phi^{AA}}{1-\phi^{BB}} + \frac{1-\phi^{BB}}{1-\phi^{AA}}\right). \end{aligned} \tag{19}$$

C.

$$\text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB}) = (\phi^{AA}+\phi^{BB}) - 1. \tag{20}$$

From Eq. (20), we see that the same autocorrelation can arise from many different $\phi^{AA}$, $\phi^{BB}$ pairs; to illustrate, any $(\phi^{AA}, \phi^{BB}) \in \{(0.5, 0.5), (0.7, 0.3), (0.05, 0.95)\}$ yields a $\text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB}) = 0$, yet the corresponding *ACT*s are, respectively, 2, 2.762, and 19.053. Independence, as measured by $\text{corr}(\phi_t^{AB}, \phi_{t+1}^{AB}) = 0$, can, in fact, be consistent with many patterns (as captured by the $ACT^A$, $ACT^B$ values).

D.

PROPOSITION 4.2. *If* $ACT\{\tilde{\gamma}_t^{AB}\} \le 2$, *then* $\text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB}) \le 0$. *If* $\phi^{AA} = \phi^{BB}$, *then* $\text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB}) \le 0$ *implies* $ACT \le 2$.[28]

[28] In Proposition 4.2, $\phi^{AA} = \phi^{BB}$ is a sufficient condition for the second claim; clearly, it is not necessary. The proposition is thus more general than it appears. For all simulations reported in this paper, $\text{corr}(\tilde{\gamma}_t^{AB}, \tilde{\gamma}_{t+1}^{AB}) \le 0$ and $ACT \le 2$ go hand-in-hand.

First, observe that Proposition 4.2 is consistent with the values presented in Table 1: the only negative autocorrelation is $\operatorname{corr}(\tilde{r}_t^e, \tilde{r}_{t+1}^e) = -0.30$ and its $ACT = 1.66$, suggesting very frequent crossings. Note that the values in Table 1 also confirm the observation formalized as Proposition 4.1: although the statistics computed there are based on the full series "$\{\tilde{\gamma}_t\}$" the $ACT$–correlation relationship expressed in Proposition 4.2 for the "$\{\tilde{\gamma}_t^{AB}\}$"series is also observed.

As modest as it is, Proposition 4.2 is a result of interest. Essentially it says that if Properties I or II are to be satisfied, the process in question must have an $ACT$ of two or less, which suggests very frequent crossings relative to the model time period. If real world data were subject to either Property I or II criteria, it is almost certain that tests for mean reversion would fail.

Figure 1 (panels A, B, and C) jointly illustrates observations C and D for various magnifications: in particular, the range of $ACT$s associated with any degree of autocorrelation, and confirms that all $ACT\text{s} \leq 2$ are identified with negative autocorrelation.[29]

E.

Proposition 4.3. *Consider two distinct, irreducible two-state Markov chains,* $\{\tilde{\gamma}^x\}$ *and* $\{\tilde{\gamma}^y\}$ *with transition probability matrices* $T^x$ *and* $T^y$*, respectively, where*

$$T^x: \begin{array}{c} \\ \gamma_1^x \\ \gamma_2^x \end{array} \begin{array}{c} \begin{array}{cc} \gamma_1^x & \gamma_2^x \end{array} \\ \begin{bmatrix} \phi_1^x & 1-\phi_1^x \\ 1-\phi_2^x & \phi_2^x \end{bmatrix} \end{array} \quad \textit{and} \quad T^y: \begin{array}{c} \\ \gamma_1^y \\ \gamma_2^y \end{array} \begin{array}{c} \begin{array}{cc} \gamma_1^y & \gamma_2^y \end{array} \\ \begin{bmatrix} \phi_1^y & 1-\phi_1^y \\ 1-\phi_2^y & \phi_2^y \end{bmatrix} \end{array}.$$

Suppose $\phi_1^x = \phi_2^x = \phi^x$ and $\phi_1^y = \phi_2^y = \phi^y$. Then

$$\operatorname{corr}(\tilde{\gamma}_t^x, \tilde{\gamma}_{t+1}^x) > \operatorname{corr}(\tilde{\gamma}_t^y, \tilde{\gamma}_{t+1}^y) \quad \text{if and only if} \quad ACT\{\tilde{\gamma}_t^x\} > ACT\{\tilde{\gamma}_t^y\}. \tag{21}$$

From the example in part (C) above, it is clear that the statement of our original motivating assertion is not generally true: a higher $ACT$ value is not necessarily equivalent to higher autocorrelation. Yet, this assertion appears to capture the relative autocorrelation-$ACT$ pattern observed in Table 1, which suggests that the underlying multistate probability transition matrices are "approximately" symmetric, a fact that follows intuitively from the technology shock symmetry.

F. By continuity, Proposition 4.3 can be generalized in the following way. Consider a family of irreducible, two state Markov chains of the form $\{\tilde{\gamma}_t^{AB}\}$. Let $\{\hat{\tilde{\gamma}}_t\}$ and $\{\bar{\tilde{\gamma}}_t\}$ be two such chains and let us associate them with transition probability pairs $(\hat{\phi}^{AA}, \hat{\phi}^{BB})$ and $(\bar{\phi}^{AA}, \bar{\phi}^{BB})$, respectively. We are interested to identify the set $A$, where

$$A = \{(\bar{\phi}^{AA}, \bar{\phi}^{BB}) : \text{ for any } (\hat{\phi}^{AA}, \hat{\phi}^{BB}), \text{ where (i) } \bar{\phi}^{AA} \leq \hat{\phi}^{AA} \text{ and (ii) } \bar{\phi}^{BB} \leq \hat{\phi}^{BB}, \\ \text{then (iii) } ACT\{\bar{\tilde{\gamma}}_t\} \leq ACT\{\hat{\tilde{\gamma}}_t\}\}.$$

[29]Figure 1 (panels A, B and) represents a parametric plot of Eq. (19) versus (20) for $0 < \phi^{AA} < 0.9$ and $0 < \phi^{BB} < 0.99$ (various magnifications). The plot was generated in Mathematica.

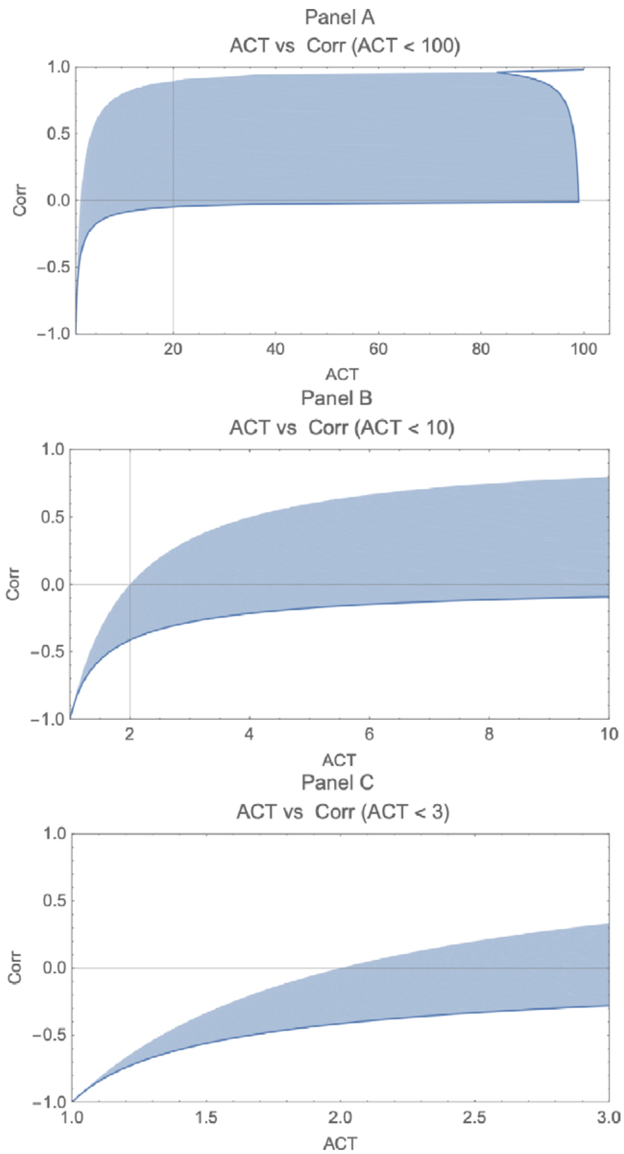


FIGURE 1. *ACT* versus correlation. Panel C clearly shows that negative autocorrelation is coincident with $ACT < 2$.

By (C), it is also true that for set $A$: (iv) $\text{corr}(\bar{\bar{\gamma}}_t, \bar{\bar{\gamma}}_{t+1}) \leq \text{corr}(\hat{\hat{\gamma}}_t, \hat{\hat{\gamma}}_{t+1})$. If either inequality (i) or (ii) is strict, then (iii) and (iv) are strict.

That the set $A$ is nonempty follows from E and continuity: there must exist a region surrounding the 45° line (where $\phi^{AA} = \phi^{BB}$) for which greater *ACTs* and greater (more positive) autocorrelations increase hand in hand. The question that remains is "how large" the region is. The answer is: quite large. This region is portrayed in Figure 2 and

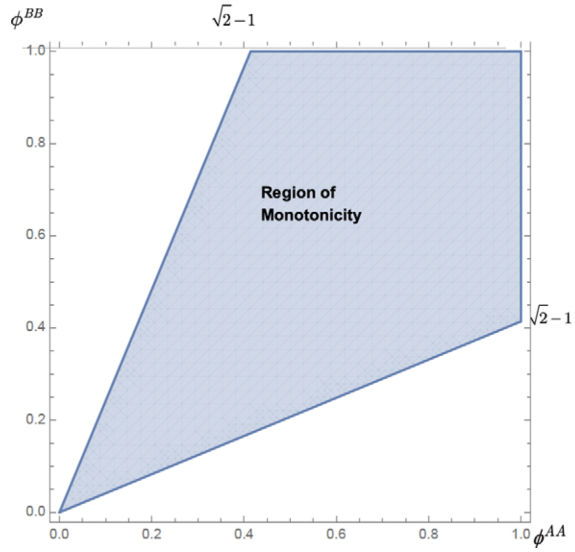


FIGURE 2. Transition matrices for which greater autocorrelation implies a larger *ACT* value.

was numerically constructed using the following alternative representation of the set $A$:

$$A = \{(\phi^{AA}, \phi^{BB}) : \partial ACT/\partial\phi^{AA} \geq 0 \text{ and } \partial ACT/\partial\phi^{BB} \geq 0\}.$$

For $\{\tilde{\gamma}^{AB}\}$ that are far from symmetric (i.e., $\phi^{AA}$ and $\phi^{BB}$ are very different; that is, the nonshaded region of Figure 2), we do not observe *ACT* and autocorrelation increasing in tandem as our original intuition suggested. While individually $\partial ACT^A/\partial\phi^{AA} > 0$ and $\partial ACT^B/\partial\phi^{BB} > 0$ uniformly, it does not necessarily follow that, for example, $\partial ACT/\partial\phi^{AA} > 0$. To illustrate, compare the *ACT*s for $\phi^{AA}$, $\phi^{BB} = (0.9, 0.05)$ and $\phi^{AA}$, $\phi^{BB} = (0.9, 0.10)$:

$$ACT_{(\phi^{AA}=0.9, \phi^{BB}=0.05)} = 9.148,$$

$$ACT_{(\phi^{AA}=0.9, \phi^{BB}=0.10)} = 9.111.$$

The phenomenon arises because the relative stationary probabilities change as $\phi^{BB}$ increases: more probability weight is placed on $ACT^B$, $ACT^B < ACT^A$ causing the overall *ACT* to decline. If $\phi^{AA} \approx \phi^{BB}$, there is little change in the corresponding $(\pi^A, \pi^B)$ when either $\phi^{AA}$ or $\phi^{BB}$ is marginally increased, so that the more intuitive relationship between the *ACT* and its corresponding autocorrelation is observed.

Observation C also suggests that to identify mean reversion solely with either Property I or II is to forgo information. Any value that this particular measurement assumes is clearly compatible with a wide range of stochastic structures $(\phi^{AA}, \phi^{BB})$. It is rather the knowledge of $ACT^A$ and $ACT^B$ that is critical to a comprehensive description of a "mean-reverting" process for investors. If the intuitive notion of "mean reversion" is the sense of "frequent crossings," then as regards the *ACT* measurement, a mean reverting process by Properties I or II is not far from a sequentially independent one.[30]

The results in Table 1 are also in the spirit of Observations E and F. Recall that the time series on which the values reported in Table 1 are generated arise by approximating the underlying real economy about its steady state. Given i.i.d. productivity shocks, the economy evolves symmetrically (about its steady state), suggesting that to a first approximation $\phi^{AA} \approx \phi^{BB}$. The lock-step increases in *ACT* and correlation that are manifest in Table 1 follow naturally from Observation F; equivalently, the series described in Table 1 all have corresponding $(\phi^{AA}, \phi^{BB}) \in A$.

Note also that the statistics computed for Table 1 were created from the basic underlying $\{\tilde{\gamma}_t\}$ processes rather than its "aggregate state" process $\{\tilde{\gamma}_t^{AB}\}$. The fact that the relationships of Proposition 4.2, which are based on the transformed $\{\tilde{\gamma}_t^{AB}\}$, are borne out in *ACT* data generated by the original $\{\tilde{\gamma}_t\}$ again confirms that $ACT\{\tilde{\gamma}_t^{AB}\} = ACT\{\tilde{\gamma}_t\}$ as per Proposition 4.1.

Certain distinctive results in Table 1 are unique to the baseline parameterization (in particular, to the $\rho = 0$, $\Omega = 1$ assumptions). In particular $\text{corr}(\tilde{p}_t^e, \tilde{p}_{t+1}^e) = \text{corr}(\tilde{p}_t^b, \tilde{p}_{t+1}^b)$ and $ACT\{\tilde{p}_t^e\} = ACT\{\tilde{p}_t^b\}$. These identities follow from the fact that $p_t^b = \frac{E(\lambda_t^{-1})}{\alpha}(p_t^e)^{1-\alpha}$ for the baseline model. Accordingly, $\{\tilde{p}_t^b\}$ exceeds its mean when $\{p_t^e\}$ does and vice versa. With identical *ACT*s, their autocorrelations must be identical. Furthermore, since $r_t^b = \frac{1}{p_t^b} - 1$, $r_t^b$ will exceed its mean if and only if $p_t^b$ falls short of its mean, and vice versa, leading to identical *ACT*s for $\{\tilde{p}_t^b\}$ and $\{\tilde{r}_t^b\}$. These relationships do not generally apply to more elaborate versions of the baseline formulation where $\rho \neq 0$.

Let us in this context reinforce our earlier remarks concerning the commonplace characterizations of mean reversion. The results of Table 1 clearly suggest that to identify a mean reverting series exclusively with negative autocorrelation is not fully informative: all the series in Table 1 mean revert (they have finite *ACT* measurements), yet only $\{\tilde{r}_t^e\}$ is negatively autocorrelated. Negative autocorrelation means "extremely frequent crossings of the mean" (a very small *ACT* value), nothing more, and the nature of these crossings can differ widely (Observation C).

We conclude this section with a few summary remarks:

1. Since multiple *ACT*s may in general, result from the same autocorrelation measure, the former is admittedly a coarser measure (and, in principle, less informative). For near to symmetric and symmetric chains, however, the mapping is one-to-one.

2. However, $ACT^A$ and $ACT^B$ together provide more information than autocorrelation, as they capture chain asymmetries.

[30] It is reasonable to propose the following question, "Is the argument for the *ACT* measure simply that two statistics $ACT^A$ and $ACT^B$ are more informative than one (autocorrelation)?" Why not show the entire transition probability matrix? While this is an entirely appropriate question, two comments may be offered in response. First, $ACT^A$ and $ACT^B$ are measures of the frequency of crossing, something that strikes us as more intuitive than the correlation measure. Second, a complex, many state transition probability matrix is, without further analysis, lacking in any intuition as to the underlying process's mean reverting characteristics. The *ACT* reduces this complexity to two numbers.

3. Properties I and II represent highly restrictive characterizations of mean reversion.

The baseline model falls short of a full-fledged business cycle model on many dimensions. In particular, none of the aggregate series is sufficiently persistent vis-à-vis the data. In the next section, we remedy this particular shortcoming and explore its consequences for the various characterizations of "mean reversion."

## 5. Model generalization: Adding persistence in the productivity shock

The "mean reverting" properties of financial time series arising from production-based asset pricing models have not been fully explored in the literature. In this section, we explore how the $ACT^A$, $ACT^B$, *ACT*, and autocorrelation of security prices and returns are affected by adding an important feature to the baseline formulation; namely, productivity shock autocorrelation.[31] It is an absolute minimal requirement for macroeconomic data replication.[32]

Table 2 and Proposition 5.2 illustrate the consequences of introducing persistence in the productivity shocks into the baseline model in a way typical of the production-based asset pricing literature. We specialize the production technology to be of the form $y_t = (k_t)^\alpha e^{\lambda_t}$ where $\tilde{\lambda}_{t+1} = \rho\lambda_t + \tilde{\varepsilon}_{t+1}$; $\{\tilde{\varepsilon}_t\}$ is i.i.d., $\tilde{\varepsilon}_t \sim N(0, \sigma_\varepsilon^2)$, and $\rho > 0$. Even with persistence in the productivity shocks of this type, the decision rules take the same form as (13)–(14).[33]

Furthermore, the addition of persistence does not alter the expressions for $p_t^e$ and $r_t^e$. However, the expressions for $p_t^b$ and $r_t^b$ are modified as follows:

$$p_t^b = \beta e^{\sigma_\varepsilon^2/2} k_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\lambda_t}/(\alpha\beta)^\alpha \quad \text{with } r_{t+1}^b = 1/p_t^b - 1. \tag{22}$$

As regards prices and dividends, the results mirror their earlier counterparts.

Proposition 5.1. *For the baseline model, where* $\rho > 0$ *the equity price series* $\{\tilde{p}_t^e\}$, *the dividend series* $\{\tilde{d}_t\}$, *and the risk-free asset price series* $\{\tilde{p}_t^b\}$ *are all mean averting by Property* I.

Our analysis of returns relies on numerical simulations of (13)–(18). Panel A of Table 2 documents the correlations while panel B gives the corresponding *ACTs*.

[31] Donaldson and Mehra (1983) showed that the resulting Markov process on output, capital stock, and consumption converges to a stationary distribution.

[32] For a full analysis of mean reversion/aversion, it is useful to resort to DSGE models to frame the discussion. Without the context of a model, it is difficult to obtain any intuition for the magnitudes involved in the *ACT*–correlation association. In many cases, small changes in autocorrelation have large consequences for the magnitude of the *ACT*.

[33] The necessary and sufficient condition for the optimal investment function is

$$u_1(c_t) = \beta \int u_1(\tilde{c}_{t+1})\alpha k_{t+1}^{\alpha-1} e^{\tilde{\lambda}_{t+1}}\, dF(\tilde{\lambda}_{t+1}; \lambda_t).$$

For the indicated functional forms and decision rules, this equation becomes

$$\frac{1}{(1-\alpha\beta)k_t^\alpha e^{\lambda_t}} = \beta \int \frac{\alpha\left(\alpha\beta k_t^\alpha e^{\lambda_t}\right)^{\alpha-1} e^{\rho\lambda_t + \tilde{\varepsilon}_{t+1}}}{(1-\alpha\beta)\left[\alpha\beta k_t^\alpha e^{\lambda_t}\right]^\alpha e^{\rho\lambda_t+\tilde{\varepsilon}_{t+1}}}\, dF(\tilde{\varepsilon}_{t+1}).$$

Table 2. Baseline model: autocorrelations, *ACT*s.[(i)]

| $u(c)=\log(c)$, $\beta=0.99$, $\Omega=1$, $\alpha=0.36$, $\tilde{\lambda}_{t+1}=\rho\lambda_t+\tilde{\varepsilon}_{t+1}$, $\sigma_\varepsilon=0.00712$.[(ii)] | | | | | | |
|---|---|---|---|---|---|---|
| | $\rho=0$ | $\rho=0.2$ | $\rho=0.4$ | $\rho=0.6$ | $\rho=0.8$ | $\rho=0.95$ |
| Panel A: Autocorrelations: Various $\rho$ | | | | | | |
| $\text{corr}(\tilde{p}_t^e, \tilde{p}_{t+1}^e)$ | 0.36 | 0.52 | 0.66 | 0.79 | 0.90 | 0.98 |
| $\text{corr}(\tilde{p}_t^b, \tilde{p}_{t+1}^b)$ | 0.36 | 0.61 | 0.80 | 0.83 | 0.66 | 0.43 |
| $\text{corr}(\tilde{r}_t^e, \tilde{r}_{t+1}^e)$ | −0.32 | −0.18 | −0.05 | 0.09 | 0.22 | 0.33 |
| $\text{corr}(\tilde{r}_t^b, \tilde{r}_{t+1}^b)$ | 0.36 | 0.61 | 0.80 | 0.83 | 0.66 | 0.43 |
| $\text{corr}(\tilde{r}_t^p, \tilde{r}_{t+1}^p)$ | −0.21 | −0.06 | 0.07 | 0.20 | 0.31 | 0.36 |
| Panel B | | | | | | |
| *ACT*s | | | | | | |
| $ACT(\tilde{p}_t^e)$ | 2.61 | 3.08 | 3.72 | 4.75 | 6.98 | 14.32 |
| $ACT(\tilde{p}_t^b)$ | 2.61 | 3.46 | 4.85 | 5.31 | 3.68 | 2.80 |
| $ACT(\tilde{r}_t^e)$ | 1.66 | 1.79 | 1.94 | 2.12 | 2.33 | 2.54 |
| $ACT(\tilde{r}_t^b)$ | 2.61 | 3.46 | 4.85 | 5.31 | 3.68 | 2.80 |
| $ACT(\tilde{r}_t^p)$ | 1.76 | 1.92 | 2.10 | 2.29 | 2.49 | 2.60 |
| *ACT*s (Above) | | | | | | |
| $ACT^A(\tilde{p}_t^e)$ | 2.60 | 3.07 | 3.70 | 4.73 | 6.93 | 14.13 |
| $ACT^A(\tilde{p}_t^b)$ | 2.61 | 3.45 | 4.84 | 5.31 | 3.68 | 2.80 |
| $ACT^A(\tilde{r}_t^e)$ | 1.65 | 1.78 | 1.93 | 2.11 | 2.33 | 2.53 |
| $ACT^A(\tilde{r}_t^b)$ | 2.61 | 3.45 | 4.84 | 5.31 | 3.68 | 2.80 |
| $ACT^A(\tilde{r}_t^p)$ | 1.76 | 1.92 | 2.09 | 2.29 | 2.48 | 2.59 |
| *ACT*s (Below) | | | | | | |
| $ACT^B(\tilde{p}_t^e)$ | 2.62 | 3.09 | 3.73 | 4.77 | 7.04 | 14.50 |
| $ACT^B(\tilde{p}_t^b)$ | 2.62 | 3.46 | 4.85 | 5.32 | 3.69 | 2.81 |
| $ACT^B(\tilde{r}_t^e)$ | 1.66 | 1.79 | 1.95 | 2.13 | 2.34 | 2.54 |
| $ACT^B(\tilde{r}_t^b)$ | 2.62 | 3.46 | 4.85 | 5.32 | 3.69 | 2.81 |
| $ACT^B(\tilde{r}_t^p)$ | 1.77 | 1.93 | 2.10 | 2.30 | 2.50 | 2.61 |

[(i)] The reported statistics represent sample averages of 10,000 independently constructed series each of length 10,000.

[(ii)] The numbers reported in this table are unaffected by the magnitude of $\sigma_\varepsilon$. They are also unaffected by the choice of $\beta$, $0<\beta<1$.

While the results of Table 2—Panel A for equity returns mirror the conclusions of Propositions 3.3 for the $\rho=0$ case, they are not robust: sufficient persistence in the random productivity disturbance yields an equity return series that is mean *averting* (by Properties I and II). The other patterns are consistent with the conclusions of Proposition 4.2: negative autocorrelated series have *ACT*s $< 2$, and Observation F; that is, for all series, *ACT*s and correlations increase and decrease in tandem. This latter fact follows from the recognition that all the financial series for the baseline model evolve roughly symmetrically about their unconditional means, with the implication that their corresponding $T^{AB}$ matrices are close to symmetric.[34]

The results of Table 2 for the $\{\tilde{r}_t^e\}$ series are partially rationalized in Proposition 5.2.

[34] We use the word "close" because the series, $\{\tilde{p}_t^e\}=\{\tilde{k}_{t+1}^e\}$, for example, is lognormally distributed, although with a very small standard deviation, and thus is not symmetric. See Footnote 15.

Proposition 5.2. *Consider the model defined by* (13) *and* (14) *with production technology and shock process specialized to* $y_t = k_t^{\alpha} e^{\lambda_t}$ *where* $\tilde{\lambda}_{t+1} = \rho\lambda_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_t\}$ i.i.d.$N(0, \sigma_\varepsilon^2)$. *A sufficient condition for bond and equity returns to be mean averting by Properties* I *and* II *is that* $\alpha + \rho > 1$.

Proposition 5.2 argues that sufficiently persistent productivity disturbances in conjunction with production function concavity results in Property I $\{\tilde{r}_t^e\}$ mean aversion (bond returns are always so). Therefore, if a model of this sort is to come close to matching the observed high persistence in output, equity returns will be mean averting by Property I and Property II.

None of these results is surprising in the least: the process on the disturbance component, $\{e^{\tilde{\lambda}_t}\}$, is itself highly mean reverting (by Property I) only if $\rho < 0$, a selection inconsistent with the behavior of its counterpart, the Solow residual. See Proposition 5.3.

Proposition 5.3.[35] *Consider a stochastic process of the form*

$$\tilde{x}_t = \rho x_{t-1} + \tilde{\varepsilon}_t, \quad \textit{where } \{\tilde{\varepsilon}_t\} \textit{ is } \text{i.i.d. } N\big(0, \sigma_\varepsilon^2\big).$$

*Define a new stochastic process by*

$$\{\tilde{\lambda}_t\} = \big\{e^{\tilde{x}_t}\big\}.$$

*Then*

$$\operatorname{cov}(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}) \begin{Bmatrix} > 0 & \textit{if } 1 > \rho > 0 \\ < 0 & \textit{if } -1 < \rho < 0 \end{Bmatrix}.$$

For calibrations customary to the macrofinance literature $(1 > \rho > 0)$ mean aversion in equity returns results as well. It is not obvious what model features would allow high persistence in aggregate series (as the data reveals) to be compatible with mean reversion in equity returns and the equity premium, at least as characterized by Properties I and II. Cogley and Nason (1995) emphasized the close relationship of the properties of the productivity process to the derived properties of model's state variables, and thus security prices and returns.

Proposition 5.3 bears upon two other unexpected features of Table 2. First, as productivity persistence $\rho$ increases, $\operatorname{corr}(\tilde{p}_t^b, \tilde{p}_{t+1}^b)$, while always positive, first increases (seeming to peak at $\rho = 0.6$) and then monotonically declines. The same pattern is observed for $\operatorname{corr}(\tilde{r}_t^b, \tilde{r}_{t+1}^b)$, although this is to be expected in view of the close association of $\{\tilde{p}_t^b\}$ and $\{\tilde{r}_t^b\}$ noted earlier. We attempt to rationalize these results as follows: for the indicated shock process, the price of the bond is specialized to the form indicated in Eq. (22) and is composed of three building blocks: the positive constant $\beta e^{\sigma_\varepsilon^2/2}/(\alpha\beta)^\alpha$, a capital stock term $k_t^{\alpha-\alpha^2}$, and a productivity shock term $e^{(1-\alpha-\rho)\lambda_t}$. While the second term is mean averting for all values of $\rho$, the third term is Property I mean averting for $(1-\alpha-\rho) > 0$, and Property I mean reverting for $(1-\alpha-\rho) < 0$ by Proposition 5.3 with

[35] We thank Sergio Villar for his help in proving Proposition 5.3.

the switch occurring for $\rho$ slightly greater than 0.6 (since $\alpha = 0.36$). Thus for low values of $\rho$, the mean aversion across the two terms is reinforcing as $\rho$ increases; for $\rho$ exceeding 0.64, the Property I mean reversion of the third term works against the Property I mean aversion of the second to bring about the observed effect.

With $\{\tilde{r}_t^b\}$ consistently mean averting, the sign of the premium's autocorrelation follows the same pattern as $\{\tilde{r}_t^e\}$. A comparison of Panels A and B also reveals that the conclusions of Proposition 4.2 are observed for all cases.

At this point, it is natural to revisit Property II L to see if it provides different insights concerning the "mean reverting nature" of these model generated series. For any arbitrary return series $\{\tilde{r}_t\}$, Campbell (2018) observed that the variance ratio $V(J)$ can be expressed as

$$V(J) = 1 + 2\sum_{k=1}^{J-1}(1 - k/J)\rho_{t-k,t}, \tag{23}$$

where $\rho_{t-k,k} = \text{Corr}(\tilde{r}_{t-k}, \tilde{r}_t)$. Fundamental to this identification is the additional requirement that $\text{Corr}(\tilde{r}_t, \tilde{r}_{t+k}) = \text{Corr}(\tilde{r}_{t+l}, \tilde{r}_{t+l+k})$ for any integer $l$.

Using equivalence (23), we next explore the extent to which any of the model series found to be mean averting under Properties I and II become mean reverting by Property II L ($\lim_{J\to\infty} V(J) < 1$) and vice versa. This is accomplished by computing $V(J)$ for various $J$ in the case of each of our series of interest, $\{\tilde{p}_t^e\}$, $\{\tilde{p}_t^b\}$, $\{\tilde{r}_t^b\}$, $\{\tilde{r}_t^e\}$, $\{\tilde{r}_t^p\}$ for various $\rho$. That is, Panel A of Table 2 is effectively represented graphically using the $V(J)$ measure with increasing $J$. Figure 3 provides the results for $\{\tilde{p}_t^e\}$; Figure 4 for series $\{\tilde{p}_t^b\}$ and $\{\tilde{r}_t^b\}$ jointly (recall that their correlation structures are identical), Figure 5 for $\{\tilde{r}_t^e\}$ and Figure 6 for $\{\tilde{r}_t^p\}$. Note first that $\{\tilde{p}_t^e\}$, $\{\tilde{p}_t^b\}$, and $\{\tilde{r}_t^b\}$ remain mean averting by property II L at all horizons, $J$, and for all shock correlations $\rho$: Properties I, II and II L thus provide identical mean aversion–reversion identifications for these series. This assertion follows from the fact that $V(J)$ is a monotone increasing function of $J$ for each of these series, for all $\rho$, and $V(J) > 1$, $\forall J$. In the case of $\{\tilde{r}_t^e\}$, however, $V(J) < 1$ for sufficiently large $J$ for all $\rho$ values: $\{\tilde{r}_t^e\}$ is Property II L mean reverting. The pattern for the equity premium conforms to that of Panel A, Table 2: mean reverting for low $\rho$ values and means averting for high values.

In summary, measure II L classifies some stationary series that repeatedly cross their respective means as mean reverting and others as mean averting, as do all the other measures we consider. For empirically relevant high $\rho$ values Properties I, II, and II L generally give identical classifications ($\{\tilde{r}_t^e\}$ is the sole exception). Turning to Property III, as shock persistence $\rho$ increases, our earlier Property III results (Proposition 3.4) are weakened for prices: Table 3 summarizes the results of extensive numerical simulations that compute $\text{corr}(\tilde{x}_s - \tilde{x}_r, \tilde{x}_u - \tilde{x}_r)$ for a wide class of $\{r, s, t, u\}$, where $r < s < t < u$.

These results are largely inconsistent with earlier results concerning $\{\tilde{r}_t^e\}$, $\{\tilde{r}_t^p\}$, and$\{\tilde{p}_t^e\}$for Property I (see Table 2, right most column), a fact that accounts for our earlier comment that Property III represents a fundamentally different measurement from either Property I or II.[36]

[36] For this reason, it deserves greater recognition and evaluation. The patterns in Table 3 are largely consistent with Proposition 3.4.

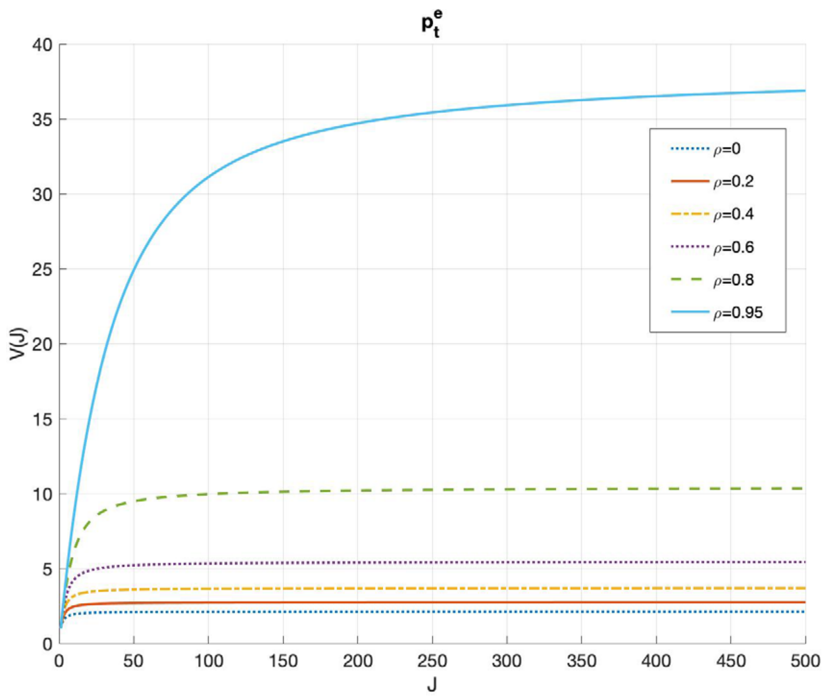


FIGURE 3. Computation of $V(J)$ for various $J$, $\{\tilde{p}_t^e\}$ series. Baseline model: $u(c) = \log(c)$, $\beta = 0.99$, $\Omega = 1$, $\alpha = 0.36$, $\sigma_\varepsilon = 0.00712$.

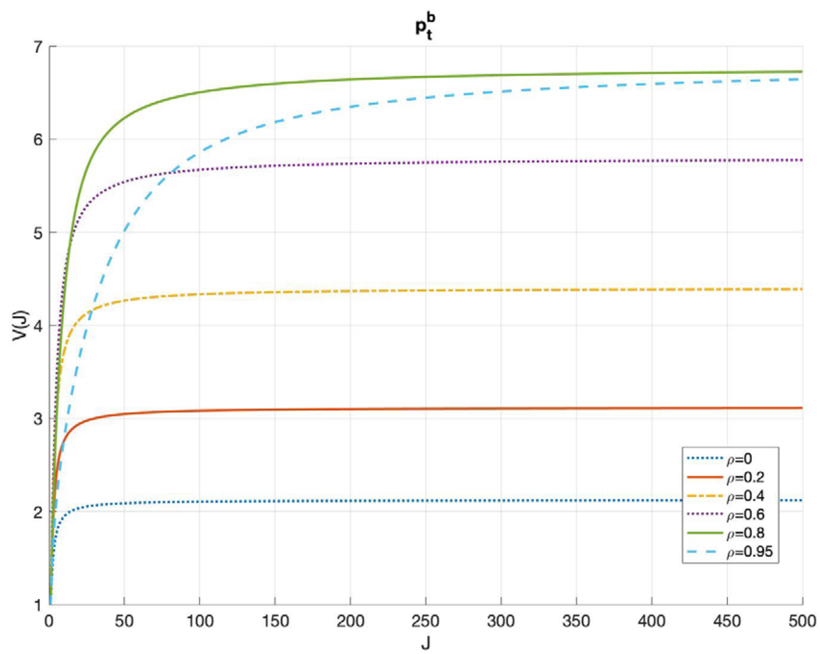


FIGURE 4. Computation of $V(J)$ for various $J$, $\{\tilde{p}_t^b\}$ and $\{\tilde{r}_t^b\}$ series. Baseline model: $u(c) = \log(c)$, $\beta = 0.99$, $\Omega = 1$, $\alpha = 0.36$, $\sigma_\varepsilon = 0.00712$.

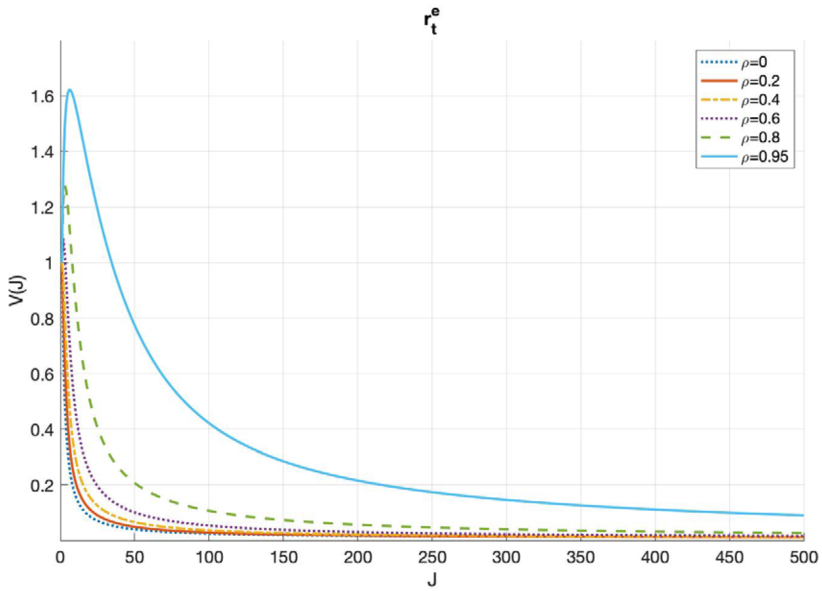


FIGURE 5. Computation of $V(J)$ for various $J$, $\{\tilde{r}_t^e\}$ series. Baseline model: $u(c) = \log(c)$, $\beta = 0.99$, $\Omega = 1$, $\alpha = 0.36$, $\sigma_\varepsilon = 0.00712$.

We close Section 5 with a summary of what we have learned: First, persistence in the productivity disturbance generically overturns specific results relative to the case of independence: equity returns appear necessarily to be Property I mean reverting only

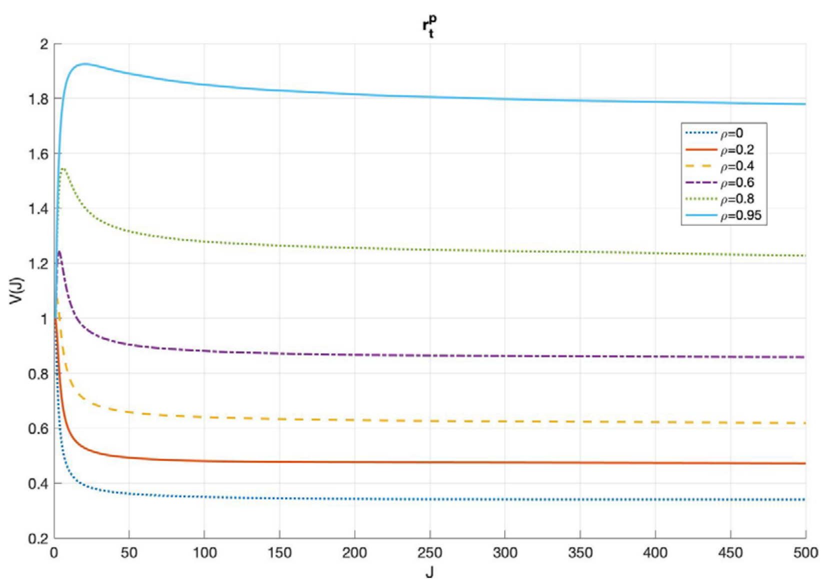


FIGURE 6. Computation of $V(J)$ for various $J$, $\{\tilde{r}_t^p\}$ series. Baseline model: $u(c) = \log(c)$, $\beta = 0.99$, $\Omega = 1$, $\alpha = 0.36$, $\sigma_\varepsilon = 0.00712$.

TABLE 3. Estimated Property III correlations: various series.[(i)]

| $u(c)=\log(c)$, $\beta=0.99$, $\Omega=1$, $\alpha=0.36$, $\tilde{\lambda}_{t+1}=\rho\lambda_t+\tilde{\varepsilon}_{t+1}$, $\sigma_\varepsilon=0.00712$, $\rho=0.95$. | |
|---|---|
| Property III correlation | Range of values across all $r, s, u, t$ |
| (i) corr$(\tilde{p}^e_s-\tilde{p}^e_r, \tilde{p}^e_u-\tilde{p}^e_t)$ | $(-0.26, 0.08)$ |
| (ii) corr$(\tilde{p}^b_s-\tilde{p}^b_r, \tilde{p}^b_u-\tilde{p}^b_t)$ | $(-0.18, 0.01)$ |
| (iii) corr$(\tilde{r}^e_s-\tilde{r}^e_r, \tilde{r}^e_u-\tilde{r}^e_t)$ | $(-0.20, 0.013)$ |
| (iv) corr$(\tilde{r}^b_s-\tilde{r}^b_r, \tilde{r}^b_u-\tilde{r}^b_t)$ | $(-0.19, 0.01)$ |
| (v) corr$(\tilde{r}^p_s-\tilde{r}^p_r, \tilde{r}^p_u-\tilde{r}^p_t)$ | $(-0.19, 0.01)$ |

[(i)]The reported range of Property III correlations reflects observations from 1000 independent time series, each of length 10,000 for all combinations of indices $i, j, k$ satisfying: (a) $s=r+i$, $t=r+j$, and $u=r+j$ and (b) $i<j<k<30$.

in the presence of low persistence productivity disturbances. Proposition 5.3 further suggests that this particular phenomenon is likely to be pervasive across many production based asset pricing formulations, implying that the search for mean reversion in equity returns and the equity premium, at least as characterized by Properties I and II is unlikely to be fruitful—if the present family of models has anything to say about actual economies. To put it differently, we find it unsurprising from a theoretical perspective that evidence for mean reversion in historical equity returns is weak if, indeed, the underlying pricing fundamentals resemble those emphasized in the present, simple macroeconomic model.

## 6. Statistical properties of the *ACT*

Although we propose the *ACT* and its $ACT^A$ and $ACT^B$ constituents as a measure by which definitions of mean reversion commonplace to the literature may be assessed, it can be viewed as an independent statistic of interest. In Section 6, we present an introduction to the sample properties of the *ACT* measure by first analyzing the case of an AR-1 process. The analogous discussion for a simple random walk is then detailed. Finally, we consider the sampling properties of the baseline model both for the case in which its technology shock follows the customary AR-1 process and for the case in which it is governed by a random walk.

### 6.1 *General procedure*

We constructed sampling distributions for the *ACTs* in the following way: for all variables of interest, $j=1,2,\dots,J$ time series were independently constructed, each $N$ elements in length. For each sample $j$ of length $N$, a sample $ACT(j,N)$ was computed and analogously for $ACT^A(j,N)$ and $ACT^B(j,N)$.[37] The set $\{ACT(j,N): j=1,2,\dots,J\}$ constituted the sampling distribution for the *ACT* with the sample mean $\widehat{ACT}$ defined by $\widehat{ACT}=\frac{1}{J-1}\sum_{j=1}^{J}ACT(j,N)$ and analogously for $\widehat{ACT^A}$ and $\widehat{ACT^B}$. We then

[37]To save space, we continue the discussion referring largely to the *ACT*, but the identical procedure was followed for $ACT^A$ and $ACT^B$.

constructed the standard error (SE) of the $\widehat{ACT}$ as the standard deviation of the set $\{ACT(j,N) : j = 1, 2, \dots, J\}$ relative to the $\widehat{ACT}$. For all cases reported in this section, $J = 1000$ and $N = 10{,}000$.

### 6.2 *ACT sampling properties for an AR-1 process*

Consider an AR-1 process:

$$\tilde{x}_{t+1} = \rho x_t + \tilde{\varepsilon}_{t+1}, \quad \{\tilde{\varepsilon}_t\} \sim N\big(0, \sigma_\varepsilon^2\big).$$

Computations are simplified because the $ACT$ of such a process can be computed directly. See Proposition 6.1.

Proposition 6.1. *Consider an AR-1 process of the form*: $\tilde{x}_{t+1} = \rho x_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_t\} \sim N(0, \sigma_\varepsilon^2)$. *Then the ACT of the process is given by*

$$ACT = \frac{\pi}{(\pi/2) - \tan^{-1}\big(\rho/\sqrt{1-\rho^2}\big)}. \tag{24}$$

Although Eq. (24) is a bit opaque, it clearly implies a positive relationship of the $ACT$ and the autoregressive parameter $\rho$, which itself coincides with $\text{corr}(\tilde{x}_t, \tilde{x}_{t+1})$. The $ACT$ and autocorrelation thus rise and fall together for an AR-1. By the symmetry of the AR-1 process about its mean of zero, $ACT = ACT^A = ACT^B$. To get an idea as to the magnitudes involved, and to confirm the validity of Propositions 6.1 and 4.1 (on which 6.1 is based) we followed the procedure outlined in Section 6.1 to compute the $\widehat{ACT}$ for a variety of persistence parameters $\rho$. The results of the exercise are presented in Figure 7. Note the near perfect congruence of the theoretical $ACT$ and $\widehat{ACT}$. Not surprisingly, both the $ACT$ and $\widehat{ACT}$ grow monotonically with persistence.

Figure 8 presents the SEs for an AR-1 process, with $\rho = 0.8$ and $\sigma_\varepsilon = 0.00712$. As is apparent, the standard errors decline rapidly with increasing sample length. More significant is their small magnitude. Confidence intervals for all $\rho \le 0.8$ will thus be extremely tight. Note that as $\rho \mapsto 1$, the $ACT$ in Eq. (24) explodes to $+\infty$. When $\rho = 1$, the AR-1 process becomes a random walk. It is to this further formulation that we next turn.

### 6.3 *ACT sampling properties for a random walk*

We consider the simplest possible random walk:

$$\tilde{x}_{t+1} = x_t + \tilde{\varepsilon}_{t+1}, \quad \{\tilde{\varepsilon}_t\} \sim N\big(0, \sigma_\varepsilon^2\big). \tag{25}$$

While such a process is not stationary ($\text{var}\,\tilde{x}_t = t\sigma_{\tilde{\varepsilon}}^2 \mapsto \infty$ as $t \mapsto \infty$), it has a well-defined unconditional mean, $E\tilde{x}_t = 0$, for all t. Furthermore, it is well known that such a process will return (cross) its mean an infinite number of times with probability one. Accordingly, for sufficient sample length, an estimated $\widehat{ACT}$ can be computed, along with its standard error. In light of formula (24), we would expect both quantities to be increasing with sample length $N$. This is indeed the case; see Figures 9 and 10.

Figures 9 and 10 suggest that the $ACT$ measure could be used as a test for non-stationarity. We consider this possibility later in Section 6.

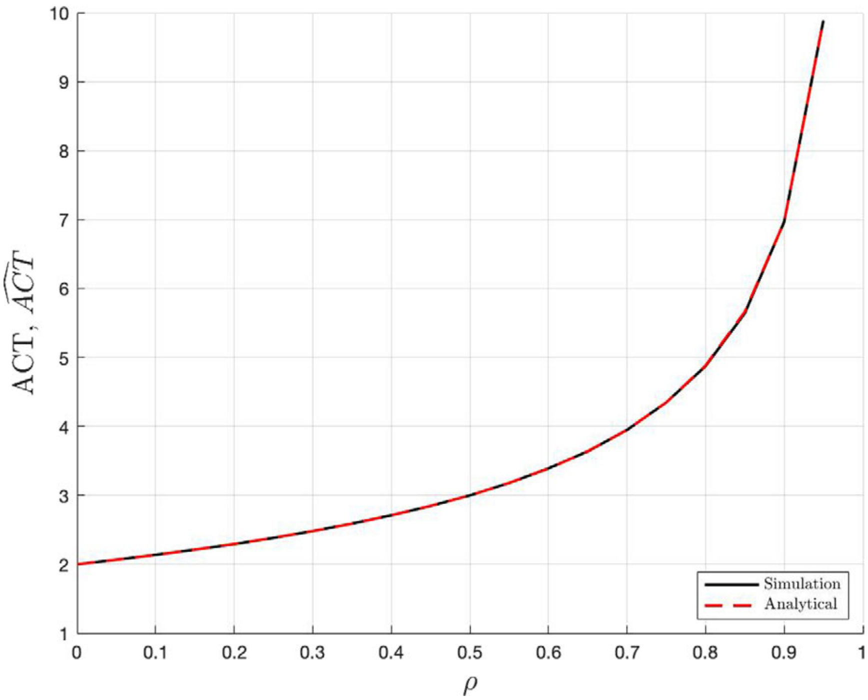


FIGURE 7. Estimated versus theoretical ACTs (Estimated ACTs based on 1000 independent simulation runs each 10,000 data points in length.). AR-1 process, various $\rho$. $\tilde{x}_{t+1} = \rho x_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_t\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$.

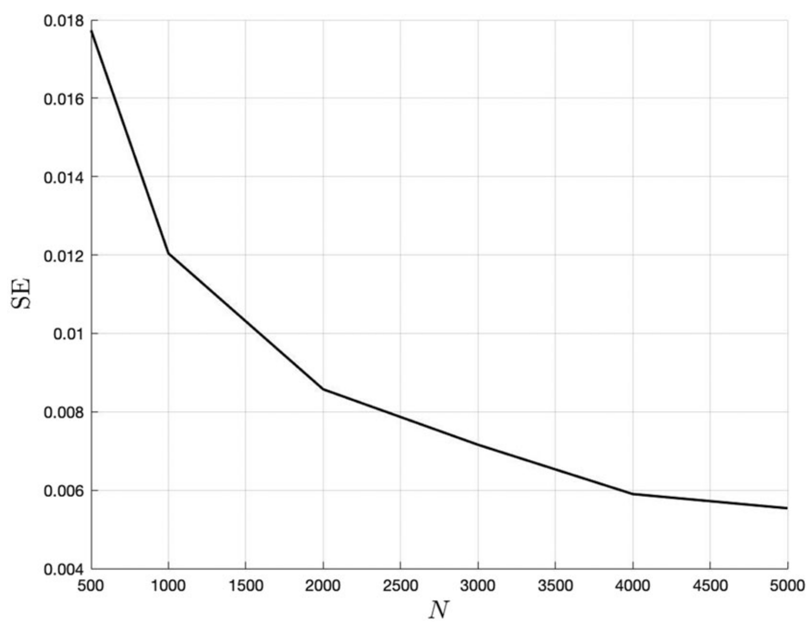


FIGURE 8. ACT standard errors versus sample length (SEs compiled on the basis of 1000 independently generated sample runs.). AR-1 process, $\rho = 0.8$. $\{\tilde{\varepsilon}_t\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$.

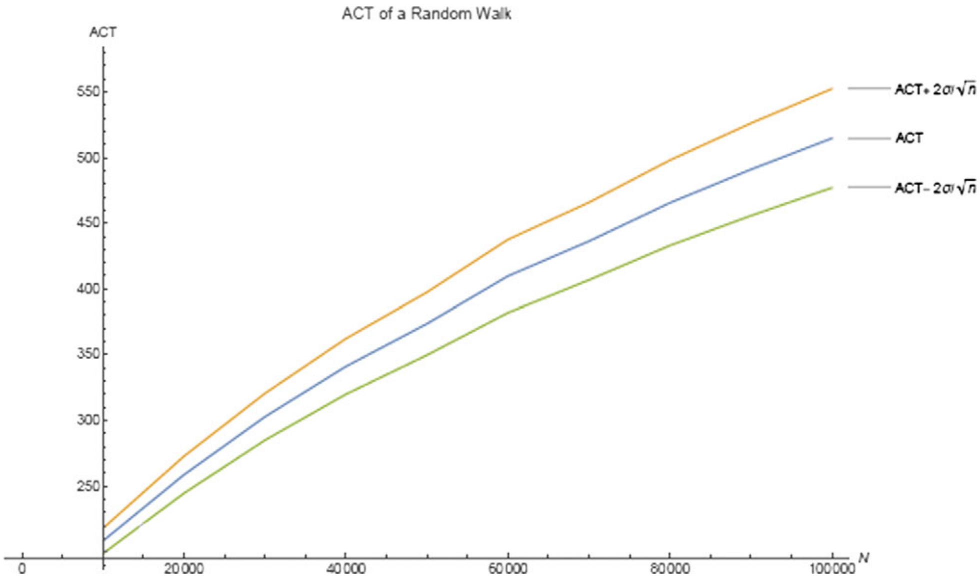


FIGURE 9. $\widehat{ACT}$ versus sample length including confidence intervals (Computations based on $J = 1000$ independent runs for each $N$). Random walk: $\tilde{x}_{t+1} = x_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_{t+1}\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$.

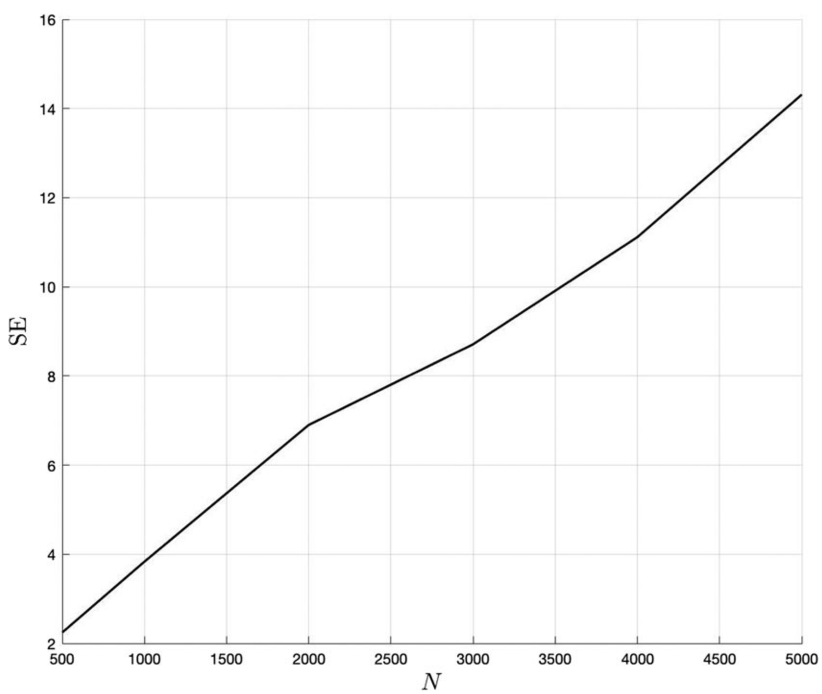


FIGURE 10. Standard errors versus sample length (Computations based on $J = 1000$ independent runs for each $N$). Random walk: $\tilde{x}_{t+1} = x_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_{t+1}\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$.

### 6.4 *Baseline model revisited: Sampling properties*

We next consider the sampling properties for each of the financial time series $\{\tilde{p}_t^e\}$, $\{\tilde{p}_t^b\}$, $\{\tilde{r}_t^e\}$, $\{\tilde{r}_t^b\}$, and $\{\tilde{r}_t^p\}$ generated by the baseline model using the same procedure detailed in Section 6.1. The assumption of $J = 1000$ and $N = 10{,}000$ is maintained, except for explorations requiring a variety of sample lengths. Two cases are considered.

6.4.1 *Baseline model: AR-1 technology shock* The first results of this exercise are presented in Table 4.

A number of observations are in order. Most importantly, the sampling distribution for the *ACT* ($ACT^A$, $ACT^B$) is very compact: the confidence intervals surrounding the sample means are small, a fact also captured by the Panel B coefficients of variation which, for all the return series, rarely exceed 2%. Comparing the corresponding entries in Tables 2 and 4 (same series and identical $\rho$), the corresponding SEs increase or decrease in tandem with autocorrelation.[38] This follows from the near symmetry of the various series about their means: high persistence, which is equivalent to a high *ACT*, suggests an increasing presence of extremely long subseries which are either continuously above or below the corresponding mean, a phenomenon leading to greater variation across the elements of the sampling distribution. This variation, however, is very small in absolute terms because the individual series themselves are all of substantial length.

Figure 11(a)–11(d) describe how *ACT* [39] standard errors computed as in Table 4, vary with sample length $N$, beginning with $N = 500$ and ending with $N = 10{,}000$ when $\rho = 0.8$. For $N = 10{,}000$, the graphical representation coincides with the corresponding values in Table 4 when $\rho = 0.8$. Because of the precise algebraic relationship relating $\{\tilde{r}_t^b\}$ and $\{\tilde{p}_t^b\}$, as detailed in Eq. (22), only the former values are reported. Our choice of $\rho = 0.8$ is arbitrary; for smaller $\rho$, the standard errors are smaller than those portrayed in the figures (for each $N$); they are slightly larger for $\rho = 0.95$.

We see that for all the series the SEs decline very rapidly with series length. But more significantly, for all series of interest their standard deviations are very small at all sample lengths. We chose $N = 500$ as the lower bound on sample length since most empirical asset pricing work has access to data sets of this magnitude or greater. In general, the standard errors for returns, for any sample length, lie below those of the corresponding price series. This is significant as financial research is largely focused on returns.

6.4.2 *Baseline model: Random walk technology shock* Similar explorations can be proposed for the baseline model when its TFP process has the form $\{e^{\tilde{\lambda}_t}\}$, with $\{\tilde{\lambda}_t\}$ following (25). See also Swanson (2019) and Kehoe et al. (2019). With the baseline model, the form of the decision rules, Eqs. (13) and (14), is unaffected, as is the identification of the equity price process $\{\tilde{p}_t^e\}$ with $\{\tilde{k}_t\}$. Pricing and return formulae (15)–(18) and (22) are similarly unchanged. While this random walk modification leads to a nonstationary capital stock

[38] As earlier, all our remarks in this section apply equally to the *ACT*, $ACT^A$, and $ACT^B$. To avoid repetition, our discussion going forward will thus be expressed only in terms of the *ACT*.

[39] Other computational nuances surrounding Figures 11(a)–11(d) are discussed in the notes to the figures.

TABLE 4. Baseline model: measures of statistical precision.

$u(c) = \log(c)$, $\beta = 0.99$, $\alpha = 0.36$, $\Omega = 1$, $\tilde{\lambda}_{t+1} = \rho\lambda_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_t\} \sim N(0, \sigma_\varepsilon)$, $\sigma_\varepsilon = 0.00712$.
Estimates for each series based on 1000 independent runs, each 10,000 elements in length.

| | $\rho = 0$ | $\rho = 0.2$ | $\rho = 0.4$ | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 0.95$ |
|---|---|---|---|---|---|---|
| Panel A: SE[(i)] | | | | | | |
| SE $\widehat{ACT}$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0348 | 0.0457 | 0.0620 | 0.0985 | 0.2069 | 0.8031 |
| $\{\tilde{p}_t^b\}$ | 0.0339 | 0.0541 | 0.0894 | 0.1123 | 0.0748 | 0.0427 |
| $\{\tilde{r}_t^e\}$ | 0.0123 | 0.0141 | 0.0177 | 0.0217 | 0.0275 | 0.0312 |
| $\{\tilde{r}_t^b\}$ | 0.0334 | 0.0548 | 0.0921 | 0.1104 | 0.0752 | 0.0422 |
| $\{\tilde{r}_t^p\}$ | 0.0147 | 0.0175 | 0.0200 | 0.0261 | 0.0296 | 0.0335 |
| SE $\widehat{ACT}^A$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0387 | 0.0504 | 0.0660 | 0.1034 | 0.2125 | 0.8126 |
| $\{\tilde{p}_t^b\}$ | 0.0368 | 0.0590 | 0.0963 | 0.1194 | 0.0786 | 0.0465 |
| $\{\tilde{r}_t^e\}$ | 0.0155 | 0.0177 | 0.0206 | 0.0253 | 0.0303 | 0.0348 |
| $\{\tilde{r}_t^b\}$ | 0.0373 | 0.0590 | 0.1003 | 0.1162 | 0.0790 | 0.0456 |
| $\{\tilde{r}_t^p\}$ | 0.0187 | 0.0203 | 0.0237 | 0.0298 | 0.0335 | 0.0373 |
| SE $\widehat{ACT}^B$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0382 | 0.0493 | 0.0679 | 0.1064 | 0.2220 | 0.8608 |
| $\{\tilde{p}_t^b\}$ | 0.0381 | 0.0579 | 0.0962 | 0.1195 | 0.0798 | 0.0456 |
| $\{\tilde{r}_t^e\}$ | 0.0161 | 0.0177 | 0.0221 | 0.0253 | 0.0312 | 0.0347 |
| $\{\tilde{r}_t^b\}$ | 0.0367 | 0.0598 | 0.0969 | 0.1194 | 0.0796 | 0.0457 |
| $\{\tilde{r}_t^p\}$ | 0.0181 | 0.0217 | 0.0238 | 0.0303 | 0.0333 | 0.0371 |
| Panel B: SE/$\widehat{ACT}$ | | | | | | |
| (SE/$\widehat{ACT}$)/$\widehat{ACT}$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0133 | 0.0149 | 0.0167 | 0.0207 | 0.0296 | 0.0560 |
| $\{\tilde{p}_t^b\}$ | 0.0130 | 0.0156 | 0.0184 | 0.0211 | 0.0203 | 0.0152 |
| $\{\tilde{r}_t^e\}$ | 0.0074 | 0.0079 | 0.0091 | 0.0103 | 0.0118 | 0.0123 |
| $\{\tilde{r}_t^b\}$ | 0.0128 | 0.0158 | 0.0190 | 0.0207 | 0.0204 | 0.0151 |
| $\{\tilde{r}_t^p\}$ | 0.0083 | 0.0091 | 0.0095 | 0.0114 | 0.0119 | 0.0129 |
| (SE $\widehat{ACT}^A$)/$\widehat{ACT}^A$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0149 | 0.0164 | 0.0178 | 0.0218 | 0.0306 | 0.0574 |
| $\{\tilde{p}_t^b\}$ | 0.0141 | 0.0171 | 0.0199 | 0.0224 | 0.0213 | 0.0166 |
| $\{\tilde{r}_t^e\}$ | 0.0094 | 0.0099 | 0.0107 | 0.0120 | 0.0130 | 0.0137 |
| $\{\tilde{r}_t^b\}$ | 0.0143 | 0.0171 | 0.0207 | 0.0219 | 0.0215 | 0.0163 |
| $\{\tilde{r}_t^p\}$ | 0.0106 | 0.0106 | 0.0113 | 0.0130 | 0.0135 | 0.0144 |
| (SE $\widehat{ACT}^B$)/$\widehat{ACT}^B$ | | | | | | |
| $\{\tilde{p}_t^e\}$ | 0.0146 | 0.0160 | 0.0182 | 0.0223 | 0.0315 | 0.0592 |
| $\{\tilde{p}_t^b\}$ | 0.0145 | 0.0167 | 0.0198 | 0.0224 | 0.0216 | 0.0163 |
| $\{\tilde{r}_t^e\}$ | 0.0097 | 0.0098 | 0.0114 | 0.0119 | 0.0133 | 0.0136 |
| $\{\tilde{r}_t^b\}$ | 0.0140 | 0.0171 | 0.0200 | 0.0224 | 0.0216 | 0.0163 |
| $\{\tilde{r}_t^p\}$ | 0.0102 | 0.0113 | 0.0113 | 0.0132 | 0.0133 | 0.0142 |

(i) SEs are computed from the *ACTs* calculated for $J = 1000$ independent runs, each of length $N = 10{,}000$.

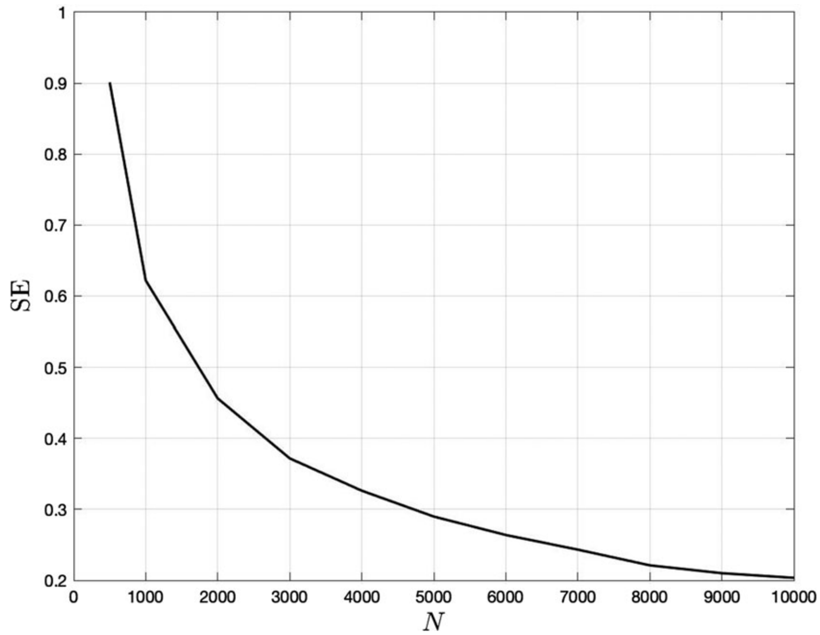


FIGURE 11(a). Baseline model: $\rho = 0.8$, $\sigma_\varepsilon = 0.00712$. ACT standard errors $\{\tilde{p}_t^e\}$ versus sample length $(N)$. (i) Same calibration of the baseline model as in Tables 2 and 4 except that $\rho \equiv 0.8$. (ii) All SEs computed on the basis of 1000 independently generated runs of indicated length $N$. (iii) When the sample length is "small," the method by which the *ACT* is computed matters. There are two methods. The first is to take the length $N$ of the sample and divide it by the number of observed crossings within it. Second, one may compute and record the sequential lengths of individual subseries that are uniformly above or below the sample mean $\mu(N, j)$ for that series, mechanically keeping track of their length and number. The *ACT* is then computed as the average lengths of these subsamples: divide the sum of the various lengths by the number of segments. The issue is how to address the ending points in a sample that may not be identified with a crossing time because the sample length is not long enough for a "final" crossing to have occurred. In the first method described above, these "unassociated data points" are included in the numerator; in the second case, they are not. Accordingly, the *ACT* computed using the first method will exceed that computed using the second. For large samples ($N \geq 1000$) the difference is negligible. For small samples it can be significant if the sample length is small. In Figure 7, we report the simple average using the two methods. Note that only the second method is appropriate for $ACT^A$ and $ACT^B$ calculations.

series (so also for the equity price), where mean and variance grow without bound, all the implied return series and the bond price series remain stationary. For the same statistical regimen that underlies Tables 2 and 4, the results may be found in Table 5.

Clearly, the $\widehat{ACT}$, $\widehat{ACT}^A$, and $\widehat{ACT}^B$ for $\{\tilde{p}_t^e\}$ are growing enormously with $N$ as are their respective standard errors, a sign of nonstationarity. For the return series, comparing the results in Table 5 with the corresponding values for $\rho = 0.95$ in Tables 2 and 4 we find little systematic difference in *ACTs*, or their standard errors, which is evidence of their continued stationarity. Most patterns are retained, though slightly exag-

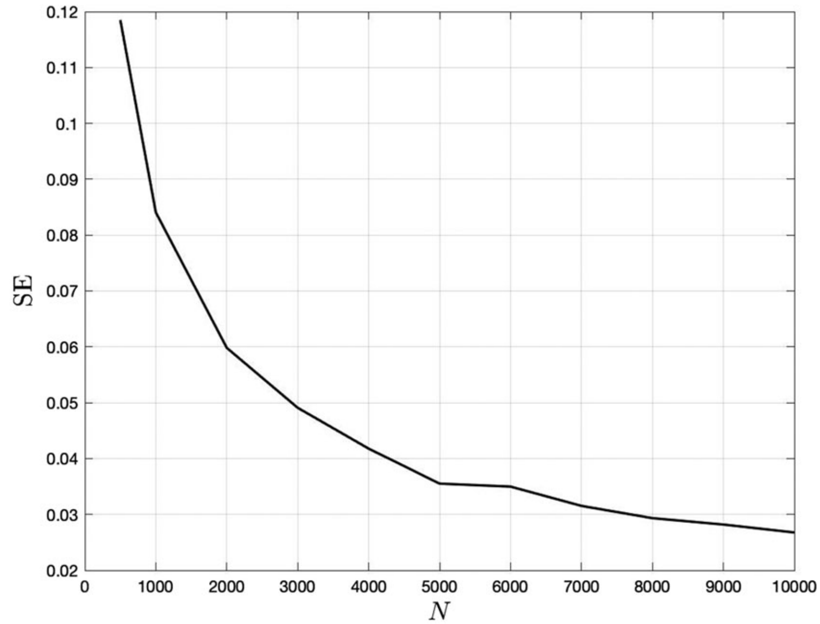


FIGURE 11(b). Baseline model: $\rho = 0.8$, $\sigma_\varepsilon = 0.00712$. ACT standard errors $\{\tilde{r}_t^e\}$ versus sample length ($N$).

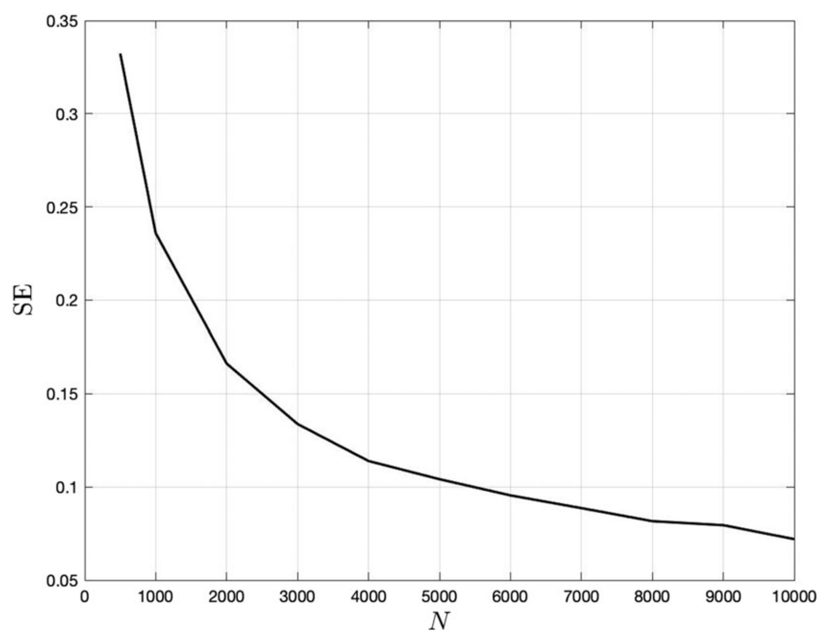


FIGURE 11(c). Baseline model: $\rho = 0.8$, $\sigma_\varepsilon = 0.00712$. ACT standard errors $\{\tilde{r}_t^b\}$ versus sample length ($N$).

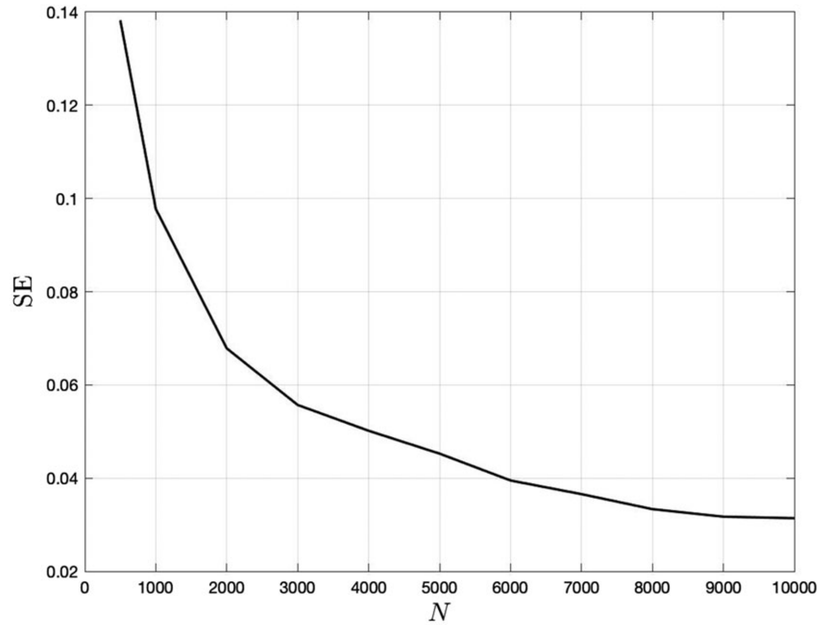


FIGURE 11(d). Baseline model: $\rho = 0.8$, $\sigma_\varepsilon = 0.00712$. ACT standard errors $\{\tilde{r}_t^p\}$ versus sample length $(N)$.

gerated in some cases. Due to the lognormality of the underlying capital stock series, $\widehat{ACT}^A < \widehat{ACT}^B$ for all series when $\rho < 1$. When $\rho = 1$, this latter feature remains the case although greatly exaggerated for the equity return series in particular.

Figure 12 presents the $\widehat{ACT}$ as a function of the sample length for the baseline random walk case. We observe that the $\widehat{ACT}$ is increasing monotonically with sample length. Figure 13 presents the corresponding standard errors which are also increasing monotonically with the sample length.

These observations suggest the possibility of using the *ACT* measure as the basis for a test for nonstationarity in economic time series. An *ACT* that steadfastly grows with sample length, for example, would be an indicator of nonstationarity. Yet, it is unclear

TABLE 5. Estimated $\widehat{ACT}$s and SE for the baseline model. $y_t = k_t^\alpha e^{\lambda_\varepsilon}$, where $\tilde{\lambda}_{t+1} = \lambda_t + \tilde{\varepsilon}_{t+1}$, $\tilde{\varepsilon}_t \sim N(0, \sigma_\varepsilon^2)$. Based on sample length $N = 10{,}000$; 1000 independent samples with $\sigma_\varepsilon = 0.00712$.

| | *ACT* | | | ACT$^A$ | | | ACT$^B$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\widehat{ACT}$ | SE | SE/$\widehat{ACT}$ | $\widehat{ACT}^A$ | SE | SE/$\widehat{ACT}^A$ | $\widehat{ACT}^B$ | SE | SE/$\widehat{ACT}^B$ |
| $\{\tilde{p}_t^e\}$ | 336.44 | 632.03 | 1.88 | 279.5 | 483.34 | 1.73 | 391.18 | 810.63 | 2.07 |
| $\{\tilde{p}_t^b\}$ | 2.611 | 0.0348 | 0.0133 | 2.608 | 0.0379 | 0.0145 | 2.614 | 0.0386 | 0.0148 |
| $\{\tilde{r}_t^e\}$ | 2.612 | 0.0334 | 0.0128 | 2.605 | 0.0366 | 0.0141 | 2.620 | 0.0374 | 0.0143 |
| $\{\tilde{r}_t^b\}$ | 2.611 | 0.0339 | 0.0130 | 2.608 | 0.0376 | 0.0144 | 2.614 | 0.0373 | 0.0143 |
| $\{\tilde{r}_t^p\}$ | 2.612 | 0.0352 | 0.0135 | 2.601 | 0.0387 | 0.0149 | 2.623 | 0.0384 | 0.0146 |

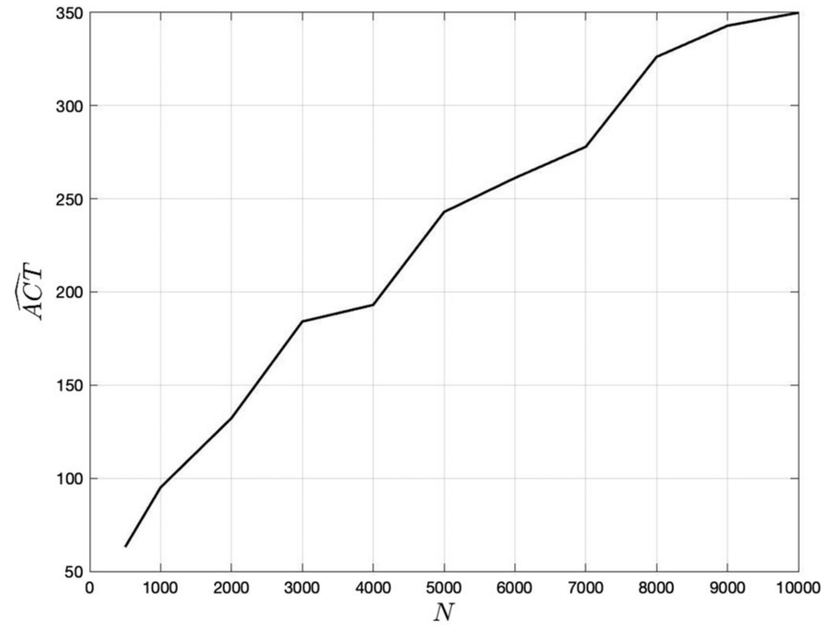


FIGURE 12. $\widehat{ACT}$ versus sample length, $\{\tilde{p}_t^e\}$ series. Baseline model with $\tilde{\lambda}_{t+1} = \lambda_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_{t+1}\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$. Computations based on $J = 1000$ independent runs for each $N$.

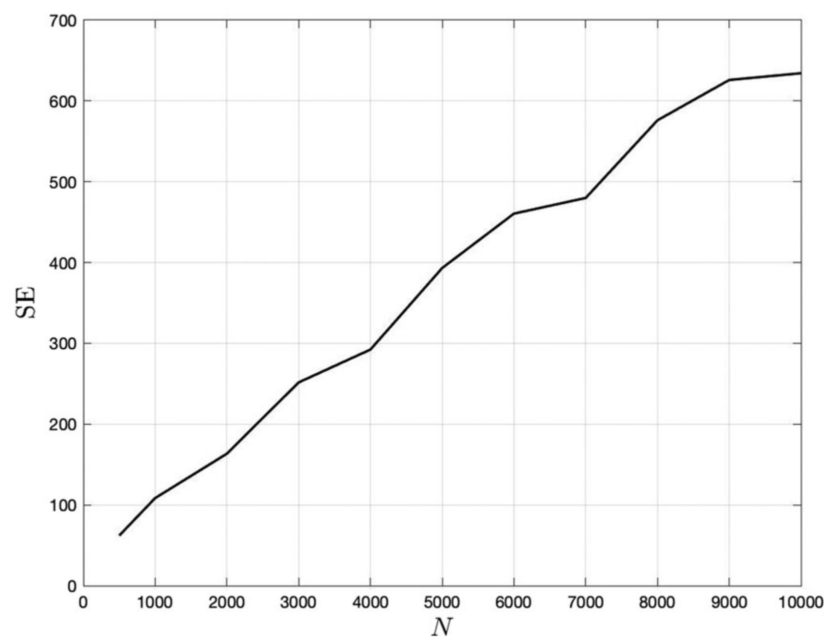


FIGURE 13. SE versus sample length, $\{\tilde{p}_t^e\}$ series. Baseline model with $\tilde{\lambda}_{t+1} = \lambda_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_{t+1}\} \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon = 0.00712$. Computations based on $J = 1000$ independent runs for each $N$.

what advantage such tests would offer beyond existing well-understood tests for unit roots. A related issue concerns the consequences for detrending. There are many detrending procedures for macroeconomic time series, the Hodrick–Prescott filter being one popular example. Detrended time series will have well-defined *ACTs*. The precise *ACT* values will, however, depend on the choice of the detrending methodology. Systematic differences in *ACT* values that may exist across detrending procedures are presently unresearched.

Another approach, especially suited to nonstationary stochastic production models is to normalize the "offending series." To illustrate, suppose the model in (6) were generalized to have a production technology of the form $k_t^\alpha(n_t\tilde{p}_t)^{1-\alpha}$ where $n_t$ is labor supplied, $0 \le n_t \le 1$, and $\{\tilde{p}_t\}$ represent the stochastic labor productivity measure, assumed to follow the process $\tilde{p}_{t+1} = \tilde{x}_{t+1}p_t$ where $\{\tilde{x}_t\}$ is level stationary with $E(\tilde{x}_t) > 1$. All series can be made stationary by normalizing them through division by $\{\tilde{p}_t\}$. For example, define $\hat{k}_t = k_t/p_t$. This normalized series will have well-defined *ACTs*. See King et al. (1988a, 1988b). Such a procedure need only be defined relative to $\{\tilde{p}_t^e\}$ (the return series are stationary in any event).

We leave the extensive explorations necessary to resolve these questions to future work.

### 6.5 *ACTs and impulse response functions*

It remains to connect the *ACT* measure to the notion of an impulse response function. The latter represents the evolution of a variable of interest, which has been subjected to a one standard deviation shock at time $t = 0$. Intuitively, we would expect stochastic processes with more drawn-out impulse response functions to have higher *ACTs*: the more slowly a process tends to return to its mean value, the more time it must spend above or below it, and hence, on average, its *ACT* should be higher. For all stationary processes considered in the present discussion, this turns out to be the case.

We choose to measure the "speed" by which a process returns to its mean by its half-life, the time horizon necessary for the process to halve its distance from its mean when perturbed away from it. We expect to find that stochastic processes with greater *ACTs* have more drawn-out impulse response functions; equivalently their half-lives are greater. In the case of the AR-1 process, if at some time $\hat{t}$, $x_{\hat{t}} = \sigma_\varepsilon$ with $\varepsilon_t = 0$ for $t > \hat{t}$, then the half-life of the process, $h$, is defined to satisfy

$$\begin{aligned} &x_{t+h} = \sigma_\varepsilon/2. \text{ Since } x_{t+h} = \rho^h x_t = \rho^h\sigma_\varepsilon, h \text{ must satisfy} \\ &x_{t+h} = \sigma_\varepsilon/2 = \rho^h\sigma_\varepsilon, \quad \text{or} \quad h = \log(0.5)/\log(\rho). \end{aligned} \tag{26}$$

In this very specialized case, with $ACT(\rho)$ defined by (24), it is evident that an increase in $\rho$ leads both to a greater half-life and a higher *ACT*. Thus a higher *ACT* and a more drawn-out impulse response go hand-in-hand.

The same relationship holds for the baseline model. Here, the key expression describes the evolution of the capital stock, $\tilde{k}_{t+1} = \alpha\beta k_t^\alpha e^{\tilde{\lambda}_t}$. Equivalently,

$$\tilde{p}_{t+1}^e = \alpha\beta\left(p_t^e\right)^\alpha e^{\tilde{\lambda}_t}. \tag{27}$$

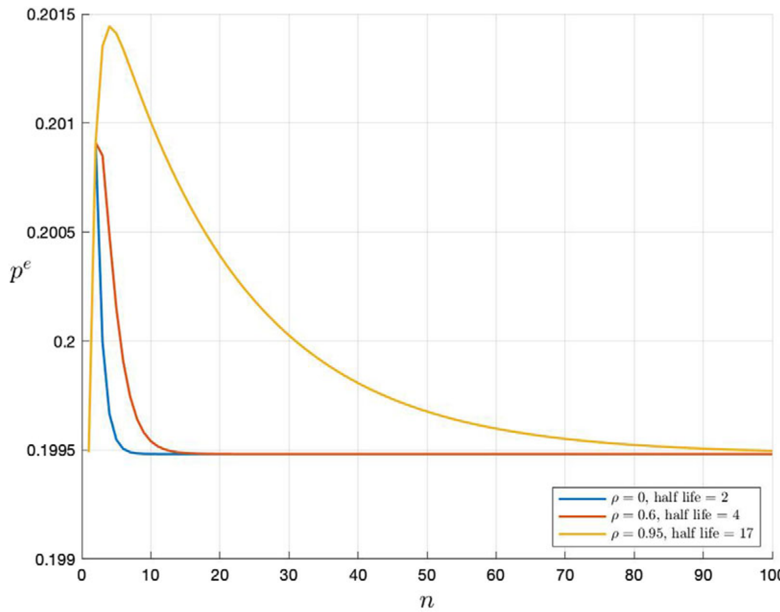


FIGURE 14. Impulse response to a one SD shock away from the mean for $\{p_t^e\}$, Baseline model, various shock autocorrelations.

The steady state capital stock is defined by $k^* = \alpha\beta(k^*)^\alpha e^{0.5\sigma_\varepsilon^2}$ implying $k^* = (\alpha\beta e^{0.5\sigma_\varepsilon^2})^{1/(1-\alpha)}$. Analagously, the steady state equity price is $(p^e)^* = (\alpha\beta e^{0.5\sigma_\varepsilon^2})^{1/(1-\alpha)}$. We subject the equity price $(p^e)^*$ to a one standard deviation shock $\lambda_t = e^{\sigma_\varepsilon}$ and then allow it to evolve in an environment where $\tilde{\varepsilon}_s = 0$ for all $s > t$. The half-life of this impulse to the equity price is defined as that $h$ for which $(p_{t+h}^e - (p^e)^*) = 0.5((p^e)^* e^{\sigma_\varepsilon} - (p^e)^*)$. Figure 14 displays the impulse response function for $\{\tilde{p}_t^e\}$ and its associated half-life for a variety of autocorrelations. It is apparent that the half-life increases as $\rho$ increases. Since the *ACT* also increases with $\rho$, it follows that there is a monotonic relationship between *ACT* and the half-life. Higher *ACTs* and more drawn-out impulse response functions go hand-in- hand. This relationship extends beyond the $\{\tilde{p}_t^e\}$ series to the other financial series discussed in this paper.

## 7. Further generalizations the baseline model

In this section, we explore a number of generalizations of the baseline model such as incomplete depreciation, greater risk aversion, etc. as regards their effects on the financial time series of interest. Generally speaking, these features only serve to increase Property I and II aversion. We substantiate this assertion in Appendix A.

## 8. Empirical support

Table 6 presents the ACTs and autocorrelations for a representative collection of U.S. financial return time series. They are: AGG (Barclays Aggregate Bond Fund), EEM (MSCI

Table 6. Average crossing times, autocorrelations and conditional probabilities: various financial return series.

| Series[(i)] | Data period | $ACT$ | $ACT^A$ | $ACT^B$ | $\rho$ | $\phi^{AA}$ | $\phi^{BB}$ |
|---|---|---|---|---|---|---|---|
| | | | Panel A: Quarterly frequency | | | | |
| SPY | 2.1993–4.2017[(ii)] | 2.3 | 2.61 | 2 | 0.092 | 0.62 | 0.5 |
| EEM | 3.2003–4.2017 | 2.23 | 2.31 | 2.15 | 0.207 | 0.57 | 0.54 |
| AGG | 4.2003–4.2017 | 1.9 | 1.87 | 1.93 | −0.21 | 0.46 | 0.48 |
| TIP | 1.2004–4.2017 | 1.87 | 1.87 | 1.87 | 0.05[(iii)] | 0.46 | 0.46 |
| VNQ | 4.2004–4.2017 | 2.30 | 2.25 | 2.36 | 0.15 | 0.56 | 0.57 |
| GLD | 1.2005–4.2017 | 1.86 | 1.86 | 1.86 | 0.04[(iii)] | 0.46 | 0.46 |
| VTI | 3.2001–4.2017 | 2.13 | 2.53 | 1.75 | 0.12 | 0.61 | 0.43 |
| USO | 3.2006–4.2017 | 2 | 2.27 | 1.75 | 0.14 | 0.56 | 0.43 |
| VIX | 2.1990–4.2017 | 1.82 | 1.43 | 2.19 | −0.31 | 0.30 | 0.54 |
| | | | Panel B: Monthly frequency | | | | |
| SPY | 02.1993–12.2017[(iv)] | 2.02 | 2.26 | 1.78 | 0.068 | 0.56 | 0.44 |
| EEM | 05.2003–12.2017 | 2 | 2.05 | 1.95 | 0.12 | 0.51 | 0.49 |
| AGG | 10.2003–12.2017 | 2.16 | 2.21 | 2.13 | 0.04 | 0.55 | 0.53 |
| TIP | 01.2004–12.2017 | 1.78 | 1.77 | 1.81 | −0.03 | 0.43 | 0.45 |
| VNQ | 10.2004–12.2017 | 1.96 | 2.10 | 1.82 | 0.05[(iii)] | 0.52 | 0.45 |
| GLD | 12.2004–12.2017 | 1.99 | 1.97 | 2 | −0.11 | 0.49 | 0.5 |
| VTI | 06.2001–12.2017 | 2.09 | 2.40 | 1.79 | 0.16 | 0.58 | 0.44 |
| USO | 05.2006–12.2017 | 2.09 | 2.18 | 2 | 0.29 | 0.54 | 0.5 |
| VIX | 02.1990–12.2017 | 1.73 | 1.47 | 2 | −0.16 | 0.32 | 0.5 |
| | | | Panel C: Daily frequency | | | | |
| SPY | 02.01.1993–12.29.2017[(vi)] | 1.92 | 1.97 | 1.87 | −0.06 | 0.49 | 0.46 |
| EEM | 04.14.2003–12.29.2017 | 1.95 | 2.02 | 1.88 | −0.10 | 0.51 | 0.47 |
| AGG | 09.29.2003–12.29.2017 | 1.85 | 1.89 | 1.81 | −0.12 | 0.47 | 0.45 |
| TIP | 12.08.2003–12.29.2017 | 1.96 | 1.98 | 1.96 | 0.01[(iii)] | 0.50 | 0.49 |
| VNQ | 09.30.2004–12.29.2017 | 1.96 | 2.00 | 1.91 | −0.18 | 0.50 | 0.47 |
| GLD | 11.19.2004–12.29.2017 | 1.92 | 1.96 | 1.87 | −0.02 | 0.49 | 0.47 |
| VTI | 06.01.2001–12.29.2017 | 1.95 | 2.04 | 1.85 | −0.06 | 0.51 | 0.46 |
| USO | 04.11.2006–12.29.2017 | 1.93 | 1.96 | 1.89 | −0.05 | 0.49 | 0.47 |
| VIX | 01.03.1990–12.29.2017 | 1.99 | 1.81 | 2.15 | −0.08 | 0.45 | 0.54 |

(i) The series corresponding to these abbreviations are found in the text.
(ii) The notation 2.1993 indicates the second quarter of 1993, etc.
(iii) Indicates departures from theory.
(iv) The notation 02.1993, signifies the 2nd month of 2003, etc.
(vi) 02.01.1993 is to be read as February 1, 1993, etc.

Emerging Markets Index ETF), SPY (SPDR S&P 500 ETF), GLD (SPDR Gold Shares), USO (United States Oil Fund ETF), TIP (Barclays TIPS Bond Fund ETF), VTI (Vanguard Total Stock Market EFT), VNQ (Vanguard Real Estate EFT), and VIX (CBOE Volatility Index). We provide the data at daily, monthly, and quarterly frequencies. The quarterly series best corresponds to the implicit time period in the model economies. Nevertheless, all three frequencies should generate ACT and autocorrelation relationships generally in accordance with Proposition 4.2 and Observations A–F.

There are a number of relevant observations. First, note that at quarterly frequencies (Panel A), the majority (5 out of 9) of return series are Property I mean averting, as was the case for all the models of Section 5, when they were subject to empirically relevant productivity autocorrelations. Unlike the present models, empirical bond returns (the AGG series) are mean reverting; this is also true of the VIX although it has no counterpart in the model-generated return series. In contrast, all series but one are Property I mean reverting at daily frequency; the sole exception (the TIPS series) being essentially independently distributed through time (note that $\phi^{AA} = \phi^{BB}$). Nothing here is surprising; it is to be expected that the underlying processes governing daily returns, whatever they are, will be largely unrelated to an economy's aggregate investment and consumption processes.

By Proposition 4.2, if $ACT \leq 2$, Property I mean reversion should be observed. At daily frequencies, this is the case for all series except for TIPs. For monthly and quarterly series, the exceptions are TIPs and GLD (quarterly) and VNG (monthly). Due to the relative lack of data at quarterly frequencies, it is not entirely surprising that the greatest number of inconsistencies are found there. At all frequencies the VIX series strongly endorses the theory.

Note that at daily frequencies all the corresponding entries in the transition matrices are close to independence with the VIX the possible exception. At monthly and quarterly frequencies, however, this is generally not so, with the VIX return series again being the most asymmetric in both cases; the VTI is the next most extreme in this regard. In particular, at quarterly frequencies, the VIX has a 0.30 probability ($\phi^{AA} = 0.3$) of remaining in the above-mean state, while only experiencing a mildly less-than-even chance of returning to it ($1 - \phi^{BB} = 0.46$). At quarterly frequencies, the VTI series remains in the above-mean state with probability 0.61 and returns to it only with probability 0.57. As a result, it has a high $ACT^A$. By comparison, the $ACT$ for the VIX is the lowest of all the series for all data frequencies. We leave the rationale for this pattern to those more familiar with its underlying determination.

Observation E makes clear that there is no necessary positive association between an increasing $ACT$ and an increasing autocorrelation across the series, and we do not observe it in the data. With $ACT$s around 2 (all of our series), the range of ($ACT$, correlation) possibilities is large (see Figures 1 and 2). Nevertheless, there is a weak positive association between a series' $ACT$ and its autocorrelation.

## 9. Conclusion

In this paper, we have argued that notions of mean reversion and mean aversion can be synthesized under one metric, the Average Crossing Time ($ACT$) with $ACT^A$ and $ACT^B$ as its underlying constituents. By the $ACT$ measure, the mean aversion/reversion distinction becomes largely artificial, with a mean reverting (stationary) process being identified only by a finite $ACT$ value. One may model mean averting processes as those with larger $ACT$s since there is nothing in the $ACT$ concept that specifies a mean reversion/aversion demarcation value. The $ACT$ concept does provide, however, a simple, intuitive sense of one time series being more strongly mean reverting than another: its $ACT^A$ and $ACT^B$ are each lower than its comparison counterparts.

As an identifying measure, the $ACT$ allows us to evaluate other time series characteristics that have been "traditional identifiers" of "mean reversion." We considered four of these, classifying them as Properties I–IV. Properties I and II were shown to be satisfied in the case of the $ACT$ being less than or equal to two, which strikes us as an extremely strong criterion for "mean reversion." Most of the analysis in the paper concerns these properties, as they are the most widely employed. A careful analysis of Property III is left to future work, for two reasons. First it does not discriminate in any way for the canonical "mean reverting" AR(1) process across autocorrelation parameters. Second, it is not often employed in the finance literature. The same should be said for Property IV; it is very restrictive, not being exactly satisfied in the traditional stationary models such as those reviewed in Section 5, or in general macroeconomic data.

Is there any real mean reversion/aversion distinction regarding stationary time series? Our analysis suggests the distinction is somewhat arbitrary, at least for Properties I and II.

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